

**Symmetric Extensions and Spectral Properties of the
Composite of Higher Order Differential and Difference
Operators on Hilbert Spaces**

Boaz Okoth Okello

A Thesis submitted in fulfilment for the requirements of the award of The Degree
of Doctor of Philosophy in Pure Mathematics, School of Science and Aerospace
Studies, Department of Mathematics, Physics and Computing.

Moi University.

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DECLARATION

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Boaz Okoth okello,
PHD/PM/5546/21

Signature:



Date: 15/08/2025

APPROVAL

This thesis has been submitted for examination with our approval as the University supervisors.

Prof. Dr.rer.nat Fredrick Oluoch Nyamwala,
Department of Mathematics, Physics and Computing,
Moi University.

Signature:



Date: 18/08/2025

Dr. David Otieno Ambogo,
Department of Pure and Applied Mathematics,
Maseno University.

Signature:



Date: 18/08/2025

ABSTRACT

The spectral theory of symmetric differential and difference equations with almost constant coefficients has been studied. The deficiency indices and spectral theory of commuting differential operators with bounded coefficients have been done extensively. However, the construction of symmetric composites and the spectral analysis of their self-adjoint extensions are lacking in the case of unbounded coefficients. In addition, this is lacking for the difference operators. This study extended the theory of the composites of differential operators with bounded coefficients to unbounded situations in addition to discrete settings. The main goal of this research was to analyze the necessary and sufficient conditions for the existence of symmetric self-adjoint operator extensions of the composites for differential and difference operators with unbounded coefficients. The specific objectives were to: Construct commutative symmetric composites of differential and difference operators, analyze the necessary and sufficient conditions for the existence of symmetric self-adjoint extensions of the composites, evaluate the deficiency indices of the self-adjoint operator extensions of the composites, and examine the spectral properties of the self-adjoint operator extensions of the composites. Commutative symmetric composites were obtained by constructing appropriate comparison algebras for a specific self-adjoint operator. Asymptotic integration and summation based on Levinson's theorems were applied to achieve the optimum growth and decay conditions on the coefficients for the existence of self-adjoint operator extensions, then square integrability and summability of the eigen solutions were determined using the inner products and the number of square-integrable and summable solutions resulted into the deficiency indices. Finally, the M-matrix was used to determine the location of the spectrum and their spectral multiplicities. If T_1 and T_2 are symmetric operators of the same order belonging to the same group of comparison algebra, then T_1 commutes with T_2 whenever the first derivative or difference of the coefficients tends to zero in the limiting sense even if the coefficients are unbounded. The deficiency indices for the $4n$ th order composite T_1T_2 lies between $2n$ and $4n$ both in the upper and lower half planes whenever f is unbounded and f'' or $\Delta^2 f$ is either integrable or summable respectively. The spectrum of the self-adjoint extension of the composite T_1T_2 is pure discrete whenever the eigen solutions are uniformly square-integrable or summable. However, absolutely continuous spectrum which is the whole real line was obtained whenever some solutions lost their square integrability or summability in the case of unbounded coefficients. The spectral multiplicity was equal to the number of solutions that lost their square integrability or summability. The achieved results are applicable in the film industries where a certain quality of image or sound is to be achieved. This purely depends on the wavelengths of the components of the white light in the electromagnetic spectrum which translates to the nature of the eigen solutions that form the spectrum. Discrete spectrum implied images of high precision. In the future, research in designing programs that would assist in analysing spectral properties of operators of order six and above is highly recommended.

DEDICATION

I dedicate this research to my wife Vallary Ogada (*min Japuonj*), sons Travis, Taniel and Tylan, dad William Okello and mum Margaret Akumu.

ACKNOWLEDGEMENT

I express my sincere gratitude to my first supervisor Prof. Dr.rer.nat. Fredrick Oluoch Nyamwala for his insightful comments, suggestions and encouragement at every stage of this research, immense knowledge, plentiful experience and above all for his patience with me during my PhD study. On the same breadth, I extend my deepest gratitude to my second supervisor Dr. David Otieno Ambogo for his invaluable advice, continuous support and constructive criticism during this research. It would have not been possible to have this dissertation in its current form without your kind supervision.

My special thanks to Deutscher Akademischer Austausch Dienst (DAAD) and Moi University for the funding opportunity under the ICIR scholarship and availing the necessary resources to undertake my studies. I equally thank all the staff members of the School of Sciences and Aerospace Studies for their treasured support which was really influential in critiquing my results.

I am deeply indebted to the Okello's family(Amor kodu Nyikwa Ger) especially my spouse Vallary, sons Travis and Taniel, my brothers and sisters for their tremendous understanding, support, encouragement and belief in me that has kept my spirits and motivation high in the past few years.

I also appreciate all the support I received from the staff members of Raliew Secondary School as well as my friends and colleagues especially Benson Ochom for the cherished times spent together in social settings and his ever willing nature to offer me his support at all times.

Lastly, it would be a remiss in not mentioning my classmates and cohort members especially Benard Owino for his moral support and encouragement during this process. I would also like to thank everybody who contributed in a way or the other to this research and have not been mentioned. Glory and thanks to the Almighty God.

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LIST OF ABBREVIATIONS AND ACRONYMS

τ	symmetric differential operator	12
$\mathcal{L}^2$	Space of square integrable sequences	12
z	spectral parameter	13
$\mathcal{J}$	symplectic matrix	14
$\mathcal{H}$	Hilbert space	14
T_r^*	maximal differential operator .	15
T	minimal differential operator .	15
ℓ^2	Space of square summable sequences	16
Δ	forward difference operator . .	16
$\mathcal{L}$	difference operator	16
L^*	maximal difference operator .	19
L	minimal difference operator . .	20
$\mathcal{N}_{L^*-\bar{z}}$	null space of $L^* - \bar{z}I$. . .	20
$\mathcal{N}_{L^*-z}$	null space of $L^* - zI$. . .	20
$\sigma(T)$	spectrum of operator T . . .	22
$\mathcal{L}^1$	Space of Integrable sequences .	62
$M(z)$	M-matrix	68
$o(\cdot)$	Landau symbol 'little-o' . . .	76
$O(\cdot)$	Landau symbol 'big-O' . . .	78
$\sigma_{ac}H$	Absolutely continuous spectrum of H	84

CHAPTER ONE

INTRODUCTION

In this chapter, we have given a background of our area of study, definitions of terms used, statement of the problem, objectives and the significance of the study.

1.1 Mathematical background

The Spectral theory of operators in a finite-dimensional space first appeared in connection with the description of the frequencies of small vibrations of mechanical systems (Arnol'd, 1985). It is noted that a simple eigenvalue problem for a differential operator arises when the vibrations of a string are considered. For a homogeneous string, the use of classical theory of Fourier series is sufficient. On the other hand, in the case of an inhomogeneous string, it is necessary to consider the general Sturm-Liouville problem, which is the eigenvalue problem for a simple one-dimensional differential operator with variable coefficients. In cases where the eigenvalue problem fails to be explicitly solvable, a qualitative and asymptotic study is employed (Egorov & Shubin, 1988).

One of the main sources of problems in spectral theory, mostly for the *Schrödinger* operators, is the quantum mechanics, in which the eigenvalues of the quantum Hamiltonian, are the possible eigenvalues of the system. Mehrabov (Mehrabov, 2022) studied the spectral properties of one eigenvalue problem for the fourth order ordinary differential equation with a spectral parameter contained in two of the

boundary conditions. In their study, it was noted that the spectral problem arises when the Fourier method is applied to a boundary value problem for the partial differential equations describing small bending vibrations of a homogeneous rod under the action of a longitudinal force in cross sections. They established the arrangement of the eigenvalues on the real axis, determined the multiplicities of the eigenvalues and studied the asymptotic behavior of the eigenvalues and eigenfunctions for this problem. Additionally, they established the necessary and sufficient conditions under which the system of root functions with two removed functions is a basis in the space of integrable functions.

The work by Kaper, Kwong, and Zettl, (Kaper, Kwong, & Zettl, 1986) addressed the problem of characterizing self-adjoint realizations of singular Sturm–Liouville differential operators, with their main focus on the Friedrichs extension. It is worth noting that the Friedrichs extension is a distinguished self-adjoint realization of an operator that arises naturally in variational problems. The authors provided multiple equivalent characterizations of this extension, which are more concrete than those in classical Weyl–Titchmarsh theory. They provided explicit and practical criteria for identifying the Friedrichs extension, which are easier to apply than the abstract conditions in Weyl’s or Titchmarsh’s theories. Their analysis mainly focussed on bounded coefficients and the singularity is of limit-circle type which are common in practice. In our study, we have extended this study to unbounded coefficients and even to limit-point cases to broaden the applicability.

Krall (Krall & Krall, 1986) explored more on bounded linear operators on Hilbert spaces in depth, building on foundational concepts introduced earlier, while introducing new tools for operator analysis, especially the integral representations. The author employed the spectral theory to decompose operators, via projection-valued measures and eigenfunction expansions and provided a proof of the spectral theorem for self-adjoint or normal operators, showing that they can be diagonalized via integrals over their spectra. Moreover, the author compared different modes of operator convergence such as norm convergence, strong convergence and the weak convergence, and their implications for limits of sequences of operators. In our work

we have extended the work by Krall to operators with unbounded coefficients and to the composites of differential and difference operators.

Marletta and Zettl, (Marletta & Zettl, 2000) studied the characterization of the Friedrichs extension for singular ordinary differential operators of order $2n$ in which they generalized the Friedrichs extension to singular $2n - th$ order differential operators, building on classical results for second-order Sturm-Liouville and regular cases. They gave the characterization in terms of principal solutions at singular endpoints, which are solutions of the differential equation with specific boundary behavior. Moreover, the authors employed Reid's theory of principal solutions for Hamiltonian systems to construct boundary conditions for the Friedrichs extension and showed that these solutions are analytic in the spectral parameter. Additionally, for the singular endpoints, the authors determined the Friedrichs extension by requiring that functions in its domain are orthogonal (in a Lagrange bracket sense) to principal solutions at each singular endpoint.

Aliev and Namazov (Z. S. Aliev & Namazov, 2020) carried out a study on the spectral problem for the ordinary differential equation of fourth order with a spectral parameter in the boundary conditions. In their study, they investigated the locations and the multiplicities of eigenvalues, studied the oscillatory properties of the eigenfunctions as well as establishing the necessary and sufficient conditions for subsystems of root functions of the ordinary differential equation under study to form a basis in the space of integrable functions.

Aliev et al (Z. Aliev & Dun'yamalieva, 2015) in their study on the defect basis property of a system of root functions of Sturm-Liouville problem with spectral parameter in the boundary conditions, investigated the spectral properties of Sturm-Liouville problem where the spectral parameter appears not only in the differential equation but also in the boundary conditions. They employed asymptotic analysis to derive the behavior of eigenvalues and eigenfunctions for a large eigenvalue. In addition, they applied the basis theory to investigate the completeness and defect basis of the root functions.

On the existence of commuting self-adjoint extensions, Slinker (Slinker, 1978) conducted a research to discuss the problem of finding commuting self-adjoint extensions of symmetric, but not necessarily self-adjoint, unbounded operators. With the motivation by quantum mechanics, where observables are represented by self-adjoint operators, and commuting operators corresponding to simultaneously measurable quantities, they investigated the conditions under which two symmetric unbounded operators admit commuting self-adjoint extensions. They employed the von Neumann's extension theory for symmetric operators to study commuting extensions. Their results showed that for the operators to admit commuting self-adjoint extensions, then the operators must strongly commute in some sense, that is, their resolvents commute. Moreover, if the operators have equal deficiency indices, then under certain conditions, they admit commuting self-adjoint extensions.

Levitan and Sargsjan (Levitan & Sargjan, 1975) carried out a study on the spectral theory of self-adjoint ordinary differential operators, particularly the Sturm-Liouville operator and the Dirac system, while also touching on higher-order ordinary differential operators. They provided a detailed spectral analysis of two central self-adjoint differential operators that is the Sturm-Liouville operator in which they considered a second-order ordinary differential operators of the form $L_y = -(py')'$ and the Dirac system in which they considered a first-order symmetric system arising in quantum mechanics. In their study, they briefly addressed n th-order ordinary differential operators, in which they acknowledged the complexity and breadth of this field. Therefore, their treatment served as an introductory overview rather than an exhaustive study.

Hinton and Lewis, (D. B. Hinton & Lewis, 1978) carried out a comprehensive study of second-order difference equations, in which they focused on their spectral properties, oscillation behavior, and classification into limit-point case or limit-circle case. They associated a linear operator T in the Hilbert space H with the difference equation and analyzed its spectral properties. They also defined the minimal and maximal operators T_0 and T and discussed their self-adjoint extensions. Moreover,

they established connections between the oscillation of solutions, the spectrum of T , and the limit point and limit circle classification.

Later on Hinton and Shaw, (D. Hinton & Shaw, 1986) investigated the nature of the spectrum, in particular the absolute continuous spectrum, of second-order differential operators with various types of potentials. They determined the conditions under which the continuous spectrum of second-order differential operators is absolutely continuous which is contrary to singular continuous or pure point spectrum. The authors also showed that the spectrum is bounded below for a certain class of equations. In their work, the authors showed that, the absolutely continuous spectrum is identified as either a ray $[a, \infty)$ if the operator is bounded below or the entire real line $\mathbb{R}$ if the operator is unbounded. Their study included operators defined on intervals with one or two singular endpoints such as, the Schrödinger operators on $(0, \infty)$ or $(-\infty, \infty)$.

A study carried out by Stephen L. Clark (S. L. Clark, 1990) investigated the conditions under which the continuous spectrum of a self-adjoint operator associated with a Hamiltonian system of ordinary differential equations is absolutely continuous. The work by Clark was an extension of the previous results by Atkinson, Hinton, and Shaw, in which he focussed on higher-order systems and provided a new criteria for absolute continuity and continuous differentiability of the spectrum. The author applied the properties of matrix-valued Titchmarsh–Weyl $M(\lambda)$ functions to analyze the spectrum in which he established the connections between the spectral measure $\rho(\mu)$ and the asymptotic behavior of solutions. Moreover, the author applied the results to a second-order system with matrix-valued coefficients and showed that the spectrum is continuously differentiable on the positive half-axis.

Elsewhere, Clark and Hinton (S. Clark & Hinton, 1993) extended the Gilbert-Pearson theory of subordinacy (which relates the subordinacy of solutions of Sturm-Liouville equations to the nature of the spectrum) to generalized Sturm-Liouville systems and introduced a stricter condition called strong nonsubordinacy, which helps in establishing stronger conclusions about the spectral function, especially its

Lipschitz properties. Additionally, they derived the sufficient conditions under which the spectral function satisfies upper and lower Lipschitz bounds, which implied the presence of absolutely continuous spectrum and also investigated a specific class of Sturm-Liouville equations that satisfy strong nonsubordinacy, demonstrating the applicability of the theoretical results.

Jan Lanke, (Lanke, 1961) carried out a study on the asymptotic distribution of eigenvalues of certain one-dimensional Sturm-Liouville operators in which he derived asymptotic formulas for the distribution of eigenvalues of one-dimensional Sturm-Liouville operators of the form $A(f) = -f'' - a(x)f$, where $a(x)$ is a real-valued potential. In his study, he established an asymptotic relation for $N(\lambda)$, the number of eigenvalues below λ as $\lambda \rightarrow -\infty$. They also generalized the previous results by extending the earlier work of Wet and Mandl to potentials $a(x)$ with slower decay rates and generalized their results to differential operators of higher orders.

The authors in (Kwong & Zettl, 1981) carried out a study on Discreteness conditions for the spectrum of ordinary differential operators. In their study, they investigated the sufficient conditions under which the spectrum of self-adjoint extensions of minimal operators associated with ordinary differential expressions is discrete. They provided sufficient conditions which involved the limits of integrals over intervals whose lengths may vary with t , controlled by a function $f(t)$ in determining when the spectrum of self-adjoint extensions of minimal operators generated by the differential expressions is discrete, that is when the essential spectrum is empty. It is worth noting that their work made significant advances in the spectral theory of ordinary differential operators by providing the necessary and sufficient conditions for the discreteness of spectra. Their innovative use of weighted norm inequalities and their ability to handle general expressions with middle terms are notable strengths.

Orochko (Orochko, 1994) investigated the deficiency indices of minimal symmetric differential operators associated with a specific differential expression on the real line. The author analyzed the deficiency index of the minimal symmetric operator generated by the differential equation and established that the deficiency index

can be any nonnegative integer, including infinity, depending on the behavior of coefficients of the differential equations. moreover, the author constructs explicit examples leading to different deficiency indices, including, finite deficiency indices (any nonnegative integer) and infinite deficiency indices which are particularly important for understanding non-trivial extensions of symmetric operators.

Brown and the authors in (B. M. Brown, McCormack, Evans, & Plum, 1999) extended Sims' results by considering more general forms of second-order differential operators with complex coefficients, including cases where the coefficients are not necessarily purely imaginary or negative. They introduced an m -function, analogous to the Titchmarsh-Weyl function, that is either unique, that is, the limit-point case or lies on a "limit-circle." They also analyzed the spectrum of the differential operators in which they showed that it lies within a closed convex set $Q(\alpha)$, which depends on boundary conditions parameterized by α . Additionally, they analyzed the essential spectrum, with results depending on the limit behavior of the coefficients at the singular endpoint.

Littlejohn and Lopez (Littlejohn & López, 2010) presented a method for constructing solutions to composite products of linear differential equations using Green's functions. The authors tackled the problem of finding a basis of solutions for the composite product of linear differential equations, given a basis of solutions for each factor equation. They employed the Green's functions associated with each differential operator to construct new solutions iteratively in which they provided a systematic way to construct a basis for the composite equation by integrating known solutions against Green's functions. Moreover, they gave numerous examples to illustrate the construction of solutions involving special functions such as the Bessel, Airy, Legendre functions and their integrals.

The authors in (Behncke & Hinton, 2005) investigated the spectral properties of a general third-order formally symmetric differential operator acting in a weighted Hilbert space $\mathcal{L}_w^2(a, \infty)$. Their focus was on operators defined on the half-line $[a, \infty)$, which introduces challenges due to the unbounded domain. They computed the de-

deficiency indices of the minimal operator L_0 associated with the differential operator L , which determines the number of independent boundary conditions needed for self-adjoint extensions. The authors applied the Kummer–Liouville Transformation which is analogous to the Liouville transformation for second-order operators, that converts the differential operator to a simpler form thereby reducing the problem to a more tractable differential operator while preserving spectral properties. Additionally, the authors provided a distinction between extensions with discrete spectra and those with continuous spectra. Moreover, they demonstrated that third-order operators can exhibit unbounded-below discrete spectra, unlike second-order cases where spectra are typically bounded below.

Behncke, (Behncke, 2006) investigated the absolutely continuous spectrum of a broad class of higher-order differential operators of order $2n$ under conditions that balance smoothness and decay of the coefficients. He studied self-adjoint differential operators of even order with coefficients that satisfy the decay and smoothness conditions in which he characterized the absolutely continuous spectrum and rule out the existence of the singular continuous spectrum under certain conditions. In his study, Behncke explicitly determined the absolutely continuous spectrum based on the asymptotic behavior of coefficients and noted that, if the coefficients decay to a periodic limit, then the absolutely continuous spectrum may resemble that of a periodic operator and that if the coefficients decay to constants, then the absolutely continuous spectrum may be a half-line.

In 2008, Behncke and Hinton (Behncke & Hinton, 2008) carried out a study on the spectral analysis of odd-order differential operators, particularly on their eigenfunction asymptotics, deficiency indices, and spectral properties. In their work, the authors determined the asymptotic behavior of eigenfunctions for a broad class of odd-order differential operators on the half-line, that is, $[0, \infty)$ and extended these results to operators defined on the entire real line ($\mathbb{R}$). They also used the derived eigenfunction asymptotics to compute the deficiency indices of the associated operators, which forms a basis for understanding the self-adjoint extensions of these operators. Additionally, they determined the spectra of the self-adjoint extensions

whenever possible and showed that the form of the eigenfunctions rules out the presence of singular continuous spectrum in all cases.

Zheng and Zhang (Zheng & Zhang, 2012) investigated the spectral properties of n -th order singular differential operators, focusing on the distribution of eigenvalues within spectral gaps. In their study, they established that for a minimal operator T_0 with a spectral gap (a, b) and deficiency index r , any finite set of points $\{\beta_i\}$ in (a, b) can be realized as eigenvalues of a self-adjoint extension $\tilde{T}$ of the minimal operator T_0 and that the multiplicities of these eigenvalues are constrained by the deficiency index r . The authors also provided explicit constructions of self-adjoint extensions of T_0 that introduces prescribed eigenvalues within the spectral gap thereby extending the classical results by Krein, Friedrichs, and Neumann, which were limited to regular or semi-bounded operators. Moreover, they showed that under certain conditions, the self-adjoint extensions have a pure point spectrum within (a, b) , meaning no continuous spectrum appears in this interval.

Gao and the other authors in (Gao, Wang, Yao, & Cao, 2024) investigated the spectral properties of a specific class of difference operators with spectral parameters in the boundary conditions. They studied the eigenvalues of a second-order left-definite difference operator where the boundary conditions contain linear spectral parameters and determined the exact number of eigenvalues and their distribution whether positive or negative. The authors proved that all eigenvalues are real and simple, that is, they are non-degenerate. They also analyzed how the number of positive/negative eigenvalues depends on the weight function (sign distribution of its elements) and the parameters in the boundary conditions. Their results showed that eigenvalues alternate under certain relationships between boundary parameters, similar to classical Sturm-Liouville theory, for example, if one parameter is varied, the eigenvalues may shift in a predictable, ordered manner.

A study on commutativity of unbounded normal and self-adjoint operators and applications by Mortad (Mortad, 2013) explored the conditions under which unbounded normal and self-adjoint operators commute, with a focus on strong com-

mutativity and its implications. They proved that if A and B are unbounded normal operators with $AB \subset BA$ and that B is invertible, then BA and $\overline{AB}$ are normal. Moreover, they showed that if $AB = BA$ and B is invertible, then A and B strongly commute. In addition, they generalized results by Devinatz, Nussbaum, and von Neumann on strong commutativity to broader classes of operators such as the invertible or semi-bounded operators.

In their recent study, Dehimi and Mortad (Dehimi & Mortad, 2023) presented significant results concerning the properties of unbounded operators, with their main focus being on those whose powers are self-adjoint, subnormal, or hyponormal. They Investigated the conditions under which a densely defined, closable operator A is necessarily closed, especially when A^2 or higher powers has a nonempty resolvent set in which they established that if A is a densely defined, closable operator such that the resolvent set of A^2 is nonempty, then A is closed. In their work, they pointed out that result is generalized to polynomials $\mathcal{P}(A)$, implying that if $\mathcal{P}(A)$ has a nonempty resolvent set, then A must be closed. Additionally, they established that if A is hyponormal and unbounded and there exist coprime integers p and q such that A^p and A^q are self-adjoint, then A must be self-adjoint.

Dan mu and the co-authors in (Mu, Sun, & Ao, 2015) studied Sturm-Liouville problems that involve coupled boundary conditions and transmission conditions. In their paper, they proved the self-adjointness of the Sturm-Liouville problem in a newly defined Hilbert space which is crucial because self-adjoint operators guarantee real eigenvalues and orthogonal eigenfunctions. The authors also described the asymptotic behavior of the eigenvalues and eigenfunctions of the Sturm-Liouville problem, which is essential while approximating solutions and analyzing their properties, especially for large eigenvalues. Moreover, they provided a necessary condition for determining eigenfunctions when a given parameter is an eigenvalue of the Sturm-Liouville problem with coupled boundary conditions.

Later on, Bairamov, Arpat and others (Bairamov, Arpat, & Mutlu, 2017) analyzed the spectral properties of a Sturm-Liouville operator L in the space $(L^2\mathbb{R}_+, H)$,

where the potential $Q(x)$ is a non-self-adjoint, compact operator in a separable Hilbert space H . They constructed the jost solutions of the differential equation $L(Y) = \lambda^2 Y$, which are useful in scattering theory and spectral analysis. Additionally, they investigated the point spectrum of L under specific conditions on the potential $Q(x)$ determining when λ is an eigenvalue. Moreover, they derived an explicit form for the resolvent of L , which is essential for studying its spectral properties and showed that L has a finite number of spectral singularities, which are points in the continuous spectrum where the resolvent behaves irregularly.

Albrecht and Vasilescu (Albrecht & Vasilescu, 2011), investigated the problem of extending unbounded operators to self-adjoint or normal operators in a Hilbert space setting. They aimed at characterizing the conditions necessary for a given family of unbounded, symmetric, or commuting operators to admit self-adjoint or normal extensions and connecting it to operator-valued moment problems, where a linear map which is defined via polynomial expressions in the operators must be represented by a spectral measure. Their results on necessary and sufficient conditions for linear operator to be representable by a spectral measure on involve positivity and boundedness constraints on polynomial expressions in the operators. Moreover, they derived the sufficient conditions necessary for a tuple $T = (T_1, \dots, T_n)$ of commuting symmetric operators to admit a self-adjoint extension with spectral measure. Their methodology was mainly based on the Naimark's Dilation Theorem which they used to construct spectral measures from positive operator-valued maps.

In his study, Feyrouz (Feyrouz, 2023) explored on the conditions under which unbounded normal and self-adjoint operators commute and the implications of such commutativity. They showed that if A and B are two strongly commuting unbounded normal operators, then their sum, $A+B$ is essentially normal. Furthermore, they showed that if A and B are two normal operators such that A is unbounded, B is bounded and that they λ -commute, that is, $BA = \lambda AB$, then AB is normal if and only if $|\lambda| = 1$. By giving explicit examples, they illustrated cases where λ -commutativity leads to normality only when $|\lambda| = 1$ and also situations where the product of self-adjoint operators fails to be self-adjoint without the given assump-

tions.

In their study on spectral approximations of unbounded operators, Marian and Lopushansky (Dmytryshyn & Lopushansky, 2019) addressed the problem of spectral approximations for unbounded linear operators in Banach spaces. Their focus was on improving Bernstein-Jackson inequalities by explicitly calculating constants on special approximation scales of analytic vectors of finite exponential types, which are generated by unbounded operators. They used subspace theory of analytic vectors of finite exponential types to build on the idea that analytic vectors of finite exponential can be used to approximate elements in the Banach space. Their Idea was rooted in the Nelson's theorem, which connects analytic vectors to spectral properties of operators with discrete spectra and self-adjoint operators. In our work we have extended these results to a more general case of unbounded operators.

1.1.1 Differential Operators

Let τ_1 and τ_2 be $2n$ th and $2m$ th order symmetric differential maps defined on $\mathcal{H} = \mathcal{L}_w^2([0, \infty))$ and given by

$$\begin{aligned}\tau_1 y &= w^{-1} \left(\sum_{k=0}^n (-1)^k (p_k y^{(k)})^{(k)} \right), \\ \tau_2 y &= w^{-1} \left(\sum_{j=0}^m (-1)^j (q_j y^{(j)})^{(j)} \right).\end{aligned}\tag{1.1}$$

We let $f(x) = p_k(x), q_j(x)$, $p_k = p_k(x)$, $q_j = q_j(x)$ which are real valued functions that are continuously differentiable, that is, $f(x) \in \mathcal{C}_0^\infty([0, \infty))$ and $w = w(x) > 0$, for all $x \in [0, \infty)$, is the weight function. Unless otherwise specified, we will assume that $w(x) = 1$ and take $\mathcal{L}^2([0, \infty)) = \mathcal{H}$.

A number of authors have been interested in the necessary and sufficient conditions for the commutativity of the differential functions τ_1 and τ_2 and for more details see the papers by Amitsur (Amitsur, 1958) and Mironov [(Mironov, 2004), (Mironov,

2014)] and the references cited therein. In the paper by Amitsur (Amitsur, 1958) the commutativity property was derived via the associated commutative ring of polynomials related to the differential equations under investigation. On the other hand, Mironov [(Mironov, 2004), (Mironov, 2014)] derived required conditions on the order of differential equations by looking at the genus of the associated polynomials. However, none of these authors was interested in the self-adjoint operator extensions of the minimal operators generated by τ_1 and τ_2 and whether such self-adjoint operator extensions commute whenever τ_1 and τ_2 commute.

H. O Cordes (Cordes, 1987) derived necessary conditions for commutativity of first and second order pseudo-elliptic differential operators by application of comparison algebras. Similar investigations were also done by S. T. Melo (Melo, 1990). However, in all the cases, none of the authors considered the case where the coefficients are unbounded and the commutativity of the associated self-adjoint operator extensions of the respective minimal differential operators. This study is, therefore, an extension of Cordes's work to higher order symmetric differential equations with unbounded coefficients thereby completing the theory in this line.

We therefore begin by constructing the minimal, maximal and self-adjoint operator extensions generated by τ_1 and τ_2 . In order to analyze the spectral properties of the operators generated by τ_1 and τ_2 , we have developed the theory in line with the results of Hinton and Shaw (D. Hinton & Shaw, 1981) which makes it easier for the application of the M -matrix. Thus, we have solved the equations $\tau_r y = zy$, where z is the spectral parameter, $r = 1, 2$. This required that the respective expressions of $\tau_r y = zy$ be converted into first order systems. In this case, we applied the quasi-derivatives as given by P. Walker (Walker, 1974) and J. Weidmann (Weidmann, 1987) that were subsequently applied many times by other authors. For details, see [(Behncke, 2006), (Behncke & Nyamwala, 2012), (M. S. P. Eastham, 1989)] and the references cited therein. The quasi-derivatives are given as follows:

$$y_l = y^{(l)}, \quad 0 \leq l \leq s-1, \quad y_s = f_s y^{(s)}, \quad y_{s+1} = -(y^{(s)})' + f_{s-1} y^{(s-1)} \quad (1.2)$$

$$y_{s+l} = -(y'_{s+l-1}) + p_{s-l} y_{s-l}, \quad 2 \leq l \leq s-1.$$

$$s = m, n, \quad f = p_k, q_j.$$

Using the quasi-derivatives, one obtains a first order system of the form

$$U' = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ C & -\mathcal{A}^* \end{pmatrix} U = \mathcal{C}U. \quad (1.3)$$

The $s \times s$ matrices A , B and C have non-zero entries given by

$$\mathcal{A}_{l,l+1} = 1, \quad \mathcal{B}_{s,s} = f_s^{-1}, \quad C_{l,l} = f_{l-1}, \quad f_0 = f_0 - wz.$$

In this case, $U = U(x, z)$ is a $2s$ -dimensional vector given by

$U = \{y_0, y_1, \dots, y_{2s-1}\}^{tr}$ and $\mathcal{C}$, $2s \times 2s$ matrix as shown above. By introducing a symplectic matrix $\mathcal{J}$ defined by

$$\mathcal{J} = \begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix},$$

the corresponding first order Hamiltonian of (1.3) is thus constructed as

$$\mathcal{J}U' = [z\mathcal{A} + P]U, \quad P = \begin{bmatrix} -C & \mathcal{A}^* \\ A & B \end{bmatrix}. \quad (1.4)$$

$\mathcal{A}$ is $2s \times 2s$ matrix with the only non-zero entry as $\mathcal{A}_{11} = w(x)$. It now remains to define the maximal and minimal differential operators which will only make sense when we introduce some regularity conditions otherwise their definitions will merely be relations on $\mathcal{H}$. Thus for $2s$ -dimensional vector $h(x)$, we require that

$$\mathcal{J}U' - PU = h(x), \quad \text{with } \|U\| = 0, \text{ if } h(x) = 0. \quad (1.5)$$

Similarly, in order to express higher order quasi-derivatives in terms of the lower ones, we demand that our system should satisfy the following regularity condition as well. The equation

$$\begin{bmatrix} 0_l & 0 \\ 0 & 0_{2s-l} \end{bmatrix} [\mathcal{J}U' - PU] = 0$$

can be solved uniquely for $y_{l+1}, \dots, y_{2s}$ in terms of $y_1, \dots, y_l$ and formal derivatives of the first l components. For this condition, see Hinton and Schneider (D. Hinton & Schneider, 1993). Thus the action of the differential operator to be defined is formally described by the differential operator τ_r

$$\tau_r y = \begin{bmatrix} \mathcal{A}^{-1}(x) & 0 \\ 0 & 0 \end{bmatrix} [\mathcal{J}U' - PU]$$

with $\mathcal{A} = \text{diag}\{w(x), 0, \dots, 0\}$ a $l \times l$ matrix and hence the equation $\tau_r y = zy$ is equivalent to the first order system (1.3) and (1.4).

In order to construct minimal and maximal differential operators generated by τ_r , we let $\{y_0, y_1, \dots, y_{2s-1}\}$ be set of vector valued functions that are square integrable and define $\mathcal{D}(T_r^*)$, the domain of maximal differential operator by

$$\begin{aligned} \mathcal{D}(T_r^*) &= \{y \in \mathcal{L}^2([0, \infty)) \mid y_0, y_1, \dots, y_{2s-1} \text{ are absolutely} \\ &\quad \text{continuous in } [0, \infty), T_r^* y \in \mathcal{L}^2([0, \infty))\}, \\ T_r^* y &= \tau_r y \text{ for all } y \in \mathcal{D}(T_r^*). \end{aligned}$$

$\mathcal{D}(T_r^*)$ is the maximum possible domain in $\mathcal{L}^2([0, \infty))$ for which the quasi-derivatives defined above make sense. T_r^* is the Hilbert adjoint of the minimal possible operator generated by τ_r . Weidmann (Weidmann, 1987) had shown that if τ is symmetric then T_r^* is symmetric, densely defined and closed. An operator defined by restricting the domain of maximal operator to only functions y with compact support is known as a pre-minimal operator. We can denote this operator by T_{r_0} and its domain is defined by

$$\mathcal{D}(T_{r_0}) = \{y \in \mathcal{D}(T_r^*) \mid y \text{ has compact support in } (0, \infty)\}.$$

$T_{r_0} y = T_r^* y = \tau y$, for all $y \in \mathcal{D}(T_{r_0})$. For unbounded domains, T_{r_0} is symmetric, densely defined but not necessarily closed but closable. The closure of pre-minimal operator T_{r_0} , $\bar{T}_{r_0}$, is the minimal operator associated with τ and will be denoted by T_r . Here, T_r is symmetric, densely defined and closed with $T_r \subset T_r^*$, in the operator sense, if T_r is not self-adjoint. The differential operators generated by τ_r are explicitly defined in terms of boundary conditions which essentially needs the aids of Green's formula which are obtained through integration by parts (Weidmann, 1987). Thus we impose boundary conditions as follows, for $y(0) = 0$ and for any two solutions $y(x)$ and $\tilde{y}(x)$, we need that $\lim_{x \rightarrow \infty} y^*(x) \mathcal{J} \tilde{y}(x) = 0$. These can also be described using two s by s complex-valued matrices α_1, α_2 as

$$(\alpha_1, \alpha_2)y(0) = 0, \tag{1.6}$$

where $\text{rank}(\alpha_1, \alpha_2) = s$ and

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I_s, \quad \alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0_s.$$

The first order system (1.3) or its Hamiltonian (1.4) have been solved using asymptotic integration.

1.1.2 Difference operators

We consider two symmetric higher order difference functions $\mathcal{L}_1$ and $\mathcal{L}_2$ defined by

$$\mathcal{L}_1 y(t) = (w(t))^{-1} \left\{ \sum_{k=0}^n (-1)^k \Delta^k (p_k \Delta^k y(t-k)) \right\}, \quad \mathcal{L}_2 y(t) = (w(t))^{-1} \left\{ \sum_{j=0}^m (-1)^j \Delta^j (q_j \Delta^j y(t-j)) \right\}, \quad (1.7)$$

on $\ell^2(\mathbb{N})$. Here, $p_k = p_k(t)$, $q_j = q_j(t)$, $t \in \mathbb{N}$, $k = 0, 1, 2, \dots, n$ and $j = 0, 1, 2, \dots, m$, are real valued functions which are assumed to have second forward difference with the assumption that $\Delta(p_k(t))$, $\Delta(q_j(t)) \rightarrow 0$ as $t \rightarrow \infty$. The operator Δ is a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$ for all $t \in \mathbb{N}$. The difference maps $\mathcal{L}_1$ and $\mathcal{L}_2$ are ordinarily defined on weighted Hilbert spaces $\ell_w^2(\mathbb{N})$ or $\ell_w^2(\mathbb{Z})$ where $w = w(t) > 0$ is a weight function. In this case, the inner products for any $\{y_1(t)\}_{t=t_0}^\infty$ and $\{y_2(t)\}_{t=t_0}^\infty$ in $\ell_w^2(\mathbb{N})$ are defined by

$$\langle y_1(t), y_2(t) \rangle = \sum_{t=t_0}^\infty y_1(t) w(t) \overline{y_2(t)}$$

where t_0 is some initial integral value. The spectral theory of difference operators generated by $\mathcal{L}_1(y)$ and $\mathcal{L}_2(y)$ have been analysed independently by different authors under various growth and decay conditions. For an overview of these results see [(Behncke & Nyamwala, 2011), (Behncke & Hinton, 2013)].

Recently, the attention has been shifted to the composites of the continuous counterparts of equations (1.7) in the case of unbounded coefficients. For the commutativity of the composites of differential equations, counterparts of (1.7), see the papers by Amitsur [(Amitsur, 1958), Mironov (Mironov, 2004), (Mironov, 2014)] and the references cited therein. Such like analysis are not available for the composites of (1.7). However, it is a well known fact that results for symmetric differential operators are normally comparable to those of their discrete counterparts under similar growth and decay conditions. For such treaties see (Behncke & Hinton, 2013). In this thesis, we have studied the solutions of the equations

$$\mathcal{L}_1(\mathcal{L}_2 y(t)) = zy(t), \quad \mathcal{L}_2(\mathcal{L}_1 y(t)) = zy(t), \quad (1.8)$$

where $z \in \mathbb{C}$ is the spectral parameter. In particular, we have been interested in cases where in (1.8) above, $\mathcal{L}_1$ commutes with $\mathcal{L}_2$. In order to simplify the computations, and since we are interested in the spectral analysis of the minimal operator generated by the composite of $\mathcal{L}_1$ and $\mathcal{L}_2$, we have assumed that $w(t) = 1$, for all $t \in \mathbb{N}$ or $t \in \mathbb{Z}$. Thus the analysis can be done on $\ell^2(\mathbb{N})$ which is isomorphic to $\ell_w^2(\mathbb{N})$ when $w(t) = 1$. The deficiency indices and the spectral results in the two spaces are similar if $w(t) = 1$ for all $t \in \mathbb{N}$.

The interest in this case is the necessary and sufficient conditions for the commutativity of the minimal and maximal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ and if so, the required conditions for the self-adjoint operator extension H generated by $\mathcal{L}_2\mathcal{L}_1$ to be expressed as H_2H_1 . Here, H_1 and H_2 are the self-adjoint operator extensions of minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

The solutions of (1.8) have been obtained by constructing the Hamiltonian systems of (1.8), construction of the equivalent first order systems and the appropriate maximal and minimal operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\ell^2(\mathbb{N})$. Thus by constructing appropriate comparison algebras and applying asymptotic summation, we have established the existence of self-adjoint operator extension of the minimal operator generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ as well as that of the composite $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$. Moreover, using the same approach, we have determined the necessary and sufficient conditions for these self-adjoint operator extensions to commute. Finally, we have also determined the location of the absolutely continuous spectrum and the spectral multiplicity of the respective self-adjoint operator extensions. In the sequel, we formulate the first order systems and construct maximal and minimal operators generated by $\mathcal{L}_r$, $r = 1, 2$.

In order to solve the equations $\mathcal{L}_r y = zy$, $r = 1, 2$, we need to convert the equations into their first order systems. To do that, we require quasi-differences [(Behncke, Hinton, & Remling, 2001), (Y. Shi, 2006)]. These are the discretized version of quasi-derivatives as given in (1.2). These are given by

$$x_i(t) = \Delta^{i-1}y(t-i), \quad u_n(t) = p_n(t)\Delta^n y(t-n), \quad (1.9)$$

$$u_k(t) = \sum_{l=k}^n (-1)^{l-k} (p_l(t) \Delta^l y(t-l)) \text{ for } \mathcal{L}_1.$$

The quasi-differences for $\mathcal{L}_2$ are done in a similar way and hence using the $2s$, $s = n, m$, vector valued function, $Y(t, z) = [x(t, z), u(t, z)]^{tr}$, where $x(t, z) = [x_1(t, z), \dots, x_s(t, z)]^{tr}$ and $u(t, z) = [u_1(t, z), \dots, u_s(t, z)]^{tr}$, we obtain the required first order system given by

$$\Delta \begin{bmatrix} x(t, z) \\ u(t, z) \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \begin{bmatrix} x(t+1, z) \\ u(t, z) \end{bmatrix} \quad (1.10)$$

Here A , B and C are $s \times s$ matrices with non-zero entries as

$$A_{l,l+1} = 1, \quad B_{s,s} = h_s^{-1}, \quad C_{l,l} = h_{l-1}, \quad l = j, k = 1, \dots, s.$$

$h_0 = h_0(t) - z$, $h = p_k, q_j$. In a more precise way, we can introduce a symplectic matrix $\mathcal{J}$ and write the above first order form in its Hamiltonian system

$$\mathcal{J} \Delta \begin{bmatrix} x(t, z) \\ u(t, z) \end{bmatrix} = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix} \begin{bmatrix} x(t+1, z) \\ u(t, z) \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix} \quad (1.11)$$

This leads to an equivalent first order of the form

$$\mathcal{J} \Delta Y(t, z) - PKY(t, z) = F(t),$$

where K is a partial shift operator defined by

$$K([x(t, z), u(t, z)]^{tr}) = [x(t+1, z), u(t, z)]^{tr}$$

,

$$P = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix}, \quad F(t) \text{ } 2s - \text{ vector function}$$

with $F(t) = [f(t), 0, \dots, 0]^{tr}$, $f(t) \in \ell^2(\mathbb{N})$.

The first order form can be written as

$$\begin{bmatrix} x(t+1, z) \\ u(t+1, z) \end{bmatrix} = S(t, z) \begin{bmatrix} x(t, z) \\ u(t, z) \end{bmatrix}. \quad (1.12)$$

where $S(t, z)$ has the block form of

$$S(t, z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}$$

and $E = (I - A)^{-1}$.

In order to properly define the maximal and minimal difference operators generated by $\mathcal{L}_r$ in (1.7), we again need some regularity conditions just like in the continuous case. The space $\ell^2(\mathbb{N})$ is described as follows using the regularity conditions.

$$\ell^2(\mathbb{N}) = \left\{ y(t) : y(t) = \{y(t)\}_{t=0}^\infty \subset \mathbb{C} \right. \\ \left. \sum_{t=0}^\infty K(Y^*(t))(KY(t)) < \infty \right\}.$$

Similarly, there exists an integral domain I_1 , $I_1 \subset \mathbb{N}$ such that

$$\sum_{t \in I_1} K(Y^*(t))(KY(t)) > 0,$$

$\ell^2(I_1) \subset \ell^2(\mathbb{N})$ and

$$\ell^2(I_1) = \left\{ y(t) \in \ell^2(\mathbb{N}) : \sum_{t \in I_1} K(Y^*(t))(KY(t)) > 0 \right\}.$$

By doing so, the scalar products are therefore defined by

$$\langle y(t), y_1(t) \rangle = \sum_{t \in I_1} y(t) \overline{y_1(t)}.$$

In addition, for any vector $F(t)$, we need

$$\mathcal{J}\Delta Y(t, z) - PKY(t, z) = F(t), \text{ with } \|Y(t, z)\| = 0, \text{ if } F(t) = 0.$$

This is the condition that is missing in (Y. Shi, 2006) which makes the definition of maximal operators in her work just mere difference relations and not properly defined maximal operators.

Next we define maximal and the minimal difference operators generated by $\mathcal{L}_r$. Define a map L_r^* via its domain $D(L_r^*)$ given by

$$D(L_r^*) = \left\{ y(t) \in \ell^2(I_1) : \text{there exists } f(t) \in \ell^2(I_1) \text{ such that} \right. \\ \left. \mathcal{J}\Delta Y(t) - PKY(t) = F(t) \right\}$$

and

$$L_r^* y(t) = f(t) \text{ if and only if } \mathcal{J}\Delta Y(t) - PKY(t) = F(t).$$

$$\mathcal{L}_r y(t) = L_r^* y(t), \quad \forall y(t) \in D(L_r^*).$$

Then $D(L_r^*)$ is the largest possible domain in which L_r^* can be defined. L_r^* is the maximal difference operator generated by $\mathcal{L}_r$ on $\ell^2(\mathbb{N})$. L_r^* is symmetric, closed and densely defined. A restriction of L_r^* to a smaller domain $D(L_{r_0})$ defined by

$$D(L_{r_0}) = \{y(t) \in D(L_r^*) : \text{there exists } n \in \mathbb{N} \\ \text{such that } y(0) = y(t) = 0 \text{ for all } t \geq n + 1\}.$$

$$L_r^* y(t) = L_{r_0} y(t), \quad \forall y(t) \in D(L_{r_0}),$$

results into a pre-minimal difference operator generated by $\mathcal{L}_r$. Here, L_{r_0} is densely defined, symmetric but not necessarily closed. Since in our analysis we require closed linear operators and because densely defined operators are closable, we therefore take the closure of pre-minimal operator $\overline{L_{r_0}}$ to be our minimal operator. This we denote by L_r . In order to have properly defined operators, we impose some boundary conditions at the left regular end point [(Behncke & Hinton, 2013), (Y. Shi, 2006)]. Define two $s \times s$ matrices (α_1, α_2) with $\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I$ and $\alpha_1 \alpha_2^* = \alpha_2 \alpha_1^*$ so that

$$(\alpha_1, \alpha_2) \begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = 0. \quad (1.13)$$

In most cases, the left regular end point is taken as t_0 for $t_0 > 1$ and $t_0 \in \mathbb{N}$. The results of Remling (Remling, 1998) can then be used to extrapolate the deficiency results to $t_0 = 0$.

Definition 1.1.1. For any z , the spectral parameter with $\text{Im} z > 0$, the pair (N_+, N_-) is defined as the deficiency indices of L_r , if $N_+ = \dim \mathcal{N}_{L_r^* - \bar{z}}$ and $N_- = \dim \mathcal{N}_{L_r^* - z}$, where $\mathcal{N}_{L_r^* - \bar{z}}$ and $\mathcal{N}_{L_r^* - z}$ are taken as the null spaces of $L_r^* - \bar{z}I$ and $L_r^* - zI$ respectively.

If $N_- = N_+ = 0$, then L_r is self-adjoint. On the other hand if $N_- = N_+ \neq 0$ then by von Neumann theorems (Schmüdgen, 2012), L_r has self-adjoint operator extension H_r defined by;

$$D(H_r) = D(L_r) \dot{+} \{y + V_r y : y \in \mathcal{N}(L_r^* - zI)\}, \quad H_r y = L_r y = \mathcal{L}_r y, \quad \forall y \in D(L_r). \quad (1.14)$$

V_r is a uniquely determined isometric mapping such that $V_r : \mathcal{N}(L_r^* - zI) \rightarrow \mathcal{N}(L_r^* + zI)$. In the case of non-limit point, added boundary conditions are imposed at

infinity in order to obtain the basis of the set of solutions of $(\mathcal{L}_r - zI)y = 0$ that are uniformly square summable.

The analysis of the first order systems of (1.7) and the existence of H_r are done using asymptotic summation which is anchored on the discretized version of Levinson's Theorem as stated below.

Theorem 1.1.2. (*Levinson's-Benzaid-Lutz Theorem (Benzaid & Lutz, 1987)*)

Let $\Lambda(t, z) = \text{diag}\{\lambda_1(t, z), \dots, \lambda_{2s}(t, z)\}$ for $t \geq t_0$. Assume

- (i) $\lambda_i(t, z) \neq 0$ for all $1 \leq i \leq 2s$ and $t \geq t_0$
- (ii) $R(t, z)$ is a $2s \times 2s$ matrix defined for all $t \geq t_0$,
satisfying $\sum_{t=0}^{\infty} \left| \frac{1}{\lambda_i(t, z)} \right| \|R(t, z)\| < \infty$, for all $i = 1, 2, \dots, 2s$
- (iii) $\Lambda(t, z)$ satisfies the following uniform dichotomy condition. For any pair of indices i and j , such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t, z)| \geq \delta$ for all $t \geq t_0$. Then, either $\left| \frac{\lambda_i(t)}{\lambda_j(t)} \right| \geq 1$ or $\left| \frac{\lambda_i(t)}{\lambda_j(t)} \right| \leq 1$ for a large t .

Then the linear system

$$Y(t+1, z) = [\Lambda(t, z) + R(t, z)]Y(t, z) \quad (1.15)$$

has a fundamental matrix satisfying,

$$Y(t, z) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l, z) \text{ as } t \rightarrow \infty.$$

Explicitly for the eigensolutions, we have

$$y_k(t, z) = (e_k(t, z) + r_k(t, z)) \prod_0^{t-1} (\lambda_k(l, z)). \quad (1.16)$$

Here, $e_k(t, z)$ is the normalized eigenvector and $r_k(t, z) \rightarrow 0$ as $t \rightarrow \infty$.

1.2 Basic concepts

In this section, we give the basic definitions of the concepts applied

Definition 1.2.1. A Hilbert space is a complete inner product space under the norm induced by its inner product and satisfies the parallelogram law.

Definition 1.2.2. An inner product space is a vector space X over $\mathbb{R}$ together with a nonnegative real valued function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$, that is, the pair $(X, \langle a, b \rangle)$, which satisfy the following properties:

- (i). $\langle a, b \rangle \geq 0$ and $\langle a, b \rangle = 0$ iff $a = b = 0$
- (ii). $\langle a, b \rangle = \overline{\langle a, b \rangle}$
- (iii). $\langle a + b, c \rangle = \langle a, c \rangle + \langle b, c \rangle$
- (iv). $\langle \alpha a, b \rangle = \alpha \langle a, b \rangle \quad \forall a, b \in X$

Definition 1.2.3. Let X be a vector space and assume that a norm $\|\cdot\|$ is defined on X , then a mapping $T : X \rightarrow Y$, Y another vector space over the same field as X is said to be an operator if T preserves the underlying algebraic structures of X and Y .

Definition 1.2.4. Let H be a Hilbert space with Hermitian inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\|\cdot\|$. Let T be an unbounded linear map $T : D_T \rightarrow H$ defined on a dense subspace D_T of H . Then T is said to symmetric if $\langle Tv, w \rangle = \langle v, Tw \rangle \quad \forall v, w \in D_T$

Definition 1.2.5. Let T_1, T_2 be symmetric differential operator, we say that T_1 and T_2 are commuting if the commutator $T_1T_2 - T_2T_1 = 0$. If $T_1T_2 - T_2T_1 \neq 0$, then T_1, T_2 are non-commutative.

Definition 1.2.6. Let $T : X \rightarrow X$ be a bounded linear operator on a normed complex space X . The spectrum $\sigma(T)$ of T is the set $\lambda \in \mathbb{C}$ such that $T - \lambda I$ has no inverse. The components of the spectrum as given in (Kreyszig, 1991) include:

- i. Point spectrum or discrete spectrum is the set of all complex numbers, λ , defined by $\sigma_p(T) = \{\lambda \in \mathbb{C} \text{ for which } (T - \lambda I)^{-1} \text{ does not exist since } T \text{ is not injective.}\}$

- ii. Continuous spectrum is the set of all complex λ defined by $\sigma_c(T) = \{\lambda \in \mathbb{C} \text{ for which } (T - \lambda I)^{-1} \text{ exists, is densely defined and unbounded} \}$
- iii. Essential spectrum is the set of all complex λ defined by $\sigma_{ess}(T) = \{\lambda \in \mathbb{C} \text{ for which } (T - \lambda I)^{-1} \text{ does not exist since } L \text{ is not semi-fredholm operator (an operator whose range is closed and is finite dimensional)}\}$
- iv. Residual spectrum is the set of all complex λ defined by $\sigma_r(T) = \{\lambda \in \mathbb{C} \text{ for which } (T - \lambda I)^{-1} \text{ exists, is densely defined and may be bounded or not.}\}$

Definition 1.2.7. An M-matrix is a matrix $A \in \mathbb{R}^{n \times n}$ of the form

$$A = sI - B \text{ where } B \geq 0 \text{ and } s > \rho(B).$$

Here, $\rho(B)$ is the spectral radius of B , that is, the largest modulus of any eigenvalue of B and $B \geq 0$ denotes that B has nonnegative entries. An M-matrix has negative off-diagonal elements and it also has positive diagonal elements.

Definition 1.2.8. Cayley transform is an operator analogue of the *möbius* transformation, $(t \rightarrow (t - \lambda)(t - \bar{\lambda})^{-1})$, which maps densely defined symmetric operators onto isometric operators V for which the range $\mathcal{R}(I - V)$ is dense, and it relates both classes of operators.

Definition 1.2.9. An isometric operator is a linear operator V on a Hilbert space H such that $\|Vx\| = \|x\|$ for all $x \in D(V)$. The domain $D(V)$ can be an arbitrary linear subspace of H . Thus if V is isometric, it is invertible and its inverse V^{-1} is also isometric. V is closed if and only if $D(V)$ is a closed linear subspace of H .

1.3 Statement of the problem

The spectral theory of symmetric differential and difference equations with almost constant coefficients has been studied. The deficiency indices and spectral theory of commuting differential operators with bounded coefficients have been done extensively. However, the construction of symmetric composites and the spectral analysis of their self-adjoint extensions are lacking in the case of unbounded coefficients. In

addition, this is lacking for the difference operators. In this study, we have considered the composites of these differential and difference operators with unbounded coefficients and have analysed their deficiency indices and spectral multiplicities of the spectrum of the self-adjoint operator extensions.

1.4 Objectives of the study

The main aim of this research was to analyze the necessary and sufficient conditions for the existence of symmetric self-adjoint operator extensions of the composites for differential and difference operators with unbounded coefficients.

The specific objectives of this study are to:

- (i) Construct commutative symmetric composites of differential and difference operators.
- (ii) Analyze the necessary and sufficient conditions for the existence of symmetric self-adjoint extensions of the composites
- (iii) Evaluate the deficiency indices of the self-adjoint operator extensions of the composites.
- (iv) Examine the spectral properties of the self-adjoint operator extensions of the composites.

1.5 Justification of the Study

This study has extended the theory of the composites of differential operators with bounded coefficients to unbounded situation. In addition, it also extends the theory of the composites of differential operators to discrete settings in which we have constructed commuting composites of difference operators, analyzed the necessary and sufficient conditions for the existence of self-adjoint operator extensions as well as examining the spectral properties of the self-adjoint operator extensions of the composites of difference operators. This work therefore, completes the theory in the

line of commuting difference and differential operators and is applicable to various sectors such as film industries as well as the field of engineering.

1.6 Significance of the study

This study was motivated by the invent of researches in electric-electronics where a system is organized as a chain of subsystems such that the output of one subsystem is the input of the subsequent subsystem. It is worth noting that main applications of differential equations arise naturally in electrodynamics, quantum mechanics and other branches of engineering. In engineering, this occurs in control theory and in particular, in electrical engineering. When the cascade of a system is connected in form of subsystems, the engineering ingenuity requires that the order of the subsystems can be changed without interfering with the quality and properties of the final product, output. The conditions necessary and sufficient for this achievement is the concept behind the theory of commutativity in Mathematics and Control theory for engineers.

On the other hand, it is noted that most observables in quantum mechanics are described using self-adjoint operators with some of the elements in the spectrum of such operators being the measured observables. This is because in quantum computing we can always model all settings of observables as a system of C^* -algebra and then one may be interested in the associated coarse graining or approximation. The observables are said to be compatible if their joint values are determined simultaneously which in the language of operator theory implies commutativity. Therefore, a measurement of intermediate results from partial computation in quantum computing implies understanding of the commutative C^* -subalgebra of the whole system.

The achieved results are applicable in the film industries where a certain quality of image or sound is to be achieved. This purely depends on the wavelengths of the components of the white light in the electromagnetic spectrum which translates to the nature of the eigen solutions that forms the spectrum. Discrete spectrum implied images of high precision.

The achieved results can also be applied in solving problems in the field of computer vision and pattern recognition such as image segmentation, routing and image classification, which is useful in forensic audits. This assists in prevention of criminal activities and hence peaceful coexistence in the region. All these shall result into economic stability within the country and the region. Finally, The results are also applicable in data mining as well as ensuring computer security, which in the end will boost more investment into ICT sector.

1.7 Structure of Thesis

This thesis has been organised as follows: Chapter One entails the mathematical background of the study, the basic concepts applied to the study, the statement of the problem, objectives of the study as well as the justification and significance of the study. In Chapter Two, the literature review of differential and difference operators have been given. A literature review on Self-adjoint operator extensions as well as that of the spectral properties of differential and difference operators has also been provided. Chapter Three gives an outline of the methodology applied in this study. The results of the study have been given in Chapter Four whereas the conclusion and recommendation for future research have been provided in Chapter Five.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Many mathematicians have conducted the study on the properties of differential and difference operators with their spectral properties being investigated. In this chapter, we give a detailed review on commuting differential operators, difference operators, their spectral properties and their self-adjoint operator extensions.

2.2 Composites of Differential and Difference Operators

The theory of commuting linear differential expressions originated from Floquet's work in 1880s (Floquet, 1883) and later on, Wallenberg (Wallenberg, 1903) and Schur (Schur, 1905) advanced on it through the contributions they made. Burchnal and Chaundy (Burchnell & Chaundy, 1928) discovered a relationship with algebraic geometry in 1920s. Later on, a number of authors have been interested in the necessary and sufficient conditions for the commutativity of the differential functions τ_1 and τ_2 and for more details see the papers by Amitsur (Amitsur, 1958) and Mironov (Mironov, 2004),(Mironov, 2014) and the references cited therein.

While building on the earlier work by Burchnell, Chaundy, Lax, and others, the

authors in (Weikard, 2000) explored on the theory of commuting linear differential expressions and their relationship with the meromorphic solutions of differential equations. They established a connection between the property that all solutions of the differential equation $Ly = zy$ are meromorphic and the algebro-geometric nature of L . This was an extension of the work by one of the authors and Gesztesy for elliptic potentials to rational and simply periodic coefficients. Additionally, they provided a necessary and sufficient condition for a differential operator L to be algebro-geometric. In particular, they established that if all solutions of $Ly = zy$ are meromorphic for every $z \in \mathbb{C}$, and if the coefficients of L are either rational or simply periodic, then L is algebro-geometric.

Van Gorder (Van Gorder, 2014), in his study on computing semi-commuting differential operators in one and multiple variables, explored on the concept of semi-commuting linear differential operators, which generalizes the notion of commuting differential operators. Their paper provided a method for constructing such operators and applied this framework to both ordinary and partial differential operators, with a focus on operators related to special functions like hypergeometric functions, Airy functions, and Bessel functions. They introduced the concept of semi-commuting operators, where two differential operators P and Q are said to semi-commute if the commutator $[P, Q]$ has a lower order than the maximum possible order $m + n - 1$, where m and n are the orders of P and Q , respectively. While focusing on finite linear operators during their study, the authors acknowledged the challenges of extending the results to infinite-dimensional operators and nonlinear operators. However, they suggested that similar methods could be applied to infinite-dimensional or nonlinear operators under certain conditions.

In the paper by Amitsur (Amitsur, 1958) the commutativity property was derived via the associated commutative ring of polynomials related to the differential equations under investigation in which he proved that the set $C[f]$ of all linear operators which commute with f is a commutative ring and that it is finitely generated over the algebra of all polynomials $f(D)$ with constant coefficients. In addition, he obtained the necessary and sufficient conditions for the existence of non-trivial differential op-

erators which commute with $f(D)$. In 1970s and 1980s, Gelfand and Dikii (Gelfand & Dikii, 1975) worked on the Gelfand-Dikii systems in which they explored more on the theory of commuting differential expressions and they noticed that these expressions had a connection with partial differential equations that are completely integrable.

On his study on a ring of commuting differential operators of rank 2, Mironov (Mironov, 2004) explored on the relationship between algebraic geometry and the theory of integrable systems. His main focus was on commuting differential operators and their connection to algebraic curves. He constructed a pair of commuting differential operators of rank 2 explicitly based on the Krichever-Novikov theory, which provides a framework for associating solutions of integrable systems with algebraic curves. In addition, he (Mironov, 2014) constructed examples of commuting differential operators of two commuting operators of rank 2 with constant coefficients. However, none of these authors was interested in the self-adjoint operator extensions of the minimal operators generated by τ_1 and τ_2 and whether such self-adjoint operator extensions commute whenever τ_1 and τ_2 commute.

H. O Cordes (Cordes, 1987) derived necessary conditions for commutativity of first and second order pseudo-elliptic differential operators by application of comparison algebras. Similar investigations were also done by S. T. Melo (Melo, 1990). However, in all the cases, none of the authors considered the case where the coefficients are unbounded and the commutativity of the associated self-adjoint operator extensions of the respective minimal differential operators. This study can be seen as extension of Cordes's work to higher order symmetric differential equations with unbounded coefficients.

In their study, Jorgensen (Jorgensen, 1987) explored on the structure and properties of algebras generated by unbounded operators acting on a common, invariant, dense domain in a Hilbert space. They extended the existing theory of commutative operator algebras for bounded operators to the unbounded case focussing on algebras generated by operators $T_1, \dots, T_n$ defined on a dense, invariant domain D

in a Hilbert space. Their results showed that any algebra generated by $T_1, \dots, T_n$ satisfying the given domain condition admits a dilation to a commutative algebra of operators $L_1, \dots, L_n$ in a larger Hilbert space where L_j 's are formally normal. Moreover, their results established that if D contains a dense set of analytic vectors for each T_j , then the commuting, formally normal extensions, L_j 's become essentially normal, and there exists a spectral measure μ such that:

$$T_j x = \int \lambda_j n \mu(\lambda) x \text{ for all } x \in D,$$

which is a generalization of the spectral theorem to unbounded operator algebras.

Bokhonov and the co-authors in (Bokhonov, 1990), investigated the conditions under which two formally commuting Hermite operators can be extended to commuting self-adjoint operators. Their aim was determine the conditions under which two formally commuting Hermite operators, that is, the differential operators with specific symmetry properties, can be extended to commuting self-adjoint operators in a Hilbert space. They proved that commuting self-adjoint extensions exist, under the given conditions that is, one operator is essentially self-adjoint while the other operator has equal defect numbers.

The authors in (Hasegawa, 1997) explored more on the construction and properties of commuting difference operators arising from Belavin's elliptic quantum R -matrix. They constructed the commuting difference operators using Belavin's elliptic quantum R -matrix and the L -operator and demonstrated that these commuting difference operators are of Macdonald type, with coefficients given by elliptic theta functions, established a connection between these operators and Ruijsenaars' operators in which they highlighted their equivalence and finally, they determined the relationship between the difference L -operator and Krichever's Lax matrix, linking algebraic and analytic approaches to integrability.

Dixmier (Dixmier, 1968) in 1968, constructed the first examples of commuting ordinary scalar differential operators of the non-trivial ranks 2 and 3 and the non-trivial genus, $g = 1$, for the nonsingular elliptic spectral curve. Krichever (Krichever, 1978) in his work gave a general classification of pairs of commuting ordinary differential

operators of rank greater than 1. Krichever together with Novikov (Krichever & Novikov, 1980) further gave a general form of pairs of commuting operators of rank 2 for an arbitrary elliptic spectral curve. By advancing on the works of Krichever and Novikov, Mokhov (Mokhov, 1990), (Mokhov, 2012) obtained a general form of pairs of commuting operators of rank 3 for an arbitrary elliptic curve. Moreover, examples of commuting differential operators of genus 1 with polynomial coefficients have been constructed for any rank.

A study carried out by Oganessian (V. S. Oganessian, 2015) showed that two operators L_n and L_m commute if there exists a non zero polynomial $R(z, w)$ such that $R(L_n, L_m) = 0$. The authors in (V. S. Oganessian, 2016) studied commuting differential operators and their associated spectral curves, with a focus on the explicit construction of common eigenfunctions for operators of rank 2. Their work was to build and improve on the prior work such as: the case of rank 1 which was studied by I. M. Krichever, in which he provided explicit expressions for the coefficients of commuting operators in terms of Riemann theta-functions, the case of rank 2 and genus 1 which was studied by S. P. Novikov and I. M. Krichever and the case of rank 3 and genus 1 which was fully classified by O. I. Mokhov. These authors utilized the method that was developed by A. E. Mironov for constructing such operators since Mironov's method is effective in generating diverse examples of commuting operators of rank 2 and higher genus and their main contribution was the explicit construction of common eigenfunctions for commuting operators with polynomial coefficients and a smooth spectral curve.

Furthermore, Oganessian (V. Oganessian, 2017) on his work on explicit characterization of some commuting differential operators on rank 2, he stated that, if the rank equals to one, then there exists explicit formulas for coefficients of commutative operators in terms of Riemann theta functions. While presenting a simplified proof of Mironov's results on commuting self-adjoint operators of rank 2, the authors in (V. S. Oganessian, 2018) also extended the methodology to construct explicit examples of matrix commuting differential operators. These examples are important as they provide a basis for understanding the structure and behavior of such operators

in practical applications. While constructing the examples, they used the simplified method to construct specific operators that satisfy the required conditions. In addition they verified the commutativity and self-adjointness of these operators through direct computation.

Zheglov and others in (Zheglov, Mironov, & Saparbayeva, 2016), explored on the existence and properties of commuting rank 2 differential operators with polynomial coefficients, specifically in the context of spectral curves of the form $w^2 = z^3 + c_2z^2 + c_1z + c_0$. In their study, they demonstrated the existence of commuting non self-adjoint differential operators of orders 4 and 6 with polynomial coefficients. They also established a relationship between the algebraic geometry of spectral curves and the analytical properties of differential operators by investigating how the structure of the spectral curve influences the existence and form of commuting differential operators. Their methodology was rooted in the Krichever–Novikov (KN) theory, which provides a framework for studying differential operators and their associated spectral curves.

Mauleshova and Mironov (Mauleshova & Mironov, 2016) investigated pairs of commuting difference operators L_k and L_m of rank one, where the coefficients depend on a functional parameter in which they focussed on operators where shift operators T appear only with positive degrees. The authors constructed and classified rank one commuting difference operators on hyperelliptic curves. The authors also provided explicit examples of such operators with polynomial coefficients linked to differential operators via the Weyl algebra and also operators with trigonometric coefficients demonstrating integrability in discrete systems. Moreover, they showed how these difference operators can be embedded into differential operators with polynomial coefficients, leading to new commutative subalgebras in the first Weyl algebra. The authors based their study the Baker-Akhiezer function, which is uniquely determined by its divisor properties and asymptotic behavior near a marked point on the spectral curve. This function serves as a common eigenfunction for the commuting operators.

Later on, Mauleshova and Mironov, (Mauleshova & Mironov, 2018) investigated one-point commuting difference operators of rank one associated with hyperelliptic spectral curves in order to understand their algebraic and spectral properties. They established a relationship between these difference operators and one-dimensional finite-gap Schrödinger operators in which they showed that difference operators can reproduce finite-gap spectral properties analogous to Schrödinger operators. Moreover, they explored how discrete operators can approximate or generalize continuous integrable systems and obtained a discrete version of finite-gap Lamé operators, which are key examples in soliton theory and integrable systems

Dehimi et al. (Dehimi, Mortad, & Bachir, 2023) investigated the conditions under which a bounded self-adjoint operator commutes with an unbounded closed symmetric operator. Their study was based within the framework of operator theory in Hilbert spaces. In particular, they focussed on the interplay between commutativity, normality, and self-adjointness. The authors established the conditions under which a bounded self-adjoint operator B commutes with an unbounded closed symmetric operator A . In particular, they established that if the product AB or BA is normal, then $BA \subset AB$, implying commutativity in a generalized sense. Additionally, the authors demonstrated that under certain conditions, the product AB or BA becomes self-adjoint. Particularly, they showed that if AB is normal and B is positive, then AB is self-adjoint and also if BA is normal, then A must be self-adjoint, and BA becomes self-adjoint. The authors used the spectral theorem for self-adjoint operators extensively, especially in defining the square root of positive operators and analyzing their properties with careful attention being paid to the domains of unbounded operators, which is crucial for ensuring the validity of commutativity and other results.

It is noted that as much as the commuting differential operators have been considered to a greater extent, the results on the commutativity of the associated self-adjoint extensions are lacking. Furthermore, there are no results on commuting difference operators. However, it is a well known fact that results for symmetric differential operators are normally comparable to those of their discrete counterparts under similar growth and decay conditions. For such treaties see (Behncke & Hinton, 2013).

Therefore, in this study we have analysed the symmetries of the composites of difference operators, the existence of the symmetric extensions of their composites in addition to the spectral theory of these composites under some suitable decay and growth conditions.

2.3 Self-adjoint Extensions

The concepts of self-adjoint and quasi self-adjoint equations were introduced by Ibragimov (Ibragimov, 2006) in which he used the concept of an adjoint equation in the construction of a Lagrangian for an arbitrary differential equation and for any system of differential equations where the number of equations is equal to the number of dependent variables. Gandarias (Gandarias, 2011), generalized the concepts of self-adjoint and quasi self-adjoint differential equations that were introduced by Ibragimov and explored on their implications for constructing conservation laws using symmetry methods. They extended Ibragimov's earlier definitions of self-adjoint and quasi self-adjoint differential equations to a broader class called weak self-adjoint equations in which they relaxed the strict conditions of self-adjointness, while retaining utility for conservation law construction. Additionally, they pointed out the importance of weak self-adjointness for deriving conservation laws associated with symmetries of differential equations, leveraging Ibragimov's general theorem.

Kochubei in their paper (Kochubei, 1975) focused on characterizing extensions of symmetric operators and symmetric binary relations using abstract boundary conditions. They classified the various types of extensions of symmetric operators with their main focus being on those with equal defect numbers, whether finite or infinite, in terms of abstract boundary conditions. They considered the analogous problem for symmetric binary relations and extended the known results from operators to a more general setting. As a result of their study, they were able to: provide a unified framework for describing extensions of symmetric operators and relations, perform explicit parametrizations of extensions via boundary conditions and derive a generalization of the classical results especially, the von Neumann extension theory, to cases with infinite defect numbers and relations.

Smith (Smith, 1991) investigated the spectral properties of operators arising from discrete second-order difference equations. He explored on the relationship between the oscillation constant, a concept from the qualitative theory of difference equations, and the bottom of the essential spectrum of self-adjoint extensions of the operator. Moreover, he derived the conditions under which the spectrum has only countably many eigenvalues below zero and established a simple criterion for the invariance of the essential spectrum under perturbations or changes in coefficients. Finally, he used the variational principles which is an analogous to the continuous Sturm-Liouville case to characterize eigenvalues.

Later on, Smith (Smith, 2012), investigated the spectral properties of a class of self-adjoint difference equations, drawing parallels to known results in differential operator theory, especially those of Shnol'. Their focus was on the study of the spectrum of the self-adjoint operator B associated with the difference equation which is assumed to be in the limit point case. They related the existence of exponentially bounded or polynomially bounded solutions of the associated difference equation to the location of the eigenvalues relative to the spectrum of the self-adjoint operator. Additionally, they extended the analysis to prove that the spectrum of the associated orthogonal polynomials contains the closure of all eigenvalues admitting a polynomially bounded solution.

A study carried out by Kurbanov and Godzhaeva (Kurbanov & Godzhaeva, 2019), focussed on the convergence properties of spectral expansions associated with a fourth-order ordinary differential operator with integrable coefficients. In their study, the authors established the sufficient conditions under which the spectral expansions of functions in Sobolev space of functions with weak first derivatives in $L^p(W_1^p(G))$ converge absolutely or uniformly. Their work is deemed as an extension of the classical spectral theory which are centered on second-order operators like Sturm-Liouville problems to higher-order cases. Moreover, their work provides explicit estimates for the rate at which the spectral expansion converges uniformly.

The study by Lomov (Lomov, 2018) on the convergence of eigenfunction expansions of a differential operator with integral boundary conditions, provides the conditions under which eigenfunction expansions for a differential operator with integral boundary conditions converge uniformly and compares them to classical Fourier expansions. In this they synthesized the sufficient conditions on the differential operator and integral boundary conditions under which the system of eigen- and associated functions forms an unconditional basis, that is a basis where convergence is independent of the order of summation. This was an important result on their work since not all differential operators with non-local (integral) boundary conditions yield such a basis. Furthermore, they derived the equiconvergence theorem, which compares the convergence rates of eigenfunction expansion and the classical trigonometric Fourier expansion. On this they established that the difference between the two expansions tends to zero at a certain rate (equiconvergence), meaning the eigenfunction expansion behaves similarly to the Fourier series in terms of convergence.

On their study, Kristkov and the co-authors in (Kritskov, 2020) investigated the convergence properties of spectral expansions associated with a self-adjoint differential operator on the real line. They established the necessary and sufficient conditions under which the spectral expansions, that is, the eigenfunction expansions of a self-adjoint differential operator converge uniformly on the entire real line $\mathbb{R}$. On this, they pointed out that, the uniform convergence is a stronger result than pointwise or the L^2 -convergence and that it ensures consistency across the entire domain. Additionally, they studied the uniform convergence of the derivatives of the spectral expansions in which they established that the derivatives of the spectral expansions also converge uniformly, provided certain regularity conditions are met. Their analysis relied more on deriving a uniform estimate for the increment of the spectral function on the diagonal which is a crucial aspect in proving the main convergence results, as it ensures that the spectral projections do not exhibit pathological behavior, enabling uniform convergence.

A qualitative spectral analysis for a class of second order difference equation was

done by Clark (S. L. Clark, 1996). The author considered a difference equation with the main focus being on the cases where the coefficient sequences have limiting values and satisfy specific bounded variation conditions. They characterized the limit point and limit circle nature of the difference equation, which are concepts analogous to those in differential equations. Their proofs relied on the method of subordinacy, which connects the asymptotic behavior of solutions to the spectral properties of the operator in which they proved that under certain conditions $\gamma < \xi$, the interval $S = (\gamma, \xi)$ is part of the absolutely continuous spectrum of the self-adjoint operator L associated with the difference equation. Their results were an extension of earlier work by Mate and Nevai, Dombrowski, and others, in which these authors assumed the weight functions to be equal to 1.

In their study, Sun and Ren (Sun & Ren, 2013) explored the theory of self-adjoint extensions for second-order linear difference equations. Their emphasis was particularly in the context of complex coefficients and the concept of J-self-adjointness. They analyze the spectrum of the J-self-adjoint extensions, which included the nature of the eigenvalues (real or complex) and the completeness of the eigenfunctions. In this study, they established that in the context of complex coefficients, the spectrum may exhibit unusual properties, such as exceptional points or spectral singularities. They employed the use of boundary triples and Krein's theory of self-adjoint extensions to classify and construct these extensions.

A study by Manfred and Bertin (Möller & Zlinsou, 2011) explores the theory of self-adjoint extensions for higher-order differential operators, particularly focusing on boundary conditions that depend linearly on the eigenvalue parameter. They applied the use of quasi-derivatives to handle the higher-order differential operators, which simplifies the analysis of boundary conditions, to characterize the self-adjoint extensions of symmetric differential operators. In their study, the eigenvalue problem was represented by an operator polynomial in which they derived the conditions under which any two operators are self-adjoint. Furthermore by studying the resolvent of the operator and its compactness properties, they provided a detailed spectral analysis of the operator polynomial and showed that under certain condi-

tions, the spectrum of operator is symmetric with respect to the imaginary axis, and eigenvalues with negative imaginary parts lie on the imaginary axis.

Moletsane and Bertin (Moletsane & Zinsou, 2017) carried out a study on Self-adjoint fourth order differential operators with eigenvalue parameter dependent and periodic boundary conditions in which they focused on fourth-order differential operators with specific boundary conditions. In their work, the boundary conditions included the periodic or anti-periodic conditions at the endpoints and one boundary condition that depends linearly on the eigenvalue parameter. The authors determined the necessary and sufficient conditions under which the operator A in the operator polynomial, $L(\lambda) = \lambda^2 M - i\alpha\lambda K - A$ is self-adjoint. Additionally, they used the quasi-derivatives, Lagrange identity, and Green's formula to analyze the spectral properties of the differential operators.

A study by Wang and Liu (Wang & Liu, 2016) on a new index theory for linear self-adjoint operator equations and its applications highlighted on the development of the new index theory and its applications to the existence and multiplicity of periodic solutions for wave equations and beam equations. Their work addressed a significant gap in the literature by developing an index theory for linear self-adjoint operators that do not have a compact resolvent which had been a big challenge because the lack of compactness means that the spectrum of the operator may not be discrete, and traditional methods for defining indices may fail. By employing their newly developed index theory, the authors established results on the existence and multiplicity of periodic solutions for these equations. This provided a systematic way to study periodic solutions in cases where traditional methods such as variational approaches or compactness-based techniques are not applicable.

Zagorodnyuk (Zagorodnyuk, 2012) focussed on addressing the problem of extending a pair of commuting symmetric operators to commuting self-adjoint operators in a Hilbert space is an important aspect in functional analysis and has implications for quantum mechanics, where commuting observables correspond to simultaneously measurable quantities. In their study, they assumed that the operators share a

dense invariant domain in a Hilbert space, which ensures that their compositions and commutativity are well-defined. Their research was based on the Godic-Lucenko theorem, which decomposes unitary operators into products of conjugations, to describe all possible commuting self-adjoint extensions systematically. In addition they used the application of Cayley transforms to convert the symmetric operators into unitary operators. and established that the commutativity of the original operators translates to commutativity of their Cayley transforms making it easier to carry out the analysis in the unitary setting. Their results provided explicit conditions necessary for the existence of commuting self-adjoint extensions, unified and generalized the earlier results through the Godic-Lucenko theorem, and offered a concrete parametrization of all possible extensions.

Later on, on their study on extensions of commuting tuples of symmetric and isometric operators, Zagorodnyuk and the co authors in (Zagorodnyuk, 2019) investigated the problem of extending commuting tuples of symmetric or isometric operators to commuting tuples of self-adjoint or unitary operators on a larger Hilbert space. In their study, they provided the sufficient conditions under which a commuting tuple of symmetric (or isometric) operators can be extended to a commuting tuple of self-adjoint (or unitary) operators. In addition, they proved an analog of the Godic–Lucenko Theorem for commuting tuples of unitary operators, in which they showed that a commuting tuple $(U_1, \dots, U_n)$ can be decomposed into a product of multidimensional conjugate symmetries.

In a study carried out by Dijksma and De Snoo (Dijksma & de Snoo, 1974) on self-adjoint extensions of closed symmetric linear manifolds beyond the original space, presented a generalization of the classical theory of self-adjoint extensions for symmetric operators in Hilbert spaces. They constructed the self-adjoint extensions of a closed symmetric linear manifold that may act in an enlarged space such as a space containing the original Hilbert space as a subspace. They based their work on the Cayley transforms of linear manifolds which is a key tool in studying symmetric and self-adjoint extensions to classify symmetric and self-adjoint extensions. Moreover, they explicitly described the resolvent of such extensions and characterized the spec-

tral measure/family associated with the self-adjoint extensions, which is essential for functional calculus and quantum mechanics applications.

De Monvel and Khorunzhy (de Monvel & Khorunzhy, 1999) carried out a study on the spectral properties of a class of large symmetric random matrices with weakly dependent Gaussian entries. They investigated the limiting behavior of the eigenvalue distribution of $N \times N$ symmetric random matrices, $H_N(x, y) = N^{-\frac{1}{2}}h(x, y)$, where $h(x, y)$ are weakly dependent Gaussian random variables. They also proved the almost sure convergence of the normalized eigenvalue counting function to a deterministic limit $\mu(\lambda)$ as $N \rightarrow \infty$. Moreover, they derived an equation for the Stieltjes transform of the limiting spectral measure $d\mu(\lambda)$ and analyzed its properties. Finally, they established that the spectral measure has a compact support.

In 2012, Trieu Le (Le, 2012) studied weighted composition operators on Hilbert spaces of analytic functions on the unit ball in $\mathbb{C}^n$. They characterized the adjoints by determining the conditions under which the adjoint of a weighted composition operator $W_{f,\psi}$ is another weighted composition operator or its inverse and extended the results from one-dimensional settings to higher dimensions. Additionally, they characterized self-adjoint, unitary, and normal weighted composition operators on Hilbert spaces H_γ with reproducing kernels of the form $(1 - \langle z, w \rangle)^\gamma$ for $\gamma > 0$. Moreover, they investigated the spectra of normal and unitary weighted composition operators, especially when ψ fixes a point in the unit ball.

Shi and Sun (Y. Shi & Sun, 2011) investigated the self-adjoint extensions for second-order symmetric linear difference equations. Their study was motivated by the challenge of defining maximal and minimal operators for singular difference equations, where the maximal operator may not be single-valued, and the minimal operator is often non-densely defined. The authors employed the Glazman-Krein-Naimark (GKN) theory for Hermitian subspaces to characterize self-adjoint extensions, addressing both subspace and operator extensions. The authors computed the defect indices and established their relationship with the limit-point/limit-circle classification of the difference equation. In addition, they provided a complete characteriza-

tions of self-adjoint subspace extensions in terms of boundary conditions, depending on whether the endpoints are regular or singular.

The authors in (Nyamwala, Ambogo, & Ngala, 2020) explored the spectral properties of self-adjoint extensions of minimal second-order difference and differential operators under specific conditions on unbounded odd coefficients. They Investigated and compared the spectral properties, that is, the discrete and continuous spectrum of self-adjoint extensions of second-order difference operators second-order differential operators while focussing on cases where the odd-order coefficient is unbounded but follows prescribed growth or decay conditions. In their work, the authors identified the specific growth or decay conditions on the odd coefficient that lead to purely discrete spectrum for difference operators and absolutely continuous spectrum of multiplicity one for differential operators.

In (B. Brown, Evans, & Wood, 2019), the authors studied the theory of positive self-adjoint extensions of symmetric operators, while focussing on applications to elliptic differential operators and the Aharonov-Bohm operator. Their work was an extension of the Krein-Vishik-Birman (KVB) theory, which establishes a one-to-one correspondence between positive self-adjoint extensions of a strictly positive symmetric operator A and pairs $\{N_B, B\}$, where N_B is a subspace of the kernel of A^* and B is a positive self-adjoint operator on N_B . The authors highlighted the Friedrichs extension (maximal) and the Krein-von Neumann extension (minimal) as extremal cases, with all other extensions lying between them in the form sense. They also linked the Krein-Vishik-Birman theory with von Neumann's theory of self-adjoint extensions, by using boundary triples to characterize extensions. Their analysis involved decomposing the operator using polar coordinates and studying its extensions via Sturm-Liouville theory and their results provided explicit boundary conditions for the extensions, with special attention to the limit-circle and limit-point cases.

Yao et al. (Yao, Sun, & Zettl, 2015) explored on the characterization of the Friedrichs extension for Sturm-Liouville operators in which they provided a complete characterization of the Friedrichs extension of the minimal operator associated with a Sturm-

Liouville equation. Unlike in the traditional approach, which describes Friedrichs extension as a restriction of the maximal domain the work of these authors characterized Friedrichs extension as an extension of the minimal domain. Their methodology leverages on the abstract Friedrichs construction in Hilbert spaces, making it more adaptable to higher-order problems, Hamiltonian systems, and difference equations rather than direct reliance on oscillation theory or regularization techniques, which are commonly used in prior works.

The authors in (Zheng & Kong, 2018) addressed the problem of characterizing and constructing Friedrichs extensions for singular Hamiltonian operators, especially in the case where the deficiency indices are intermediate, that is when the deficiency indices are neither minimal nor maximal. They used principal solutions, which are solutions of the Hamiltonian system that exhibit specific behavior at singular endpoints to construct explicit boundary conditions for the Friedrichs extensions. Their approach of refined LC-type (limit-circle-type) solutions at singular endpoints as a key tool in the Friedrichs construction allows for a more precise analysis of the behavior of solutions near singular points. Moreover, by unifying and generalizing the known results, the paper establishes a more comprehensive theory for Friedrichs extensions in the singular case.

Later on, Zheng and others in (Zheng, Qi, & Shao, 2023) investigated the Friedrichs extensions of Sturm–Liouville operators with complex coefficients, following the classification scheme introduced by Brown and others (B. M. Brown et al., 1999). They studied Friedrichs extensions for Sturm–Liouville operators with complex coefficients, which are generally non-self-adjoint. It is worth noting that unlike the classical real-coefficient case, complex coefficients introduce challenges in determining extensions due to the lack of symmetry. They explored spectral properties of these extensions, in which they located eigenvalues, essential spectrum and provided resolvent estimates. Additionally, they provided explicit descriptions of Friedrichs extensions at regular endpoints, via boundary conditions and at singular endpoints, via growth or decay conditions on functions. In our study however, we are only interested in operators with real-coefficients hence operators with symmetric operator

extensions.

The authors in (Stempak, 2021) studied the spectral properties of differential operators that can be decomposed into compositions of first-order delta-derivatives and their adjoints in which they analyzed the spectral characteristics such as, the eigenvalues, eigenfunctions and the essential spectrum of ordinary differential operators that admit special decompositions. In particular, they examined the Sturm-Liouville operators that can be decomposed as a product of the delta-derivative and its formal adjoint or similar forms, which arise in various applications, including orthogonal polynomial expansions. Additionally, the authors provided the conditions under which the operator L has a discrete spectrum, essential spectrum, or mixed spectral structure.

Stempak and the authors in (Stempak, 2023) investigated the spectral properties of symmetrized differential-difference operators, focusing on self-adjoint extensions, Friedrichs extensions, and associated Sobolev spaces. The authors introduced a symmetrization method for differential operators which leads to skew-symmetric operators and associated Laplacian-like operators. They also studied the self-adjoint extensions of the symmetrized operators, particularly the Friedrichs extension, which they characterized in terms of Sobolev spaces. Moreover, they defined weak derivatives and associated Sobolev spaces which are used to describe the domains of the minimal and maximal operators and established the relationships between weak derivatives and classical weak derivatives are established, leveraging the structure of the weak derivative and its coefficients.

Bao et al. (Bao, Wei, & Zettl, 2022) focused on the study of symmetric operators arising in the context of singular Sturm-Liouville problems and their Friedrichs extensions. Their main objective was to explicitly determine the boundary conditions that characterize the symmetric extensions of the minimal operator associated with singular Sturm-Liouville problems. Moreover, they identified the specific boundary condition that determines its Friedrichs extension for every symmetric extension that is bounded below. In their work, they addressed the challenges that are posed

by singular Sturm-Liouville problems, where the coefficients of the differential operator may have singularities or the interval defined may be unbounded. The authors derived these results using detailed asymptotic analysis of solutions near singular points and their relationship to the operator's spectrum.

In their recent paper which was an extension of their previous results, Bao et al. (Bao, Wei, & Zettl, 2023) obtained the matrices which determine all symmetric extensions of the minimal operators for singular even order symmetric differential operators. Moreover, they determined the boundary condition of the Friedrichs extension of each of the symmetric operator which is bounded below. On this, they established that there exists a Friedrichs extension for every symmetric extension of the operator of regular problems which are bounded below. None of these authors was interested in the minimal operators generated by the composite and their self-adjoint extensions. Therefore, in our study we extend these results to the composite operators and also obtain the discretized version.

2.4 Spectral Properties of Differential and Difference Operators

The spectrum of a bounded linear operator usually generalizes the idea of eigenvalues associated with that operator. With its applications being useful in various field of operator theory, functional analysis and even in numerical analysis, many researchers have been continuously providing their important and valuable inputs in the field of spectral theory by considering different types of operators under varied conditions imposed on their coefficients. Spectral theory was initially studied by Wenger (Wnger, 1975) for the cesaro operators and was further advanced by Rhoades (Rhoades, 1983).

Subsequently, a number of researchers have continued making their contributions in the field of spectral theory as discussed herein: Remling [(Remling, 1998), explored on the generalization of the classical Titchmarsh–Weyl m -function which is a

fundamental object in the spectral theory of second-order differential operators, to higher-order differential operators. The authors noted that while such generalizations exist, they often become unwieldy in practical applications. They therefore, proposed a Hilbert space interpretation of the m-function, and demonstrated that the classical m-function can be recovered as a component of its higher-order counterpart. In their work, the authors showed that the higher-order M-matrix contains the classical m-function as a sub-component, making it possible to extract simpler spectral information. They also provided new insights into the multiplicity of spectra for higher-order operators, linking it to the structure of the M-matrix and established the conditions under which the spectrum remains stable under perturbations, using the M-matrix's properties.

Additionally, Remling (Remling, 1999)] focused on analyzing the spectrum of a fourth-order self-adjoint differential operator under specific asymptotic conditions on the coefficients in which he investigated the location and nature of the spectrum of a fourth-order self-adjoint differential operator. The author also investigated how the asymptotic behavior of the coefficients influences the spectral properties. Moreover, the author extended the spectral theory techniques, typically developed for second-order operators such as, the Sturm-Liouville problems, to fourth-order equations, which are more complex due to increased dimensionality and boundary condition dependencies.

Shi and Yan (G. Shi & Yan, 2008) investigated the spectral properties of second-order left definite difference boundary value problems. The authors define a Hilbert space $H[1, N]$ equipped with an inner product $\langle ., . \rangle_H$ that incorporates the boundary conditions and coefficients p_n, q_n . They also introduced a self-adjoint operator L associated with the eigenvalue problem and proved its self-adjointness using a Dirichlet-type formula. They established that the spectrum of this self-adjoint operator L consists of N real eigenvalues $\lambda_1, \dots, \lambda_N$, with corresponding orthogonal eigenfunctions and that these eigenfunctions form a complete orthogonal set in $[H, 1]$, analogous to the continuous case.

Behncke, Hinton and Remling (Behncke et al., 2001) carried out a study on the spectral properties of higher-order ordinary differential operators with coefficients that are small perturbations of constants. Their work generalized classical results for second-order differential operators to higher-order operators and helped to understand how the spectrum, that is, the absolutely continuous, singular continuous, or point spectrum is affected when the coefficients slightly deviate from constants. The authors also extended the Weidmann's result for second-order Schrödinger operators to higher-order operators in which they established for the unperturbed case (constant coefficients), the spectrum is purely absolutely continuous and can be described via Fourier analysis. Their results showed that under small perturbations, the essential spectrum remains unchanged, and the absolutely continuous spectrum is preserved, though additional point spectrum may arise. Moreover, the authors established that the perturbed operator retains the same absolutely continuous spectrum as the unperturbed operator, with possible changes in multiplicity and that the singular continuous spectrum is empty under the given perturbations.

Elsewhere in (Behncke, 2002), Behncke investigated the spectral properties of higher-order differential operators with coefficients that are asymptotically constant. The author considered a differential operator whose coefficients are assumed to satisfy certain asymptotic conditions with an aim to characterize the absolutely continuous spectrum of self-adjoint extensions of this operator. The author employed the asymptotic integration methods to transform the eigenvalue equation into a system of first-order differential equations. He also used the Kummer-Liouville transformation as a central tool to simplify the system in which it is used diagonalize the system, simplifying the analysis of eigenfunctions and spectra. The author also established that under certain smoothness and decay conditions on the coefficients, the absolutely continuous part of any self-adjoint extension is unitarily equivalent to multiplication by a polynomial. Moreover the author characterized the essential spectrum in terms of the limiting behavior of the coefficients and gave the conditions for the absence of singular continuous spectrum and for the operator to have a compact resolvent.

Nyamwala (Nyamwala, 2010) explores the spectral properties of unbounded differential and difference operators under certain smoothness and decay conditions. The author studied the limit-point/limit-circle classification at infinity for difference operators with coefficients that are "almost constant" that is, asymptotically constant. He also analyzed the absolutely continuous spectrum of their self-adjoint extensions, particularly its relation to the limiting operator, the operator with constant coefficients in which he established that absolutely continuous spectrum of the self-adjoint extensions matches that of the limiting constant-coefficient operator.

Thereafter, Behncke and Nyamwala (Behncke & Nyamwala, 2012) focused on understanding how the eigenvalue distribution of the characteristic polynomial influences the spectrum, deficiency indices, and other spectral characteristics of the operator. The authors considered even-order linear differential operators where the coefficients are not necessarily bounded. They showed that the eigenvalues can be grouped into distinct clusters, meaning they can be partitioned into classes with similar asymptotic behavior. The clustering of the eigenvalues allows the spectral analysis to be decomposed into contributions from each group thereby simplifying the problem. The authors also showed that the spectral properties such as the deficiency indices and the essential spectrum of the operator can then be determined by the superposition of contributions from each cluster. They employed the techniques from asymptotic analysis based on a refined version of Levinson's theorem, to study the behavior of solutions to the differential equation as the independent variable tends to infinity.

In 2013, Behncke and Hinton (Behncke & Hinton, 2013) provided a comprehensive exploration of the spectral properties of higher-order differential operators. The authors employed the technique of asymptotic integration as a primary tool to analyze deficiency indices, absolutely continuous spectra, and singular spectra, while also addressing operators with unbounded coefficients. The authors gave an outline of a systematic approach to studying higher-order differential operators, converting them into first-order systems and diagonalizing them to analyze their spectral properties. Their use of quasi-derivatives provided a simplified way of handling the operators whose coefficients have limited differentiability, making the analysis more simplified.

Additionally, the authors demonstrated how the deficiency indices of operators can be determined using asymptotic integration, particularly for operators with bounded and unbounded coefficients. Moreover, they showed that the absolutely continuous spectrum can be derived from the characteristic polynomial of the operator, with its multiplicity linked to the number of real roots of the polynomial.

Ren and Shi (Ren & Shi, 2010) showed that the maximal difference operator risk being multi-valued and not densely defined. In order to rescue such a scenario, Ren and Shi [(Ren & Shi, 2011), (Ren & Shi, 2014)] reverted to using subspace theory to analyse the deficiency indices and spectrum of such difference equations with the only option of reverting to operator theory approach if and only if the maximal difference operator are densely defined and single-valued. In their study, Ren and Shi (Ren & Shi, 2011) proved that the defect index of the minimal subspace and the number of linearly independent solutions that are square summable are equal if and only if the definiteness condition is satisfied.

Sun and Ren (Sun & Ren, 2013) investigated the properties and extensions of second-order linear difference equations with complex coefficients, focusing on their J -self-adjoint extensions. In their study, they Characterized J -self-adjoint subspace extensions and J -self-adjoint operator extensions of the minimal subspace associated with the difference operator. additionally, they provided the boundary conditions that describe these extensions. Moreover, they introduced the maximal, pre-minimal, and minimal subspaces and analyzed their properties, including defect indices and boundary forms. With the concrete examples they provided, their paper was able to give illustrations on such cases where the equation is in the "limit circle" or "limit point" case at infinity, and how boundary conditions like Dirichlet or Neumann arise.

In a study carried out on the Positive and Negative Deficiency Indices of Formally Self-Adjoint Difference Equations, the authors in (Ren, 2022), explored on the properties of formally self-adjoint difference equations, focusing on their deficiency indices and the implications for self-adjoint extensions. In their study, they established that the order of such equations must be even, which contrasts with differential equations

where odd-order formally self-adjoint equations can exist under specific conditions. Moreover, their results showed that for formally self-adjoint difference equations, the positive and negative deficiency indices are always equal ($d_+ = d_-$) which is a significant departure from differential equations, where d_+ and d_- can differ. Additionally, their paper highlighted a fundamental difference between difference and differential equations in which they established that for differential equations, the deficiency indices need not be equal, leading to more restrictive conditions for self-adjoint extensions.

In contrast, difference equations always admit such extensions due to the equality of deficiency indices. Baloev, (Baloev, 2019) in their study on the universal form of a solution to a stationary linear difference equation, derived a universal form of the general solution to a stationary linear difference equation that works for any type of roots (that is: real, complex, distinct, or repeated) of the characteristic equation, except zero roots. This they did in an attempt to make the solution more practical by eliminating the need for case-by-case analysis based on root properties which is particularly useful in computational implementations where a single formula can be programmed without conditional checks. Moreover, they simplified they simplified the algorithm for obtaining the general solution, thereby, reducing computational complexity and improving efficiency in solving such equations.

Later on, in their study on the General Solution of Difference Equations with Constant Coefficients, Baloev and Lazareva, (Baloev & Lazareva, 2021) proposed a method for expressing solutions to linear difference equations with constant coefficients, which aimed to simplify the traditional approach by eliminating the need to account for root multiplicities in characteristic equations. Their method simplified the solution process, particularly for higher-order equations or equations with repeated roots. It involved reparameterizing the solution space using techniques from linear algebra or generating functions, bypassing the need for direct root analysis.

Ibrogimov and other authors (Ibrogimov, Siegl, & Tretter, 2016), while analysing the essential spectrum of singular matrix differential operators, focused on the study

of the essential spectrum of a specific class of matrix differential operators that exhibit singular behavior. They employed a combination of asymptotic analysis, Schur complement techniques and Nevanlinna behavior which enabled them to carry out a deep and rigorous investigation of the essential spectrum. In their study, they provided a complete and explicit description of the singular part of the essential spectrum, developed criteria for establishing the presence or absence of singular spectrum, which is a specific component of the essential spectrum arising from the singular behavior of the operator, analyzed the topological structure and essential spectral radius and finally applied the obtained results to solve open problems, such as the conjecture on stellar equilibrium models.

In their study of behavioral properties of solutions of differential equations near infinity, Agarwal et al (Agarwal et al., 2002), presented a contribution on the asymptotic behavior of solutions of second order differential equations as well as the behavioral properties of positive solutions of singular Emden-Flowler-type equations. In this, they showed that besides the existence of nonoscillatory solutions, an explicit asymptotic form of the nonoscillatory solutions may be provided. They further provided the necessary and sufficient conditions for the solutions to have a specified asymptotic behavior as $t \rightarrow \infty$ by imposing restrictions on the sign of the integrable coefficients of the equation.

Dhaoudadi (Dhaouadi, 2007) explored on the spectral theory associated with second order q -difference operators. In his study, they applied Green's function to analyse the boundary value problems associated with the q -difference operator which involves obtaining solutions to the q -difference equation that satisfy certain boundary conditions. Additionally, they discussed the completeness property of the eigenfunctions which ensures that any function in the relevant function space can be expressed as a series of eigenfunctions.

Gingold and Xue (Gingold & Xue, 2007) proposed and studied a new technique for asymptotically summing solutions to linear difference systems that provides sufficient conditions for asymptotic summation. Their work demonstrates that well-

known theorems that is the Levinson's and Hartman-Wintner's Theorems emerge as special cases of the proposed framework. The authors introduced a summation equation that captures the asymptotic structure of solutions rather than relying on direct estimates or transformation techniques such as diagonalization and successive approximations. In addition, they showed that Levinson's theorem which relates to diagonalizable systems with dichotomies and Hartman-Wintner's results which are as a result of asymptotic integration can be derived as corollaries of the new framework.

Eastham (M. Eastham, 1982) considered a critical class of fourth order equations which are characterized by specific structural properties that make their asymptotic analysis challenging. In this study, Eastham employed the Wentzel-Kramers-Brillouin techniques to derive the asymptotic expansions of differential equations. These asymptotic expansions for the solutions describe the behavior of solutions at infinity. Moreover, Eastham provided a precise error estimates for the asymptotic approximations which ensures that the results obtained are accurate. Additionally, he used perturbation methods to handle critical case, where small changes in the coefficients can lead to significant changes in the asymptotic behavior of the solutions. In our work, we have used these Eastham's approximations to obtain the roots of the fourier polynomial generated by the differential and difference equations.

On their advancement on the results on spectral theory, Kozko and Alexander Pechentsov (Kozko & Pechentsov, 2010) carried out a study on the spectral function of a singular differential operator of order $2m$. On this, their main focus was on analyzing the spectral function of a self-adjoint, semibounded differential operator on a Hilbert space, particularly under general boundary conditions at zero. They provided an explicit formulae for the spectral function of a singular differential operator under general boundary conditions which enables one to directly compute spectral densities. Their work offers a gap in which we exploit by extending these results to higher-order differential and difference operators in addition to their composites.

Chebli and Mahmoud, (Chebli & Mahmoud, 2004) investigated the asymptotic anal-

ysis of spectral objects associated with a singular second-order differential operator with matrix-valued coefficients. They analyzed the asymptotic behavior of the Harish-Chandra function which is a key object in harmonic analysis and representation theory in the context of a singular matrix-valued differential operator. They based their study on a detailed asymptotic analysis of certain eigenvectors of the operator, which helps in understanding the spectral properties. Their results shows how the Harish-Chandra function decays or grows at infinity, in terms of exponential or power-law asymptotics, depending on the operator's structure.

Garkavenko et al (Garkavenko, Zgolich, & Uskova, 2018) carried out a study of the spectral properties of a second order difference operator with a growing potential in which they explored on the spectral properties of a second-order difference operator acting in the Hilbert space of square-summable sequences. They employed the method of similar operators to derive asymptotic estimates for eigenvalues, eigenvectors, and spectral projections. Additionally, they used perturbation theory to derive asymptotic expansions for eigenvalues and eigenvectors, relying on the growing nature of the potential to ensure small error terms. While their theoretical results are robust, their paper lacks numerical illustrations to demonstrate the practical utility of the estimates.

A study carried out by Bichegkuev (Bichegkuev, 2013), investigated the spectral properties of difference and differential operators acting in weighted sequence and function spaces. Their study extended the results to differential operators with an unbounded operator coefficient acting in weighted function spaces. They characterized the spectrum of a difference operator of the form: $(Lx)_n = x_{n+1} - Ax_n$ where A is a bounded linear operator in a Banach space B , and L acts on weighted spaces $(\mathbb{Z}, B, w)$ of two-sided sequences with values in B and analyzed the spectrum in terms of the spectrum of A and the properties of the weight function w . Furthermore, they investigate how the choice of the weight function w influences the spectrum of L

In their study on Spectral analysis of differential operators with unbounded operator-

valued coefficients, difference relations and semigroups of difference relations, Baskakov (Baskakov, 2009) investigated the differential operators with operator-valued coefficients in a function space setting, with special attention given to their invertibility and Fredholm properties. They examined the linear differential operators where the coefficients are themselves unbounded operators acting on a space of functions defined on the semi-axis. They derived the necessary and sufficient conditions for such operators to have a bounded inverse or to be Fredholm, that is, to have finite-dimensional kernel and cokernel with closed range.

Later on, in the paper by Baskakov, (Baskakov, 2013) explored on the connections between linear differential equations with unbounded operator coefficients and the spectral properties of the associated differential and difference operators. In their paper they established a correspondence between a linear differential operator with unbounded coefficients and an associated linear difference operator in which they showed that their spectral properties, such as invertibility, Fredholmness, and stability, are closely related. They also introduced the concept of a state of a linear relation which is a generalization of multivalued linear operators. This captures the properties like the dimension of the kernel, codimension of the range, and the existence of complemented subspaces and are used to classify the behavior of solutions such as boundedness, almost periodicity, stability. Moreover, they derived the necessary and sufficient conditions for the differential operator and its associated difference relation to have the Fredholm property.

Additionally, Baskakov and the authors in (Baskakov & Duplischeva, 2015), investigated the second-order linear difference operators, with their focusing being on their invertibility, Fredholm properties, and asymptotic behavior of solutions. They determined the necessary and sufficient conditions under which a second-order linear difference operator is invertible or Fredholm. Moreover, they derived the asymptotic formulas for solutions of homogeneous second-order difference equations which are helpful in understanding the long-term behavior of solutions. Their work provides an illustration on the conversion of second-order difference operators to first-order matrix systems, enabling one to address questions of invertibility, Fredholmness,

and asymptotics in the second-order difference operators.

A study carried out by Campbell et al. (Weikard, 2020) on the spectral theory for first-order systems without the unique continuation property, investigated the spectral theory of first-order differential systems where the coefficients are distributions of order zero (measures), and the unique continuation property for solutions may fail. They considered the first-order system of the form $Ju' + qu = wf$ on an interval (a, b) , where: J is a constant, invertible skew-Hermitian matrix, q is a Hermitian matrix of distributions (measures) and w is a non-negative matrix of distributions in which they provided explicit formulas for solutions in intervals between discontinuities and derived the matching conditions at jump points to piece together global solutions. They further established that the adjoint of the minimal operator $T_{\min}$ (solutions with compact support) is the maximal operator $T_{\max}$ (all solutions), even when unique continuation fails.

Mutlu (Mutlu, 2020) investigated the spectral properties of a discrete Sturm-Liouville operator defined on the whole axis with self-adjoint operator coefficients. The author's major focus was on determining the continuous and discrete spectrum of the discrete operator L generated by a second-order difference equation with self-adjoint operator coefficients. Mutlu identified the continuous spectrum as $[-2, 2]$, while the discrete spectrum established to be lying outside this interval. The author also derived explicit expressions for the Jost solutions (which are crucial for understanding the spectral properties of the operator) of the difference equation and analyzed their analytical and asymptotic properties. Moreover, the author proved the condition on the coefficient under which the operator L has a finite number of eigenvalues.

The authors in (Mehrabov, 2022) studied the spectral properties of one eigenvalue problem for the fourth-order differential equations with a spectral parameter contained in two of the boundary conditions. In this study, it was established that the spectral problem arises when the fourier method is applied to a boundary value problem for partial differential equations that describes small bending vibrations of a homogenous rod under the action of longitudinal force in cross sections. The

arrangement of multiplicities of the eigenvalues on the real axis were determined, the asymptotic behaviour of these eigenvalues and the eigenfunction of this spectral problem were studied and finally, the sufficient conditions under which the system of root functions with two removed functions is a basis in L_p space were found.

The authors in (Polyakov, 2020) focused on the spectral analysis of a fourth-order differential operator with matrix coefficients, subject to quasi-periodic boundary conditions. They studied the spectral properties of a fourth-order differential operator with matrix coefficients by analyzing the distribution and behavior of its eigenvalues. In particular, they found the asymptotics of the arithmetic mean of the eigenvalues and, in certain special cases, the asymptotics of the individual eigenvalues. Moreover, they investigated the spectral characteristics of the operator under periodic and antiperiodic boundary conditions, which are special cases of quasi-periodic boundary conditions.

In their study on Spectral Estimates for the Fourth-Order Differential Operator with Periodic Coefficients, the authors in (Polyakov, 2022) studied the spectral properties of a self-adjoint fourth-order differential operator defined on the unit interval. They considered an operator which has real, 1-periodic coefficients, and its domain determined by Neumann–Dirichlet boundary conditions. They focused on deriving the asymptotic behavior of the eigenvalues of the operator, specifically in the high-energy regime (that is, for large eigenvalues) which involved understanding how the eigenvalues grow or distribute as their magnitude increases. The asymptotic formulas which the authors explicitly derived for the eigenvalues of the fourth-order differential operator in the high-energy regime provide insights into the growth rate and distribution of the eigenvalues of which in addition to Eastham’s approximations have been applied in this work.

Additionally, in a study on The Spectral Properties of a Two-Term Fourth-Order Operator with a Spectral Parameter in the Boundary Condition, Polyakov (Polyakov, 2023b) investigated the spectral problem for a fourth-order differential operator with a non smooth potential and boundary conditions that include a spectral parameter.

On this, they defined the operator on an interval, and its boundary conditions involved a spectral parameter. This work by Polyakov's contributed to the spectral theory of higher-order differential operators by deriving eigenvalue asymptotics and establishing a regularized trace formula for a two-term fourth-order operator with a non smooth potential and boundary conditions involving a spectral parameter.

Later on, in a study of Eigenvalue Asymptotics and a Trace Formula for a Fourth-Order Differential Operator, Polyakov (Polyakov, 2023a), focused on the spectral analysis of a self-adjoint fourth-order differential operator with real, 1-periodic, nonsmooth coefficients, defined on the unit interval with Neumann-type boundary conditions. They investigated the asymptotic behavior of the eigenvalues of the operator in the high-energy regime and established a trace formula which relates the eigenvalues (spectral data) to the coefficients of the operator (geometric or physical data), providing a deeper connection between the operator's structure and its spectral properties. The authors also addressed the challenges posed by the non-smooth nature of the periodic coefficients, which adds complexity to the analysis compared to the smooth case.

Motivated by applications in quantum mechanics and other areas where non-self-adjoint operators arise naturally, Behncke and Hinton (Behncke & Hinton, 2020) explored on the spectral theory of second-order linear differential operators with complex coefficients, focusing on operators that are C -symmetric. They determined the spectrum of C -symmetric second-order differential operators on half-intervals or the real line in which they showed that under certain conditions, the essential spectrum is empty, and the operator has a compact resolvent. Moreover, they established that the spectrum is discrete, and the eigenvalues can have higher algebraic multiplicities, unlike the self-adjoint case where multiplicities are always one. The authors also constructed the resolvent (Green's function) using the Titchmarsh-Weyl m -function, which is defined via square-integrable solutions rather than the traditional nesting circle analysis and went ahead to analyze the behavior of the resolvent and m -function near their poles, with explicit representations provided for higher-order poles.

Recently, Behncke and Hinton (Behncke & Hinton, 2023) in their study on the spectral analysis of differential operators with complex coefficients, explored the spectral properties of higher-order differential operators with complex coefficients. The authors extended the spectral analysis from Sturm-Liouville operators (second-order) to higher-order differential operators with complex coefficients. They gave Conditions for the essential spectrum for an operator T to be empty or non-trivial. For example, if coefficients lie in a convex cone, then the essential spectrum $\sigma_{ess}(T) \neq \mathbb{C}$. They also analyzed the existence and compactness of the resolvent in which they discussed the Hilbert-Schmidt resolvents under specific coefficient decay conditions. Additionally, they used the numerical range $\mathcal{N}(T)$ to localize the spectrum, with $\sigma_{ess}(T) \subseteq \overline{\mathcal{N}(T)}$. These results were obtained for independent differential and difference operators. In addition, there was no focus on the self-adjoint extensions of these operators. However, in this study we have considered the composites of differential and difference operators, established the existence of their self-adjoint operator extensions and have synthesized their spectral properties in cases of unbounded coefficients.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, we have given an outline of the methods that we have applied to achieve the objectives of this research study. In particular, an illustration on how the symmetric composites for differential and difference operators are constructed is given. We have given an overview of the system formulation, Asymptotic integration and Asymptotic summation, the existence of the self-adjoint operator extension and finally, an outline of how an M-matrix is used to carry out the spectral analysis have been discussed.

3.2 Construction of Symmetric Operator Extensions

The construction of symmetric operator extensions has been done via the Cayley transform as given in (Schmüdgen, 2012) and illustrated as follows: Suppose L is a densely defined symmetric linear operator on H and $\lambda \in \mathbb{C}$ such that $\text{Im}\lambda > 0$, then we have that $\bar{\lambda}$ is in the resolvent of L . Therefore, $L - \bar{\lambda}I$ is invertible. The Cayley transform of L is the operator

$$V_L = (L - \lambda I)(L - \bar{\lambda}I)^{-1}$$

whose domain is given by $D(V_L) = \mathcal{R}(L - \bar{\lambda}I)$ and the range given by $\mathcal{R}(V_L) = (L - \lambda I)$. Thus for any $x \in D(L)$, we have that $V_L(L - \bar{\lambda}I)x = (L - \lambda I)x$. The Cayley transform helps to reduce the work of describing all the symmetric extensions of L to just a matter of finding all the isometric extensions of the Cayley transform of V_L . Thus if K is a symmetric extension of L , then V_K is an isometric extension of V_L . Conversely, if V is an isometric operator on H such that $V_L \subseteq V$, then V is the Cayley transform of the symmetric operator K and $L \subseteq K$ since, $\mathcal{R}(I - V_L) \subseteq \mathcal{R}(I - V)$ and $\mathcal{R}(I - V_L)$ and $\mathcal{R}(I - V)$ are dense. The following properties of the Cayley transform as given in the proposition below, have been useful in the construction of the symmetric operator.

Proposition 3.2.1. (i) *The Cayley transform V_L is an isometric operator on H with dimension $D(V_L) = \mathcal{R}(L - \bar{\lambda}I)$ and the range $\mathcal{R}(V_L) = \mathcal{R}(L - \lambda I)$*

(ii) *$\mathcal{R}(I - V_L) = D(V_L)$ and $L = (\lambda I - \bar{\lambda}V_L)(I - V_L)^{-1}$*

(iii) *L is closed if and only if V_L is closed.*

(iv) *If K is another symmetric operator on H , then $L \subseteq K$ if and only if $V_L \subseteq V_K$.*

Hence, upon the construction of the composites of differential and difference operators, we have applied the Cayley transforms in the construction of the symmetric operator extensions of the composites.

Additionally, we have applied the concept of Green's function in constructing the appropriate isometric isomorphisms corresponding to H_r and comparison algebras. This is constructed as follows:

$$H_r^{-1}y(t) = \sum_0^{\infty} G_r(t, t_1)y(t_1), \quad y(t) \in \ell^2(I_1).$$

and the corresponding isometric isomorphism operator W_r by

$$W_r = H_r^{-\frac{1}{2n}} = \sum_0^{\infty} (-1)^l \binom{\frac{1}{2n}}{l} (I - H_r^{-1})^l, \quad r = 1, 2, \quad n = m. \quad (3.1)$$

Let $\sigma(H_r)$, $\sigma_{ac}(H_r, l)$, $\sigma(H)$ and $\sigma_{ac}(H, l)$ denote the spectrum of H_r , absolutely continuous spectrum of H_r with spectral multiplicity of l , spectrum of H and absolutely continuous spectrum of H with spectral multiplicity of l respectively. In

order to determine the self-adjoint extensions after the computation of the deficiency indices, we have applied von Neumann theorem which states that if the deficiency indices are equal in both the upper and the lower half planes, that is, $N_+(L) = N_-(L) = n$, $n \neq 0$, then the operator (L) has infinitely many self-adjoint extensions.

Theorem 3.2.2. *(von-Neumann Theorem, see [(Schmüdgen, 2012), Theorem 13.10])*

A densely defined symmetric operator L on a Hilbert space H possesses a self-adjoint extension on H if and only if $N_+(L) = N_-(L)$. If $N_+(L) = N_-(L)$, then all self-adjoint extensions of L are given by the operators L_U , where U is an isometric linear mapping of $N(L^* - \lambda I)$ onto $N(L^* - \bar{\lambda}I)$, and

$$\begin{aligned} D(L_U) &= D(\bar{L}) \dot{+} (I - U)N(L^* - \lambda I) \\ L_U(x + (I - U)z) &= \bar{L}x + \lambda z - \bar{\lambda}Uz \end{aligned}$$

for $x \in D(L)$ and $z \in N(L^* - \lambda I)$.

Here, $L_U x = Lx$, for all $x \in D(L)$. In addition, by constructing the appropriate comparison algebras of the corresponding self-adjoint operator extensions and application of asymptotic integration under suitable decay and growth conditions imposed on the coefficients, we have deduced that the operators commute if they are of the same order and the absolutely continuous spectrum of their self-adjoint extensions is the whole of real line. Using some examples, we have shown that some of these necessary conditions cannot be weakened further.

3.3 System Formulation

In order to achieve the second, third and fourth objectives, we converted the composites of the higher order differential and difference operators into their first order systems and computed their quasi-derivatives and quasi-differences respectively as developed by Shi (Y. Shi, 2006). These are given in (1.2) and (1.9) in Chapter 1, Section 1.1.1 and 1.1.2 respectively. For the second objective, we applied the quasi-derivatives and quasi-differences to describe maximal and the minimal operators generated by the composites after which we applied the Cayley-transforms

and von-Neumann theorems to investigate on the existence of the symmetric operator extensions as well as the self-adjoint operator extensions of the composites. To achieve the third objective, we obtained the characteristic polynomial of the respective differential and difference operators whose roots we have approximated using Eastham's approach (M. S. P. Eastham, 1989). The deficiency indices have been read off from the asymptotics of the eigenfunctions.

3.4 Asymptotic Integration

It is worth noting that asymptotic integration is a perturbation result and since singular continuous spectrum is not stable even under finite rank perturbation, it will be assumed to be absent in our case. Asymptotic integration is anchored on Levinson's Theorem which states that the solutions of the system $Y' = (\Lambda + R)Y$ looks like the solutions of $Y' = \Lambda Y$ if $\Lambda = \text{dia}(\lambda_k(x))$ satisfies some dichotomy conditions and entries of R are absolutely integrable. If we paraphrase the spectral parameter version of the Levinson theorem as given in (Behncke et al., 2001) we have

Theorem 3.4.1. *Let*

$$Y'(x, z) = (\Lambda(x, z) + R(x, z))Y(x, z) \quad (3.2)$$

be a first order form of $\tau_r y = zy$, then the system has solutions of the form

$$y_l(x, z) = (e_l(x, z) + r_l(x, z)) \exp \left(\int_0^x \lambda_l(t, z) dt \right) \quad (3.3)$$

if $\lambda_l(x, z)$, of first order satisfies the z -uniform dichotomy condition and $R_{ij}(x, z)$ is absolutely integrable. Here, $\Lambda(x, z) = \text{diag}(\lambda_l)_{l=1,2,\dots,2s}$ and $R_{ll}(x, z) = 0$. $e_l(x, z)$ is the normalized eigenvector, $r_l(x, z)$ is the correction term as a result of repeated diagonalizations. The dichotomy condition implies that for any $l \neq j$, $\text{Re}(\lambda_l(x, z) - \lambda_j(x, z))$ is of constant sign modulo integrable terms.

For one to apply the results of Theorem 3.4.1 in order to obtain form of solutions of (1.3), it requires that we determine the eigenvalues of the matrix $\mathcal{C}(x, z)$ which are

the zeros of the characteristic polynomial $\mathcal{P}(x, \lambda, z) = \det(\mathcal{C}(x, z) - \lambda I_{2s})$. Multiplying this characteristic polynomial by f_s , $f = p_k, q_j$, $s = m, n$ we obtain a Fourier polynomial of the form

$$f_s \mathcal{P}(x, \lambda, z) = \mathcal{F}(x, \lambda, z) = \sum_{l=0}^s (-1)^l f_l \lambda^{2l}, \quad (3.4)$$

whose zeros can now be solved either by approximation or explicitly depending on the growth conditions of the coefficients f_l and the value of s . Additionally, one also requires to establish the z -uniform dichotomy condition which state that the real parts for any two eigen values λ_l and λ_j , $l \neq j$ are distinct. By application of results in (Behncke & Nyamwala, 2011), this is done only for those λ -roots that are pure imaginary. Finally, we will have to transform (1.3) into (3.2) through diagonalizations which requires computation of diagonalizing matrix $Q(x, z)$ normally obtained from the corresponding eigenvectors. For the eigenvectors, in the quasi-derivatives (1.2), a *derivative* is replaced by λ and thus for eigenvalue λ_l one has

$$v_l = \{1, \lambda_l^1, \dots, \lambda_l^{s-1}, f_s \lambda_l^s, \dots, \}.$$

To achieve the form (3.2), we require some decay conditions which are given by

$$\frac{f'_l}{f_l} \in \mathcal{L}^2, \quad \frac{f''_l}{f_l}, \left(\frac{f'_l}{f_l} \right)^2 \in \mathcal{L}^1. \quad (3.5)$$

3.5 Asymptotic Summation

Asymptotic summation is a straight forward method that is anchored on Levinson's Benzaid-Lutz Theorem 1.1.2 where we compute the eigenvalues of the matrix $S(t, z)$ in equation (1.12), establishing the uniform dichotomy condition and finally transforming (1.12) into the almost diagonal system depending on the decay conditions of the coefficients. This shall require computation of eigenvalues of $S(t, z)$, the corresponding eigenvectors and some diagonalizations. Thus for the eigenvalues, we need the characteristic polynomial $\mathcal{P}(t, \lambda, z) = \det[S(t, z) - I_{2n}]$. If we multiply $\mathcal{P}(t, \lambda, z)$ by $(-1)^s h_s \lambda^{-s}$, $h_s = p_n, q_m$, $s = n, m$, we obtain the fourier polynomial

$$\mathcal{F}(t, \gamma, z) = (-1)^s h_s \lambda^{-s} \mathcal{P}(t, \lambda, z) = \sum_{l=0}^s h_l \gamma^l, \quad (3.6)$$

where $\gamma = (1 - \lambda)(1 - \lambda^{-1})$.

3.6 Dichotomy Condition

The uniform dichotomy condition for the eigenvalues of the differential and difference operator are established off the real axis. As stated earlier in Levinson-Theorem, the uniform dichotomy condition needed amounts to: for any unequal pair of indices i and j , $\text{Sign } \text{Im}(\lambda_i(x, z) - \lambda_j(x, z))$ is constant modulo $\mathcal{L}^1$. This is because the roots of the polynomial $\mathcal{P}(x, \lambda, z)$ are calculated from the characteristic Fourier polynomial, $\mathcal{F}(x, \lambda, z)$. Similarly, for the difference operator, Levinson's-Benzaid-Lutz theorem states that the solution of a system

$$y(t+1) = [\Lambda(t) + R(t)]y(t)$$

where $\Lambda(t)$ is diagonal and invertible, look like the solutions of the unperturbed system $y(t+1) = [\Lambda(t)]y(t)$ if $R(t)$ is absolutely summable. That is, for all $i = 1, \dots, 2n$, $\lambda^{-1}R(t) \in \ell$ and $\Lambda(t) = \text{diag}(\lambda_i(t))$ satisfy the dichotomy condition. In this case, the dichotomy condition needed will amount to: for any pair of unequal indices i and j , and assume that there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t)| \geq \delta$ for all $t \geq t_0$, then either, $|\frac{\lambda_i(t)}{\lambda_j(t)}| \geq 1$ or $|\frac{\lambda_i(t)}{\lambda_j(t)}| \leq 1$ for a large t . A result that simplifies the proof of the uniform dichotomy condition as given by Nyamwala is as illustrated below.

Theorem 3.6.1. *Let*

$$y(t+1, z) = [\Lambda(t, z) + R(t, z)]y(t, z) \quad t \geq t_0, \quad (3.7)$$

$$\Lambda(t, z) = \text{diag}(\lambda_1(t, z), \dots, \lambda_{2n}(t, z))$$

be asymptotically constant difference equation such that

$$\sum_{t=t_0}^{t-1} \|R(t, z)\| |\lambda_i^{-1}(t, z)| < \infty.$$

Assume that the eigenvalues $\lambda_i(t, z)$ for $i = 1, \dots, 2n$, satisfy $\lambda_i(t, z) = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2}$, with $\lambda_{i,0}$ constant, $\lambda_{i,1}(t, z) \rightarrow 0$ as $t \rightarrow \infty$, $\lambda_{i,2}$ is conditionally summable and $\lambda_{i,0}$ distinct. Let $h(t) > 0$ be a nonsummable, monotonic function on $\mathbb{N}$ and assume that the eigenvalues can be sorted into classes $C_1, \dots, C_n$ so that if $\lambda_i(t, z), \lambda_j(t, z) \in C_k$, then $(\frac{|\lambda_i(t, z)|}{|\lambda_j(t, z)|} - 1) = o(h(t))$; if $\lambda_i(t, z) \in C_k, \lambda_j(t, z) \in C_l, k \neq l$ then $(\frac{|\lambda_i(t, z)|}{|\lambda_j(t, z)|} - 1) \leq$

$1 - h(t)$ or $(\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|} - 1) \geq 1 + h(t)$. For each $\lambda(t, z)$, now write $|\lambda(t, z)| = 1 + \nu(t, z)$ with $\nu_+ = \max(0, \nu)$ and $\nu_- = \min(0, \nu)$ and for each class k , define,

$$a_k(t, z) = \max \nu_+(t, z) \text{ and } b_k(t, z) = \max \nu_-(t, z).$$

The above conditions are assumed to hold uniformly for $z \in \mathcal{K}$. Then associated to each C_k there are $|C_k|$ ($|C_k|$ is the number of elements in the k^{th} class) solutions satisfying

$$K_1 \prod_{s=t_o}^{t-1} (1 - b_k(s, z)) \leq \|y(t, z)\| \leq K_2 \prod_{s=t_o}^{t-1} (1 + a_k(s, z))$$

for all $t \geq t_o$.

The implication of this theorem is that the eigenvalues $\lambda_k(t, z)$ such that $|\lambda_k(t, z)| > 1$, for $t \geq t_o$ lead to eigensolutions that are non-square summable whereas the eigenvalues which are such that $|\lambda_k(t, z)| < 1$ lead to solutions that are uniformly square summable. For these sets of eigenvalues, the establishment of the uniform dichotomy condition is not required. The uniform dichotomy condition is therefore only established off the real axis for the eigenvalues that are pure imaginary by letting $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$ for some small $\eta > 0$

3.7 Diagonalisation of the System

The appropriate eigenvectors can be evaluated directly from the quasi-differences (1.9) by replacing Δ by $(\lambda - 1)$ and $y(t - l)$ by λ^{-l} .

The matrix $T(t, z) = [v_1, \dots, v_s]^{tr}$ is now used to diagonalise equation (1.12). We make the transformation $\chi(t, z) = T(t, z)Y(t, z)$ which results into

$$\begin{aligned} \chi(t+1, z) &= T^{-1}(t+1, z)S(t, z)T(t, z)\chi(t, z) \\ &= [\Lambda(t, z) + \mathfrak{R}(t, z)]\chi(t, z) \end{aligned}$$

where,

$$\mathfrak{R}(t, z) = -T^{-1}(t+1, z)\Delta T(t, z)\Lambda(t, z) \quad (3.8)$$

and

$$\Lambda(t, z) = \text{diag}(\lambda_l(t, z)), l = j, k = 1, \dots, s.$$

Here, $\mathfrak{R}(t, z)$ consists of terms in $\ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N})$. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t, z)$, $k = 1, 2$. In (3.8), $\Delta T(t, z) = T(t+1, z) - T(t, z)$. The second diagonalisation is done by use of the eigenvectors of the matrix $[\Lambda(t) + \mathfrak{R}(t, z)]$. Using the results of Behncke and Hinton (Behncke & Hinton, 2011), a matrix of the form $[I + B(t, z)]$ with

$$B_{kk}(t, z) = 0, \quad B_{kj}(t, z) = (\lambda_j - \lambda_k)^{-1}\mathfrak{R}_{kj},$$

$$k \neq j, \quad k, j = 1, \dots, s., \quad t \geq t_0,$$

will be required for the second diagonalisation. The second diagonalisation results into correction terms added to the diagonals given by $(\Lambda_2)_{kk} = \text{diag}((\mathfrak{R}B)_{kk})$. The second diagonalisation is thus done using the transformation

$$\psi(t, z) = [I + B(t, z)]\chi(t, z)$$

and which results into a system of the form

$$\psi(t+1) = \{[\Lambda(t, z) + \Lambda_2(t, z)] + [I + B(t+1, z)]^{-1}\mathfrak{R}(t)[I + B(t, z)]\}\psi(t, z)$$

where $(\Lambda_2) = \text{diag}(\mathfrak{R}B)_{kk}$.

One therefore obtains Levinson-Benzaid-Lutz (LBL) form of

$$\psi(t+1, z) = [\Lambda(t, z) + R(t, z)]\psi(t, z)$$

to which we apply Theorem 1.1.2 to obtain the solutions of the form

$$Y(t, z) = T(t, z)[I + B(t, z)][E(t, z) + o(I_s)] \prod_{t_0}^{t-1} \Lambda(t, z)$$

if we apply backward transformations. Note that the matrices $T(t, z)[I + B(t, z)]$ shall be bounded and the asymptotics of the eigensolutions shall depend on the matrix $\prod_{t_0}^{t-1} \Lambda(t, z)$. Explicitly, we have

$$y_k(t, z) = \rho_k(t, z)[e_k(t, z) + r_{kk}(t, z)] \prod_{t_0}^{t-1} \lambda_k(t, z)$$

where $\rho_k(t, z)$ is a bounded function given by $(T(t, z)[I + B(t, z)])_{kk}$, $e_k(t, z)$ is a normalised eigenvector while $r_{kk}(t, z) \in \ell^1(\mathbb{N})$ and tends to zero as $t \rightarrow \infty$.

The spectral results of H_s in the case of the composite operators, are now obtained via the M -matrix [(D. Hinton & Shaw, 1981), (Remling, 1998), (Y. Shi, 2006)]. In higher dimensions, the M -matrix is equivalent to the classical Titmarsh-Weyl m -function. In order to construct the M -matrix associated with the first order system of the composite operators, we let

$$\Omega_r(t, z) = \begin{bmatrix} \Omega_{r_1}(t, z) \\ \Omega_{r_2}(t, z) \end{bmatrix}$$

be the set of square summable solutions satisfying boundary conditions (1.13) at $t_0 = 0$ with $\alpha_1 = I_s$, the identity matrix of order s , and $\alpha_2 = 0_s$. Then applying boundary conditions at the left regular endpoint at t_0 , we have

$$M_r(t_0, z) = \Omega_{r_2}(t_0, z)\Omega_{r_1}^{-1}(t_0, z).$$

Here, $\Omega_{r_1}(t_0, z)$ is the Vandermonde's matrix for the eigenvalues of square summable solutions and it is easy to show that $\Omega_{r_1}^{-1}(t_0, z)$ exists and $M_r(t_0, z)$ is continuous and bounded. The $M_r(t_0, z)$ is the Herglotz function. This implies that the spectrum is not singular continuous apart from at the isolated poles and hence the density of spectral measure of H_r is absolutely continuous. In order to determine the continuity of $M_r(t_0, z)$, we compute the scalar product of square summable solutions with $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$, $\eta > 0$, small, while taking limits as $\eta \rightarrow 0^+$ as given below

$$\lim_{\eta \rightarrow 0^+} \langle y_l(t_0, z), y_l(t_0, 0) \rangle.$$

The M -matrix can then be shown to be bounded absolutely since the above limit exists boundedly and this allows one to represent the resolvent of any arbitrary self-adjoint operator extension H_r of the generated minimal operator L_r using Green's function as

$$(H_r^{-1} - zI)y(t) = \sum_0^\infty G_r(t, t_1, z)y(t_1), \forall y(t) \in \mathcal{H},$$

where the Green function $G_r(t, t_1, z)$ is defined by,

$$G_r(t, t_1, z) = \begin{cases} \chi_r(t, z)R(\Phi)_r^*(s, \bar{z}) : & 0 \leq s \leq t-1 \\ \Phi_r(t, z)R(\chi_r)^*(s, \bar{z}) : & t \leq s < \infty \end{cases}.$$

$\Phi_r(t, z)$ are the fundamental solutions of the first order systems and $\chi_r(t, z)$ are the $s \times 2s$ matrices consisting of those solutions satisfying boundary conditions $\alpha(\alpha_1, \alpha_2)$ with $\alpha_1 = I_n$ and $\alpha_2 = 0$ at the regular point $t = 0$ or t_0 and are z -uniformly square summable as $t \rightarrow \infty$. This is also true for the composites of differential operators.

The rank of the M -matrix is, therefore, the spectral multiplicity of the absolutely continuous spectrum of H_r and this loosely translates to the number of solutions that lose their square summability as $\eta \rightarrow 0^+$.

3.8 Deficiency Indices and Self-adjoint extensions

Let T_r be the minimal differential operators and z , the spectral parameter with $\text{Im} z > 0$, in most cases a majority of authors would prefer taking $z = i$, then the pair (N_+, N_-) is known as deficiency indices of T_r , if $N_+ = \dim \mathcal{N}(T_r^* - \bar{z}I)$ and $N_- = \dim \mathcal{N}(T_r^* - zI)$, where $\mathcal{N}(T_r^* - \bar{z}I)$ and $\mathcal{N}(T_r^* - zI)$ are null spaces of $T_r^* - \bar{z}I$ and $T_r^* - zI$ respectively. Suppose that $N_- = N_+ = 0$ then T_r is self-adjoint otherwise for the operators we are dealing with, we have generally that $s \leq N_- = N_+ \leq 2s$. In such a case, the operator T_r has self-adjoint Operator extension H_r by application of von-Neumann theorems (Weidmann, 1980) and its domain described by

$$D(H_r) = D(T_r) + \{y + V_r y : y \in \mathcal{N}(T_r^* - zI)\}. \quad (3.9)$$

Note that V_r is a uniquely determined isometric mapping such that $V_r : \mathcal{N}(T_r^* - zI) \rightarrow \mathcal{N}(T_r^* + zI)$. For non-limit point cases, added boundary conditions are necessary at infinity in order to obtain the basis of set of solutions of $(T_r - zI)y = 0$ that are uniformly square integrable, for details see Hinton and Shaw (D. Hinton & Shaw, 1981).

3.9 M-Matrix for Difference Operators

The theory of the M -matrix has been applied in analysing the spectral properties of the symmetric composites for differential and difference operator extensions. As

defined earlier, the M-Matrix is a generalisation of the Weyl-Titchmarsh m-function and so it relates the asymptotics of the eigenfunction of higher order difference operators to the spectrum of their self-adjoint extensions.

We take $n \times n$ matrices α_1 and α_2 , that is, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{n \times n}$, with

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I_n, \quad \alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0_n$$

and set the boundary condition for the left regular end point a as

$$(\alpha_1, \alpha_2)y(a) = 0.$$

Let $Y_\alpha(., z) = (U_\alpha(., z), V_\alpha(., z))$ be the fundamental matrix with initial values

$$Y_\alpha(a, z) = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix}$$

with α_1 and α_2 satisfying the above boundary conditions. U_α and V_α are $2n$ by n matrices whose every column solves $Ly(t) = zy(t)$ and $V_\alpha(., z)$ satisfies the self-adjoint boundary conditions at a . In the limit point case with $Imz > 0$, we have a matrix $M \in \mathbb{C}^{n \times n}$ such that

$$Y_\alpha(t, z) \begin{pmatrix} I_n \\ M(z) \end{pmatrix} = U_\alpha(t, z) + V_\alpha(t, z)M(z)$$

The matrix $M(z)$ is determined from the solutions that stay absolutely square summable as $Imz \searrow 0$. Given that L is limit point at $t = \infty$, then we can construct the M-matrix $M(z)$ for the Hamiltonian system restricted to $[a, \infty)$ with Dirichlet boundary conditions. In this case, we let $\begin{pmatrix} X_1(a, z) \\ X_2(a, z) \end{pmatrix}$ be a system of n square summable solutions for $Imz > 0$. From the work of Hinton and Shaw (D. Hinton & Shaw, 1981) which was extended to the discrete setting by Shi (Y. Shi, 2006), these solutions also arise from $Y(t, z) \begin{pmatrix} I_n \\ M(z) \end{pmatrix}$ where $Y_a(t, z)$ is the fundamental solution of the system satisfying the appropriate boundary conditions at a . If we compare the n sets of solutions, it is evident that there exists an invertible n by n matrix C such that

$$\begin{pmatrix} X_1(a, z) \\ X_2(a, z) \end{pmatrix} C = Y(t, z) \begin{pmatrix} I_n \\ M(z) \end{pmatrix}.$$

This implies that $M(z) = X_2(a, z)X_1^{-1}(a, z)$. If we now let $F_\alpha(., z)$ be a $2n$ by n system of square summable solutions of the Hamiltonian system satisfying boundary conditions at a and infinity and $z, z' \in \mathbb{R}$, then $F_\alpha(., z) - F_\alpha(., z') \in D(H)$. Thus, by the results of Remling (Remling, 1998), Lemma 3.1] and Shi (Y. Shi, 2006), Theorem 6.3], we have that $\langle F_\alpha(., z), F_\alpha(., z) \rangle = (Imz)^{-1}ImM(z)$ if $z' = z$. Therefore, if $z = z_0 + i\eta$ for $\eta > 0$, then we have for $z_{0+} = \lim_{\eta \rightarrow 0+} z_0 + i\eta$,

$$\begin{aligned} ImM(z_{0+}) &= \lim_{\eta \rightarrow 0+} ImM(z_0 + i\eta) \\ &= \lim_{\eta \rightarrow 0+} \eta \langle F_\alpha(., z_{0+} + i\eta), F_\alpha(., z_{0+} + i\eta) \rangle. \end{aligned}$$

We note that the absolutely spectrum of the self-adjoint extension of the minimal difference operator exists if the M-matrix is finite and bounded. Moreover, its spectral multiplicity is equal to the rank of the M-matrix.

3.10 M-Matrix for Differential Operators

In order to obtain the spectral properties of H_r , we apply the M -matrix which is the equivalent of the classical Weyl m -function and in higher dimensions is obtained from those functions that are uniformly square integrable [(D. Hinton & Shaw, 1981), (Remling, 1998)]. Let

$$\chi_r(x, z) = \begin{bmatrix} \chi_{r1}(x, z) \\ \chi_{r2}(x, z) \end{bmatrix}$$

be the set of square integrable solutions satisfying boundary conditions at $\alpha_{r1} = I$ and $\alpha_{r2} = 0$. Then the M -matrix at the left end regular point x_0 is given by

$$M_r(x_0, z) = \chi_{r2}(x_0, z)\chi_{r1}^{-1}(x_0, z).$$

Here, $\chi_{r1}(x_0, z)$ is the Vandermonde's matrix for the eigenvalues of square integrable solutions and therefore $\chi_{r1}^{-1}(x_0, z)$ exists and $M_r(x_0, z)$ is continuous and bounded and spectrum is not singular continuous apart from isolated poles. The density of spectral measure of H_r is absolutely continuous. Thus the continuity of $ImM_r(x_0, z)$ is determined via

$$\lim_{\eta \rightarrow 0} \eta \langle y_l(x_0, z), y_l(x_0, 0) \rangle, \text{ for } z = z_0 + i\eta.$$

Finally the rank of the M -matrix is the spectral multiplicity of the continuous spectrum of H_r and this loosely translate to the number of solutions that lose their square integrability as $\eta \rightarrow 0^+$.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

In this chapter, we have discussed the results in line with the objectives stated in chapter 1. The chapter is arranged as follows: Section 4.2, we have analysed the necessary and sufficient conditions for the commutativity of differential operators. The spectrum of the associated self-adjoint operator extensions of the commuting differential operators have been obtained in section 4.3. In section 4.4, we have analysed the commutativity of generated difference operators as well as their spectral properties and finally in section 4.5, the spectral analysis of the symmetric composites of order two and four have been done.

4.2 Commutativity of Self-adjoint Operator Extensions of Differential Operators

In this section, we have proved the necessary and sufficient conditions for the commutativity of differential operators generated by τ_r in (1.1). We have also obtain the spectrum of the associated self-adjoint operator extensions. The method employed here, basically is by constructing the appropriate comparison algebras of the corresponding self-adjoint operator extensions and the application of asymptotic integration under suitable decay and growth conditions imposed on the coefficients.

First we define almost constant coefficients.

Let $f(x) \in \mathcal{C}_0^\infty([0, \infty))$ be a real valued function defined on $[0, \infty)$, then the function $f(x)$ is said to be almost constant function if there exists a constant c and a function $g(x) \in \mathcal{C}_0^\infty([0, \infty))$ such that $f(x) = c + g(x)$ with $g(x) \rightarrow 0$ as $x \rightarrow \infty$. In the following results we will assume throughout that the coefficients of $\tau_1 y$ and $\tau_2 y$ satisfy the following conditions:

$$m = n, \quad p_n, q_n > 0, \quad f'_l(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \quad (4.1)$$

$$f_l(x) = p_l(x), q_l(x), \quad l = 0, 1, 2, \dots, n.$$

Theorem 4.2.1. *Let τ_1 and τ_2 be symmetric differential functions given in (1.1) defined on $\mathcal{L}^2([0, \infty))$ with (3.5) and (4.1) satisfied. Moreover, assume that*

$$p_k \rightarrow c_k, \quad q_j \rightarrow d_j, \quad \text{as } x \rightarrow \infty,$$

where c_k and d_j are constants. The self-adjoint operator extensions H_1 and H_2 of minimal differential operators T_1 and T_2 respectively are commutative, their spectra agrees with that of the operators generated by constant coefficient functions and the set of their commutators contains the set of compact operators defined on $\mathcal{H}$.

Proof. The proof of existence of self-adjoint operator extensions of T_1 and T_2 follow closely from the results of Behncke [(Behncke, 2002), (Behncke, 2006)] and therefore we shall give an outline of the same and rather be detailed in the commutativity aspect. Each of the differential equations is converted into their first order system and the associated eigenvalues determined. In the results of Behncke and Nyamwala (Behncke & Nyamwala, 2011), z -uniform dichotomy condition is only established for those eigenvalues $\lambda_l(x, z)$ which are pure imaginary since the other roots with non-zero real parts shall satisfy uniform dichotomy condition irrespective of the value of spectral parameter z . Thereafter, as explained in Section 3.4, the transforming matrix $Q(x, z)$ can be determined from the eigenvectors and the first order system diagonalized repeatedly to obtain Levinson's form (3.2). Application of Levinson's Theorem, Theorem 3.4.1, now gives the form of solutions (3.3) that are required for analysis.

The solutions that stay uniformly square integrable shall contribute to deficiency indices of the operators T_r , $r = 1, 2$. By the results of Naimark (Naimark, 1967), $\text{def} T_r = (\beta, \beta)$, $n \leq \beta \leq 2n$. In case we have non-limit point case, a limit point case is obtained if we apply extra boundary conditions $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = I_n$ and $\alpha_2 = 0$ at infinity (D. Hinton & Shaw, 1981). Thus we obtain $\text{def} T_r = (n, n)$ to which we can apply the theory of the M -matrix. Each of the minimal differential operator T_r has self-adjoint operator extension H_r which can be described by uniquely determined partial isometries (in some cases isometric *-isomorphisms) $V_r, V_r : \mathcal{N}(T_r^* - zI) \rightarrow \mathcal{N}(T_r^* + zI)$ and their domains defined by (3.9).

In the remaining part of the proof, we have applied techniques of comparison algebras as constructed by H. O. Cordes (Cordes, 1987) and also applied in the work of S. T. Melo (Melo, 1990). Now define the operators H_r^{-1} using the appropriate Green's function $G_r(t, t_1)$ given by

$$H_r^{-1}y(t) = \int_0^\infty G_r(x, t)y(t)dt, \quad y(t) \in \mathcal{H},$$

where,

$$G_r(x, t) = \begin{cases} \chi_r(x, z)R(\Phi)_r^*(x_1, \bar{z}) : & 0 \leq x_1 \leq x \\ \Phi_r(x, z)R(\chi_r)^*(x_1, \bar{z}) : & x \leq x_1 < \infty \end{cases}$$

$\Phi_r(x, z)$ are the fundamental solutions of the first order systems and $\chi_r(x, z)$ are the $n \times 2n$ matrices consisting of those solutions satisfying boundary conditions $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1 = I_n$ and $\alpha_2 = 0$ at the regular point $x = 0$ and are uniformly square integrable as $x \rightarrow \infty$. Thus H_r^{-1} , $r = 1, 2$, are bounded self-adjoint operators which are linear and continuous. The boundedness property of the operator H_r is required since the theory of comparison algebras are anchored on the theory of C^* -algebras which requires that the generating operators must be bounded. We, therefore, construct isometric *-isomorphisms W_r by

$$W_r = H_r^{-\frac{1}{2n}} = \sum_0^\infty (-1)^l \binom{\frac{1}{2n}}{l} (1 - H_r^{-1})^l, \quad r = 1, 2.$$

It follows, therefore, that the operators $W_r^n H_r W_r^n$ are equally bounded in $\mathcal{H}$. Now for the operator H_r , we define the set $\{H_r, [0, \infty), \mu\}$ as the comparison triple from which we construct the H_r -comparison algebra. The comparison algebras are actually C^* -algebras by construction. Because the underlying set from which we define

our smooth functions is half-line, the appropriate measure will be dx , that is, $\mu = dx$. Thus the coefficient functions are elements of $\mathcal{C}_0^\infty([0, \infty))$. Let $\mathcal{C}_0^\infty([0, \infty))$ be the class of all bounded real-valued functions which includes also constant functions. To be more inclusive, assume for any function $f(x) \in \mathcal{C}_0^\infty([0, \infty))$ that can be expressed in the form $f(x) = f_0(x) + \gamma$ such that $f_0(x) \in \mathcal{C}_0^\infty([0, \infty))$ and $\gamma \in \mathbb{R}$. In our case, the function $f(x)$ represents $p_k(x)$ and $q_j(x)$, $k, j = 0, 1, \dots, n$.

Of great significance too, is the class of all linear symmetric differential operators of order $2n$ which we shall denote by $\mathcal{W}$. The class $\mathcal{W}$ should contain the operators T_r, T_r^*, H_r having compact support within the half-line $[0, \infty)$. In order to construct the appropriate comparison algebra, we will be interested in the algebra generated by classes of operators of the form $T = W_r^n H_r W_r^n$ together with the corresponding classes of multiplication operators $Ty(x) = f(x)y(x)$ in addition to the commutators of two generating operators $[W_r^n H_r W_r^n, f(x)H_r]$. The required algebra is obtained by norm-closing the algebra generated by the function $f(x)$ and the operator $W_r^n H_r W_r^n$ for all functions $f(x) \in \mathcal{C}_0^\infty([0, \infty))$ and $W_r^n H_r W_r^n \in \mathcal{W}$ having compact support. This results into minimal comparison algebra of H_r . The comparison algebra is non-unital C^* -algebra since $[0, \infty)$ is non-compact. It can be made unital by adjoining $\{1\}$ to it. Every minimal algebra, that is, every comparison algebra of self-adjoint operator H_r contains the entire ideal of compact operators $\mathcal{K}(\mathcal{H})$ defined on $\mathcal{H}$ (Cordes, 1987).

We now show that every element in $\mathcal{K}(\mathcal{H})$ is in the set of commutator and that the commutator set is non-empty. Suppose that $f(x) \in \mathcal{C}_0^\infty([0, \infty))$, with $f(x)$ not vanishing to zero identically as $x \rightarrow \infty$, then one can consider the operator defined by $f(x)W_r = f(x)H_r^{-\frac{1}{2n}}$. This operator is compact and is in the set of commutator since in general any differential function of the form $\frac{d}{dx}$ generates a differential operator of the form $H_r^{-\frac{1}{2n}}$ and $[f(x), \frac{d}{dx}]$ is in the set of commutator if and only if $\frac{d}{dx}f(x) = f'(x) \rightarrow 0$ as $x \rightarrow \infty$ and this is one of the assumptions made. Thus $f(x)W_r \neq 0$. It follows, therefore, by construction that $HW_r^n = W_r^n H$ is defined on the space $H\mathcal{C}_0^\infty([0, \infty))$.

Moreover, there exist compact operators C_1 and C_2 defined on $\mathcal{H}$ such that $W_r^n C_1 W_r^n = W_r^n C_2 W_r^n$.

Finally, we show that the operators $H_1 W_1^n$ commutes with $H_2 W_2^n$. First note that by Gelfand-Naimark theorem, the H_1 -comparison algebra is isomorphic to H_2 -comparison algebra. Secondly, the operators W_1 and W_2 are isometric $*$ -isomorphisms by construction and they are constructed from the same underlying manifold this is because the deficiency indices of T_1 and T_2 are equal and hence $\mathcal{N}(T_1^* - zI) = \mathcal{N}(T_2^* - zI)$. One, therefore, concludes that W_1 and W_2 are commutative since isometries of the same manifold form abelian group under composition. Moreover, by construction, the operators $H_r W_r^n$ and W_r^n are commutative and compact. Thus, there exists a compact operator C_0 such that $H_r W_r^{2n} - W_r^n H_r W_r^n = C_0 W_r^n$, where $C_0 = H_r W_r^n - W_r^n H_r$. This implies that for any function $y(x) \in D(W_r^{-n})$ we have $H_r W_r^n y(x) = W_r^n H_r y(x) + C_0 y(x)$ so that $W_r^n H_r$ with domain $D(W_r^{-n})$ must have a continuous extension $(W_r^n H_r)^{**}$. This extension is a linear continuous operator in $\mathcal{H}$ and $H_r W_r^n = (W_r^n H_r)^{**} + C_0$. Computing the adjoints of both sides of this relation we get $(H_r W_r^n)^* = H_r^* W_r^n + C_0^*$ implying that $(H_r W_r^n)^* - H_r^* W_r^n$ and $H_r W_r^n - (W_r^n H_r)^{**}$ are compact. Using similar techniques, there exists a compact operator C_1 such that $C_1 = W_1^n [H_1, H_2] W_2^n$ since the operator $W_1^n [H_1, H_2] W_2^n$ is bounded in $\mathcal{H}$. Applying the fact that $H_r W_r^n - (W_r^n H_r)^{**}$ is compact, for $y(x) \in H_r \mathcal{C}_0^\infty([0, \infty))$, we obtain

$$\begin{aligned}
H_1 W_1^n H_2 W_2^n y(x) &= (W_1^n H_1)^{**} H_2 W_2^n y(x) + C_0 y(x) \\
&= C_0 y(x) + W_1^n H_1 H_2 W_2^n y(x) \\
&= C_1 y(x) + W_2^n H_2 H_1 W_1^n y(x) + C_2 y(x) \\
&= C_3 y(x) + (W_2^n H_2)^{**} H_1 W_1^n y(x) \\
&= H_2 W_2^n H_1 W_1^n y(x) + C_4 y(x),
\end{aligned}$$

for some C_2, C_3, C_4 compact operators in $\mathcal{H}$. This implies that H_1 and H_2 commutes and thus their respective minimal differential operators. $\square$

Note that when all the conditions set in Theorem 4.2.1 are met and τ_1 and τ_2 are commutative, then the expression $\tau_2(\tau_1 y) = \tau_1(\tau_2 y)$ is generally expressed in its

symmetric form given by

$$\tau_2(\tau_1 y) = \sum_{j=0}^m (-1)^j \left(\sum_{k=0}^n (-1)^k (q_j(p_k y^{(k)})^{(k+j)})^{(j)} \right). \quad (4.2)$$

By construction, if $\text{def}T_1 = \text{def}T_2$, then the operators V_1 and V_2 are defined on the same subspace and by composition, V_1 and V_2 commutes with their composite leading to another isometry.

4.3 Deficiency Indices and Spectrum of Commutative Differential Operators

In Theorem 4.2.1, the results were obtained only for coefficients $f(x)$ that were almost constant. There are non-trivial examples of unbounded coefficients with their first derivatives decaying to zero as $x \rightarrow \infty$. For example, consider $f(x) = \ln x$, $x > 0$. In the next Theorem, we allow the coefficients to be unbounded. In addition to the conditions given in (3.5) and (4.1), we let

$$\frac{f_{l-1}}{f_l} < 0, \quad f_{l-1}f_{l+1} = o(f_l^2), \quad (4.3)$$

Theorem 4.3.1. *Let T_r be minimal differential operator generated by τ_r in (1.1) and assume that (3.5), (4.1) and (4.3) are satisfied. If $|f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}}$ is absolutely integrable then $\text{def}T_r = (2n, 2n)$ and $\sigma_{ac}(H_r)$ is pure discrete. If $|f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}}$ is not absolutely integrable then $\text{def}T_r = (n, n)$ and $\sigma_{ac}(H_r, n) = \mathbb{R}$. The operators H_1 and H_2 commutes.*

Proof. In this proof we assume that all the conditions are satisfied. Then the characteristic polynomial will be given by (3.7) with the appropriate fourier polynomial after modification through multiplication by the leading coefficient, that is, q_n or p_n . Because of the assumptions in (4.3) and applying techniques in Eastham (M. S. P. Eastham, 1989) which were later used by Nyamwala and others [(Agure,

Ambogo, & Nyamwala, 2013),(Behncke & Nyamwala, 2012)] the zeros of the polynomial (3.7) can be approximated from

$$f_l \lambda^2 - f_{l-1} + \mathcal{R}(x, \lambda, z) = 0,$$

$$\mathcal{R}(x, \lambda, z) = \sum_{\nu=l+1}^n (-1)^{\nu-l} f_\nu \lambda^{2(\nu-l+1)} + \sum_{\nu=0}^{l-2} (-1)^{\nu-l} f_\nu \lambda^{2(\nu-l+1)}.$$

Here, the term $\mathcal{R}(x, \lambda, z) \rightarrow 0$ as $x \rightarrow \infty$. This is easy to see since in the absolute value sense we have $|\lambda_l(x, z)| \approx |\frac{f_{l-1}}{f_l}|^{\frac{1}{2}}$ and therefore by assumptions (4.3) on the growth conditions of the coefficients, for a specific $\lambda_l(x, z)$ the absolute values of the terms of $\mathcal{R}(x, \lambda, z)$ are approximately given by

$$|f_\nu| \left| \frac{f_{l-1}}{f_l} \right|^{\nu-l+1} = o(1), \quad \nu = 1, \dots, l-2, l+1, \dots, n.$$

It follows that the λ -roots of the characteristic polynomial $\mathcal{P}(x, \lambda, z)$ can be approximated by $\lambda_l(x, z) \approx \left(\frac{f_{l-1}}{f_l} \right)^{\frac{1}{2}} + o(1)$ and since $\frac{f_{l-1}}{f_l} < 0$, one has $\lambda_l(x, z) \approx \pm i \left| \frac{f_{l-1}}{f_l} \right|^{\frac{1}{2}} + o(1)$. By application of Banach fixed point theorem and using some iteration in some restricted interval with suitable initial value, these λ -roots approximations can be shown to be equal to the actual λ -roots which implies that all the λ -roots are pure imaginary. Now define

$$\mathcal{K} = \{z : |z - z_0| < \epsilon\}$$

to be the set where $\mathcal{F}(x, \lambda, z)$ has distinct zeros. This set can be obtained and for more details, see [(Behncke, 2006), (Behncke et al., 2001)]. The z -uniform dichotomy condition has to be established off-real line, that is, we take $z = z_0 + i\eta$, z_0, η real with $\eta > 0$, small, $0 < \eta < \epsilon$, then the correction term in this case is approximated as

$$z \left| \frac{\partial \mathcal{P}(x, \lambda, z)}{\partial \lambda} \right|^{-1} \approx z |f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}}.$$

Off the real axis, we have $\text{Re} \lambda_{l+} \approx -\eta |f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}} < 0$ so that the associated eigensolution is square integrable while on the other hand $\text{Re} \lambda_{l-} \approx \eta |f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}} > 0$ with the associated eigensolution non-square integrable. This is the uniform dichotomy condition required for the two eigenvalues with almost equal magnitude. For eigenvalues determined by different indices $l - \nu$, l and $l + \nu$, we note that the correction terms have different magnitudes since

$$\begin{aligned} |f_{l+\nu}|^{l+\nu-\frac{3}{2}} |f_{l+\nu-1}|^{-l-\nu+\frac{1}{2}} &\ll |f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}} \\ &\ll |f_{l-\nu}|^{l-\nu-\frac{3}{2}} |f_{l-\nu-1}|^{-l+\nu+\frac{1}{2}}, \end{aligned}$$

are approximately the leading terms of $|\frac{\partial \mathcal{P}(x, \lambda, z)}{\partial \lambda}|^{-1}$ in each case. The correction terms associated to z have different magnitudes and therefore, the uniform dichotomy condition will be satisfied by application of Bezout's theorem since all the tangents to the characteristic polynomial at different points can only be equal if and only if the polynomials as a result of distinct λ -roots are identical. Thus $\lambda_{l-\nu}(x, z)$, $\lambda_l(x, z)$ and $\lambda_{l+\nu}(x, z)$ satisfy the required uniform dichotomy condition.

The system can then be diagonalized to put it into Levinson's form. This can be done using the eigenvectors as explained earlier. After the first diagonalization, the off diagonal terms will be $O(\frac{f_l'}{f_l})$ and these are $\mathcal{L}^2$ -terms implying that the system can be diagonalized for the second time in order to obtain terms that are absolutely integrable. Therefore a second diagonalization gives off diagonal terms of either $O(\frac{f_l''}{f_l})$ or $O((\frac{f_l'}{f_l})^2)$ which are $\mathcal{L}^1$ -terms. In this case we obtain a system of the form

$$V' = [\Lambda + \tilde{R}].$$

Here, $\Lambda = \Lambda(x, z) = \text{diag}(\lambda_l(x, z) + o(1))$, where, $o(1)$ is the correction term to the diagonal because of approximations and two diagonalizations. The matrix $\tilde{R}$ has entries $\tilde{R}_{ll} = 0$ and its off diagonal terms are either $O(\frac{f_l''}{f_l})$ or $O((\frac{f_l'}{f_l})^2)$. Application of Levinson's theorem now gives solutions of the form (3.3).

Suppose that $|f_l|^{l-\frac{3}{2}}|f_{l-1}|^{-l+\frac{1}{2}}$ is integrable for all $l = 1, 2, \dots, n$, then all the solutions are uniformly square integrable and $\text{def}T_r = (2n, 2n)$ implying limit cycle case (l.c.c.). T_r has self-adjoint operator extension with pure discrete spectrum. Application of results obtained in Theorem 4.2.1 shows that H_1 and H_2 commutes and so are the respective minimal operators. On the other hand if $|f_l|^{l-\frac{3}{2}}|f_{l-1}|^{-l+\frac{1}{2}}$ is not integrable for all $l = 1, 2, \dots, n$, then $\text{def}T_r = (n, n)$. Still T_r has self-adjoint operator extension and application of Theorem 4.2.1 results imply that H_1 and H_2 commutes with each other and so do the respective minimal differential operators. However, in this case, there are n -solutions that lose their square integrability as $\eta \rightarrow 0^+$. It is these solutions that lose their square integrability that contribute to absolutely continuous spectrum. It remains to show that the associated M -matrix is bounded non-trivially. For that we use the expression for determining $\text{Im}M(., z)$

as given by Remling (Remling, 1998)

$$\begin{aligned}\text{Im}M(., z) &= \lim_{\eta \rightarrow 0^+} \eta \langle y_{l+}(x, z), y_{l+}(x, z) \rangle \\ &\approx \int_0^\infty \rho(x, z) (\exp -2\eta \int_a^x |f_l|^{l-\frac{3}{2}} |f_{l-1}|^{-l+\frac{1}{2}} (\phi, z) d\phi) dt\end{aligned}$$

which is bounded. This shows that the spectral measure is continuous. $\square$

We consider two examples to illustrate these results. In the first case using order two symmetric operators, we show that the condition that $f'_l(x) \rightarrow 0$ as $x \rightarrow \infty$ cannot be weakened further. In the second case, we obtain the necessary and sufficient conditions for the differential operators generated by Sturm-Liouville and Eckart equations shall lead to commutative self-adjoint operator extensions.

Example 4.3.2. Consider second order symmetric differential equations of the form

$$\tau_1 y = -y'', \quad \tau_2 = -((\sin x)y')'.$$

In this particular case, $p_1(x) = 1$ while $q_1(x) = \sin x$. When we generate the commutators $[\sin x \frac{d}{dx}]y = -y \cos x$ which cannot be zero in the whole of $[0, \infty)$. This implies that the two differential equations are non-commutative. Even direct computation results into

$$\tau_1(\tau_2 y) - \tau_2(\tau_1 y) = 2y''' \cos x - 3y'' \sin x - y' \cos x.$$

This clearly shows that the condition that $f'_l(x) \rightarrow 0$ as $x \rightarrow \infty$ is necessary for the results of Theorem 4.2.1 to be true.

In Example 4.3.3 below, we show that the second order symmetric Sturm-Liouville and Eckart equations generate commutative differential operators if the potential in Sturm-Liouville equation is constant and the constant δ in Eckart equation is zero.

Example 4.3.3. Consider two second order symmetric differential operators generated by Sturm-Liouville and Eckart equations of the form

$$\tau_1 y = -y'' + py, \quad \tau_2 y = -y'' - \left[\frac{\alpha\eta}{1+\eta} + \frac{\beta\eta}{(1+\eta)^2} + \gamma \right] y,$$

where $\eta = \exp(\delta x)$ and α, β, γ and δ are constants. A quick computation of the commutator requires that we have $p(x) = c$ where c is a constant while in the Eckart

equation we have either $\eta = -1$ which is an impossibility or $\delta = 0$. The latter case is mathematically sound and therefore we have constant coefficients. The composite $\tau_2\tau_1$ on y is given by

$$\begin{aligned}\tau_2(\tau_1 y) &= \tau_1(\tau_2 y) \\ &= y^{(iv)} - \left[\frac{\alpha}{2} + \frac{\beta}{4} + \gamma + c \right] y'' + c \left[\frac{\alpha}{2} + \frac{\beta}{4} + \gamma \right] y\end{aligned}$$

which is a symmetric fourth order differential expression defined on

$\mathcal{H} = \mathcal{L}^2([0, \infty))$. The independent analysis for each of differential operators generated by τ_1 , τ_2 and their composite can be done. In the case that $p(x) = c > 0$, then $\text{def}T_1 = (1, 1)$ and the spectrum of H_1 is purely discrete. In the case of Eckart operator, a similar observation is obtained when α , β and γ are all negative since $\text{def}T_2 = (1, 1)$ and the spectrum of H_2 is at most discrete. If $c < 0$ and α , β and γ are all positive, then we have H_1 and H_2 commuting with each other, and $\sigma_{ac}(H_1)$ and $\sigma_{ac}(H_2)$ non-empty with spectral multiplicity of one.

In the remaining part of this section, we develop a similar analysis to that of $\tau_1 y(x)$ and $\tau_2 y(x)$ in section 4.2 and so far in section 4.3 and extend it to the composite of $\tau_1 y(x)$ and $\tau_2 y(x)$. We denote this composite $\tau_1(\tau_2 y(x))$ by $\tau y(x)$ as given in (4.2) defined on $\mathcal{L}^2(\mathbb{N})$ with $\sum_{k=0}^n (\sum_{j=0}^m (p_k q_j)) > 0$. We thus develop the structural properties of the minimal operator generated by the first order form of $\tau y(x) = zy$ for some spectral parameter z . The coefficients p_k and q_j are assumed to satisfy the growth and decay conditions as set in (3.5), (4.1) and (4.3). We therefore solve the equation $\tau y(x) = zy$. Just like in section 1.1.1, $\tau y(x) = zy$ is reduced into first order form using the quasi-derivatives as developed by Walker (Walker, 1974). We thus define $2(n + m)$ -dimensional vector valued functions given by the set $\{y^{[0]}, y^{[1]}, \dots, y^{[2(n+m)-1]}\}$ that are square integrable. The maximal domain $D(T^*)$ of operator generated by the composite is therefore defined by

$$\begin{aligned}D(T^*) &= \{y \in \mathcal{L}^2([0, \infty)) \text{ such that } \{y^{[0]}, y^{[1]}, \dots, y^{[2(n+m)-1]}\} \\ &\quad \text{are absolutely continuous in } [0, \infty), \tau^* y \in \mathcal{L}^2([0, \infty))\} \\ T^* y &= \tau^* y \text{ for all } y \in D(T^*).\end{aligned}$$

$D(T^*)$ is the maximum possible domain in $\mathcal{L}^2([0, \infty))$ for which the quasi-derivatives makes sense. T^* is the Hilbert adjoint of the minimal possible operator generated by

the composite τ^*y . T^* is densely defined and closed since τ^*y is symmetric. We thus define a pre-minimal operator T_0 by restricting the domain of the maximal operator to only functions y with compact support. The domain of this operator is by

$$\begin{aligned} D(T_0) &= \{y \in D(T^*) \text{ such that } y \text{ has compact support in } [0, \infty)\} \\ T_0y &= T^*y = \tau^*y \text{ for all } y \in D(T_0). \end{aligned}$$

For unbounded domains, T_0 is symmetric, densely defined but not closed. Since we need closed operators, we take the closure of this pre-minimal operator denoted by T with $T \subset T^*$ in the operator sense. In line with the results of Walker (Walker, 1974) and Weidmann (Weidmann, 1987), $\tau_1y(x)$ and $\tau_2y(x)$ generate symmetric differential operators while the differential operator generated by $\tau y(x)$ is not necessarily a symmetric operator. However, it is a well known fact and as verified by the results in Theorem 4.2.1 that the composite of two commuting symmetric operators is symmetric and thus generate a symmetric operator under the given growth and decay conditions on the coefficients hence the following lemma holds.

Lemma 4.3.4. *Let T_1 , T_2 and T be the minimal operators generated by τ_1y , τ_2y and τy respectively. Suppose T_1 and T_2 are symmetric operators with T_1 commuting with T_2 then $T = T_1T_2$ is symmetric.*

Deficiency indices results require proper understanding of the asymptotic behaviour of the solutions associated to these differential equations. It is therefore necessary to start by showing that the basis of homogeneous equation $\tau y(x) = 0$ can be determined from the bases of the solutions of $\tau_1y(x) = 0$ and $\tau_2y(x) = 0$ as given by Littlejohn and Lopez (Littlejohn & López, 2010).

Theorem 4.3.5. *Let $\tau y(x) = 0$, $n = m$ be a homogeneous differential equation, then the basis of the solution of this composite can be determined from the bases of the solutions $\tau_1y(x) = 0$ and $\tau_2y(x) = 0$*

Proof. Suppose that $\{y_1, y_2, \dots, y_{2m}\}$ is a basis of solutions of $\tau_2y(x) = 0$ and $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{2n}\}$ is the basis of the solutions of $\tau_1y(x) = 0$, then the basis of the solutions of the composite is obtained using the Green's function $G_2(x, t)$ associated with

$\tau_2 y(x) = 0$ and the corresponding Wronskian determinant $W_2 = \{y_1, y_2, \dots, y_{2m}\}(t)$. For a fixed left regular end point $t_0 \in \mathbb{N}$, we define functions

$$y_{2m+s}(x) = \int_{t_0}^x G_2(x, t) \tilde{y}_s(t) dt \quad s = 1, 2, \dots, 2n.$$

Then, the set $\{y_1, y_2, \dots, y_{2m}, y_{2m+1}, y_{2m+2}, \dots, y_{2(n+m)}\}$ is a basis of solutions of the composite equation $\tau y(x) = 0$ for $x \in [0, \infty)$. To prove this, consider the equations $\tau_2 y_k(x) = 0$, $k = 1, 2, \dots, 2m$, $\tau_2 y_k(x) = \tilde{y}_k(x)$, $k = 2m+1, 2m+2, \dots, 2(n+m)$ and $\tau_1 \tau_2 y_k(x) = 0$, $k = 2m+1, 2m+2, \dots, 2(n+m)$. This implies that

$$\tau_1 \tau_2 y_k(x) = \tau_1(\tau_2 y_k(x)) = \tau_1 y_k(x) = 0 \quad k = 2m+1, 2m+2, \dots, 2(n+m)$$

Thus, for a set of constants $\{\beta_k : k = 1, 2, \dots, 2(n+m)\}$ with the following linear combination

$$\beta_1 y_1(x) + \beta_2 y_2(x) + \dots + \beta_{2m} y_{2m}(x) + \beta_{2m+1} y_{2m+1}(x) + \dots + \beta_{2(n+m)} y_{2(n+m)}(x) = 0$$

Successive application of τ_1 and τ_2 starting with τ_2 on both sides with the right assumptions of the basis elements for solutions of each differential map, it follows that all the constants β_k , $k = 1, 2, \dots, 2(n+m)$ are zero. $\square$

Note that as stated before, the composite operator $T = T_1 T_2$ is symmetric but not necessarily densely defined. In the next result therefore, we give the necessary and sufficient conditions for the composite operator to be densely defined.

Theorem 4.3.6. *Let T_1 and T_2 be the minimal differential operators generated by τ_1 and τ_2 respectively. Assume that (4.1) is satisfied, T_1 and T_2 are injective and that $\mathcal{R}\tau_1$ and $\mathcal{R}\tau_2$ are closed, then the operator $T = T_1 T_2$ is closed and densely defined.*

Proof. T_1 and T_2 are closed by construction as given in Section 1.1.1. Thus, their ranges are closed and there exists a closed subspace Γ of $\mathcal{H} \times \mathcal{H}$ defined by

$$\Gamma = \{(y, T_1 T_2 y) : y \in D(T_1 T_2)\}.$$

By injectivity and linearity of T_1 and T_2 which imply that of $T_1 T_2$, Γ is a graph of $T_1 T_2$ since for any $(0, y) \in \Gamma$, it follows that $y = 0$. For any $y \in D(T_1 T_2)$, there is

at most $\hat{y}$ such that $(y, \hat{y}) \in \Gamma$ and $\hat{y} = T_1 T_2 y$, $y = y(x)$ and $\hat{y} = \hat{y}(x)$. The graph of $T_1 T_2$ is closed and the operator $T_1 T_2$ is closed as required. Now consider $D(T_1 T_2)$ which can be expressed as

$$D(T_1 T_2) = D(T_2) \cap T_2^{-1} D(T_1).$$

Since $D(T_1)$ and $D(T_2)$ are dense in $\mathcal{H}$ by construction, $T_2^{-1} D(T_1)$ is a dense subspace of $\mathcal{H}$ and thus by Baire's Category Theorem, the intersection of $D(T_1)$ and $T_2^{-1} D(T_1)$ is dense in $\mathcal{H}$ hence the operator $T_1 T_2$ is densely defined. $\square$

In the next result, we show that the deficiency index of $T_1 T_2$ is equal to the sum of the deficiency indices of T_1 and T_2

Theorem 4.3.7. *Assume that all the conditions in Theorem 4.3.6 are satisfied. Moreover, assume that the $\dim \mathcal{R}(T_2)^\perp$ is finite, then, $\text{def} T_1 T_2 = \text{def} T_1 + \text{def} T_2$*

Proof. By Closed-Range Theorem, if all the conditions in Theorem 4.3.6 and Theorem 4.3.7 above are satisfied, then the range of T^* is closed. Thus by application of Rank-Nullity Theorem, we have that

$$\dim \mathcal{N}(T^* - zI) = \dim \mathcal{N}(\overline{T_1 T_2^*} - zI) = \dim \mathcal{N}(T_1^* - zI) + \dim \mathcal{N}(T_2^* - zI).$$

Thus,

$$\text{def} T_1 T_2 = \text{def} T_1 + \text{def} T_2$$

$\square$

Finally, we consider an example in which we compute the deficiency indices to prove the results of Theorem 4.3.7 as well as carry out the spectral analysis of the composite operator $T = T_1 T_2$ generated by $\tau = \tau_1 \tau_2$.

Example 4.3.8. Consider two differential operators T_1 and T_2 generated by order two functions τ_1 and τ_2 respectively, defined by

$$\tau_1 = -\alpha y'' \quad \text{and} \quad \tau_2 = p(x) y''$$

A computation of the composite leads to a fourth order symmetric differential equation given by $\tau_1\tau_2 = -\alpha(py'')$. Here, $p = p(x)$ is assumed to be continuously differentiable and bounded. Taking z as the spectral parameter, we solve the equation $\tau_1\tau_2 = zy$. By the assumptions in (3.5), (4.1) and (4.3), we convert the fourth order differential equation $\tau_1\tau_2$ into first order form using the quasi-derivatives as discussed in section 1.1.1. The associated characteristic polynomial is given by $\alpha p\lambda^4 + z = 0$.

Suppose $\alpha p < 0 \forall x \in [0, \infty)$, then the four eigenvalues are given by

$$\lambda_{1\pm}(x, z) = \pm \left| \frac{z}{\alpha p} \right|^{\frac{1}{4}} \quad \text{and} \quad \lambda_{2\pm}(x, z) = \pm i \left| \frac{z}{\alpha p} \right|^{\frac{1}{4}}.$$

The dichotomy conditions is then applied off the real axis only to those eigenvalues that are pure imaginary. Thus, we let $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$ for some small $\eta > 0$. The dichotomy condition will be satisfied for $\lambda_{2\pm}(x, z)$ since the correction term which is given by $|\frac{\partial p(\lambda, x)}{\partial \lambda}|^{-1} \approx \frac{\eta}{|\alpha p|^{\frac{1}{4}}}$ and has different signs for each pair $\lambda_{2\pm}(x, z)$ as the real part. If $|p|^{-\frac{1}{4}}$ is integrable, then the solutions associated with $\lambda_{1-}(x, z)$ and those of $\lambda_{2\pm}(x, z)$ will all be uniformly square integrable and the deficiency index of T_1T_2 , $\text{def}T_1T_2 = (3, 3)$. Hence any self-adjoint extension of T_1T_2 has discrete spectrum at most.

Suppose $|p|^{-\frac{1}{4}}$ is not integrable, then the solutions associated with $\lambda_{1-}(x, z)$ and $\lambda_{2+}(x, z)$ will stay uniformly square integrable while the solution associated with $\lambda_{2-}(x, z)$ will lose its square integrability as $\eta \rightarrow 0^+$. Thus for T_1T_2 , $\text{def}T_1T_2 = (2, 2)$. The solution that lose its square integrability will thus contribute to the absolutely continuous spectrum H , (the self-adjoint operator extension of T_1T_2). This implies that the absolutely continuous spectrum of H is given by $\sigma_{ac}H \subset [(\alpha \underline{p})^{-1}, \infty)$, where, $\underline{p} = \liminf p(x)$ and is of spectral multiplicity one.

The deficiency indices of the individual minimal operators T_1 and T_2 is done in a similar fashion. If $\alpha > 0$, the eigenvalues of T_1 are $\lambda \pm(x, z) = \pm i | \frac{z}{\alpha} |^{\frac{1}{2}}$, thus, $\text{def}T_1 = (2, 2)$. Hence, the absolutely continuous spectrum can only be obtained within the interval $[-\alpha^{-1}, \alpha^{-1}]$ and is of spectral multiplicity one as $\eta \rightarrow 0^+$. For $\alpha < 0$, the eigenvalues are given by $\lambda \pm(x, z) = \pm | \frac{z}{\alpha} |^{\frac{1}{2}}$ hence $\text{def}T_1 = (1, 1)$ and only discrete spectrum can be realised. Similarly for T_2 , if $p(x) > 0 \forall x \in [0, \infty)$, the eigenvalues are given by $\lambda \pm(x, z) = \pm | \frac{z}{p} |^{\frac{1}{2}}$ and $\text{def}T_2 = (1, 1)$. Suppose $p(x) < 0$, the

eigenvalues are given by $\lambda_{\pm}(x, z) = \pm i|z|^{\frac{1}{2}}$. thus if $|p|^{\frac{-1}{2}}$ is integrable, $def T_2 = (2, 2)$. Conversely if $|p|^{\frac{-1}{2}}$ is not integrable, then $def T_2 = (1, 1)$. Thus, the deficiency index of $T_1 T_2$ is equal to the sum of the deficiency indices of T_1 and T_2 if the growth and decay conditions given in (3.5) and (4.1) are satisfied.

4.4 Commutativity of Generated Difference Operators

The results of this section are the discretized version of the results in section 4.3. For any real valued function $h(t)$, we shall define the commutator $[\Delta, h(t)]$ to be the relation

$$[\Delta, h(t)]y(t) = \Delta(h(t)y(t)) - h(t)(\Delta y(t)), \quad \forall t \in \mathbb{N}$$

The commutator vanishes identically to zero if and only if $\Delta(h(t)) \rightarrow 0$ as $t \rightarrow \infty$. It is therefore true like in the continuous case, that in the discrete case we replace *derivative* with Δ to obtain our commutator. For $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$, there are many examples that satisfy this condition. For example, constant functions, almost constant functions, that is, the functions of the form $h(t) = c + g(t)$ such that $g(t) \rightarrow 0$ as $t \rightarrow \infty$. However, there exists also unbounded functions that satisfy the condition. For example, take $h(t) = \ln t$, $t \in \mathbb{N}$, it follows that $\Delta h(t) = \ln(1 + \frac{1}{t}) \rightarrow 0$ as $t \rightarrow \infty$. These show that there exist non-trivial functions that satisfy the commutator condition. Therefore, we work with the following decay conditions:

$$p_n, q_m > 0, \quad \Delta h \rightarrow 0, \quad \text{as } t \rightarrow \infty, \quad h = p_k, q_j, \quad k, j = 1, 2, \dots, \max(n, m). \quad (4.4)$$

and

$$\frac{\Delta h_l}{h_l} \in \ell^2(I_1), \quad \left(\frac{\Delta h_l}{h_l}\right)^2, \quad \frac{\Delta^2 h_l}{h_l} \in \ell^1(I_1), \quad h_l = p_l, q_l, \quad l = k, j. \quad (4.5)$$

In the theorem below, we have shown that at limiting point, the spectrum of H_r , the self-adjoint operator extension in (1.14), has no singular continuous spectrum and the absolutely continuous spectrum agree with that of constant coefficient limiting operators.

Theorem 4.4.1. *Consider the minimal difference operators L_1 and L_2 generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.7) respectively and assume (4.4) and (4.5) are satisfied. Then L_1 , and L_2 are at limit points at infinity if the boundary conditions defined by matrices α_r (1.13) at $t_0 = 0$ are satisfied. The corresponding self-adjoint operator extension H_r has no singular continuous spectrum $\sigma_{sc}(H_r) = \emptyset$. The absolutely continuous spectra of the self-adjoint operator extensions agree with those of constant coefficient limiting operators. In particular, the spectral measure μ for H_r belongs to absolutely continuous spectra having multiplicities equal to half the number of roots of the corresponding characteristic polynomials with unit magnitude.*

Proof. The proof follows closely from that of Theorem 4.7 of Behncke and Nyamwala (Behncke & Nyamwala, 2011) with some modification in line with the formulation of $\mathcal{L}_1$ and $\mathcal{L}_2$. Here, we define $\mathcal{L}_1$ and $\mathcal{L}_2$ on the Hilbert space $\ell_w^2(I_1) \cong \ell^2(I_1)$, $w = w(t) = 1$, for all $t \in \mathbb{N}$. Each of the difference equations is converted into their first order system. The respective propagator form (1.12) can be computed explicitly and the associated eigenvalues determined. Uniform dichotomy condition can then be established uniformly for those eigenvalues $\lambda(t, z)$ of unit magnitude since the other eigenvalues shall satisfy uniform dichotomy condition irrespective of the value of spectral parameter z (Behncke & Nyamwala, 2011). Thereafter, the transforming matrix $T(t, z)$ can be determined from the eigenvectors and the first order system diagonalized repeatedly to obtain Levinson-Benzaid-Lutz (LBL) form as explained in Section 1.1.2. The assumptions in (4.5) imply that two diagonalizations will bring the first order form into the required Levinson-Benzaid-Lutz form (1.15). Application of the discretized version of Levinson's Theorem now gives the form of solutions (1.16) that are required for analysis.

We consider the zeros of $\mathcal{F}(t, \gamma, z)$ in (3.6). The roots of the polynomial in (3.6) will be considered within a set $\mathcal{K} = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$ with relevant spectral parameters which can be adjusted until $\mathcal{P}(\lambda, t, z)$ has distinct roots. Suppose the roots of the characteristic polynomial $\mathcal{P}_r(\lambda, t, z)$ are $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{2l}, \tilde{\lambda}_{2l+1}, \tilde{\lambda}_{2l+2} + \dots + \tilde{\lambda}_{2s}$, $s = n, m$, $l \leq s$. Here $\lambda_l = \lambda_l(t, z)$, $\tilde{\lambda}_l = \tilde{\lambda}_l(t, z)$ where $|\lambda_k| \neq 1$, $k = 1, 2, \dots, 2l$, $|\tilde{\lambda}_k| = 1$, $k = 2l + 1, \dots, 2s$, then by the results of Behncke (Behncke, 2010) and

Behncke and Nyamwala (Behncke & Nyamwala, 2011), the dichotomy condition is only required for those characteristic roots (eigenvalues of $S(t, z)$ with magnitude 1, that is $\tilde{\lambda}_k(t, z)$). Using complex analysis, represent these eigenvalues by $\tilde{\lambda}_k(t, z) = \exp(i\beta_k)$ for some angle β_k . It follows that $\tilde{\lambda}_k(t, z) = \exp(-i\beta_k)$ will also be a root. Then off the real axis and taking $z = z_0 + i\epsilon$, $\epsilon > 0$, $z_0, \epsilon \in \mathbb{R}$, $\epsilon > 0$ is small, we have,

$$\tilde{\lambda}_k(z_0 + i\epsilon) + \overline{\tilde{\lambda}_k}(z_0 + i\epsilon) = 2 \cos \beta_k \quad k = l + 1, \dots, (s - l).$$

Thus

$$\tilde{\lambda}_k(z_0 + i\epsilon) = \tilde{\lambda}_k(z_0) + \frac{i\epsilon}{\partial_{\lambda_k} \mathcal{P}_r(\lambda, t, z)}.$$

It is the correction term $i\epsilon(\partial_{\lambda_k} \mathcal{P}_r(\lambda, t, z))^{-1}$ which will lead to either $\tilde{\lambda}_k(z_0 + i\epsilon)$ having magnitude greater than 1 or less than 1 off the real axis. The number of roots with magnitude almost one at limiting point but with magnitude greater than 1 off the real axis will be half the roots $\tilde{\lambda}_l(t, z)$ while the others will have magnitude less than 1 off the real axis. This is the required uniform dichotomy condition on the eigenvalues of $\mathcal{P}_r(\lambda, t, z)$. The roots $\lambda_l(t, z)$ that off the real axis have magnitude greater than 1 lead to eigensolutions that lose their square summability as $\epsilon \rightarrow 0^+$ and hence contributes to absolutely continuous spectrum. The spectral multiplicity can thus be analysed via the M-matrix.

The Titchmarsh-Weyl functions for the minimal difference operators L_1 and L_2 are the respective M-matrices $M_r(z) = M_r$, $r = 1, 2$ and which are Borel-transforms of the spectral measures μ_r , $r = 1, 2$. The density of the absolutely continuous spectrum of each H_r is given by

$$\left(\frac{1}{\pi}\right) \lim_{\epsilon \rightarrow 0^+} M_r(\mu_r + i\epsilon) = \left(\frac{1}{\pi}\right) M_r(\mu_+) = \varrho(\mu_r).$$

The spectrum is absolutely continuous if $M_r(\mu_r + i\epsilon)$ has finite limits $M_r(\mu_{r+})$. The eigenvalues of H_r correspond to the poles of M_r . The $M_r(z)$ are determined off the real axis and can be constructed from the eigenfunctions of H_r that are square summable. In this case, note that the deficiency indices for L_1 and L_2 are (n, n) and (m, m) respectively at the limiting point with α -boundary conditions also imposed as $t \rightarrow \infty$. Application of von-Neumann theorems now show that L_1 and L_2 have

self-adjoint operator extensions that are described by $D(H_r)$ in (1.14).

Assume that

$$V_r(t, z) = \begin{pmatrix} V_{r_1}(t, z) \\ V_{r_2}(t, z) \end{pmatrix},$$

it follows that

$$V_r(t_0, z) = \begin{pmatrix} Y \end{pmatrix}_r(t_0) \begin{pmatrix} I_s \\ M_{t_0}(z) \end{pmatrix}$$

satisfy boundary conditions at t_0 and also at infinity. The boundary conditions are those given in (1.13) and hence

$$\begin{pmatrix} M_{t_0} \end{pmatrix}_r(z) = V_{r_2}(t_0, z)V_{r_1}^{-1}(t_0, z)$$

In this case, $V_{r_1}(t_0, z)$ are the Vandermonde matrices for eigenvalues of the square summable solutions. $V_{r_1}^{-1}(t_0, z)$ exists boundedly and hence $(M_{t_0})_r(z)$ are continuous for all z within the region of consideration. The continuous spectrum for H_r is not singular continuous spectrum but absolutely continuous spectrum. The left end regular point t_0 can then be extended to $t_0 = 0$ using the results of Remling (Remling, 1998). $\square$

Suppose that $p_k(t)$ and $q_j(t)$ are almost constant coefficients and $m = n$, then as shown in the next result, the set of commutators generated by the self-adjoint operator extensions of L_1 and L_2 contain the compact operators on $\ell^2(\mathbb{N})$, that is, $\mathcal{K}(\ell^2(\mathbb{N})) \subseteq W_1^n[H_1, H_2]W_2^n$, where $\mathcal{K}(\ell^2(\mathbb{N}))$ is the set of compact operators defined on $\ell^2(\mathbb{N})$.

Theorem 4.4.2. *Let L_1 and L_2 be the minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively in (1.7). Assume (4.4) and (4.5) are satisfied in addition to the following conditions*

$$m = n, \quad p_k(t) \rightarrow c_k, \quad q_j(t) \rightarrow d_j,$$

where, c_k and d_j are constants. Then:

- (i) *At limiting point, the deficiency indices of L_1 and L_2 are (n, n) . L_1 and L_2 have self-adjoint operator extensions; H_1 and H_2 , respectively.*

- (ii) The set of commutators $W_1^n[H_1, H_2]W_2^n$ contains the set of compact operators defined on $\ell^2(\mathbb{N})$.
- (iii) The self-adjoint operator extensions H_1 and H_2 commutes.

Proof. (i) Follows immediately from Theorem 4.4.1.

(ii) In order to construct the set of compact operators from the set of commutators $W_1^n[H_1, H_2]W_2^n$ with W_1 and W_2 as defined in (3.1), we apply the techniques of comparison algebras as constructed by H. O. Cordes (Cordes, 1987) and applied in the work of S. T. Melo (Melo, 1990) to construct self-adjoint operator extensions with common domain and show that the set of commutators contain the set of compact operators on $\mathcal{H} = \ell^2(\mathbb{N})$. Note that the operators $W_r^n H_r W_r^n$ are bounded in $\mathcal{H}$. The boundedness properties of these operators are required since the theory of comparison algebras are embedded in the theory of C^* -algebras.

Let $\{H_r, \mathbb{N}, \mu\}$ be a comparison triple for each self-adjoint operator extension H_r . It is this triple that we use to construct the H_r -comparison algebras. The comparison algebras are actually C^* -algebras. Here, μ is the counting measure in $\mathbb{N}$. Similarly, let $\mathcal{C}_0^\infty(\mathbb{N})$ be the class of all bounded complex-valued functions which includes also constant functions. This implies that any function $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ can be thought of as $h(t) = h_0(t) + c$ such that $h_0(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ and $c \in \mathbb{C}$. In our case, the function $h(t)$ represents $p_k(t)$ and $q_j(t)$, $k, j = 0, 1, \dots, n$.

On the other hand, assume that $\mathcal{W}$ is the class of all linear symmetric difference operators of order $2n$ generated by $\mathcal{L}_r$ such that L_r, L_r^*, H_r and those generated by complex conjugates of $\mathcal{L}_r$ all having compact support are contained in $\mathcal{W}$. The interest in this case is the algebra generated by classes of operators of the form $T = W_1^n H_r W_2^n$ together with the classes of multiplication operators $Ty(t) = h(t)y(t)$ in addition to the commutators of two generating operators $[W_1^n H_r W_2^n, h(t)H_r]$. Thus the required H_r -comparison algebras are obtained by norm-closing the algebra generated by the function $h(t)$ and the operator $W_1^n H_r W_2^n$ for all functions $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ and $W_1^n H_r W_2^n \in \mathcal{W}$ having compact support. This results into minimal comparison algebra of H_r which we will denote by $\mathcal{C}$.

The comparison algebra is a non-unital C^* -algebra since $\mathbb{N}$ is non-compact. It can be made unital by adjoining $\{e\}$ to it, e is the identity element. Every minimal algebra, that is, every comparison algebra of self-adjoint operator H_r contain the entire ideal of compact operators $\mathcal{K}(\mathcal{H})$ defined on $\mathcal{H}$ (Cordes, 1987).

We now show that every element in $\mathcal{K}(\mathcal{H})$ is in the set of commutator and that the commutator set is non-empty. Suppose that $f(t) \in \mathcal{C}_0^\infty(\mathbb{N})$, $h(t) = p_k(t)$, $q_j(t)$ with $h(t)$ not vanishing to zero identically as $t \rightarrow \infty$, then one can consider the operator defined by $h(t)W_r = h(t)H_r^{-\frac{1}{2n}}$. This operator is compact and is in the set of commutator since in general any operator of the form Δ generates an operator of the form $H_r^{-\frac{1}{2n}}$ and $[h(t), \Delta]$ is in the set of commutator if and only if $\Delta h(t) \rightarrow 0$ as $t \rightarrow \infty$ and this is one of the assumptions made. Thus $h(t)W_r \neq 0$. It follows, therefore, by construction that $H_r W_2^n = W_1^n H_r$ and they are defined on the space $H_r(\mathcal{C}_0^\infty(\mathbb{N}))$. Moreover, there exists compact operators C_1 and C_2 defined on $\mathcal{H}$ such that $W_1^n C_1 W_2^n = W_1^n C_2 W_2^n$.

Finally, we show that the operator $H_1 W_1^n$ commutes with $H_2 W_2^n$. First note that by Gelfand-Naimark theorem, the H_1 -comparison algebra is isomorphic to H_2 -comparison algebra and since W_1 and W_2 are isometric isomorphisms constructed from the same underlying manifold, they are commutative since isometries of the same manifold form abelian group under composition. Secondly, by construction, the operators $H_r W_2^n$ and $W_1^n H_r$ are commutative and since they are compact, there exists a compact operator C_0 such that $H_r W_2^{2n} - W_1^n H_r W_2^n = C_0 W_2^n$, where $C_0 = H_r W_2^n - W_1^n H_r$ and this implies that for any function $y(t) \in D(W_r^{-n})$, we have $H_r W_2^n y(t) = W_1^n H_r y(t) + C_0 y(t)$ so that $W_1^n H_r$ with domain $D(W_r^{-n})$ has a continuous extension $(W_r^n H_r)^{**}$ which is a linear continuous operator in $\mathcal{H}$. Similarly, $H_r W_2^n = (W_1^n H_r)^{**} + C_0$. Computing the adjoints of both sides of the last relation, we get $(H_r W_2^n)^* = H_r^* W_2^n + C_0^*$ which implies that $(H_r W_2^n)^* - H_r^* W_2^n$ and $H_r W_2^n - (W_1^n H_r)^{**}$ are compact.

Using similar arguments, there exists a compact operator C_1 such that $C_1 = W_1^n [H_1, H_2] W_2^n$.

This is because the operator $W_1^n[H_1, H_2]W_2^n$ is bounded in $\mathcal{H}$. Applying the fact that $H_r W_2^n - (W_1^n H_r)^{**}$ is compact, for $y(t) \in H_r \mathcal{C}_0^\infty(\mathbb{N})$, we obtain

$$\begin{aligned}
H_1 W_1^n H_2 W_2^n y(t) &= (W_1^n H_1)^{**} H_2 W_2^n y(t) + C_0 y(t) \\
&= C_0 y(t) + W_1^n H_1 H_2 W_2^n y(t) \\
&= C_1 y(t) + W_2^n H_2 H_1 W_1^n y(t) + C_2 y(t) \\
&= C_3 y(t) + (W_2^n H_2)^{**} H_1 W_1^n y(t) \\
&= H_2 W_2^n H_1 W_1^n y(t) + C_4 y(t),
\end{aligned}$$

for some C_2, C_3, C_4 compact operators in $\mathcal{H}$. This implies that the set of commutator $W_1^n[H_1, H_2]W_2^n$ contains the set of compact operators on $\mathcal{H}$ and it is non-empty.

(iii) We apply Zagorodnyuk results (Zagorodnyuk, 2012) to prove that H_1 and H_2 commute. From the construction of the commutator sets $W_1^n[H_1, H_2]W_2^n$ in (ii) above, it follows that $H_1 W_1^n$ commutes with $H_2 W_2^n$ and hence we need to show that $D(H_1) = D(H_2)$ so that H_1 commutes with H_2 . From the assumptions in (4.4) and the comparison algebra $\mathcal{C}$ constructed in (ii), it follows that H_1 and H_2 have common domain, that is, $D(H_1) = D(H_2)$ and since H_r , $r = 1, 2$ is self-adjoint, $D(H_r)$ is densely defined, $\overline{D(H_r)} = \mathcal{H} = \ell^2(\mathbb{N})$. It is also true that $H_r(D(H_r)) \subseteq D(H_r)$ and the restriction of H_2 to $(H_1 - iI)D(H_1)$ or H_1 to $(H_2 - iI)D(H_2)$ is not only essentially self-adjoint but also self-adjoint. Because $n = m$ by assumption, the symplectic matrix $\mathcal{J}$ in (1.11) results into $\mathcal{J}H_r = H_r\mathcal{J}$. By the results of (Zagorodnyuk, 2012), it follows that H_1 commutes with H_2 and so are their respective minimal and maximal difference operators.

This implies that the commutator algebra generated by H_r contains the set of compact operators in $\mathcal{H}$ and that the operators H_1 and H_2 commute and so are the respective minimal difference operators. $\square$

The example below verifies the results of Theorem 4.4.2. This result gives the necessary and sufficient conditions on the coefficients of difference operators of the same order to commute. The condition that $\Delta(h(t)) \rightarrow 0$ as $t \rightarrow 0$ is necessary and cannot be omitted as the example also shows.

Example 4.4.3. As an immediate example, consider the operators generated by difference equations

$$\mathcal{L}_1 y(t) = -\Delta(p(t)\Delta y(t-1)), \quad \mathcal{L}_2 y(t) = -\Delta(q(t)\Delta y(t-1)),$$

$$\Delta p(t), \Delta q(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

It can be shown that $\mathcal{L}_1(\mathcal{L}_2 y(t)) - \mathcal{L}_2(\mathcal{L}_1 y(t)) \rightarrow 0$ as $t \rightarrow \infty$ and hence the minimal operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ as well as their respective self-adjoint operator extensions are commutative. Similarly, assume that $p(t)$ and $q(t)$ are almost constant coefficients, then commutativity shall be achieved. On the other hand, if we assume for simplicity, that $p(t) = 1$ for all $t \in \mathbb{N}$ and $q(t) = \cos \frac{\pi t}{2}$, then $\Delta q(t)$ shall be 1 or -1 depending on whether t is odd or even. A simple computation shows that $\mathcal{L}_1 y(t)$ does not commute with $\mathcal{L}_2 y(t)$. Hence the condition that $\Delta h(t) \rightarrow 0$ as $t \rightarrow \infty$ in (4.4) is necessary.

In the remaining Section, we will assume that (4.4) and (4.5) hold and in addition, we will also assume that $n = m$. We now develop a similar analysis to that of $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ as done so far in Chapter *FOUR*, Section 4.4 for the composite function $\mathcal{L}_1(\mathcal{L}_2 y(t))$. Thus we now consider the composite of $\mathcal{L}_1$ and $\mathcal{L}_2$, denoted by $\mathcal{L}$ defined on $\ell^2(\mathbb{N})$. The difference equation for $n = m$ is a $4n$ th order symmetric difference equation given by

$$\begin{aligned} \mathcal{L}y &= \mathcal{L}_1(\mathcal{L}_2 y) = \mathcal{L}_2(\mathcal{L}_1 y) \\ &= \sum_{k=0}^n (-1)^k \left\{ \sum_{j=0}^m (-1)^j \Delta^k (p_k \Delta^{k+j} (q_j \Delta^j y(t-j))) \right\} \\ &= \sum_{j=0}^m (-1)^j \left\{ \sum_{k=0}^n (-1)^k \Delta^j (q_j \Delta^{j+k} (p_k \Delta^k y(t-k))) \right\}, \end{aligned} \tag{4.6}$$

defined on $\ell^2(\mathbb{N})$ with $\sum_{k=0}^n (\sum_{j=0}^m (p_k q_j)) > 0$. The interest in this analysis is to develop structural properties of the minimal operator generated in the Hamiltonian form as well as the first order form of $\mathcal{L}_1(\mathcal{L}_2 y) = zy$ for some spectral parameter z . The coefficients p_k and q_j are assumed to satisfy similar growth and decay conditions that have been set in (4.4) and (4.5) so that at the end, we determine all the necessary and sufficient conditions for $H = H_1 H_2$ where H is the self-adjoint

operator extension generated by minimal operator L corresponding to $\mathcal{L}_1(\mathcal{L}_2)$. We thus solve the equation $\mathcal{L}y = zy$, where z is the spectral parameter. First, we need to reduce $\mathcal{L}y = zy$ into its first order form using quasi-differences of the form

$$\begin{aligned}
x_j(t) &= q_j(t)\Delta^{j-1}y(t-j) & 1 \leq j \leq m \\
x_{m+1}(t) &= q_m(t)\Delta^m(y(t-m)) \\
x_{m+k}(t) &= -\Delta(q_m(t)\Delta^m y(t-m)) - p_{n-k}(t)(q_{n-k}(t)\Delta^{n-k}y(t-n+k)) \\
&& 2 \leq k \leq n \\
x_{n+m+r}(t) &= -(\Delta(x_{n+m+r-1}(t)) + p_{n-r}(t)x_{m+n-r}(t)) & 2 \leq r \leq n+m-1
\end{aligned} \tag{4.7}$$

In this case, we define $2(n+m)$ -dimensional vector

$$Y(t, z) = [x_1(t), x_2(t), \dots, x_{2(n+m)}(t)]^{tr}.$$

Just like in Section 1.1.2, we use symplectic matrix

$$\mathcal{J} = \begin{bmatrix} 0_{n+m} & -I_{n+m} \\ I_{n+m} & 0_{n+m} \end{bmatrix}.$$

Therefore, the Hamiltonian system for (4.6) is given by

$$\mathcal{J} \triangle Y(t) = P(t)K_1 Y(t), \tag{4.8}$$

where K_1 is the forward partial shift operator and $P(t)$ is a $2(n+m) \times 2(n+m)$ matrix that can be written in a block form

$$P(t) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

with A, B, C as $(n+m) \times (n+m)$ matrices having non-zero entries given by

$$\begin{aligned}
A_{l,l+1} &= 1 & B_{n+m,n+m} &= (p_n q_m)^{-1}, \\
C_{l,l} &= f_l = \sum_{k=0}^l \left(\sum_{j=0}^l (p_k q_j) \right), & l &= 0, 1, 2, \dots, n+m.
\end{aligned}$$

Here, the term $p_0 q_0$ should be interpreted as $p_0 q_0 - z$. The first order can now be written explicitly as

$$\begin{aligned}
Y(t+1, z) &= \begin{bmatrix} E & EB \\ CEB & I - A^* + CEB \end{bmatrix} Y(t, z) \\
&= S(t, z)Y(t, z).
\end{aligned} \tag{4.9}$$

Here, we note that $E = (I_{n+m} - A)^{-1}$.

In order to properly define the maximal and minimal difference operators generated by $\mathcal{L}$ in (4.6), we still need some regularity conditions just like in Section 1.1.2. These are

$$\ell^2(\mathbb{N}) = \left\{ y(t) : y(t) = \{y(t)\}_{t=0}^\infty \subset \mathbb{C} \right. \\ \left. \sum_{t=0}^\infty K_1(Y^*(t))K_1(Y(t)) < \infty \right\}.$$

in addition, there exists an integral domain $I_2 \subset \mathbb{N}$ such that

$$\sum_{t \in I_2} K_1 Y^*(t) K_1 Y(t) > 0,$$

$\ell^2(I_2) \subset \ell^2(\mathbb{N})$ and

$$\ell^2(I_2) = \left\{ y(t) \in \ell^2(\mathbb{N}) : \sum_{t \in I_2} K_1 Y^*(t) K_1 Y(t) > 0 \right\}.$$

we define, therefore, the scalar products for $y(t), y_1(t) \in \ell^2(I_2)$ by

$$\langle y(t), y_1(t) \rangle = \sum_{t \in I_2} y(t) \overline{y_1(t)}.$$

From the results of Theorem 4.4.2, it follows that $I_2 \subseteq I_1$ and hence $\ell^2(I_2)$ is a subspace of $\ell^2(I_1) \subset \ell^2(\mathbb{N})$.

Now if $F_1(t)$ is $(2(n+m))$ -dimensional vector, then we also need a similar condition like that stated in section 1.1.2 in order for the operators generated to be properly defined.

$$\mathcal{J}\Delta Y(t, z) - PK_1 Y(t, z) = F_1(t), \text{ with } \|Y(t, z)\| = 0, \text{ if } F_1(t) = 0.$$

The construction of the associated maximal and minimal difference operators via their domains is done as follows. The maximal difference operator L^* defined via its domain $D(L^*)$ is given by

$$D(L^*) = \left\{ y(t) \in \ell^2(I_2) : \text{there exists } F_1(t) \in \ell^2(I_2) \text{ such that} \right. \\ \left. \mathcal{J}\Delta Y(t) - PK_1 Y(t) = F_1(t) \right\}$$

and

$$L^*Y(t, z) = F_1(t) \text{ if and only if } \mathcal{J}\Delta Y(t, z) - PK_1Y(t, z) = F_1(t).$$

The corresponding pre-minimal operator is now given by

$$\begin{aligned} D(L_0) &= \{y(t) \in D(L^*) : \text{there exists } n \in \mathbb{N} \\ &\quad \text{such that } y(0) = y(t) = 0 \text{ for all } t \geq n + 1\}. \end{aligned}$$

$$L^*y(t) = L_0y(t), \quad \forall y(t) \in D(L_0).$$

L_0 is densely defined, symmetric but not necessarily closed. Since we need closed operators, we take the closure of this pre-minimal operator to be the minimal difference operator which we denote by L . The boundary conditions at the regular left end point [(Behncke & Hinton, 2013), (Y. Shi, 2006)] can now be defined using $(n + m) \times (n + m)$ matrices $(\tilde{\alpha}_1, \tilde{\alpha}_2)$ with $\tilde{\alpha}_1\tilde{\alpha}_1^* + \tilde{\alpha}_2\tilde{\alpha}_2^* = I_{n+m}$ and $\tilde{\alpha}_1\tilde{\alpha}_2^* = \tilde{\alpha}_2\tilde{\alpha}_1^*$ so that at the regular left end point t_0

$$(\tilde{\alpha}_1, \tilde{\alpha}_2) \begin{bmatrix} Y(t_0) \end{bmatrix} = 0. \quad \text{where } Y(t) = [x_1(t), x_2(t), \dots, x_{2(n+m)}]^{tr} \quad (4.10)$$

In solving the equation $\mathcal{L}y = zy$, we shall apply techniques of Eastham (M. S. P. Eastham, 1989) which have been applied extensively by Nyamwala and other authors, see [(Behncke & Hinton, 2013), (Behncke & Nyamwala, 2012), (Behncke & Nyamwala, 2011), (Nyamwala, 2017), (Nyamwala, 2015)] for details. We assume the following growth conditions for the remaining part of the analysis

$$h_{l-1}h_{l+1} = o(h_l^2), \quad h_l = p_l, q_l, \quad l = 1, 2, \dots, n = m. \quad (4.11)$$

We note now that the growth conditions in (4.11) results into $f_{l-1}f_{l+1} = o(f_l^2)$, where

$$f_l = \sum_{k,j=0}^l p_k q_j, \quad l = 0, 1, 2, \dots, n + m. \quad (4.12)$$

In order to transform (4.6) into the required first order form, we need the eigenvalues of the matrix $S(t, z)$ which we obtain from the zeroes of the characteristic polynomial $\mathcal{P}(t, \lambda, z) = \det(S(t, \lambda, z) - \lambda \cdot I_{2(n+m)})$. By multiplying $\mathcal{P}(t, \lambda, z)$ by $(-1)^{n+m} p_n q_m \lambda^{-(n+m)}$ and taking $\gamma = 2 - (\lambda + \lambda^{-1})$, we obtain a reduced polynomial

of the form

$$\begin{aligned}\mathcal{F}(t, \gamma, z) &= (-1)^{n+m} p_n q_m \lambda^{n+m} \mathcal{P}(t, \lambda, z) \\ &= \sum_{l=0}^{n+m} f_l \gamma^l.\end{aligned}\tag{4.13}$$

Before we proceed with our analysis, we start by proving that the basis of the homogeneous equation $\mathcal{L}_1 \mathcal{L}_2 y = 0$ can be determined from the bases of solutions of equations $\mathcal{L}_1 y = 0$ and $\mathcal{L}_2 y = 0$.

Theorem 4.4.4. *Suppose $\mathcal{L}_1 \mathcal{L}_2 y = 0$, $n = m$, then the basis of the solutions of this equation can be determined from the bases of the solutions of $\mathcal{L}_1 y = 0$ and $\mathcal{L}_2 y = 0$.*

Proof. The proof in this case follow from the results of (Littlejohn & López, 2010) with obvious modification to suit difference equations. Suppose that $\{y_1, y_2, \dots, y_{2n}\}$ is a basis of solutions of $\mathcal{L}_2 y = 0$ and $\{\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{2n}\}$ is the basis of solutions of $\mathcal{L}_1 y = 0$, then the basis of the solutions of the composite is obtained using the Green's function $G_2(t, t_1)$ associated with $\mathcal{L}_2 y = 0$, and the corresponding Wronskian determinant. For a fixed left regular end point $t_0 \in \mathbb{N}$, define functions

$$y_{2n+s}(t) = \sum_{t_1=t_0}^{t-1} G_2(t, t_1) \tilde{y}_s(t_1) \quad s = 1, 2, \dots, 2n.$$

Then, the set $\tilde{Y} = \{y_1, y_2, \dots, y_{2n}, y_{2n+1}, y_{2n+2}, \dots, y_{4n}\}$ is a basis of solutions of the composite equation $\mathcal{L}_1 \mathcal{L}_2 y = 0$ for $t \in \mathbb{N}$. To prove this, consider the equations $\mathcal{L}_2 y_k = 0$, $k = 1, 2, \dots, 2n$, $\mathcal{L}_2 y_k = \tilde{y}_k$, $k = 2n + 1, \dots, 4n$ and $\mathcal{L}_1 \mathcal{L}_2 y_k = 0$, $k = 2n + 1, \dots, 4n$. This implies that

$$(\mathcal{L}_1 \mathcal{L}_2) y_k = \mathcal{L}_1 (\mathcal{L}_2 y_k) = \mathcal{L}_1 \tilde{y}_k = 0, \quad k = 2n + 1, \dots, 4n.$$

The rest of the proof now follows from the usual linear algebra techniques as illustrated in Theorem 4.3.5. □

In the next result, we now prove that if $\mathcal{L}_1$ and $\mathcal{L}_2$ commute, then their composite is symmetric and so is the operator $L_1 L_2$.

Theorem 4.4.5. *Assume that (4.4) is satisfied such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative. Then the minimal difference operators L_1 and L_2 are commutative and the composite $\mathcal{L}_1\mathcal{L}_2$ is symmetric as well as L_1L_2 .*

Proof. From Theorem 4.4.2, we have that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative and so are L_1 and L_2 . Similarly, the operators L_1^* and L_2^* commute and to see this, take $y, y_1 \in D(L_2)$ and we have

$$\langle y, L_1^*L_2^*y_1 \rangle = \langle y, (L_2L_1)^*y_1 \rangle = \langle y, (L_1L_2)^*y_1 \rangle = \langle y, L_2^*L_1^*y_1 \rangle.$$

Hence $L_1^*L_2^* = L_2^*L_1^*$.

It remains to show that $D(L_1L_2) \subseteq D((L_1L_2)^*)$. Since L_1 and L_2 are closed symmetric operators, it follows that $\overline{D(L_2)} = \mathcal{H}$ and since L_1 commutes with L_2 , we have $D(L_1L_2) = D(L_2)$ such that $D(L_1) \subseteq \mathcal{R}(L_2)$. Suppose now that $\tilde{y} \perp D(L_2)$ such that $\langle \tilde{y}, y \rangle = 0$, for all $y \in D(L_2)$, then for any solution $\hat{y}$ of $\mathcal{L}_1\mathcal{L}_2\hat{y} = \tilde{y}$ and since $D((L_1L_2)^*) = D(L_1^*L_2^*) = D(L_2^*)$, it implies that

$$\langle \hat{y}, \mathcal{L}_1\mathcal{L}_2y \rangle = \langle L_2^*L_1^*\hat{y}, y \rangle = \langle L_1^*L_2^*\hat{y}, y \rangle = \langle \mathcal{L}_1\mathcal{L}_2\hat{y}, y \rangle = \langle \tilde{y}, y \rangle = 0.$$

This implies that $y \in \mathcal{R}(L_1L_2)^\perp = \mathcal{N}(L_2^*L_1^*) = \mathcal{N}((L_1L_2)^*)$ and hence $\tilde{y} = \mathcal{L}_1\mathcal{L}_2\hat{y} = (L_1L_2)^*\hat{y} = 0$. The operator L_1L_2 is symmetric and so is the function $\mathcal{L}_1\mathcal{L}_2$ that generates it on $\mathcal{H} = \ell^2(\mathbb{N})$. $\square$

The composite operator L_1L_2 is now symmetric but not necessarily densely defined. We now state the necessary and sufficient conditions for the composite to be densely defined.

Theorem 4.4.6. *Let L_1 and L_2 be minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Assume (4.1) is satisfied and that $\mathcal{R}(L_1)$, $\mathcal{R}(L_2)$ are closed. Then the operator L_1L_2 is densely defined.*

Proof. By construction of L_1 and L_2 in Section 1.1.2, the two operators are closed. This implies that their ranges are closed and hence there exists a closed subspace $\mathcal{D}$ of $\mathcal{H} \times \mathcal{H}$ defined by

$$\mathcal{D} = \{(y, L_1L_2y) : y \in D(L_1L_2)\}.$$

By injectivity and linearity of L_1 and L_2 which imply that of L_1L_2 , $\mathcal{D}$ is graph of L_1L_2 because for any $(0, \tilde{y}) \in \mathcal{D}$, it follows that $\tilde{y} = 0$. On the other hand, for any $y \in D(L_1L_2)$, there is at most one $\hat{y}$ such that $(y, \hat{y}) \in \mathcal{D}$ and $\hat{y} = L_1L_2y$. The operator L_2^{-1} can now be defined on $\mathcal{R}(L_2)$ because $\mathcal{R}(L_2)$ is closed and L_2 is injective hence $D(L_1) = \mathcal{R}(L_2)$. For the operator L_2L_1 , now consider $D(L_1L_2)$ which can be expressed as

$$D(L_1L_2) = D(L_2) \cap L_2^{-1}D(L_1).$$

Since $D(L_1)$ and $D(L_2)$ are dense in $\mathcal{H}$ by construction, $L_2^{-1}D(L_1)$ is a dense subspace of $\mathcal{H}$ and thus by Baires Category theorem, the subspace $D(L_2) \cap L_2^{-1}D(L_1)$ is dense in $\mathcal{H}$. The operator L_1L_2 is densely defined.

□

The operator L_1L_2 together with conditions in Theorem 4.4.6 is now symmetric and densely defined but not necessarily closed. We may take its closure, that is, $\overline{L_1L_2}$ and this we shall denote by L . This allows for the analysis of spectral theory of self-adjoint operator extension of the composite of operators L_1 and L_2 . Note that by Closed-Range Theorem, if all the conditions in Theorem 4.4.6 are satisfied, then the range of L^* is closed. Thus by application of Rank-Nullity theorem we have

$$\dim \mathcal{N}(L^* - zI) = \dim \mathcal{N}(\overline{L_1L_2}^* - zI) = \dim \mathcal{N}(L_1^* - zI) + \dim \mathcal{N}(L_2^* - zI).$$

This implies that $\text{def} L = \text{def} L_1 + \text{def} L_2$.

4.5 Symmetries and Extensions of Composites of Order Two and Four Difference Operators

In this section, by allowing the leading coefficients to be unbounded and the other coefficients as constant functions, we have shown that the composite $\mathcal{L}_1\mathcal{L}_2$ is symmetric if the leading coefficients are scalar multiple of each other and the common divisor of their orders is 1. Using examples, we have shown that these conditions cannot be weakened. If in (1.7), $n = m$, $q_m(t) = \alpha p_n(t)$ with $\alpha \neq 0$, a real constant

and all other coefficients are constant valued functions, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ and their composites are symmetric as shown below.

Theorem 4.5.1. *Suppose that $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ are symmetric difference equations given in (1.7) with $p_n, q_m \neq 0$, $q_m(t)$ is a constant multiple of $p_n(t)$, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ if the common divisor of m and n is 1 and all the other coefficients are constants or when the rank of $\mathcal{L}_1$ and $\mathcal{L}_2$ is 2, $m = n$, q_m is a constant multiple of p_n while all other coefficients q_j , $j = 0, 1, 2, \dots, m-1$ and p_k , $k = 0, 1, 2, \dots, n-1$ are all constants. In particular, the composite $\mathcal{L}_1 \mathcal{L}_2 y(t) = \mathcal{L}_2(\mathcal{L}_1 y(t))$ as given in (4.6), that is,*

$$\begin{aligned} \mathcal{L}_1(\mathcal{L}_2 y(t)) &= \mathcal{L}_2(\mathcal{L}_1 y(t)) \\ &= \sum_{k=0}^n (-1)^k \sum_{j=0}^m (-1)^j \{ \Delta^k (p_k \Delta^j (q_j \Delta^{j+k} y(t-j-k))) \} \end{aligned}$$

Proof. In the proof, we shall write $\mathcal{L}_1$ and $\mathcal{L}_2$ to imply $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ respectively. $\mathcal{L}_1$ and $\mathcal{L}_2$ are of orders $2n$ and $2m$ respectively and since by assumption the common divisors of m and n is 1, it implies that the common divisors of their orders is 2. By the results of Mironov [(Mironov, 2004), (Mironov, 2014)] and those of Amitsur (Amitsur, 1958), the dimension of the space of common eigenfunctions of $\mathcal{L}_1$ and $\mathcal{L}_2$ is 2 and hence $\mathcal{L}_1$ and $\mathcal{L}_2$ commute. On the other hand, if we assume $m = n$ and all other coefficients apart from p_n and q_m are constants, then by expanding $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\mathcal{L}_2(\mathcal{L}_1 y(t))$ using the definition of Δ or Δ as forward or backward difference operators and comparing all the terms of these expansions, it suffices to check the equality on the coefficients of the terms $y(t+2n-1)$ which is similar to those of the term $y(t-2n+1)$ albeit with some shift in the independent coefficients as a result of successive applications of shift operators since $\Delta = E - I$ where E is a forward shift, $E(t) = t+1$ and I is the appropriate identity. Here we always note that $q_m(t) = \alpha p_n(t)$ for all $t \in \mathbb{N}$. The coefficients of the term $y(t+2n-1)$ for both composites are given by

$$(-1)^n \{ q_{n-1} p_n(t+n-1) + \alpha [p_{n-1} p_n(t+n-1) + p_n(t) p_n(t+n-1) + 2(p_n(t+n-1))^2 + p_n(t+n-1) p_n(t+n)] \},$$

where q_{n-1} and p_{n-1} are constants as per the assumptions made. Now by mathematical induction, it follows that the composite can be written as claimed in the theorem. $\square$

We now demonstrate using examples that the results of Theorem 4.5.1 are true and that some of the conditions given in Theorem 4.5.1 cannot be weakened.

Example 4.5.2. This example verifies the results of Theorem 4.5.1. We assume that all the conditions as outlined in Theorem 4.5.1 are satisfied and consider second order difference equations of the form:

$$\begin{aligned}\mathcal{L}_1(y) &= -\Delta(p(t)\Delta y(t-1)) + \beta_1 y(t) \\ \mathcal{L}_2(y) &= -\Delta(\alpha p(t)\Delta y(t-1)) + \beta_2 y(t)\end{aligned}$$

where α , β_1 and β_2 are real valued constants, $\alpha \neq 0$.

By expansion and comparing the coefficients of the leading terms in the composites $\mathcal{L}_1(\mathcal{L}_2 y)$ and $\mathcal{L}_2(\mathcal{L}_1 y)$, we have that for the term $y(t+2)$, the coefficients are given as $\alpha p(t+1)p(t+2)$ for $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\alpha p(t+1)p(t+2)$ for $\mathcal{L}_2(\mathcal{L}_1 y(t))$. The term $y(t+1)$ has the coefficients given as $-\beta_2 p(t+1) - \alpha[\beta_1 p(t+1) + p(t)p(t+1) + 2\{p(t+1)p(t+1)\} + p(t+1)p(t+2)]$ for both composites. For the term $y(t)$, the coefficients of the two composites are given by $\beta_1\beta_2 + \beta_2(p(t)p(t+1)) + \alpha[2\{p(t)p(t)\} + 2\{p(t)p(t+1)\} + 2\{p(t+1)p(t+1)\} + \beta_1(p(t)p(t+1))]$ while for the term $y(t-1)$, we have, $-\beta_2 p(t) - \alpha[\beta_1 p(t) + p(t)p(t-1) + 2\{p(t)p(t)\} + p(t+1)p(t)]$ as the coefficients and for the term $y(t-2)$ the coefficients are given by; $\alpha p(t)p(t-1)$ in both composites. These clearly show that $\mathcal{L}_1(\mathcal{L}_2 y)$ is equal to $\mathcal{L}_2(\mathcal{L}_1 y)$

Example 4.5.3. In this example, we confirm that some of the conditions imposed on the arbitrary leading coefficients in Theorem 4.5.1 are necessary. Assume that $n = m = 2$ with $p_2 = p$, $q_2 = q$, $p_1 = q_1 = 0 \forall t \in \mathbb{N}$ and let $p_0 = r_1$, $q_0 = r_2$ where r_i , $i = 1, 2$ are non-zero constants. By expansion of the composites $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\mathcal{L}_2(\mathcal{L}_1 y(t))$, the coefficients of $y(t \pm l)$, $l = 0, 1, 2, 3, 4$ shows some symmetry with the coefficients of $y(t+l)$ matching those of $y(t-l)$ with some kind of a shift which is as a result of successive application of shift operator, $Ef(t) = f(t+1)$. Note that the forward difference operator Δ can be re-written as $\Delta = E - I$ where I is the identity operator. On checking the coefficients of $y(t+4)$ for the composites, we obtain either $p(t+2)q(t+4)$ or $q(t+2)p(t+4)$ and for $y(t-4)$ either the coefficients are $p(t)q(t-2)$ or $q(t)p(t-2)$. The coefficients of the term $y(t+3)$ are given by $-[2p(t+1)q(t+3) + 4p(t+2)q(t+3) + 2p(t+2)q(t+4)]$ or $-[2q(t+1)p(t+3) +$

$4q(t+2)p(t+3) + 2q(t+2)p(t+4)]$. The coefficients of the other middle terms, that is, $y(t \pm l)$, $l = 0, 1, 2$ have similar patterns showing that $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ if and only if $q(t) = \alpha p(t)$ for some non-zero constant α and for all $t \in \mathbb{N}$. This confirms the requirement in Theorem 4.5.1 that the leading coefficients must be constant multiple of each other.

Remark 4.5.4. We point out that if $m \neq n$, $q_m(t)$ and $p_n(t)$ are not constants, then even if $q_m(t)$ is a constant multiple of $p_n(t)$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are not commutative as shown in the next example.

Example 4.5.5. In this example we show that if the orders of the difference equations in (1.7) are not equal, then even if the leading coefficients are scalar multiple of each other, the results in Theorem 4.5.1 will not hold. For simplicity, we consider the following order two and four symmetric difference equations with leading coefficients as scalar multiple of the other given by

$$\mathcal{L}_1(y) = \Delta^2(p(t)\Delta^2 y(t-2))$$

and

$$\mathcal{L}_2(y) = -\Delta(\alpha p(t)\Delta y(t-1)).$$

By expansion and comparing the coefficients of the leading terms of their composites, we have for the term $y(t+2)$ as $\alpha\{p(t)p(t+2) + 3p(t+1)p(t+2) + 2p(t+1)p(t+3)\}$ for $\mathcal{L}_1(\mathcal{L}_2 y)$ and $\alpha\{2p(t+1)p(t+2) + 3[p(t+2)]^2 + p(t+2)p(t+3)\}$ for $\mathcal{L}_2(\mathcal{L}_1 y)$ which shows that $\mathcal{L}_1(\mathcal{L}_2 y)$ is not equal to $\mathcal{L}_2(\mathcal{L}_1 y)$.

Now suppose that L_1 and L_2 are minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ respectively, by the results of G. Ren and Y. Shi (Ren & Shi, 2011), they had shown that the respective maximal difference operators risk being multi-valued and not densely defined. In order to rescue such a scenario, G. Ren and Y. Shi [(Ren & Shi, 2010), (Ren & Shi, 2011), (Ren & Shi, 2014)] had reverted to using subspace theory to analyse the deficiency indices and spectrum of such difference equations with the only option of reverting to operator theory approach if and only if the respective maximal difference operators are densely defined and single-valued.

This situation can be navigated by working on a subspace of $\ell^2(\mathbb{N})$ since as outlined in (Ren & Shi, 2011) and eventually applied by Behncke and Hinton (Behncke & Hinton, 2013), there exists a finite interval $I_1 \subset \mathbb{N}$ such that for any non-trivial solution $y(t)$ of (1.7), one has

$$\sum_{t \in I_1} Ry^*(t)Ry(t) > 0.$$

Here, $Ry(t)$ is a partial shift operator for which if $y(t) = [y_1(t), y_2(t), \dots, y_{2n}(t)]^{tr}$, then $Ry(yt) = [y_1(t+1), \dots, y_n(t+1), y_{n+1}(t), \dots, y_{2n}(t)]^{tr}$. Now the deficiency indices and spectral results obtained in $\ell^2(I_1)$ can then be easily be extrapolated to $\ell^2(\mathbb{N})$ using Remling's results (Remling, 1998) which were later validated in the discrete setting by Y. Shi (Y. Shi, 2006). With these clarifications, we therefore show that the operators generated by the composites of $\mathcal{L}_1\mathcal{L}_2$ are symmetric and have symmetric operator extensions when the conditions outlined in Theorem 4.5.1 are satisfied.

Theorem 4.5.6. *Assume that all the conditions of Theorem 4.5.1 are satisfied, then the minimal operator generated by (4.6) is symmetric and has symmetric operator extensions.*

Proof. Suppose all the conditions of Theorem 4.5.1 are satisfied, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$, and therefore, for all $y(t), y_1(t) \in \ell^2(\mathbb{N})$, we have

$$\begin{aligned} \langle \mathcal{L}_1\mathcal{L}_2y(t), y_1(t) \rangle &= \langle \mathcal{L}_2y(t), \mathcal{L}_1y_1(t) \rangle = \langle y(t), \mathcal{L}_2\mathcal{L}_1y_1(t) \rangle \\ &= \langle y(t), \mathcal{L}_1\mathcal{L}_2y(t) \rangle = \langle (\mathcal{L}_1\mathcal{L}_2)^*y(t), y_1(t) \rangle. \end{aligned}$$

This implies that $D(\mathcal{L}_1\mathcal{L}_2) \subseteq D(\mathcal{L}_1\mathcal{L}_2)^*$, and hence the composites $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{L}_2\mathcal{L}_1$ are symmetric. We now construct the respective minimal and maximal symmetric operators generated by the composite $\mathcal{L}_1\mathcal{L}_2y(t)$.

In line with the quasi-differences given by Y. Shi (Y. Shi, 2006), the quasi-differences for the composites $\mathcal{L}_1\mathcal{L}_2y$ are constructed as illustrated in (4.7). In order to apply the quasi-differences in (4.7) to convert $\mathcal{L}_1\mathcal{L}_2y(t)$ into its first order form, we define the vector valued functions $Y(t) = [x_1(t), x_2(t), \dots, x_{2(n+m)}(t)]^{tr}$. Just like in the case of non-composites given in (Y. Shi, 2006) and applied in (Behncke & Nyamwala,

2011), as before, we also use symplectic matrix defined by:

$$\mathcal{J} = \begin{bmatrix} 0_{n+m} & -I_{n+m} \\ I_{n+m} & 0_{n+m} \end{bmatrix}.$$

It follows that

$$\mathcal{J}\Delta Y(t) = P(t)RY(t),$$

where $P(t)$ is a $2(n+m) \times 2(n+m)$ matrix that can be written in a block form

$$P(t) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

with A, B, C as $(n+m) \times (n+m)$ matrices with non-zero entries given by

$$\begin{aligned} A_{k,k+1} &= 1 & B_{n+m,n+m} &= (p_n q_m)^{-1} \\ C_{k,k} &= \sum_{k=1}^n \sum_{j=1}^m (p_{k-1} q_{j-1} + p_{n-1} q_{m-1}). \end{aligned}$$

Always note that by construction, $n = m$, p_n and q_m are scalar multiple of each other with the other coefficients as constants. $p_n(t), q_m(t) \neq 0$ for all $t \in \mathbb{N}$. R as mentioned earlier is a partial shift operator and in this case,

$$RY(t) = [x_1(t+1), x_2(t+1), \dots, x_{n+m}(t+1), x_{n+m+1}(t), \dots, x_{2(n+m)}(t)]^{tr}$$

. Thus for the Hilbert space $\ell^2(\mathbb{N})$, the space is now defined by

$$\ell^2(\mathbb{N}) = \{y(t) : y(t) = \{y(t)\}_{t=0}^\infty \subset \mathbb{C} \text{ and } \sum_{t=0}^\infty R(y^*(t))(Ry(t)) < \infty\}.$$

As before, there exists a finite interval $I_1 \subset \mathbb{N}$ such that

$$\sum_{t \in I_1} Ry^*(t)Ry(t) > 0,$$

so that we define $\ell^2(I_1)$ as a subspace of $\ell^2(\mathbb{N})$ by

$$\ell^2(I_1) = \{y(t) \in \ell^2(\mathbb{N}) : \sum_{t \in I_1} Ry^*(t)Ry(t) > 0\}.$$

The scalar product is defined in the usual way. Therefore, the maximal and the pre-minimal difference operator generated by (4.6) are defined via their domains as done before in 4.4. Thus the pre-minimal operator generated is densely defined, symmetric but not necessarily closed. Again, since densely defined symmetric operators defined on Hilbert spaces are closable, if the pre-minimal is not closed, then we can take its closure which now gives the minimal operator generated by $\mathcal{L}_1 \mathcal{L}_2$

on $\ell^2(\mathbb{N})$. This we now denote by L . The minimal symmetric operator L is densely defined symmetric operator, and therefore, has symmetric operator extensions, see the results in ((Schmüdgen, 2012), chapter 13).

In order to have properly defined operators, boundary conditions at the left regular end point are imposed as discussed earlier. Usually, the left regular end point is taken at t_0 for $t_0 > 1$ and $t_0 \in \mathbb{N}$. The results of Remling (Remling, 1998) can then be used to extrapolate the deficiency results to $t_0 = 0$. Thus, to construct the symmetric operator extension, we note that the operator L has an isometric Cayley-transform V_L because L is linear and densely defined symmetric operator on $\ell^2(\mathbb{N})$. Hence, for any $z \in \mathbb{C}$ such that $\text{Im}z > 0$, we have $V_L = (L - zI)(L - \bar{z}I)$ with $D(V_L) = \mathcal{R}(L - \bar{z}I)$ so that $V_L(L - \bar{z}I)y = (L - zI)y$ for all $y \in D(L)$. Thus one can also define the inverse Cayley-transform by $(zI - \bar{z}V_L)(I - V_L)^{-1}$ with its domain as $\mathcal{R}(I - V_L)$. Therefore, the extension of V_L to an everywhere defined bounded operator on $\ell^2(\mathbb{N})$ denoted by $V\tilde{y} = 0$ for all $\tilde{y} \in D(V_L)^\perp$ is a partial isometry with $D(V) = \mathcal{R}(L - \bar{z}I)$ and $\mathcal{R}(V) = \mathcal{R}(L - zI)$. Hence, the Cayley-transform $L \rightarrow V$ is a bijective mapping of the densely defined closed linear operators on $\ell^2(\mathbb{N})$ onto the set of partial isometries V on $\ell^2(\mathbb{N})$. The inverse Cayley-transform $\tilde{L} = (zI - \bar{z}V)(I - V)^{-1}$ is the required symmetric operator extension of L . In this case, $D(L) \subseteq D(\tilde{L}) \subseteq D(L^*)$, where $Ly = \tilde{L}y = L^*y$, for all $y \in D(L)$ with $\langle \tilde{L}y, y \rangle = \langle y, \tilde{L}y \rangle$. $\square$

4.6 Spectral Analysis of L_1 , L_2 and L with unbounded coefficients.

In this section, we will assume that (4.4), (4.5) and (4.11) are satisfied in addition to those conditions in Theorem 4.4.6 so that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative and the operators L_1 and L_2 are commutative as well. As before, we will take the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ and for simplicity take $n = m$ though the analysis can still be carried out even if $n \neq m$. We will also allow the coefficients p_k and q_j to be unbounded. The reader is cautioned that the set up in (Nyamwala, 2017) is slightly

different from ours since in the mentioned reference, the author had non-zero odd order coefficients. We now formulate results which are extensions of those in Theorem 4.4.1 under unbounded coefficients. We begin with the approximation of the eigenvalues if (4.11) is assumed.

Lemma 4.6.1. *Suppose that (4.11) is satisfied, then the eigenvalues of the first order system (1.12) computed from the zeros of the polynomial $\mathcal{F}(t, \gamma, z)$ are approximately given by*

$$\lambda_l(t, z) \approx 1 + \frac{h_{l-1}}{2h_l} \pm \left\{ \frac{h_{l-1}}{h_l} + \frac{h_{l-1}^2}{4h_l^2} \right\}^{\frac{1}{2}} + o(1), \quad h_l = p_l, q_l.$$

Proof. In this case, we consider the polynomial $\mathcal{F}(t, \gamma, z)$ in (3.6) and without loss of generality, we may assume that $h_l = p_l$ since the proof using the coefficients q_l is done in a similar way and will thus follow at once. Now we can equate $\mathcal{F}(t, \gamma, z)$ to zero and by right scaling, we multiply all through by γ^{-l+1} to obtain a reduced polynomial of the form

$$p_l \gamma + p_{l-1} + \mathcal{R}(t, \gamma, z), \quad (4.14)$$

where

$$\mathcal{R}(t, \gamma, z) = \gamma^{-l+1} \left\{ \sum_{\nu=l+1}^n p_\nu \gamma^\nu + \sum_{\nu=0}^{l-2} p_\nu \gamma^\nu \right\}.$$

We need, therefore, to show that $\mathcal{R}(t, \gamma, z) \rightarrow 0$ as $t \rightarrow \infty$. In order to show this, we invoke the coefficient growth conditions in (4.11) and only show this for the coefficients $\nu = l+1$ and $\nu = l-2$ since as one moves further away from the l index, the summands tend to zero rapidly faster as $t \rightarrow \infty$. Assume, therefore, that $\nu = l+1$ then we have the magnitude of leading terms for $\mathcal{R}(t, \gamma, z)$ as either $O(|\frac{p_{l-1}}{p_l}|^2 |p_{l+1}|)$ or $O(|p_{l-2}| |\frac{p_l}{p_{l-1}}|)$. These terms go to zero as $t \rightarrow \infty$ as shown here.

$$\begin{aligned} |\frac{p_{l-1}}{p_l}|^2 |p_{l+1}| &\approx |\frac{p_{l-1}p_{l+1}}{p_l^2}| |p_{l-1}| = o(1). \\ |p_{l-2}| |\frac{p_l}{p_{l-1}}| &\approx |p_{l-2} \frac{p_{l-1}}{p_{l-1}} \frac{p_l}{p_{l-1}}| = |\frac{p_{l-2}p_l}{p_{l-1}^2}| |p_{l-1}| = o(1). \end{aligned}$$

Now substitute the value of $\gamma = 2 - (\lambda_l(t, z) + \lambda_l^{-1}(t, z))$ in (4.14) and apply completing square formula. Finally, one can easily show that these λ -roots are actually the approximate value of the eigenvalues of $S(t, z)$ in (1.12). For this, take $\lambda_l(t_0, z)$, t_0 left regular end point, as the initial approximate of the zeros of

(3.6) and within a suitably restricted region, perform infinite iterations and since the underlying space is a Hilbert space, application of Banach fixed point theorem will complete the proof. For more details, see (Behncke & Nyamwala, 2012) and the references cited therein.

□

The next result gives the spectral results of the operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ under unbounded coefficients and thus could be considered as extension of Theorem 4.4.1. Similarly, the results extend those of Theorem 4.3.1 into the discrete setting. In this case, the location of the absolutely continuous spectrum is explicit unlike in Theorem 4.4.1.

Theorem 4.6.2. *Assume that (4.4), (4.5) and (4.11) are satisfied and $n = m$. Then the minimal difference operators L_1 and L_2 are commutative, have self-adjoint operator extensions H_1 and H_2 with absolutely continuous spectrum $\mathbb{R}$ of various multiplicities. Moreover, the operators H_1 and H_2 are commutative and defined via separated boundary conditions at infinity.*

Proof. The proof of this theorem follow closely from Theorem 4.4.1, Theorem 4.4.2, Lemma 4.6.1 and the main results of (Nyamwala, 2017). For completion, we will therefore give an outline of the proof and detailed computations where required. Commutativity of the generated minimal difference operators are immediate from Theorem 4.4.2. By the results of Lemma 4.6.1 above, the eigenvalues of each of the first order systems generated by $\mathcal{L}_r y = zy$ can be approximated. Now one needs to show that these eigenvalues satisfy the z -uniform dichotomy condition. For the eigenvalues generated from the same root γ_l , the dichotomy condition is established via the correction term $z \mid \frac{\partial \mathcal{F}(t, \gamma_l, z)}{\partial \gamma} \mid^{-1}$ for $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$, $\eta > 0$, small. For the coefficients of $\mathcal{L}_1 y = zy$, explicit approximation of these λ -roots lead to the following approximations as $t \rightarrow \infty$

$$\lambda_{l\pm}(t, z) \approx 1 \mp (z_0 + i\eta) \mid p_l \mid^{l-2} \mid p_{l-1} \mid^{-l+1}$$

and hence at the limit point, we have $\mid \lambda_{l+}(t, z) \mid < 1$ and $\mid \lambda_{l-}(t, z) \mid > 1$. The analysis for the roots of $\mathcal{L}_2 y = zy$ is done in a similar way. This will separate the

magnitude of $\lambda_{l+}(t, z)$ and $\lambda_{l-}(t, z)$ such that one root has magnitude greater than one and the other less than one. For the eigenvalues belonging to different γ -roots, say, γ_l and $\gamma_{l\pm\nu}$, then note that by application of (4.11) we have the relation

$$\left| \frac{\partial \mathcal{F}(t, \gamma_{l-\nu}, z)}{\partial \gamma} \right|^{-1} \gg \left| \frac{\partial \mathcal{F}(t, \gamma_l, z)}{\partial \gamma} \right|^{-1} \gg \left| \frac{\partial \mathcal{F}(t, \gamma_{l+\nu}, z)}{\partial \gamma} \right|^{-1}.$$

This will result into correction terms with different magnitudes and hence will lead to distinct asymptotic behaviour of eigensolutions as required. The two arguments now settle the uniform dichotomy condition.

The first order systems can then be diagonalized appropriately and be transformed into the Levinson-Benzaid-Lutz form like in Section 1.1.2 and thus the solutions of the respective first order are given by (1.16). The deficiency indices, here, will depend on the nature of the γ_l -roots. The two solutions associated to the γ_l -roots such that $|\gamma_l| \rightarrow 2$ as $t \rightarrow \infty$ will both be square summable if $|h_l|^{l-2} |h_{l-1}|^{-l+1}$ is summable. In this particular case, the operators L_r will be at non-limit point at infinity. The self-adjoint operator extensions can only be described by separated boundary conditions. We will do this for the operator L_1 since the analysis for the second operator L_2 is done in a similar way. For simplicity, we may assume that $\text{def} L_1 = (\zeta, \zeta)$, $n \leq \zeta \leq 2n$. This implies that the set of all square summable solutions of $L_1 y = zy$ is linearly dependent. In order to obtain the basis of this set, we let $\omega_1, \dots, \omega_{\zeta-n}$ be the set of solutions which are linearly independent modulo $D(L_1)$ at infinity and these we may choose as the eigensolutions of $L_1^* \omega_k^* = z \omega_k$, $k = 1, 2, \dots, \zeta - n$. Thus we set $\zeta - n$ extra boundary conditions at infinity by demanding that

$$\lim_{t \rightarrow \infty} \omega_k^*(t) \mathcal{J} \omega_{k_1}(t) = 0, \quad \text{for } k, k_1 = 1, 2, \dots, \zeta - n.$$

This will result into a set of linearly independent square summable solutions for the equation $\mathcal{L}_1 y = zy$. The self-adjoint operator extension of the operator L_1 is defined using separated boundary conditions given here below

$$\begin{aligned} D(H_{1\infty}) = & \{y(t) \in D(L_1^*) | (\alpha_1, \alpha_2)y(0) = 0, \\ & \lim_{t \rightarrow \infty} \omega_k^*(t) \mathcal{J} y(t) = 0, \quad \text{for } k = 1, 2, \dots, \zeta - n. \} \end{aligned}$$

It follows, therefore, that for any self-adjoint operator extension of L_1 defined this way, we obtain the relation,

$$\dim(D(H_{1\infty})/D(L_1)) = \dim \mathcal{N}(L_1^* - zI) = n$$

so that $H_{1\infty}$ is n -dimensional operator extension of L_1 . One has, therefore, $L_1 \subset H_1 = H_{1\infty} = L_1^*$ in the operator sense. Here, the operator H_1 is the self-adjoint operator extension of L_1 at limit point which is defined in *ONE*, Section 1.1.2 and also similar to that constructed under conditions imposed on the coefficients in Theorem 4.4.2. The analysis and construction of the self-adjoint operator extension of L_2 , $H_{2\infty}$, when $|q_j|^{j-2}|q_{j-1}|^{-j+1}$ are summable is done in a similar way.

Those eigensolutions of L_r that lose their square summability as $\eta \rightarrow 0^+$ contribute to absolutely continuous spectrum. Just like in the case of differential operators, as well as in Theorem 4.4.1, we need to check that the associated M -matrix is bounded so that the spectral measure is continuous. For that, we apply the relation for computing $\text{Im}M(t, z)$ of the M -matrix as used in [(Behncke & Nyamwala, 2011), (Remling, 1998), (Y. Shi, 2006)]. Here, we do it for the solution associated with the eigenvalue $\lambda_{l+}(t, z)$.

$$\text{Im}M(t, z) = \lim_{\eta \rightarrow 0^+} \eta \langle y_{l+}(t, z), y_{l+}(t, z) \rangle = \lim_{\eta \rightarrow 0^+} \eta |\rho_{l+}(t, z)|^2 \prod_{s=t_0}^t |\lambda_{l+}(s, z)|^2$$

and by application of Euler logarithmic relation, this can be approximated by

$$\lim_{\eta \rightarrow 0^+} \int_{t_0}^{\infty} |\rho_{l+}(t, z)|^2 \exp \left(-2\eta \int_{t_0}^t |h_l|^{l-2} |h_{l-1}|^{-l+1}(s, t) ds \right) dt,$$

which is bounded and the limit exists non-trivially. Here, $\rho_{l+}(t, z)$ is the normalized eigenvector. Since the coefficients are allowed to be unbounded, the absolutely continuous spectrum will be the whole of $\mathbb{R}$. The spectral multiplicity will be the number of solutions with such asymptotic behaviour.

The case when $|p_k|^{k-2}|p_{k-1}|^{-k+1}$ and $|q_j|^{j-2}|q_{j-1}|^{-j+1}$ are summable for all $k = 1, 2, \dots, n$ and for all $j = 1, 2, \dots, m$ is similar to that in Chapter *FOUR*, Section 4.4 in Theorem 4.4.1, since both the operators L_1 and L_2 will be at limit point with discrete spectrum at most and the self-adjoint operator extension given by (1.14).

□

We now extend the results of Lemma 4.5.1 to the case of the composite operators by obtaining the expression for approximating the roots of the polynomial in (4.13).

Lemma 4.6.3. *Suppose that (4.11) is satisfied and that $\Delta f_l \rightarrow 0$ as $t \rightarrow \infty$ as required by (4.4), f_l defined in (4.12), then the γ -roots of $\mathcal{F}(t, \gamma, z)$ in (4.13) can be approximated from the relationship*

$$f_l \gamma + f_{l-1} + \mathcal{R}(t, \gamma, z), \quad (4.15)$$

where

$$\mathcal{R}(t, \gamma, z) = \gamma^{-l+1} \left\{ \sum_{\nu=l+1}^{n+m} f_\nu \gamma^\nu + \sum_{\nu=0}^{l-2} f_\nu \gamma^\nu \right\}.$$

Proof. Just like in Lemma 4.6.1, it suffices to show that $\mathcal{R}(t, \gamma, z) \rightarrow 0$ as $t \rightarrow \infty$. This can be shown for the coefficients $\nu = l+1$ and $\nu = l-2$. Assume, therefore, that $\nu = l+1$ then we have that the leading term for $\mathcal{R}(t, \gamma, z)$ is either $O(|\frac{f_{l-1}}{f_l}|^2 |f_{l+1}|)$ or $O(|f_{l-2}| |\frac{f_l}{f_{l-1}}|)$. It is easy to show that these terms go to zero as $t \rightarrow \infty$ using similar approach like in Lemma 4.6.1 □

The eigenvalues of $\mathcal{L}y = zy$ resulting from polynomial (4.13) can then be approximated using (4.15). The dichotomy conditions and the form of solutions follow closely from the results in Section 4.4 with the obvious change of the appropriate coefficients. The continuity of the M -matrix follows immediately from the results of Theorem 4.4.1 in the case of approximate constant coefficients and Theorem 4.6.2 in the case of unbounded coefficients. Only those γ values such that the associated λ -roots satisfy the relation $|\lambda(t, z) + \lambda^{-1}(t, z)| \leq 2$ shall lead to some solutions that lose their square summability and hence contribute to absolutely continuous spectrum. These can now be summarised in the following two theorems. Theorem 4.6.4 extends the results of Theorem 4.4.1 to the composite case while Theorem 4.6.5 extends the results of Theorem 4.6.2 to the composite case.

Theorem 4.6.4. *Consider the minimal difference operator L generated by $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$ in (4.6) and assume that (4.4) and (4.5) in addition to the conditions in Theorem*

4.4.6 are satisfied. Then L is at limit point at infinity if the boundary conditions defined by the matrix $\tilde{\alpha}$ at t_0 are satisfied. The corresponding self-adjoint operator extension H has no singular continuous spectrum and the absolutely continuous spectrum agrees with that of the constant coefficient limiting operator with the spectral multiplicity equals to half the number of eigenvalues with magnitude 1.

Proof. The proof follows closely from that of Theorem 4.4.1 with some modification in line with the composite $L_1 L_2$. Here, we define $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$ on the Hilbert space $\ell^2(\mathbb{N}) \cong \ell_w^2(\mathbb{N})$, $w = w(t) = 1$, for all $t \in \mathbb{N}$. The first order system of the composite is obtained as in (4.9) and the corresponding characteristic polynomial $\mathcal{P}(\lambda, t, z)$ computed. The roots of the polynomial in (4.13) will be considered within a set $\mathcal{K} = \{z \in \mathbb{C} \mid |z - z_0| < \epsilon\}$ with relevant spectral parameters which can be adjusted until $\mathcal{P}(\lambda, t, z)$ has distinct roots. The roots are then approximated using (4.15) and one can apply $\gamma = 2 - (\lambda + \lambda^{-1})$ to obtain λ -roots from the corresponding γ -roots.

By superimposition principle, the roots of $\mathcal{P}(\lambda, t, z)$, the characteristic polynomial generated by $\mathcal{L} = \mathcal{L}_1 \mathcal{L}_2$, will have $\lambda_1(t, z), \lambda_2(t, z), \dots, \lambda_{4s}, \tilde{\lambda}_{4s+1}, \tilde{\lambda}_{4s+2} + \dots + \tilde{\lambda}_{2(m+n)}$, where $|\lambda_l(t, z)| \neq 1$, $l = 1, 2, \dots, 4s$, $|\tilde{\lambda}_l(t, z)| = 1$, $l = 4s + 1, \dots, 2(m + n)$. Just like in Theorem 4.4.1, the dichotomy condition is only required for those characteristic roots (eigenvalues of $S(t, z)$) with magnitude 1, that is $\tilde{\lambda}_l(t, z)$. Using complex analysis, represent these eigenvalues by $\tilde{\lambda}_l(t, z) = \exp(i\beta_l)$ for some angle β_l . It follows that $\tilde{\lambda}_l(t, z) = \exp(-i\beta_l)$ will also be a root. Then off the real axis and taking $z = z_0 + i\epsilon$, $\epsilon > 0$, $z_0, \epsilon \in \mathbb{R}$, $\epsilon > 0$ is small, we have,

$$\tilde{\lambda}_l(z_0 + i\epsilon) + \tilde{\lambda}_l(z_0 + i\epsilon) = 2 \cos \beta_l \quad l = 2s + 1, \dots, 2(s - l).$$

Thus

$$\tilde{\lambda}_l(z_0 + i\epsilon) = \tilde{\lambda}_l(z_0) + \frac{i\epsilon}{\partial_{\lambda_l} \mathcal{P}(\lambda, t, z)}(z_0, \lambda_l).$$

It is the correction term $i\epsilon(\partial_{\lambda_k} \mathcal{P}(\lambda, t, z))^{-1}$ which will lead to either $\tilde{\lambda}_l(z_0 + i\epsilon)$ having magnitude greater than 1 or less than 1 off the real axis. As before, the number of roots that will have magnitude greater than 1 off the real axis will be half the roots $\tilde{\lambda}_l(t, z)$ while the others will have magnitude less than 1 which establishes the uniform dichotomy condition.

The M-matrix of L again is the Borel-transform of the spectral measure μ . The density of the absolutely continuous spectrum of H is given by

$$\left(\frac{1}{\pi}\right) \lim_{\epsilon \rightarrow 0^+} M(\mu + i\epsilon) = \left(\frac{1}{\pi}\right) M(\mu_+) = \varrho(\mu).$$

The spectrum is absolutely continuous if M has finite limits $M(\mu_+)$. The eigenvalues of H correspond to the poles of M . The $M(z)$ is determined off the real axis and can be constructed from the eigenfunctions of H that are square summable. Thus one has to estimate the zeros of the characteristic polynomial $\mathcal{P}(\lambda, t, z)$ and then apply the results of Theorem 1.1.2 in order to obtain the right form of solutions as in (1.16) which can be summed over the integral domain I_2 given in Section 1.1.2 and with $r_{kk}(t, z) \rightarrow 0$ as $t \rightarrow \infty$.

In this case note that the deficiency indices for L are $(n+m, n+m)$ at the limiting point with α -boundary conditions also imposed as $t \rightarrow \infty$.

Assume that $V(t, z) = \begin{pmatrix} V_1(t, z) \\ V_2(t, z) \end{pmatrix}$ is the set of square summable eigenfunctions of L with Dirichlet boundary conditions given in (4.10) with

$$\tilde{\alpha}_1 \tilde{\alpha}_1^* + \tilde{\alpha}_2 \tilde{\alpha}_2^* = I_{n+m}, \quad \tilde{\alpha}_1 \tilde{\alpha}_2^* - \tilde{\alpha}_2 \tilde{\alpha}_1^* = 0.$$

Then the square summable eigenfunctions are given by $\begin{pmatrix} Y_{t_0} \end{pmatrix} (t) \begin{pmatrix} I_{n+m} \\ M_{t_0}(z) \end{pmatrix}$.

and one can then show that

$$\begin{pmatrix} M_{t_0} \end{pmatrix} (z) = V_2(t_0, z) V_1^{-1}(t_0, z).$$

$V_1^{-1}(t_0, z)$ exists boundedly because again $V_1(t_0, z)$ is the Vandemonde's matrix. $(M_{t_0})(z)$ is continuous for all z within the region of consideration. The continuous spectrum for H is not singular continuous spectrum but absolutely continuous spectrum. The left end regular point t_0 can then be extended to $t_0 = 0$ using the results of Remling (Remling, 1998). $\square$

We now prove a result that extends those of Theorem 4.6.2 to the composite case. This result therefore completes the spectral analysis of operator L_1 , L_2 and their respective composite L .

Theorem 4.6.5. *Let L be the difference operator generated by the difference function $\mathcal{L}$ in (4.6). Assume that (4.4), (4.5) and (4.11) are satisfied in addition to conditions in Theorem 4.6.2. Then*

- (i) *If the coefficients p_k and q_j are almost constant coefficients and $n = m$, then at limit point $\text{def}L = (n + m, n + m)$ and the spectrum of H coincides with that of the constant coefficients case.*
- (ii) *Suppose that $n = m$ and (4.11) is satisfied. Moreover, if $|f_{l-1}|^{-l+1}|f_l|^{l-2}$, for all $l = 1, 2, \dots, n + m$, is not summable, then $\text{def}L = (n + m, n + m)$ and the absolutely continuous spectrum of H is the whole of real line, $\sigma_{ac}(H) = \mathbb{R}$.*

Proof. (i) The proof follows immediately from Theorems 4.4.1, 4.4.2, 4.6.2 and 4.6.4. We note here, that by construction, the operator H is $n+m$ -extension of the operator L and it is constructed as

$$D(H) = D(L) \dot{+} \{y + Vy : y \in \mathcal{N}(L^* - zI)\},$$

where $V = (zI - L)(\bar{z}I - L)^{-1}$.

(ii) We note that if $|f_{l-1}|^{-l+1}|f_l|^{l-2}$ is not summable, then each of the γ -roots of $\mathcal{F}(\gamma, t, z)$ in (4.13) will have λ -roots with magnitude almost equal to one as explained in Theorem 4.6.4. It is half of the roots that lose their square summability as $t \rightarrow \infty$ and $\eta \rightarrow 0^+$ that contributes to absolutely continuous spectrum. Since the coefficients are unbounded, the spectrum is the whole of $\mathbb{R}$. $\square$

In our subsequent results which are on the existence of the self-adjoint operator extension of the minimal operator generated by $\mathcal{L}_1\mathcal{L}_2$ and the absolutely continuous spectrum of the self-adjoint extension in the case of unbounded coefficients, we shall assume the following growth conditions on the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$.

$$p_k q_j = o(p_n^2 \cdot p_0 q_0)^{\frac{1}{2n}}, \quad k = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m-1, \quad n = m. \quad (4.16)$$

Theorem 4.6.6. *Assume that the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.7) satisfy (4.16) in addition to the conditions in Theorem 4.5.1, then the deficiency index of the minimal difference operator generated by $\mathcal{L}_1\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ is (l, l) , where $2n \leq l \leq 4n$*

and the absolutely continuous spectrum of the self-adjoint operator extension is non-empty if $|p_n|^{\frac{-1}{n}}$ is not absolutely summable and has spectral multiplicity equal to the number of pairs of eigenvalues with magnitude one.

Proof. In this case we use the simplifications in Theorem 4.5.6 which gives the first order form of $\mathcal{L}_1\mathcal{L}_2y(t)$, that is, $\mathcal{J}\Delta Y(t) = P(t)RY(t)$. We now introduce a spectral parameter z , $z \in \mathbb{R}$. The operator L generated by $\mathcal{L}_1\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ is symmetric by the results of Theorem 4.5.6 and therefore, its spectrum $\sigma(L)$, is a subset of $\mathbb{R}$. We now solve the equation $\mathcal{L}_1\mathcal{L}_2y(t) = zy(t)$, $z \in \mathbb{R}$, or $(L - zI)y = 0$. We shall absorb z into p_0q_0 , that is, p_0q_0 will be interpreted as $p_0q_0 - z$. Therefore, using techniques of Shi (Y. Shi, 2006), this first order can be expressed as in (1.12). The characteristic polynomial of the $2n \times 2n$ matrix $S(t, z)$ in (1.12) can be computed explicitly. This is determined by $\mathcal{P}(t, \lambda, z) = \det(S(t, z) - \lambda I_{2n})$. Multiplying $\mathcal{P}(t, \lambda, z)$ by $\frac{p_n q_n}{\lambda^{2n}}$ in order to obtain a simpler version and assuming that $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$, we obtain a fourier polynomial $\mathcal{F}(\gamma, t, z)$ given by

$$\mathcal{F}(\gamma, t, z) = \frac{p_n q_n}{\lambda^{2n}} \mathcal{P}(t, \lambda, z) = \sum_{k=0}^n \sum_{j=0}^n (p_k \gamma^k) (q_j \gamma^j) = \sum_{k,j=0}^n p_k q_j \gamma^{k+j}.$$

From our construction, note that $q_n = \alpha p_n$ for some constant $\alpha \neq 0$. Similarly, $p_{n-1}, p_{n-2}, \dots, p_0$ and $q_{n-1}, q_{n-2}, \dots, q_0$ are real valued constants. Suppose $|p_n(t)| \nearrow \infty$ as $t \rightarrow \infty$ and the other coefficients are such that $p_k q_j = o(p_n^2 \cdot p_0 q_0)^{\frac{1}{2n}}$ as assumed in (4.16), then the γ -roots of the Fourier polynomial are approximately

$$\gamma \approx \left| \frac{p_0 q_0}{\alpha p_n^2} \right|^{\frac{1}{2n}} \text{sign arg} \left(-\frac{p_0 q_0}{\alpha p_n^2} \right) \quad (4.17)$$

Here, $\gamma = \gamma(t, z)$ are analytical functions of t and z . One needs to show that the γ -roots of $\mathcal{F}(\gamma, t, z)$ are given by the relation in (4.17). In order to see this, consider a polynomial of the form $\mathcal{F}(t, z, \gamma) = \alpha p_n^2 \gamma^{2n} + p_{n-1} q_{n-1} \gamma^{2n-1} + \dots + p_1 q_{n-1} \gamma + p_1 q_{n-1} \gamma^{2m-1} \dots + p_1 q_1 \gamma + p_0 q_0$. By application of Kummer-Liouville transforms and scaling the coefficients, one can easily transform the Fourier polynomial $\mathcal{F}(t, z, \gamma)$ into a polynomial of the form $P(\gamma) = a_{2n} \gamma^{2n} + a_{2n-1} \gamma^{2n-1} + \dots + a_1 \gamma + a_0$ such that $|a_0|, |a_{2n}| = 1$, $a_i = p_k q_j$, $(k, j = 1 \dots, 2n)$ and a_i , $(1 \leq i \leq 2n - 1)$ are bounded. This can be achieved because we have the freedom of choice of α . It follows that by taking $C = \max\{a_i : i = 0, 1, \dots, 2n\}$, there exists a positive constant M , $M > C + 1$

such that magnitude of all the γ - roots are bounded between M^{-1} and M . In so doing, the roots of $\mathcal{F}(t, z, \gamma)$ shall all be bounded within a certain interval. Once the γ - roots have been obtained for the Kummer-Liouville transformed polynomial, the γ - roots of the original polynomial $\mathcal{F}(\gamma, t, z)$ can be obtained via backward transformation that involves the coefficients.

It remains therefore to show that any γ - root given by (4.17) is asymptotically equal to the γ - roots of $\mathcal{F}(t, z, \gamma)$. It suffices therefore to show that if the particular γ - root is substituted into the polynomial, then all the middle terms tend to zero as $t \rightarrow \infty$. We only do this for the term $p_n \cdot q_{n-1} \gamma_k^{2n-1}$ since the others are done in a similar way. It is worth noting that even though in the polynomial $\mathcal{F}(\gamma, t, z)$, when expanded, apart from the leading term, will have some coefficients with p_n . In these cases, p_n will be of power one. Thus comparatively, we have $p_n = o(p_n^2)$ and this can easily be seen in the case of power coefficients since in the limit sense $t = o(t^2)$ as $t \rightarrow \infty$.

In order to transform (1.12) into Levinson's Benzaid-Lutz (LBL) form (1.15), we will need two diagonalisations. The appropriate eigenvectors can be evaluated directly from the quasi-differences by replacing Δ by $(\lambda - 1)$ and $y(t - k)$ by λ^{-k} . Thus from (4.7) we have,

$$\begin{aligned}
x_j(t) &= q_j(t)(\lambda - 1)^{j-1} \lambda^{-j} & 1 \leq j \leq m \\
x_{m+1}(t) &= q_m(t)(\lambda - 1)^m \lambda^{-m} \\
x_{m+k}(t) &= -(\lambda - 1)(q_m(t)(\lambda - 1)^m \lambda^{-m}) - p_{n-k}(t)(q_{n-k}(t)(\lambda - 1)^{n-k} \lambda^{-(n+k)}) \\
& & 2 \leq k \leq n \\
x_{n+m+r}(t) &= -((\lambda - 1)(x_{n+m+r-1}(t)) + p_{n-r}(t)x_{m+n-r}(t)) & 2 \leq r \leq n + m - 1
\end{aligned}$$

The matrix $T(t, z)$ formed with the eigenvectors as columns will diagonalise $S(t, z)$. Thus one transforms the system by $\chi(t, z) = T(t, z)Y(t, z)$ which results into a system of the form,

$$\begin{aligned}
\chi(t+1, z) &= T^{-1}(t+1, z)S(t, z)T(t, z)\chi(t, z) \\
&= [\Lambda(t, z) + \mathfrak{R}(t, z)]\chi(t, z)
\end{aligned}$$

where, $\mathfrak{R}(t, z)$ is as defined in 3.7. Note that $\mathfrak{R}(t, z)$ consists of ℓ^1 and ℓ^2 terms. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t, z)$, $k = 1, 2$. The second diagonalisation is done by use of the eigenvectors of the matrix $[\Lambda(t) + \mathfrak{R}(t, z)]$ as discussed in 3.7. The second diagonalisation will thus result into correction terms added to the diagonals given by $(\Lambda_2)_{kk} = \text{diag}((\mathfrak{R}B)_{kk})$ as in 3.7. One therefore obtains LBL form (1.15) to which we apply Theorem 1.1.2 to obtain the solutions of the form (1.16).

The asymptotics and summability of $y_k(t, z)$ will depend on $\prod_{l=t_0}^{t-1} \lambda_k(l, z)$. These are analysed off the real axis based on the magnitude of the corresponding eigenvalues since all the γ - roots are such that $|\gamma| \approx |\frac{p_0 q_0}{\alpha p_n^2(t)}|^{\frac{1}{2n}}$ and as $t \rightarrow \infty$, it implies that $|\gamma_r| < 2 \forall r = 1, 2, \dots, 2n$ and thus we have $2n$ pairs of eigenvalues with absolute value almost equal to 1. Thus by application of LBL theorem, the dichotomy condition can only be established off the real axis. Therefore, the number of roots that will have a magnitude greater than 1 off the real axis will be half the roots while the others will have a magnitude less than 1.

Now let $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$, $\eta > 0$ and small, then, for $|\lambda_{r+}(t, z) + \lambda_{r-}^{-1}(t, z)| < 2$ as $t \rightarrow \infty$, which is the required uniform dichotomy condition, it follows that if $|p_n|^{\frac{-1}{n}}$ is absolutely summable since by construction, α, p_0, q_0 are constants, then all the associated eigenfunctions are z -uniformly square summable and $\text{def } \mathcal{L} = (4n, 4n)$ (limit circle case). On the other hand if $|p_n|^{\frac{-1}{n}}$ is not absolutely summable, then half of the associated eigenfunctions will lose their square summability and $\text{def } \mathcal{L} = (2n, 2n)$ (limit point case). The minimal operator generated by $(\mathcal{L}_1 \mathcal{L}_2 - z)y(t) = 0$ has a self-adjoint operator extension $\mathcal{H}$ by application of von-Neumann theorems ((Weidmann, 1987), Theorem 8.11 and 8.12) which can be described via its domain $D(\mathcal{H})$ as follows

$$D(H) = D(L) \dot{+} N(L^* - z) \dot{+} N(L^* + z) \text{ or}$$

$$D(H) = D(L) \dot{+} \{y + Vy : y \in N(L^* - z)\}$$

The roots that off the real axis have magnitude greater than 1 lead to the eigen-solutions that lose their square summability as $\eta \rightarrow 0^+$ and hence contributes to the continuous spectrum. The spectral multiplicity can thus be analysed via the M-matrix.

The M-matrix of L again is the Borel-transform of the spectral measure μ . The density of the absolutely continuous spectrum of H is given by

$$\left(\frac{1}{\pi}\right) \lim_{\epsilon \rightarrow 0^+} M(\mu + i\epsilon) = \left(\frac{1}{\pi}\right) M(\mu_+) = \varrho(\mu).$$

The spectrum is absolutely continuous if M has finite limits $M(\mu_+)$. The eigenvalues of H will correspond to the poles of M . The M-matrix $M(z)$ is determined off the real axis and can be constructed from the eigenfunctions of H that are square summable. Applying the Dirichlet boundary conditions as discussed earlier, we have that $(M_{t_0})(z)$ is continuous for all z within the region of consideration and the spectrum is discrete at most, otherwise, we have $\sigma_{ac}(H) = \mathbb{R}$ if $p_n(t) < 0$ and if $p_n(t) > 0$, $\sigma_{ac}(H) \subseteq [\bar{c}, \infty)$ where $\bar{c} = \limsup \frac{p_0 q_0}{\alpha p_n(t)}$, in both cases the spectral multiplicity is $2n$.

□

Finally we relate the operator H and the composites of H_1 and H_2 .

Theorem 4.6.7. *Let L_1 , L_2 and L be minimal difference operators generated by $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ respectively and defined on $\ell^2(\mathbb{N})$. Assume that (4.4), (4.5) and (4.11) are satisfied. Moreover, assume that L_1 and L_2 are injective with closed ranges, then*

(i) $H = H_1 H_2$ with $\sigma_{ac}(H) = \mathbb{R}$ if $|p_k|^{k-2} |p_{k-1}|^{-k+1}$ and $|q_j|^{j-2} |q_{j-1}|^{-j+1}$ are not summable for all $k, j = 1, 2, \dots, n$.

(ii) $H = H_1 H_2$ with $\sigma(H)$ discrete if $|p_k|^{k-2} |p_{k-1}|^{-k+1}$ and $|q_j|^{j-2} |q_{j-1}|^{-j+1}$ are summable for all $k, j = 1, 2, \dots, n$.

Proof. The proof of this theorem now follows from those of Theorems 4.4.2, 4.6.2, 4.6.4 and 4.6.5. We only note that by Rank-Nullity theorem, $\text{def} \tilde{L} = \text{def} L_1 + \text{def} L_2$

and in the two cases considered in the theorem, L has self-adjoint operator extension which we shall denote by H . This is a $2n$ -dimensional extension of L . From the results of Theorem 4.6.4, without loss of generality, we may even take $H = H_1 H_2$. The rest of the proof are now straight forward.

□

Remark 4.6.8. In Sections 4.5 and 4.6, we have obtained the necessary and sufficient conditions required for the following commutative diagram which we have summarised in Theorem 4.6.7 when the coefficients are allowed to be unbounded.

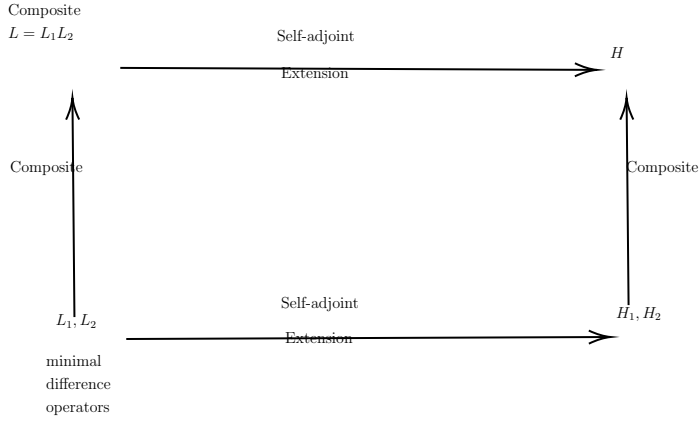


Figure 4.1: L_1 commutes with L_2 , L_1, L_2 symmetric and injective, $R(L_1)$ and $R(L_2)$ closed

The example below now verifies the results of Theorems 4.6.4 and 4.6.5 by computing the composite of two second order symmetric difference operators with unbounded coefficients and then analysing their deficiency indices and spectra.

Example 4.6.9. We consider two second order difference operators generated by Sturm-Liouville equation with potential $p(t) = \ln(t)$ and a second order square well given by

$$\mathcal{L}_1 y(t) = -\Delta^2 y(t-1) + (\ln t)y(t), \quad \mathcal{L}_2 y(t) = -\Delta^2 y(t-1) + t^{-\epsilon} y(t), \quad \epsilon > 0 \quad (4.18)$$

defined on $\ell^2(\mathbb{N})$. The functions $\mathcal{L}_1$ and $\mathcal{L}_2$ at limit point commute and their composite can be written as

$$\mathcal{L}y(t) = \Delta^4 y(t-2) - \Delta [(\ln t + t^{-\epsilon})\Delta y(t-1)] + t^{-\epsilon} \ln t y(t).$$

The conversion of the composite equation $\mathcal{L}y = zy$ into its first order form is done using quasi-differences of the form

$$x_1(t) = y(t-1), \quad x_2(t) = \Delta y(t-2),$$

$$u_1(t) = (\ln t + t^{-\epsilon})\Delta y(t-1) - \Delta^3 y(t-2), \quad u_2(t) = \Delta^2 y(t-2).$$

The characteristic polynomial $\det(S(t, z) - \lambda \cdot I_4)$ when multiplied by λ^{-2} and equated to zero leads to

$$\gamma^2 + (\ln t + t^{-\epsilon})\gamma + t^{-\epsilon} \ln t - z = 0,$$

where $\gamma = 2 - (\lambda + \lambda^{-1})$. Thus we fix our regular left end point at $t_0 > 1$ and hence we have approximate values of γ as

$$\gamma_1 \approx -t^{-\epsilon} + t^{-2\epsilon}(\ln t)^{-1} + O(\ln t)^{-2}$$

$$\gamma_2 \approx -\ln t - t^{-2\epsilon}(\ln t)^{-1} + O(\ln t)^{-2}.$$

Now approximation for the λ -roots for this system gives

$$\lambda_{1\pm}(t, z) \approx 1 \pm t^{-\frac{\epsilon}{2}} + O(t^{-\epsilon}(\ln t)^{-1})$$

and

$$\lambda_{2+}(t, z) \approx (\ln t)^{-1} + O(\ln t)^{-2}, \quad \lambda_{2-}(t, z) \approx 2 + \ln t - (\ln t)^{-1} + O(\ln t)^{-2}.$$

The solutions associated with the λ -roots derived from γ_2 contribute $(1, 1)$ to the deficiency indices because of the nature of the λ -roots. On the other hand, the two solutions associated with γ_1 will all be square summable if the term $t^{-\frac{\epsilon}{2}}(\ln t)^{-1}$ is summable and thus contributes $(2, 2)$ to the deficiency indices. All the solutions will be uniformly square summable. Thus $\text{def} L = (3, 3)$ and the spectrum of H is discrete at most. Meanwhile, if the term $t^{-\frac{\epsilon}{2}}(\ln t)^{-1}$ is not summable, the γ_1 also contributes $(1, 1)$ to the deficiency indices with one solution losing its square summability as $\eta \rightarrow 0^+$ and hence the absolutely continuous spectrum of H is non-empty. Thus $\text{def} L = (2, 2)$ and $\sigma_{ac}(H) \subset [0, \infty)$ of spectral multiplicity one.

4.7 Further Examples

In this section we give further examples on determining the deficiency indices and spectral multiplicities of composites of order two and four operators that are gen-

erated by (4.6) as a justification of the claim in Theorem 4.6.6. Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative so that the corresponding difference expression is given by (4.6). In order to obtain the comparative analysis of the deficiency indices and spectral results of the associated difference operators, we solve the equations $(\mathcal{L}_1 y) = zy$, $(\mathcal{L}_2 y) = zy$ and $\mathcal{L}_1(\mathcal{L}_2 y) = zy$ as illustrated in the following examples.

Example 4.7.1. We consider second order difference equations of the form:

$$\begin{aligned}\mathcal{L}_1(y) &= -\Delta p(t)\Delta y(t-1) + b_0 y(t) \\ \mathcal{L}_2(y) &= -\Delta \alpha p(t)\Delta y(t-1) + c_0 y(t)\end{aligned}$$

where α is a real-valued constant, $\alpha \neq 0$. In this example we have that $p(t) \nearrow \infty$. We compute the deficiency indices of the minimal operator generated by the composite of the above difference equations.

The composite $\mathcal{L}_1 \mathcal{L}_2 y(t)$ is given by,

$$\mathcal{L}_1 \mathcal{L}_2 y(t) = \Delta^2(\alpha p^2 \Delta^2 y(t-2)) - \Delta p(b_0 + c_0) \Delta y(t-1) + b_0 c_0 y(t) \quad (4.19)$$

Reducing (4.19) to first order, obtaining its characteristic polynomial and assuming that $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$ then scaling by $\frac{\alpha p^2}{\gamma^2}$ we obtain the Fourier polynomial

$$\mathcal{F}(\gamma, t) = \alpha p^2 \gamma^2 + (c_0 p + \alpha b_0 p) \gamma + b_0 c_0 \quad (4.20)$$

Solving (4.20) above explicitly, we obtain,

$$\begin{aligned}\gamma_1 &= -\left(\frac{c_0 + \alpha b_0}{2\alpha p}\right) + \frac{1}{2\alpha p}(c_0 - \alpha b_0) = \frac{-b_0}{p} \\ \gamma_2 &= -\left(\frac{c_0 + \alpha b_0}{2\alpha p}\right) - \frac{1}{2\alpha p}(c_0 - \alpha b_0) = \frac{-c_0}{\alpha p}\end{aligned}$$

Since $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$, substituting and solving for λ we have that,

$$\begin{aligned}\lambda_{1/2} &= \lambda^2 + \left(\frac{-b_0}{p} - 2\right)\lambda + 1 \\ &= \lambda^2 - \left(\frac{b_0}{p} + 2\right)\lambda + 1\end{aligned}$$

and

$$\begin{aligned}\lambda_{3/4} &= \lambda^2 + \left(\frac{-c_0}{\alpha p} - 2\right)\lambda + 1 \\ &= \lambda^2 - \left(\frac{c_0}{\alpha p} + 2\right)\lambda + 1\end{aligned}$$

Which results to

$$\begin{aligned}
\lambda_1 &\approx 1 + \left| \frac{b_0}{P} \right|^{\frac{1}{2}} + \frac{b_0}{2p} + \frac{b_0}{8p} \left| \frac{b_0}{P} \right|^2 - \frac{b_0^2}{128p^2} \left| \frac{b_0}{P} \right|^2 + O(p^{-3}) \\
\lambda_2 &\approx 1 - \left| \frac{b_0}{P} \right|^{\frac{1}{2}} + \frac{b_0}{2p} - \frac{b_0}{8p} \left| \frac{b_0}{P} \right|^2 + \frac{b_0^2}{128p^2} \left| \frac{b_0}{P} \right|^2 + O(p^{-3}) \\
\lambda_3 &\approx 1 + \left| \frac{c_0}{\alpha P} \right|^{\frac{1}{2}} + \frac{c_0}{2\alpha p} + \frac{c_0}{8\alpha p} \left| \frac{c_0}{\alpha P} \right|^2 - \frac{c_0^2}{128\alpha p^2} \left| \frac{c_0}{\alpha P} \right|^2 + O(p^{-3}) \\
\lambda_4 &\approx 1 - \left| \frac{c_0}{\alpha P} \right|^{\frac{1}{2}} + \frac{c_0}{2\alpha p} - \frac{c_0}{8\alpha p} \left| \frac{c_0}{\alpha P} \right|^2 + \frac{c_0^2}{128\alpha p^2} \left| \frac{c_0}{\alpha P} \right|^2 + O(p^{-3})
\end{aligned}$$

By replacing Δ by $(\lambda - 1)$ and $y(t - k)$ by λ^{-k} in the quasi-differences and normalizing the first component through multiplication by λ , we obtain the eigenvectors given as:

$$v_k = \{1, (\lambda_k - 1)\lambda_k^{-1}, (c_0 p + \alpha b_0 p)(\lambda_k - 1)^3 \lambda_k^{-1}, \alpha p^2 (\lambda_k - 1)^2 \lambda_k^{-1}\}^{tr}$$

Approximating the eigenvectors using the leading terms only, we obtain the eigenvectors of the form:

$$\begin{aligned}
v_1 &= [1, \left(\frac{b_0}{p}\right)^{\frac{1}{2}} + \left(\frac{b_0}{p}\right), \left(\frac{c_0^2 b_0^3}{p}\right)^{\frac{1}{2}} + \left(\frac{b_0^5}{p}\right)^{\frac{1}{2}}, \alpha p b_0 + \alpha \left(\frac{b_0^3}{p}\right)^{\frac{1}{2}}]^{tr} \\
v_2 &= [1, \left(\frac{b_0}{p}\right)^{\frac{1}{2}} - \left(\frac{b_0}{p}\right), \left(\frac{c_0^2 b_0^3}{p}\right)^{\frac{1}{2}} - \left(\frac{b_0^5}{p}\right)^{\frac{1}{2}}, \alpha p b_0 - \alpha \left(\frac{b_0^3}{p}\right)^{\frac{1}{2}}]^{tr} \\
v_3 &= [1, \left(\frac{c_0}{p}\right)^{\frac{1}{2}} + \left(\frac{c_0}{p}\right), \left(\frac{b_0^2 c_0^3}{p}\right)^{\frac{1}{2}} + \left(\frac{c_0^5}{p}\right)^{\frac{1}{2}}, \alpha p b_0 - \alpha \left(\frac{b_0^3}{p}\right)^{\frac{1}{2}}]^{tr} \\
v_4 &= [1, \left(\frac{c_0}{p}\right)^{\frac{1}{2}} - \left(\frac{c_0}{p}\right), \left(\frac{b_0^2 c_0^3}{p}\right)^{\frac{1}{2}} - \left(\frac{c_0^5}{p}\right)^{\frac{1}{2}}, \alpha p b_0 + \alpha \left(\frac{b_0^3}{p}\right)^{\frac{1}{2}}]^{tr}
\end{aligned}$$

Thus the matrix of transformation $T(t, \lambda, z)$ is given by:

$$\begin{pmatrix}
1 & 1 & 1 & 1 \\
O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) \\
O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) & O(p^{-\frac{1}{2}}) \\
O(p) & O(p) & O(p) & O(p)
\end{pmatrix}$$

As $t \rightarrow \infty$, the absolute value of λ_k tends to 1 and by application of Levinson's Benzaid-Lutz theorem, the dichotomy conditions can only be established off the real axis. $|\lambda_1|, |\lambda_3| > 1$ while $|\lambda_2|, |\lambda_4| < 1$ as $t \rightarrow \infty$ which is the required dichotomy condition. Performing the first diagonalisation as outlined in Theorem 4.6.6, equation (3.8) results to

$$\mathfrak{R}(t, z) = \begin{pmatrix}
O(\Delta p) & O(\Delta p) & O(\Delta p) & O(\Delta p) \\
O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) \\
O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) & O(p^{\frac{1}{2}} \Delta p) \\
O(p^{-1} \Delta p) & O(p^{-1} \Delta p) & O(p^{-1} \Delta p) & O(p^{-1} \Delta p)
\end{pmatrix}$$

The correction terms are of the form:

$$\begin{aligned} \mathfrak{R}_{k,k}(t, z) = & \{O(p^{-\frac{1}{2}}) + O(\Delta p), O(p^{-\frac{1}{2}}) + O(p^{\frac{1}{2}}\Delta p), \\ & O(p^{-\frac{1}{2}}) + O(p^{\frac{1}{2}}\Delta p), O(p^{-\frac{1}{2}}) + O(p^{-1}\Delta p)\} \end{aligned}$$

Therefore, since p is unbounded, $\Delta p \rightarrow 0$ as $t \rightarrow \infty$ so that the correction terms after the first diagonalisation are bounded and the off diagonal terms given by $\{O(\Delta p), O(p^{\frac{1}{2}}\Delta p)$ and $O(p^{-1}\Delta p)\}$ are ℓ^2 terms. After the second diagonalisation, the correction terms are of the form:

$$\begin{aligned} \mathfrak{R}B_{k,k}(t, z) = & \{O(p^{-\frac{1}{2}}) + O(\Delta p) + O(p(\Delta p)^2), O(p^{-\frac{1}{2}}) + O(p^{\frac{1}{2}}\Delta p) + O(p^{\frac{3}{2}}(\Delta p)^2), \\ & O(p^{-\frac{1}{2}}) + O(p^{\frac{1}{2}}\Delta p) + O(p^{\frac{3}{2}}(\Delta p)^2), O(p^{-\frac{1}{2}}) + O(p^{-1}\Delta p) + O((\Delta p)^2)\} \end{aligned}$$

in which both $(\Delta p) \rightarrow 0$ and $(\Delta p)^2 \rightarrow 0$ as $t \rightarrow \infty$ so that the correction terms can remain bounded. The off diagonal terms $\{O(p(\Delta p)^2), O(p^{\frac{3}{2}}(\Delta p)^2)$ and $O((\Delta p)^2)\}$ are ℓ^1 terms. Thus with two diagonalisations, (1.12) is converted into LBL form and an explicit computation shows that

$$\lim_{\eta \rightarrow 0} \langle y_k(t, z), y_k(t, z) \rangle \approx \prod_{l=t_0}^{t-1} \lambda_k(l, z)$$

hence if $|O(p)^{-\frac{1}{2}}|$ is absolutely summable, then all the four solutions are square summable and $\text{def} L = (4, 4)$, thus a limit circle case and the spectrum is discrete. Conversely, if $|O(p)^{-\frac{1}{2}}|$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability as $\eta \rightarrow 0^+$ and we have that $\text{def} L = (2, 2)$, a limit point case. We now compute the M-matrix using the eigenvectors associated with $\lambda_2(t, z)$ and $\lambda_4(t, z)$ whose eigensolutions remain square summable. Their eigensolutions are of the form:

$$y_k(t, z) = \rho_k(t, z)[e_k(t, z) + o(1)] \prod_{l=t_0}^{t-1} \lambda_k(l)$$

where $\rho_k(t, z)$ are the diagonal terms of $T(t, z)[I + B(t, z)]$ which are bounded. For $z = z_0 + i\eta$, $\eta > 0$, small and $z_0 \in \mathbb{R}$, then from the relation

$$\text{Im} M(t, z) = \lim_{\eta \rightarrow 0^+} \langle F(t, z), F(t, z) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|\alpha p_n(l)|^{\frac{1}{2}}} \approx -2\eta \int_{t_0}^{t-1} |\alpha p_n(l)|^{-\frac{1}{2}} dl.$$

The one for $y_4(t)$ follows in a similar fashion. These eigenfunctions $y_1(t)$ and $y_3(t)$ that lose their square summability contribute to the absolutely continuous spectrum. We note that the leading term associated with spectral parameter z is given by $|\frac{p_0 q_0 - z}{\alpha p_n}|^{\frac{1}{2}}$ so that as $p_n(t) \nearrow \infty$ and $\bar{c} < z < \infty$ where $\bar{c} = \limsup(\frac{p_0 q_0}{\alpha p_n})$. hence, the absolutely continuous spectrum is a subset of $[\bar{c}, \infty)$ and of spectral multiplicity 2.

Finally, we compute the deficiency indices of a minimal difference operator generated by the composites of order four difference equations.

Example 4.7.2. In this example we consider the fourth order difference equations of the form:

$$\begin{aligned}\mathcal{L}_1(y) &= \Delta^2(t^\epsilon \Delta^2 y(t-2)) + c_1 y(t) \\ \mathcal{L}_2(y) &= \Delta^2(\alpha t^\epsilon \Delta^2 y(t-2)) + c_2 y(t)\end{aligned}$$

where α is a real-valued constant, $\alpha \neq 0$ and $\epsilon > 0$.

The composite of the above equations as given by:

$$\mathcal{L}_1 \mathcal{L}_2 y = \Delta^4(\alpha t^{2\epsilon} \Delta^4 y(t-4)) - \Delta^2(b_0 t^\epsilon + c_0 t^\epsilon) \Delta^2 y(t-2) + b_0 c_0 y(t) \quad (4.21)$$

and the quasi-differences of (4.21) above are given as:

$$\begin{aligned}x_1(t) &= y(t-1) & x_3 &= \Delta^2 y(t-3) \\ x_2 &= \Delta y(t-2) & x_4 &= \Delta^3 y(t-4) \\ u_1(t) &= -\Delta(c_2 t^\epsilon + \alpha c_1 t^\epsilon) \Delta^2 y(t-2) - \Delta^3 \alpha t^{2\epsilon} \Delta^4 y(t-4) \\ u_2(t) &= (c_2 t^\epsilon + \alpha c_1 t^\epsilon) \Delta^2 y(t-2) + \Delta^2 \alpha t^{2\epsilon} \Delta^4 y(t-4) \\ u_3(t) &= -\Delta \alpha t^{2\epsilon} \Delta^4 y(t-4) & u_4(t) &= \alpha t^{2\epsilon} \Delta^4 y(t-4)\end{aligned}$$

Reducing (4.21) to first order system using the above quasi-differences, obtaining the characteristic polynomial and thereafter performing the right scaling and taking $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$, we obtain the fourier polynomial given by:

$$\mathcal{F}(\gamma, t) = \alpha t^{2\epsilon} \gamma^4 + (c_2 t^\epsilon + \alpha c_1 t^\epsilon) \gamma^2 + c_1 c_2 \quad (4.22)$$

solving the zeros of (4.22) we have that:

$$\begin{aligned}\gamma_1 &\approx i \left| \frac{c_1}{t^\epsilon} \right|^2 \\ \gamma_2 &\approx -i \left| \frac{c_1}{t^\epsilon} \right|^2 \\ \gamma_3 &\approx i \left| \frac{c_2}{t^\epsilon} \right|^2 \\ \gamma_4 &\approx -i \left| \frac{c_2}{t^\epsilon} \right|^2\end{aligned}$$

and since $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$ it implies that:

$$\begin{aligned}\lambda_{1/2} &= 1 - \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} \pm \left\{ \frac{c_1}{4t^\epsilon} - i \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\ \lambda_{3/4} &= 1 + \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} \pm \left\{ \frac{c_1}{4t^\epsilon} + i \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\ \lambda_{5/6} &= 1 - \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} \pm \left\{ \frac{c_2}{4t^\epsilon} - i \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} \right\}^{\frac{1}{2}} \\ \lambda_{7/8} &= 1 + \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} \pm \left\{ \frac{c_2}{4t^\epsilon} + i \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} \right\}^{\frac{1}{2}}\end{aligned}$$

Which results to:

$$\begin{aligned}\lambda_1 &\approx 1 - \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_2 &\approx 1 - \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_3 &\approx 1 + \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_4 &\approx 1 + \frac{i}{2} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \left| \frac{c_1}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_5 &\approx 1 - \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_6 &\approx 1 - \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_7 &\approx 1 + \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon}) \\ \lambda_8 &\approx 1 + \frac{i}{2} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \left| \frac{c_2}{t^\epsilon} \right|^{\frac{3}{4}} + O(t^{-\frac{5}{4}\epsilon})\end{aligned}$$

By obtaining the transforming matrix as outlined in example 4.7.1 above and performing the two diagonalisations to convert (1.12) into LBL form and by mathematical induction, we have that:

$$\lim_{\eta \rightarrow 0} \langle y_k(t, z), y_k(t, z) \rangle \approx \prod_{l=t_0}^{t-1} \lambda_k(l, z)$$

hence if $|O(t^\epsilon)^{-\frac{1}{4}}|$ is absolutely summable, then all the eight solutions are square summable and $defL = (8, 8)$, thus a limit circle case and the spectrum is discrete.

Conversely, if $|O(t^\epsilon)^{\frac{-1}{4}}|$ is not absolutely summable, then $y_1(t)$ and $y_4(t)$, $y_5(t)$ and $y_8(t)$ lose their square summability as $\eta \rightarrow 0^+$ and we have that $\text{def} L = (4, 4)$, a limit point case. The M-matrix can then be computed using the eigenvectors associated with $\lambda_2(t, z)$, $\lambda_3(t, z)$, $\lambda_6(t, z)$ and $\lambda_7(t, z)$ whose eigensolutions remain square summable. Their eigensolutions are of the form:

$$y_k(t, z) = \rho_k(t, z)[e_k(t, z) + o(1)] \prod_{l=t_0}^{t-1} \lambda_k(l)$$

where $\rho_k(t, z)$ are the diagonal terms of $T(t, z)[I + B(t, z)]$ which are bounded. Again for $z = z_0 + i\eta$, $\eta > 0$, small and $z_0 \in \mathbb{R}$, and from the relation

$$\text{Im} M(t, z) = \lim_{\eta \rightarrow 0^+} \langle F(t, z), F(t, z) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|t^\epsilon(l)|^{\frac{1}{4}}} \approx -2\eta \int_{t_0}^{t-1} |t^\epsilon(l)|^{-\frac{1}{4}} dl.$$

The one for $y_3(t)$, $y_6(t)$, and $y_7(t)$ are done in a similar fashion. These eigenfunctions $y_1(t)$, $y_4(t)$, $y_5(t)$ and $y_8(t)$ that lose their square summability contribute to the absolutely continuous spectrum. The leading term associated with spectral parameter z is given by $|\frac{c-z}{t^\epsilon}|^{\frac{1}{4}}$ so that as $t^\epsilon \nearrow \infty$ and $\bar{q} < z < \infty$ where $\bar{q} = \limsup(\frac{c}{t^\epsilon})$. Hence, the absolutely continuous spectrum is a subset of $[\bar{q}, \infty)$ and of spectral multiplicity 4.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

This chapter provides a summary of the main results of this thesis, limitations as well as recommendations for future research.

5.1 Conclusion

In Theorem 4.2.1, we have proved the existence of self-adjoint operator extensions H_1 and H_2 by application of asymptotic integration and von-Neumann theorems. We have also proved that these self-adjoint extensions commute by constructing their appropriate comparison algebras. This was done by allowing the leading coefficients of each differential operator to be constant functions. Thus, we have established that when the orders of the respective differential operators are equal and that the first derivative of these coefficients tends to zero as x tends to infinity, then the composite of these differential operators are symmetric and generate self-adjoint operator extensions which are commutative.

By allowing the coefficients to be unbounded in Theorem 4.3.1, we have proved the existence of symmetric self-adjoint operator extensions, the deficiency index have been shown to be $(2n, 2n)$ and the absolutely continuous spectrum for a $2n - th$ order differential operator if all the correction terms after diagonalizations are in-

tegrable, otherwise, the deficiency index is (n, n) and the absolutely continuous spectrum is the whole real line. Examples 4.3.2 and 4.3.3 shows that the condition of first derivative tending to zero as x tends to infinity cannot be weakened further. Lemma 4.3.4, Theorem 4.3.5, Theorem 4.3.6, Theorem 4.3.7 and example 4.3.8 gives the results on the composites of differential operators with unbounded coefficients. In particular, we have established that the minimal differential operator generated by the composite differential operator is symmetric and densely defined if the respective minimal operators generated by differential operators τ_1 and τ_2 are symmetric, injective, commute with one another and have closed ranges. We have further showed that under the optimal growth and decay conditions stated in (4.1), the deficiency index of the minimal differential operator generated by the composite is expressible as the sum of the deficiency indices of the independent minimal operators generated by the respective differential operators. Example 4.3.8 verifies this claim.

Theorem 4.4.1 shows that at limiting point, the spectrum of the self-adjoint operator extension of the minimal difference operator generated has no singular continuous spectrum and that its absolutely continuous spectrum agree with that of constant coefficient limiting operators. The conditions for commutativity of the self-adjoint operator extension are proved in Theorem 4.4.2 for which we have showed that for the self-adjoint extensions generated by the minimal difference operator to commute, then their orders must be equal and the first difference of their coefficients tends to zero in the limiting sense. These conditions are verified in example 4.4.3.

Theorems 4.4.5 and 4.5.1 gives the results on the symmetry of the composite of two difference operators as well as the symmetry of the generated minimal operator in which we have proved that for symmetry to hold, the difference operators must commute, their orders must be equal and the first difference of their coefficients tends to zero in the limiting sense. In addition, we have showed that the deficiency index of the minimal operator generated by the composite is equal to the sum of the deficiency indices of the minimal operator generated by the respective difference operators. Examples 4.5.2 and 4.5.3 have been used to illustrate that the conditions

for symmetry cannot be weakened further. Additionally, in theorem 4.5.6, we have applied Cayley transforms and established that for the minimal operator generated by the composite to have symmetric operator extensions, the orders of the difference operators must be equal, the leading coefficient of one difference operator must be a scalar multiple of the leading coefficient of the other difference operator whereas all other coefficients are constant functions.

In section 4.6, we have carried out the spectral analysis of the minimal operators generated by the difference operators as well as their composites. In particular, we have established that the minimal operator L generated by the composite is at limit point at infinity if the boundary conditions at $t_0 > 1$ are satisfied, the corresponding self-adjoint operator extension H has no singular continuous spectrum and that the absolutely continuous spectrum agrees with that of the constant coefficient limiting operator with the spectral multiplicity equal to half the number of eigenvalues with magnitude 1. We have further showed that if L_1 , L_2 and L are the minimal operators generated by $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ respectively, where $\mathcal{L}$ is the composite of $\mathcal{L}_1$ and $\mathcal{L}_2$, then H which is the self-adjoint operator extension of L can be expressed as a composite of H_1 and H_2 , where H_1 and H_2 are the self-adjoint operator extensions of L_1 and L_2 respectively. Finally we have given examples to verify the results on difference operators by computing the symmetric composites of commuting difference operators with unbounded coefficients and analysed their deficiency indices and spectra.

The results in this work have all been published in peer-reviewed journals (Okello, Nyamwala, & Ambogo, 2025), (Okello, Nyamwala, & Ambogo, 2024) and (Nyamwala & Okello, 2023).

5.2 Recommendations

In our work however, we were faced with difficulties in computing the eigenvalues and deficiency indices of higher degree polynomials (polynomials of degree greater than four) which forced us to use approximations as given by Eastham

(*M. S. P. Eastham*, 1989) to our results on polynomials of degree four to analyse the deficiency indices and the spectra of the composite of two order four operators. We therefore recommend further research in this area to come up with a model that can be of help in the computation of eigenvalues as well as the deficiency indices of such higher degree polynomials. This research can also be extended to the case of non-commuting composites of differential and difference operators.

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