

**Enumeration of skew 2-Dyck paths, non-decreasing 2-plane
trees, non-decreasing 2-noncrossing trees and related
combinatorial structures**

by

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DECLARATION

Declaration by the Candidate

I declare that this thesis is my original work and has not been presented for any academic award in any institution; and shall not be reproduced in part or full, or in any format without prior written permission from the author and/or Moi University.

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DEDICATION

*This thesis is dedicated to my family, with a special dedication to my mother,
Jesinta Wambui.*

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ABSTRACT

Graphs, lattice paths and trees have a widespread application in various fields such as Biology, Computer Science and Physics. While the study of these structures has gained significant attention, several gaps in understanding specific variants remain, limiting the ability to fully leverage their potential. The main goal of this study was to enumerate Dyck paths, plane trees and noncrossing trees in addition to characterizing their properties. The specific objectives of this study were to: Derive enumeration formulas for skew 2-Dyck paths, Evaluate the structural properties of non-decreasing 2-plane trees by analyzing their number of vertices, root degree and label of eldest child, Examine and assess the combinatorial behavior of weakly labelled k -plane trees through occurrences of vertices of certain labels, root degree, label of eldest child, number of leaves, forests and outdegree sequence, Synthesize counting methods for non-decreasing 2-noncrossing trees and non-decreasing 2-noncrossing increasing trees based on their number of vertices, root degree, forests and label of leftmost child of the root and Create bijections between plane Husimi graphs and various combinatorial structures. The generating functions of skew 2-Dyck paths were obtained by path decomposition followed by application of Lagrange inversion formula and various identities. Symbolic method was applied to derive generating functions for non-decreasing 2-plane trees. For non-decreasing 2-noncrossing trees and their increasing counterparts, the study utilized tree butterfly decomposition method, symbolic method and the Lagrange-Burmann theorem. Weakly labelled k -plane trees were analyzed by symbolic method and bijections between sets were established through sequential injective and surjective mappings. Categorized by length, skew 2-Dyck paths were enumerated by generalized Narayana numbers. Non-decreasing 2-plane and 2-noncrossing trees with roots labeled 1 or 2 corresponded to and extend large and little Schröder numbers, respectively. A bijection between weakly labelled k -plane trees, enumerated by vertex count, and k -ary Husimi graphs was then constructed. Notably, many of the studied structures were found to be enumerated by Little and Large Schröder paths and their extensions. These findings established connections between the examined structures and various combinatorial constructs of similar cardinality, addressing Stanley's open question on interpretations of structures enumerated by little and large Schröder numbers through bijections. This study expanded combinatorial knowledge, with potential applications in various fields, including genealogy through the introduction of non-decreasing sibling labels in plane and noncrossing trees. Further research is thus recommended on the enumeration of skew k -Dyck paths and non-decreasing k -noncrossing trees where $k \geq 2$.

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LIST OF ABBREVIATIONS AND ACRONYMS

G	Graph	1	$W(x)$	generating function for non-decreasing 2-plane trees with roots labelled 1	91
V	set of vertices	1	$B(x)$	generating function for non-decreasing 2-plane trees with roots labelled 2	91
E	Set of edges	1	$P_i(x, u_1, \dots, u_k)$	multivariate generating function for weakly labelled k -plane trees whose root was labelled i	114
BFS	Breadth First Search	10	$18u_j$	a vertex labelled j for $j = 1, \dots, k$	114
DFS	Depth First Search	10	$P(x)$	generating function for weakly labelled k -plane trees	123
OEIS	On-line Encyclopaedia of Integer Sequences	13	$d_0, d_1, \dots$	Outdegree sequence	124
U	step $(1, 1)$	14	$C(x, u)$	bivariate generating function for non-decreasing 2-noncrossing trees with roots labelled 2	144
D	step $(1, 1)$	14	$D(x, u)$	bivariate generating function for non-decreasing 2-noncrossing trees with roots labelled 1	144
L	step $(-1, -1)$	14	$D(x)$	generating function for non-decreasing 2-noncrossing trees labelled 1	147
$SYT(k, k, 1^{n-k})$	Standard Young tableau	18	$C(x)$	generating function for non-decreasing 2-noncrossing trees labelled 2	147
$Inc_k(2 \times n)$	Increasing tableaux	18	$I(x, u)$	bivariate generating function for non-decreasing 2-noncrossing increasing trees with roots labelled 2	156
$RInc_k(n \times k)$	row-increasing tableau	18	$J(x, u)$	bivariate generating function for non-decreasing 2-noncrossing increasing trees with roots labelled 1	156
$N(n, k)$	Narayana Numbers	20			
$S(n, k)$	Stirling number of the second kind	21			
$L(x)$	Generating function of large Schröder numbers	22			
$l(x)$	Generating function of little Schröder numbers	22			
$\binom{n}{k}$	Binomial coefficient	22			
$[x^n]f(x)$	n^{th} coefficient of $f(x)$	24			
OGF	Ordinary generating function	24			
EGF	Exponential generating function	24			
C_k	The k^{th} Catalan number	56			
z	Atom	72			
$N(n, i)$	Narayana number	80			
$S'(z)$	Differentiation of $S(z)$ w.r.t. z	82			
$UDLD$	A factor	85			
$DUDLD$	A box	85			

resp respectively 165

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CHAPTER ONE: INTRODUCTION

Chapter ONE is devoted to introducing basic concepts which are key to presenting subsequent sections and chapters. Statement of the problem, Objectives of the study and Justification of the study are listed here.

1.1 Basic concepts

Graph theory and the study of lattice paths are two broad areas of combinatorics which are interconnected. Specifically, a tree which is a special type of a graph, is a discrete structure that represents hierarchical relationships between individual elements or nodes with applications in fields such as computer science. Trees give a pictorial way of representing data and lattice paths are mostly used in solving various diversified problems especially in probability and statistics. In this section, we give various definitions or concepts that will enable us to present the content of the subsequent sections and chapters.

1.1.1 Graph theoretic concepts

For the definitions provided here, the text (Diestel, 2005) is an excellent reference material.

Definition 1.1. A *graph* G is an ordered pair (V, E) where V is a set of vertices and E is a set of edges. A graph is *connected* if there is a path between any two vertices.

Definition 1.2. A *loop* is an edge from a vertex to itself. If there are more than one edge connecting two given vertices, then the edges are said to be *multiple*. A *simple graph* is a graph with no loops or multiple edges.

From the set of simple graphs, we are able to obtain trees defined as;

Definition 1.3. A *tree* is a simple connected graph without cycles. When one of the vertices of a tree is identified as a root, then the tree is referred to as a *rooted tree*.

The structure in Figure 1.1(a) is a tree because it is connected, has no loops, multiple edges and cycles while the structure in Figure 1.1(b) is not a tree since it has a cycle.

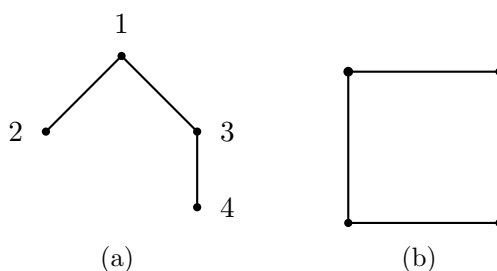


Figure 1.1: A tree and a non-tree on 4 vertices

Definition 1.4. Consider a rooted tree P . A vertex u is said to be *adjacent* to a vertex v in P if (u, v) is an edge in P . The *degree* of a vertex v is the number of vertices that are adjacent to it. The *outdegree* of v is the number edges which are oriented away from v if we orient all edges away from the root. A vertex of outdegree 0 is called a *leaf* and a vertex which is not a leaf is called an *internal vertex*. The *outdegree sequence* is the arrangement of outdegrees of all vertices in the tree.

Definition 1.5. A *labelled tree* with n vertices is a tree whose vertices are assigned numbers $\{1, 2, 3, \dots, n\}$.

Figure 1.1(a) is an example of a rooted labelled tree where vertex 1 is a root, 2 and 4 are leaves and 3 is an internal vertex. Vertex 3 has degree two but outdegree one and vertex 1 is adjacent to vertices 2 and 3 but not vertex 4.

Definition 1.6. Consider a rooted labelled tree. An edge (a, b) is an *ascent* (resp. *descent*) if $a < b$ (resp. $a > b$) where a is closest to the root. In a rooted tree, the *depth* or *level* of a vertex v is the number of edges in the path from the root to v .

We now discuss various trees that are enumerated in this thesis.

1.1.1.1 Plane trees and d -ary trees

Definition 1.7. A *plane tree*, also called *ordered tree*, is a rooted tree such that all subtrees are ordered.

A well known result is that plane trees on $n + 1$ vertices are counted by the n^{th} Catalan number (Dershowitz & Zaks, 1980), In Figure 1.2(a) and Figure 1.2(b), we get two different plane trees on 9 vertices.

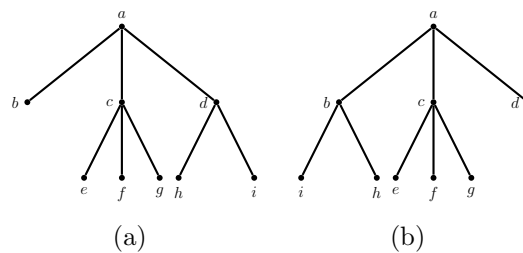


Figure 1.2: Two plane trees on 9 vertices

Definition 1.8. A vertex v of a plane tree is said to be a *child* (resp. *parent*) of a vertex u if the two vertices are adjacent and vertex v is at a lower level (resp. higher level) than u . Children that share a parent are called *siblings* and a child which appears on the far left is the *eldest* while the one on the far right is the *youngest*.

In Figure 1.2(a), f is a child of c and while e , f and g are siblings with e being their eldest sibling and g is the youngest.

Definition 1.9. (Gu, Prodinger, & Wagner, 2010) If vertices of a plane tree are given labels from the set $\{1, 2, \dots, k\}$ such that the sum of labels of any two adjacent vertices is not more than $k + 1$, then the tree is called a k -plane tree.

When $k = 2$ in Definition 1.9, we obtain a 2-plane tree and a 1-plane tree is just an ordinary plane tree. An example of a 2-plane tree is given in Figure 1.3.

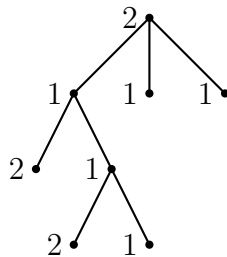


Figure 1.3: A 2-plane tree on 8 vertices with root labelled 2

Definition 1.10. A d -ary tree is a plane tree in which each internal vertex has at most d children. They are known to be enumerated by generalized Catalan numbers. Setting $d = 2$ and $d = 3$ we obtain *binary trees* and *ternary trees* respectively. If each internal vertex of a d -ary tree has exactly d children then we get a *complete d -ary tree*. Moreover, each internal vertex in a complete binary tree or complete ternary tree has exactly 2 or 3 children respectively.

An example of a 3-ary tree on 6 vertices is given in Figure 1.4. Consider a ternary

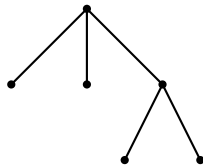


Figure 1.4: A ternary tree on 6 vertices

tree, then for each internal vertex, an edge which is incident to this vertex and appears on the far left, middle and far right are called *left-edge*, *middle-edge* and *right-edge* respectively.

Definition 1.11. (Pang & Lv, 2010) A k -labelled tree is a plane tree whose vertices other than the root are assigned labels from the set $\{1, 2, \dots, k\}$. When such a k -labelled tree has all children of the root labelled with a fixed integer i for $i \in \{1, 2, \dots, k\}$ and the labels of all vertices sharing a parent are non-decreasing from left to right, then a *weakly labelled k -tree* is obtained.

In (Nyariaro & Okoth, 2024), it was proved that the set of weakly labelled k -trees on n vertices is enumerated by

$$\frac{1}{n} \binom{(k+1)n}{n-1}.$$

Figure 1.5 illustrates a weakly labelled 2-tree

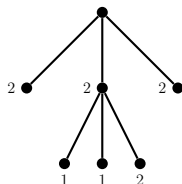


Figure 1.5: A weakly labelled 2-tree on 7 vertices

Definition 1.12. A k -binary tree is defined in (Gu, Li, & Mansour, 2008) as a binary tree whose vertices are given labels from the set $\{1, 2, \dots, k\}$ such that the labels of the right children of each vertex are non-decreasing as one moves away from the root.

From [(Nyariaro & Okoth, 2024, Corollary 1.12)], we find that the set of k -binary trees on $n - 1$ vertices with root labelled k is enumerated by

$$\frac{1}{n} \binom{(k+1)n}{n-1}.$$

1.1.1.2 Noncrossing trees

Definition 1.13. (Noy, 1998) A *Noncrossing tree* is a tree whose vertices lie on the circumference of a circle and their edges do not intersect inside the circle.

In Figure 1.6, we find an example of a noncrossing tree.

Definition 1.14. A k -noncrossing tree is a noncrossing tree whose vertices are labelled from the set $\{1, 2, \dots, k\}$ such that if (i, j) is an ascent in the path from the

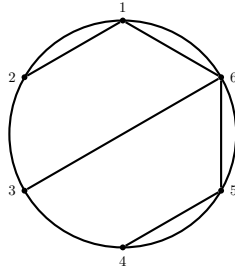


Figure 1.6: A noncrossing tree on 6 vertices.

root then the sum of the labels of endpoints of the ascent is no more than $k + 1$. A k -noncrossing increasing tree is a k -noncrossing tree whose labels are increasing if we consider a path from the root (Okoth, 2021a).

An example of a 4-noncrossing tree is presented in Figure 1.7.

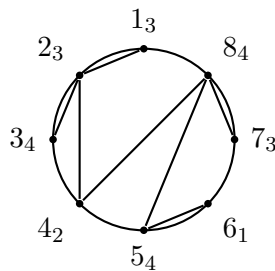


Figure 1.7: A 4-noncrossing tree on 8 vertices where subscripts are the labels of the vertices.

When $k = 2$, the set of 2-noncrossing trees is obtained defined as;

Definition 1.15. (Yan & Liu, 2009) A 2-noncrossing tree is a noncrossing tree whose vertices are labelled 1 or 2 such that there are no ascent whose endpoints are both labelled 2.

The set of 2-noncrossing trees on n vertices is enumerated by the same formula enumerating the set of 5-ary trees with $n - 1$ internal vertices (Yan & Liu, 2009).

A noncrossing tree can be represented as a plane tree by means of (l, r) -representation that was introduced by Panholzer and Prodinger in (Panholzer & Prodinger, 2002).

Given a noncrossing tree, if (i, j) is an ascent (resp. a descent) in a path from the root then j is marked r (resp. l). The (l, r) -representation is particularly valuable for exploring the characteristics of noncrossing trees and establishing bijections with related combinatorial structures. Such connections can lead to deeper insights into the enumeration and properties of these trees, enhancing our understanding of their role in combinatorial mathematics. Overall, the (l, r) -representation serves as a powerful tool for analyzing and visualizing the intricate structures of noncrossing trees. In Figure 1.8, we find a noncrossing tree and its (l, r) -representation.

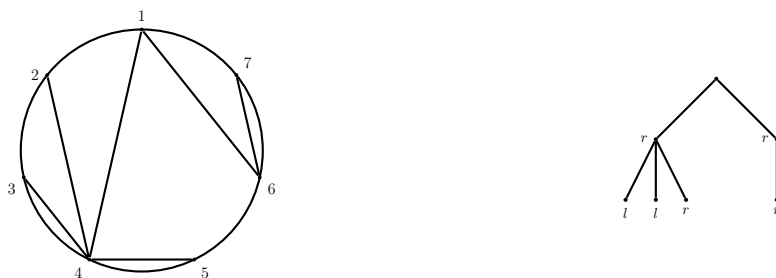


Figure 1.8: A noncrossing tree on 7 vertices with its (l, r) -representation.

Orientation of edges of labelled trees and graphs is categorised into; *global orientation* where all edges in a rooted labelled tree point towards the root and *local orientation* where each edge is directed towards the vertex that has the higher label (Okoth, 2015).

Definition 1.16. (Okoth, 2015) In oriented trees, we define two important metrics for each vertex: the indegree and the outdegree. The *indegree* of a vertex is the count of edges pointing towards it, while its *outdegree* is the number of edges directed away from it. These metrics, when collected for all vertices, form the indegree and outdegree sequences of the tree respectively. We represent these sequences in a format as $a_1^{b_1}, a_2^{b_2}, \dots$, where a_i denotes the in/outdegree and b_i indicates the number of vertices with that particular in/outdegree.

Definition 1.17. (Okoth & Wagner, 2015) *Locally oriented noncrossing (k -noncrossing) trees* are noncrossing (k -noncrossing) trees whose edges are oriented from a vertex of lower label towards a vertex of higher label. If all edges incident to a given vertex

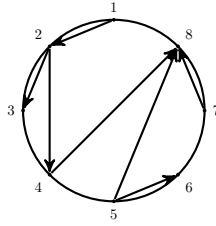


Figure 1.9: A locally oriented noncrossing tree on 8 vertices

v are directed towards (resp. away from) v , then such a vertex is said to be a *sink* (resp. *source*).

Figure 1.9 depicts an illustration of a locally oriented noncrossing tree where vertices 3, 6, 8 are sinks, vertices 1, 5 and 7 are sources and vertices 2 and 4 are neither sources nor sinks.

1.1.2 Tree-like structures

A *tree-like structure* is obtained when the edges of a tree are replaced with graphs or polygons, without changing the properties of these trees, i.e. connectedness and cycle-freeness.

Definition 1.18. A *partition* of $n \in \mathbb{N}$ is a way of representing n as an unordered sum of positive integers (Okoth, 2015). For a finite set S , where P is a set partition of S , the elements of S are called *blocks*. When such blocks replace all edges of a tree, we create a *Husimi graph* (Husimi, 1950). If the blocks of this connected graph are replaced by cycles, then the graph is called a *cactus* (Harary & Uhlenbeck, 1953) and if these blocks are oriented cycles, then an *oriented cactus* is created.

Definition 1.19. A *plane Husimi graph* is a Husimi graph drawn on a plane and its blocks are ordered (Okoth, 2015). When a plane Husimi graph (respectively cactus) is drawn with the block vertices lying along the boundary of a circle such that the blocks are ordered and not intersecting, then a *noncrossing Husimi graph* (respectively a noncrossing cactus) is obtained.

Definition 1.20. A block b attached to a vertex i in a plane Husimi graph is called a *block child* of i and the *outdegree* of a vertex i is the number of its block children (Onyango, Okoth, & Kasyoki, 2023). The *degree* of vertex i in a Husimi graph is the number of blocks incident to i .

Okoth in (Okoth, 2021b) found the number of plane Husimi graphs on $n \geq 2$ vertices to be

$$\frac{1}{n} \sum_{k=1}^{n-1} \binom{n+k-1}{k} \binom{n-2}{k-1}.$$

In Figure 1.10 and 1.11, we get a plane Husimi graph and a cactus respectively.

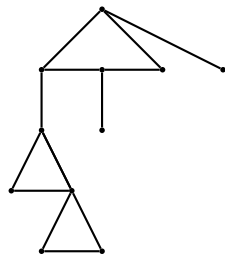


Figure 1.10: plane Husimi graph

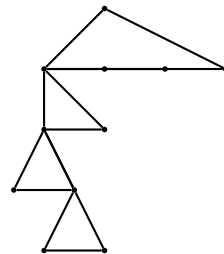


Figure 1.11: Cactus

An example of a noncrossing Husimi graph is depicted in Figure 1.12.

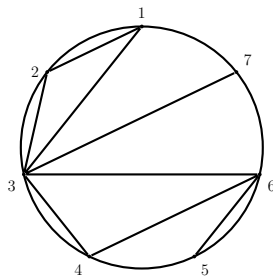


Figure 1.12: A noncrossing Husimi graph on 7 vertices with 4 blocks.

Definition 1.21. (Onyango et al., 2023) A k -ary Husimi graph is a plane Husimi graph whose vertices have at most outdegree k .

An illustration of a k -ary Husimi graph is given in Figure 1.13

The number of k -ary Husimi graphs on n vertices is given by

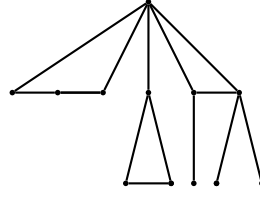


Figure 1.13: A 3-ary Husimi graph on 12 vertices

$$\frac{1}{n} \binom{(k+1)n-2}{n-1}$$

1.1.3 Tree traversals

We consider two broad categories of tree traversals which can be extended to tree-like structures.

Definition 1.22. (Rosen & Krithivasan, 1999) *Breadth first search* (BFS) is a graph traversal algorithm that explores all vertices of a graph or tree at the present depth before moving to vertices at the next depth level moving from left to right while *Depth first search* (DFS) is an algorithm that explores as far as possible along each branch before backtracking.

The main variations of the DFS are;

Definition 1.23. *Preorder traversal* of a tree, where in the algorithm, the root of the tree is visited first followed by its left most subtree, moving towards its right most subtree. *Postorder traversal* of a tree, where the left most subtree of the root is visited first followed by the next subtree on the right until the right most subtree is visited. The root is visited last. *Inorder traversal* of a tree, where the left most subtree of the root is visited first followed by the root then the next subtree on the right until the right most subtree is visited.

In the plane tree in Figure 1.14, the sequence of vertices traversed in BFS is $a, b, c, d, e, f, g, h, i$ while in DFS preorder is $a, b, c, e, f, g, d, h, i$, postorder is

$b, e, f, h, c, h, i, d, a$ and inorder is $b, a, e, f, g, c, h, , i, d$.

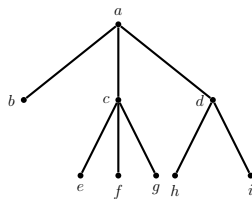


Figure 1.14: An example of a plane tree on 4 vertices

1.1.4 Lattice path theory

Definition 1.24. A *lattice path* is path that starts at a given point on the xy -plane and uses horizontal, vertical and/or diagonal steps. The number of steps is the length of the path. A horizontal step or vertical step or diagonal step of length k is represented by $(0, k)$ or $(k, 0)$ or (k, k) respectively.

Definition 1.25. A *non-negative lattice path* is a lattice path that lies entirely on the first quadrant of the xy -plane.

1.1.4.1 Dyck paths

We now look at various examples of lattice paths.

Definition 1.26. (Deutsch, 1999) A *Dyck path* of length $2n$ or semilength n is a non-negative lattice path starting from the origin to the point $(2n, 0)$ with the step set $\{(1, 1), (1, -1)\}$.

Definition 1.27. A *peak* of a Dyck path is a point where an up-step is immediately followed by a down-step. Visually, it looks like the top of a mountain, shape $(\wedge)$. A *valley* is the opposite of a peak. It's a point where a down-step is immediately followed by an up-step. Visually, it looks like the bottom of a $(\vee)$ shape. The *height* of a point in a Dyck path is its vertical distance from the x -axis (or the starting line) while the height of the entire Dyck path is the maximum height reached by

any point in the path. The *level* of a step in a Dyck path refers to the height at which that step occurs. For example, a step at level 2 means it's two units above the x -axis.

All Dyck paths of length 4 are given in Figure 1.15.

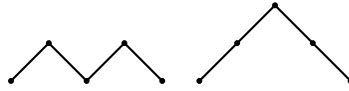


Figure 1.15: Dyck paths of length 4.

Dyck paths of length $2n$ are counted by the Catalan number

$$\frac{1}{n} \binom{2n-2}{n-1},$$

as found in (Deutsch, 1999).

1.1.4.2 k -Dyck paths

A generalization of Dyck paths by using $(1, -k)$ steps instead of $(1, -1)$ was done in (Selkirk, 2019; Heuberger, Selkirk, & Wagner, 2023) which we now introduce:

Definition 1.28. A k -Dyck path of length $(k+1)n$ is a lattice path with steps $(1, 1)$ and $(1, -k)$ that lie weakly above the line $y = 0$ and end on the x -axis.

The set of k -Dyck paths of length $(k+1)n$ are enumerated by the Fuss-Catalan number,

$$\frac{1}{kn+1} \binom{(k+1)n}{n}$$

(Selkirk, 2019).

A bijection between the set of k -Dyck paths of semilength n and the set of $(k+1)$ -ary trees with $(k+1)n$ nodes was constructed in (Selkirk, 2019). All 2-Dyck paths of length 6 are given in Figure 1.16.



Figure 1.16: 2-Dyck paths of length 6.

Definition 1.29. (Selkirk, 2019). A k_t -Dyck path of semilength n as a Dyck path that starts at the origin and uses unit up steps $((1, 1)$ steps) and down steps of length k $((1, -k)$ steps) such that the path stays weakly above the line $y = -t$ and ends at $((k + 1)n, 0)$. When $t = 0$, then the path is a k -Dyck path.

1.1.4.3 Skew and dual skew Dyck paths

Definition 1.30. When a south west step $(-1, -1)$ is introduced to Dyck paths, *skew Dyck paths* are obtained so long as the paths do not intersect with each other and they remain above the x -axis (Deutsch, Munarini, & Rinaldi, 2010a).

In Figure 1.17, we get skew Dyck paths of length 4.

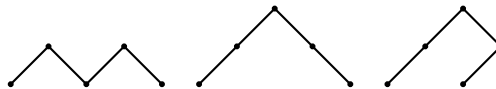


Figure 1.17: Skew Dyck paths of length 4

Skew Dyck paths are enumerated by sequence A002212 in OEIS (Sloane et al., 2018).

Definition 1.31. The mirror version of the skew Dyck paths, where the paths are read from right to left are *dual skew Dyck paths* (Prodinger, 2021c). These paths have two up-steps $(-1, 1)$ and $(1, 1)$ unlike the skew Dyck paths which has two down steps $(1, -1)$ and $(-1, -1)$. *Partial skew Dyck paths* are skew Dyck paths whose south-west step $(-1, -1)$ is replaced with the path $(1, -1)$ but colored differently and ending at a level j instead of level 0 (Prodinger, 2022).

The enumeration sequence of the dual skew Dyck paths is given as A002212 in the

OEIS (Sloane et al., 2018) which also enumerates skew Dyck paths.

1.1.4.4 K -box paths

A subset of skew Dyck paths called box paths was introduced by Zhang and Zhuang in (Zhang & Zhuang, 2024).

Definition 1.32. (Zhang & Zhuang, 2024) A k -box path of size n is a skew Dyck path of semilength $(k + 2)n - 1$ comprised of n UD^kL factors where U represents an up-step $(1, 1)$, D is a $(1, -1)$ step and L is a $(-1, -1)$ step.

Definition 1.33. (Zhang & Zhuang, 2024) A *box* is a factor of the skew Dyck path comprising of three consecutive steps UDL . These UDL factors protrude from the path in a box-like shape hence the motivation behind their name.

We create an extension of these boxes into blocks in our results and define a block as

Definition 1.34. In this work, we define a *block* as a factor of the skew Dyck path comprised of four consecutive steps $UDLD$.

The set of k -box paths with n UDL factors is enumerated by

$$\frac{1}{(k + 2)n - 1} \binom{(k + 2)n - 1}{n}.$$

In Figure 1.18, we get a box path of size 2.

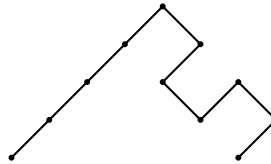


Figure 1.18: A box path of size 2

1.1.4.5 Motzkin paths

Definition 1.35. (Aigner, 1998) A *Motzkin path* is a lattice path whose step set is $\{(1, 0), (1, 1), (1, -1)\}$, starting at the origin $(0, 0)$, does not go below the x -axis and terminates after $2n$ steps on the x -axis.

Motzkin paths of length 3 are given in Figure 1.19.

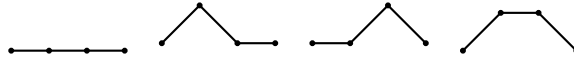


Figure 1.19: Motzkin paths of length 3.

Definition 1.36. *Skew Motzkin paths* are Motzkin paths with an extra south west steps $(-1, -1)$ not intersecting other steps.

By introducing certain restrictions to the Motzkin paths (Prodinger, Selkirk, & Wagner, 2020), the authors studied the following two kinds of Motzkin paths;

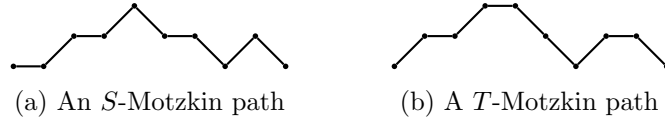
Definition 1.37. An *S-Motzkin path* of length $3n$ is a Motzkin path that consists of n up-steps, n down-steps and n horizontal steps which starts with an initial horizontal step and each horizontal step and up-step alternate.

Definition 1.38. A *T-Motzkin path* of length $3n$ on the other hand is defined as a Motzkin path starting with an up-step and any subsequent horizontal step alternates with an up-step not considering the down-steps. *T-Motzkin paths* of length $3n$ have n steps of each kind similar to the *S-Motzkin paths*.

Figure 1.20(a) and 1.20(b) give examples of an *S-Motzkin* and a *T-Motzkin* path of length 9.

1.1.4.6 Schröder paths

Definition 1.39. (Stanley, 1999) A *Schröder path* is an extension of Motzkin paths where the up-steps and down-steps remain of unit length but the horizontal steps

Figure 1.20: S -Motzkin and T -Motzkin paths of length 9

are $(2, 0)$ steps.

Schröder paths are classified into large and little Schröder paths defined as follows:

Definition 1.40. (Ardila, 2015) *Large Schröder paths* of length $2n$ are lattice paths from $(0, 0)$ to $(2n, 0)$ that comprise of up-steps $(1, 1)$, down-steps $(1, -1)$ and horizontal steps $(2, 0)$ which do not go below the x -axis while *little Schröder paths* are lattice paths from $(0, 0)$ to $(2n, 0)$ comprised of up-steps $(1, 1)$, down-steps $(1, -1)$ and horizontal steps $(2n, 0)$ not going below the x -axis such that there are no horizontal steps on the x -axis

Figure 1.21 gives all the 6 large Schröder paths of length 4.

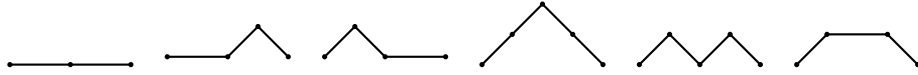


Figure 1.21: Large Schröder paths of length 4.

1.1.4.7 m -Schröder paths

A generalization of Schröder paths into m -Schröder paths was done by Yang et.al. in (S.-L. Yang & Jiang, 2021).

Definition 1.41. (S.-L. Yang & Jiang, 2021) An m -Schröder path of length mn is a lattice path lying in the first quadrant from $(0, 0)$ to $(mn, 0)$ consisting of $(m-1, 1)$, $(m-2, 2)$ and $(1, -1)$ steps. If the path starts from the x -axis with a $(m-2, 2)$ then we get a *small m -Schröder path*.

Setting $m = 3$ in Definition 1.41, we obtain a 3-Schröder path and a small 3-Schröder path. The set of 3-Schröder paths and small 3-Schröder paths are enumerated by

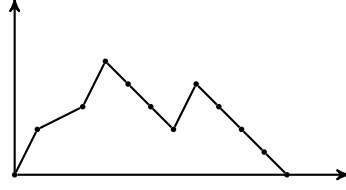


Figure 1.22: A 3-Schröder path of length 12

sequences *A027307* and *A034015* respectively of the OEIS (Sloane et al., 2018).

Figure 1.22 gives a 3-Schröder path.

1.1.4.8 Deutsch paths

Definition 1.42. A *Deutsch path* of length n is a non-negative lattice path comprising of n steps which are unit $(1, 1)$ steps and $(1, -j)$ steps for any $j \geq 1$ and end on level i (Prodinger, 2021a, 2021b). If $i = 0$, then the Deutsch paths end on the x -axis and the paths are referred to as *closed Deutsch paths* and if i is unspecified, then they are called *open Deutsch paths*.

Figure 1.23 lists all the six Deutsch paths of length 5

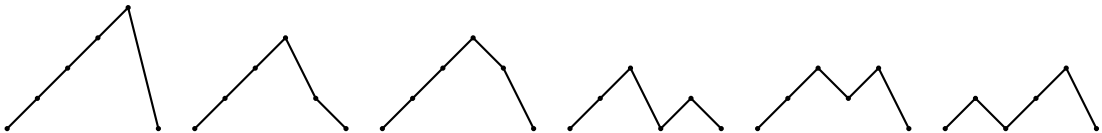


Figure 1.23: Deutsch paths of length 5

1.1.4.9 Related combinatorial structures

Definition 1.43. A *Young tableau (diagram)* of shape λ is defined as a collection of cells in left-justified rows such that there are i boxes in the i^{th} row from the bottom, i.e. length of the rows is weakly decreasing. A *semi-standard Young tableau* of shape λ satisfies the conditions that; entries in each row are weakly increasing and each column are strictly increasing (Adin & Roichman, 2015). The entries in these tables

can be any alphabets but are mostly integers. Table 1.1 represents a semi-standard Young tableau of shape $(4, 3, 2, 1)$.

Table 1.1: A semi-standard Young tableaux

1	2	2	3
3	4	4	
5	6		
7			

Definition 1.44. (Fulton, 1997) A *standard Young tableau* $SYT(k, k, 1^{n-k})$, is defined as a tableau whose first two rows have k partitions and the first column has $n - k$ partitions excluding those of the first two rows such that any value appears only once and the total number of partitions is $n + k$.

Table 1.2 gives a standard Young tableau, $SYT(4, 4, 1^2)$.

Table 1.2: A standard Young tableaux

1	2	4	6
3	5	7	10
8			
9			

Definition 1.45. An *Increasing tableau* is defined in (Pechenik, 2014) as a tableau comprising of 2 rows and n columns, with both rows and columns having strictly increasing values, such that for every value i contained in the table, all the smaller positive integer values less than i also appear in the table. Every value in the table can be repeated at most twice and k represents the number of repeated terms. We denote the set of increasing tableaux with n columns, 2 rows and k repeated terms by $Inc_k(2 \times n)$.

Consider the illustration in Table 1.3 of an increasing tableau.

Definition 1.46. A *row-increasing tableau* denoted as $RInc_k(n \times k)$ is a modification of the increasing tableau, having 2 rows with strictly increasing entries and

Table 1.3: An example of an element in $Inc_2(2 \times 4)$.

1	2	3	4
2	4	5	6

n columns with weakly increasing entries. Every value in the table can be repeated at most twice and k represents the number of repeated terms. This set is also enumerated by large Schröder numbers.

Consider Table 1.4 below. This is a row-increasing tableau but not an increasing tableau.

Table 1.4: An example of an element in $RInc_3(2 \times 4)$.

1	2	3	4
1	2	4	5

1.1.5 Special numbers

The *Catalan numbers* form a sequence of natural numbers that occur in various counting problems, often involving recursively defined objects. The generating function for Catalan numbers satisfies a simple quadratic equation:

$$C(x) = 1 + xC(x)^2 \quad (1.1)$$

The n^{th} Catalan number is represented as;

$$\frac{1}{n+1} \binom{2n}{n}$$

and recursively as

$$C_0 = 1, C_{n+1} = \sum_{i=0}^n C_i C_{n-i} \quad \text{for } n \geq 0.$$

The first few Catalan numbers are 1, 1, 2, 5, 14, ... and are represented as sequence A000108 in OEIS (Sloane et al., 2018). The Catalan numbers have numerous other applications in combinatorics, computer science, and other areas of mathematics. They often appear in problems involving recursively defined structures and are closely related to many other combinatorial concepts.

The *Narayana number* $N(n, k)$ counts the number of structures of a specific type among the structures counted by the n^{th} Catalan number (Soykan, 2020). Specifically, it counts those structures with i occurrences of a certain feature. They were first introduced in the 14th century by an Indian mathematician (Allouche & Johnson, 1996). These numbers were first applied to solve a probability question and have since received wide attention in their generalizations and applications such as in (Soykan, 2020). The n^{th} number is expressed as

$$N(n, k) = \frac{1}{n} \binom{n}{i} \binom{n}{i-1}.$$

Fine numbers are a sequence of integers closely related to Catalan numbers (Deutsch & Shapiro, 2001). The n^{th} Fine number, typically denoted as F_n has its recursive formula as;

$$\begin{cases} F_0 = 1 \\ F_1 = 0 \\ F_n = F_{n-1} + 2F_{n-2} \quad \text{for } n \geq 2 \end{cases}$$

and explicit formula as;

$$F_n = \frac{(n+1)C_{n-1} - nC_n}{2}.$$

for the n^{th} Catalan number C_n .

Stirling numbers are categorized into the first kind and the second kind defined as follows;(Gould, 1960) The Stirling number of the first kind, denoted as $s(n, k)$,

counts the number of permutations of n elements with exactly k cycles. It is recursively represented as

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$$

with initial conditions $\begin{bmatrix} n \\ 0 \end{bmatrix} = \delta_{n,0}$ and $\begin{bmatrix} n \\ n \end{bmatrix} = 1$. The Stirling number of the second kind, denoted as $S(n, k)$, counts the number of ways to partition a set of n elements into k non-empty subsets. Its recursive formula is

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = k \begin{Bmatrix} n-1 \\ k \end{Bmatrix} + \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}$$

with initial conditions $\begin{Bmatrix} n \\ 0 \end{Bmatrix} = \delta_{n,0}$ and $\begin{Bmatrix} n \\ n \end{Bmatrix} = 1$.

Motzkin numbers are related to various counting problems. The n^{th} Motzkin number, typically denoted as M_n has an explicit formula involving sums of binomial coefficients represented as;

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

where C_k is the k^{th} Catalan number (Aigner, 1998). The generating function for Motzkin numbers satisfies the functional equation:

$$M(x) = 1 + xM(x) + x^2M(x)^2$$

which can be simplified to;

$$M(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

The first few Motzkin numbers are 1, 1, 2, 4, 9, ... represented as sequence A001006 in the OEIS (Sloane et al., 2018).

Schröder numbers are categorized into *large Schröder numbers* and *little Schröder numbers*. From (Qi & Guo, 2017), we find that the generating function of large Schröder numbers is

$$L(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2} \quad (1.2)$$

and that of little Schröder numbers is

$$l(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4}. \quad (1.3)$$

The sequence of large Schröder numbers is recorded as A006318 in the OEIS (Sloane et al., 2018) while that of little Schröder numbers is recorded as sequence A001003.

1.1.6 Multinomial Theorem and other identities

The multinomial theorem is a fundamental principle in algebra that provides a formula for expanding the n^{th} power of a sum of k terms. For any positive integer n and any non-negative integer k , it states that:

Theorem 1.1 ((Comtet, 1974, Theorem B)). *For any integer $n \geq 0$*

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{m_1 + m_2 + \dots + m_k = n \\ m_i \geq 0}} \frac{n!}{m_1! m_2! \dots m_k!} \prod_{i=1}^k x_i^{m_i}$$

where

$$\prod_{i=1}^k x_i^{m_i} = x_1^{m_1} x_2^{m_2} \dots x_k^{m_k}.$$

Setting $m = 2$, we get the binomial theorem

Theorem 1.2. (Comtet, 1974) *For any integer $n \geq 0$,*

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

Definition 1.47. The binomial coefficient $\binom{n}{k}$ is the number of ways of getting k

unordered outcomes out of n possibilities.

The *Pascal's rule identity*

Identity 1.1.

$$\binom{n}{k} = \binom{n}{n-k}$$

is one of several interesting properties of the binomial coefficient (Comtet, 1974). It satisfies the recursive formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

The *binomial identity for negative integral components* states: (Comtet, 1974)

Identity 1.2.

$$(x-a)^{-n} = \sum_{k=0}^n \binom{-n}{k} (-a)^k = \sum_{k=0}^n \binom{n+k-1}{k} a^k.$$

The *Vandermonde's convolution* formula is an identity that relates the product of two generating functions (Comtet, 1974). It states;

Identity 1.3.

$$\binom{n+m}{k} = \sum_{j=0}^k \binom{n}{j} \binom{m}{k-j}.$$

The following generalization of Vandermonde type convolution was obtained in [(Kucukoglu & Simsek, 2017, Corollary 5.5)].

Identity 1.4. Let m_1 and m_2 be positive integers and n a non-negative integer

then,

$$\sum_{j=0}^n \binom{m_1 + j - 1}{j} \binom{m_2 + n - j - 1}{n - j} = \binom{m_1 + m_2 + n - 1}{n}.$$

The *hockey stick identity* (Comtet, 1974) comes from sum of terms in the Pascal's triangle and is expressed as:

Identity 1.5.

$$\sum_{a=b}^n \binom{a}{b} = \binom{n+1}{b+1}.$$

The *Rothe-Hagen identity* (Comtet, 1974) provides a relationship between sums of products of binomial coefficients. It states that;

Identity 1.6. For non-negative integers n , k and r :

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \binom{k + n - i}{r} = \binom{k}{r - n}.$$

These identities are applied in simplifications of extended binomial expressions.

1.1.7 Generating functions

Generating functions are defined as;

Definition 1.48. (Wilf, 2005) A *generating function* is a formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

whose coefficients give the sequence $\{a_1, a_2, \dots, a_n\}$. Note that the variable x only serves to keep track of the coefficient of x^n .

It is a common combinatorial practice to denote the n^{th} coefficient of the gener-

ating functions $f(x)$ by $[x^n]f(x)$. There are various types of generating functions such as ordinary generating functions (OGF's) and exponential generating functions (EGF's) .

Definition 1.49. Given a sequence $\{a_1, a_2, \dots\}$, the *ordinary generating function* is given as

$$f(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{k=0}^{\infty} a_k x^k.$$

Definition 1.50. Given a sequence $\{a_1, a_2, \dots\}$, the *exponential generating function* is given as

$$f(x) = a_0 + a_1x + a_2\frac{x^2}{2!} + \dots = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}.$$

OGFs are useful for studying and manipulating sequences, as well as solving recurrence relations while EGFs are particularly useful for counting objects such as permutations, partitions, and compositions (McBride, 2012).

For example, given a sequence of 1's, i.e. $1, 1, 1, 1 \dots$, its OGF is

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n$$

and the EGF is

$$e^x = \sum_{n \geq 0} \frac{x^n}{n!}.$$

Generally, $\frac{1}{1-qx} = \sum_{n=0}^{\infty} q^n x^n$ is an ordinary generating function. Equations that relate a generating function to itself are referred to as *functional equations* and often arise when solving recurrence equations of combinatorial structures (McBride, 2012).

1.2 Statement of the problem

Study of lattice paths has provided a wider interpretation of their applications mainly in solving statistical questions. Dyck paths have received more attention

in comparison to the other lattice paths as in (Deutsch, 1999; Burstein, 2023). Extensions of Dyck paths have also been studied, such as k -Dyck paths (Selkirk, 2019). Skewness in lattice paths was first introduced to Dyck paths by Deutsch (Deutsch et al., 2010a), where he enumerated them by introducing a south west step $(-1, -1)$ without crossing the other steps in a Dyck path. Application of skewness on the extended class of Dyck paths, i.e. the k -Dyck paths has not yet been studied. In this work, we enumerated the set of skew 2-Dyck paths and obtained bijections to other combinatorial objects.

The study of trees and graphs has similarly gained popularity across the years due to their applications in various fields such as computer science, biology and physics. Specifically, Husimi graphs have a wide application in physics (Husimi, 1950; Leroux, 2004). Similarly, k -plane trees and k -noncrossing trees were recently studied and their enumeration formulas found by various parameters in (Okoth, 2021b; Okoth & Wagner, 2024). We extended the study of these trees by introducing the non-decreasing condition on the labels of sibling vertices and enumerated them by various parameters such as number of vertices, root degree, labels of children and forests. By restricting their labelling, they can find applications in fields such as data storage and genealogy.

Richard Stanley, in his illustrious book (Stanley, 1999, Exercise 6.39) posed a problem for combinatorial interpretations of various structures enumerated by little and large Schröder numbers. In this thesis, we set out to obtain several such structures enumerated by these numbers and discuss their relationships with already existing structures including plane and k -ary Husimi graphs by method of bijections.

1.3 Objectives of the study

1.3.1 General objective

The main objective of this study was to develop comprehensive enumeration methods for skew 2-Dyck paths, non-decreasing 2-plane trees and non-decreasing 2-noncrossing trees while establishing their structural relationships with other combinatorial objects.

1.3.2 Specific objectives

The specific objectives of this research were to;

- (i) Derive enumeration formulas for skew 2-Dyck paths
- (ii) Evaluate the structural properties of non-decreasing 2-plane trees by analyzing their number of vertices, root degree and label of eldest child
- (iii) Examine and assess the combinatorial behavior of weakly labelled k-plane trees through occurrences of vertices of certain labels, root degree, label of eldest child, number of leaves, forests and outdegree sequence.
- (iv) Synthesize counting methods for non-decreasing 2-noncrossing trees and non-decreasing 2-noncrossing increasing trees based on their number of vertices, root degree, forests and label of leftmost child of the root.
- (v) Create novel bijections between plane Husimi graphs and related combinatorial structures.

1.4 Justification of the study

The study of skew 2-Dyck paths, non-decreasing 2-plane trees, non-decreasing 2-noncrossing trees, weakly labelled k -plane trees and plane Husimi graphs is justified by its potential in advancing knowledge in the field of enumerative combinatorics. It uncovers deep connections between related combinatorial structures. The techniques applied in their enumeration such as the Lagrange inversion formula, showcase the power of combinatorial tools in deriving explicit formulas and enumerations. Enumeration of these structures by the various labelling constraints and patterns avoidance provides an avenue for new applications such as in restricted growth models, permutation patterns and genealogical reconstructions. These enumerative results could be useful in other combinatorial problems involving tree-like structures with certain labelling constraints.

Combinatorial structures considered in this thesis, such as Schröder paths, Dyck paths, increasing tableaux and plane trees have connections to representation theory, algebraic geometry, and related areas of mathematics. Specifically, trees have a wide application in data systems mainly used by Computer Scientists while further application of Husimi graphs can be traced to theoretical physics. The bijections established could facilitate transfers of ideas between them.

This study therefore contributes to the growing body of knowledge in combinatorics, with direct applications in algorithm design (computer science), modelling of physical systems (theoretical physics), enumeration of geometric objects (algebraic geometry), and probabilistic modelling of branching processes (probability theory).

CHAPTER TWO: LITERATURE REVIEW

Chapter TWO provides a review of related literature on the combinatorial structures in our objectives. An introduction is found in Section 2.1, literature on trees is found in Section 2.2, on tree-like structures in Section 2.3, on lattice paths in Section 2.4 and on Schröder numbers in Section 2.5.

2.1 Introduction

Trees, graphs, and lattice paths are fundamental structures in discrete mathematics and computer science, with far-reaching applications across various fields. These structures have been the subject of extensive research for decades, serving as powerful tools for modeling complex systems, solving algorithmic problems, and understanding combinatorial relationships. The study of these structures not only provides insights into abstract mathematical concepts but also has practical implications in areas such as network analysis, data structures, and computational biology.

The history of tree structures in mathematics can be traced back to the 19th century, with early work by Arthur Cayley on the enumeration of chemical compounds (Cayley, 1857). Since then, trees have become an integral part of graph theory and computer science, playing crucial roles in algorithms, data organization, and problem-solving. Plane trees, in particular, have garnered significant attention due

to their connections with various combinatorial objects and their elegant enumerative properties.

Graph theory, which encompasses the study of trees, has evolved into a rich and diverse field with applications ranging from social network analysis to operations research. Within this broader context, specialized structures such as Husimi graphs, cacti, and oriented cacti have emerged as important objects of study, offering unique insights into graph properties and relationships. These tree-like structures bridge the gap between trees and more general graphs, providing a fertile ground for mathematical exploration.

Lattice paths, another key focus of this research, have their roots in probability theory and combinatorics. The classic Bertrand's ballot problem, posed in the 19th century (Takács, 1962), laid the foundation for the systematic study of lattice paths. Since then, various types of lattice paths, including Dyck paths, Motzkin paths, and Schröder paths, have been extensively studied, revealing deep connections with related combinatorial structures and algebraic identities.

The Schröder numbers, named after the German mathematician Ernst Schröder (Gessel, 2009), have emerged as a fascinating sequence with numerous combinatorial interpretations. These numbers, which come in two varieties (large and small), enumerate a wide array of mathematical objects, from certain types of lattice paths to polygon dissections. The appearance of Schröder numbers across different mathematical contexts highlights the interconnectedness of seemingly disparate areas of combinatorics.

This literature review aims to provide a comprehensive overview of the current state of research in these interrelated areas. It explores the enumerative properties of various tree structures, delves into the intricacies of tree-like graphs, examines the combinatorial richness of lattice paths, and investigates the diverse manifestations of Schröder numbers. By synthesizing results from numerous studies, we seek to iden-

tify common threads, highlight recent advancements, and point towards promising directions for future research.

As we navigate through this complex landscape of mathematical structures, we will encounter a wealth of elegant formulas, bijective proofs, and surprising connections. From the generalized Catalan numbers that enumerate k -ary trees to the intricate bijections between lattice paths and tableau structures. Through this exploration, we aim to contribute to the ongoing dialogue in the field and inspire further investigations into these fundamental combinatorial objects.

2.2 Trees

The study of plane trees and their generalizations continues to be an active and fruitful area of research, with far-reaching connections to various branches of combinatorics, probability theory, and theoretical computer science. The concept of plane trees was first introduced by de Bruijn and his collaborators in 1972 (de Bruijn, Knuth, & Rice, 1972), marking a significant milestone in the field. Their groundbreaking work focused on determining the average height of planted plane trees, laying the foundation for future investigations into these fundamental structures.

Building upon this initial research, Dershowitz and Zaks conducted an extensive study on the enumeration of plane trees, considering various parameters such as the number of edges, root degree, and internal vertices (Dershowitz & Zaks, 1980). One of their main results was; the number of plane trees on $n \geq 2$ vertices and having k leaves is given by the $(n - 1)^{th}$ Narayana number,

$$\frac{1}{n - 1} \binom{n - 1}{k} \binom{n - 1}{k - 1}.$$

This formula not only provided a precise count of these trees but also revealed deep connections between plane trees and related combinatorial structures. Narayana

numbers were first introduced in the 14th century by an Indian mathematician (Allouche & Johnson, 1996). These numbers were first applied to solve a probability question and have since received wide attention in their generalizations and applications such as in (Soykan, 2020). The n^{th} number is expressed as

$$N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

A well known result is the relationship between plane trees and Catalan numbers. The summation of the n^{th} Narayana number over k gives the n^{th} Catalan number,

$$\frac{1}{n+1} \binom{2n}{n},$$

This formula counts plane trees with $n+1$ vertices (Dershowitz & Zaks, 1980), showcasing the fundamental role of Catalan numbers in tree enumeration. The discovery of this connection laid the groundwork for extensive research into generalizations of plane trees, opening up new avenues for exploration in combinatorial mathematics.

The enumeration of 2-plane trees, a natural extension of plane trees, was introduced by Gu and Prodinger in (Gu & Prodinger, 2009). They found that 2-plane trees on $n+1$ vertices with root labelled 2 is given by

$$\frac{1}{2n+1} \binom{3n}{n}.$$

Interestingly, they proved that this formula also counts ternary trees with n internal vertices, hinting at a deeper structural relationship between these seemingly different objects. When the root is labelled 1, the authors found them to be enumerated by

$$\frac{1}{n+1} \binom{3n+1}{n}.$$

This formula, in turn, counts incomplete ternary trees with n internal nodes, further expanding the web of connections between various tree structures.

Further relations between the set of ternary trees with n internal nodes and 2-binary trees satisfying certain conditions, noncrossing trees with $n + 1$ vertices, Schröder paths, Motzkin paths and Dyck paths were done in (Gu et al., 2008). These connections not only enriched our understanding of 2-plane trees but also provided new insights into the interplay between different combinatorial objects.

Later, Lumumba et.al. did a comprehensive study on the enumeration of 2-plane trees, which they defined as plane trees where vertices are colored either white or black, with the restriction that no two black vertices can be adjacent (Lumumba, Okoth, & Kasyoki, 2024). The authors focused on categorizing these trees based on the color of the first child of the root, the degree of the root, and the level of vertices of a given color. They applied concepts such as Lagrange inversion formula, decomposition of trees so as to use Rothe-Hagen identity and the butterfly decomposition and found key results such as the derivation of formulas for counting 2-plane trees with a white root, depending on whether the first child is black or white, as well as establishing recurrence relations for sequences of paths colored under specific conditions. The authors also discussed the implications of these findings in the context of combinatorial structures and their connections to the Catalan numbers, providing a refined enumeration that enhanced the understanding of 2-plane trees and their properties.

Gu, Prodinger and Wagner generalized the set of 2-plane trees in (Gu et al., 2010). In this paper, they investigated plane trees whose vertices are labeled from the set $\{1, 2, \dots, k\}$ with the constraint that the sum of the labels along any edge does not exceed $k + 1$, which they referred to as k -plane trees. These authors found that the number of k -plane trees on n vertices with the root labelled i is

$$\frac{k - i + 1}{kn - i + 1} \binom{(k + 1)n - i - 1}{n - 1}.$$

When $k = 2$, then the set of 2-plane trees on n vertices with root labelled i is

enumerated by

$$\frac{2-i+1}{2n-i+1} \binom{3n-i-1}{n-1}.$$

This result generalizes the Catalan numbers, as the special case when $i = k$ such trees are enumerated by,

$$\frac{1}{k(n-1)+1} \binom{(k+1)(n-1)}{n-1}.$$

which relates to the enumeration of $(k+1)$ -ary trees with $n-1$ internal vertices and lattice paths consisting of $n-1$ up-steps of size k and $k(n-1)$ down-steps of size 1.

Additionally, the authors established bijections between the class of labeled plane trees and $(k+1)$ -ary trees, as well as generalized Dyck paths, extending classic results that connect binary trees to plane trees and Dyck paths (Gu et al., 2010). These bijections illustrate the interconnectedness of these combinatorial structures and provide a framework for further exploration of new counting problems by modifying the definitions of k -plane trees. The findings not only enhance the understanding of labeled plane trees but also open avenues for future research in combinatorial enumeration.

Since then, there has been substantial recent research on labelled plane trees. For example; Mlotkowski and Penson (Mlotkowski & Penson, 2013) analyzed the probability measure corresponding to 2-plane trees in 2013. They examined sequences related to 2-plane trees, particularly focusing on the Fuss-Catalan numbers of order 3, which are linked to the number of 2-plane trees on $n+1$ vertices. The authors investigated the moment sequence corresponding to this measure and proved that the associated probability measure μ_0 is absolutely continuous, providing an explicit density function for it. Additionally, they demonstrate that μ_0 is infinitely divisible with respect to the additive free convolution, connecting it to the Marchenko-Pastur distribution. The study utilized advanced mathematical tools, including the beta

distribution, Meijer G -function, and concepts from free probability theory, to uncover the properties of this measure and its relevance in combinatorics and probability theory.

The concept of natural correspondence, introduced by Donald Knuth in "The Art of Computer Programming," revolutionized the representation of forest structures in computer science and combinatorics (Knuth, 1997). Knuth's approach provided a method to represent any forest as a binary tree, bridging the gap between these fundamental data structures. In his formulation, Knuth considered a forest of two trees and applied a series of transformations. He first linked together the children of each family within the forest, then removed all edges connecting children to their parents, except for the leftmost edge. This was followed by a 45° anti-clockwise rotation of the entire structure. The result was a rooted binary tree with no right subtree, effectively transmuting the forest while preserving its essential information.

The elegance of Knuth's natural correspondence lies not only in its ability to transform forests into binary trees but also in its extensibility. An extension of this basic correspondence yielded a one-to-one relationship between trees with $n + 1$ vertices and binary trees with n vertices. This bijection established a deep connection between two seemingly distinct classes of trees. The implications of this correspondence were far-reaching, providing new ways to store and manipulate tree-like data in algorithms, and allowing results from binary tree theory to be applied to more general tree structures. Knuth's work sparked further research into tree representations and transformations, finding applications in areas such as parse tree manipulation, hierarchical clustering algorithms, and even in certain problems in computational biology.

Flajolet and Sedgewick further developed the rotation correspondence, using it to establish bijections between certain tree structures and cyclic permutations (Flajolet & Sedgewick, 2009). The correspondence worked by mapping a rooted tree to a sequence of vertex labels or edge descriptions, which were then cyclically rotated to

generate equivalent representations. This approach led to significant advancements in the analysis of various tree families, the study of properties of random trees, and the derivation of precise asymptotic results for tree-based algorithms. The rotation correspondence yielded notable results, including efficient algorithms for random tree generation, exact and asymptotic enumeration of various tree classes, and detailed analysis of parameters like path length in trees. These developments significantly advanced the field of analytic combinatorics and provided a systematic approach to solving previously intractable problems in combinatorics and algorithm analysis.

In (Okoth, 2022a), Okoth presented a detailed enumeration of k -plane trees based on the number of vertices labeled by different integers. A significant result was the formula derived for counting k -plane trees with a specific labeling of the root and other vertices, which also connects to the enumeration of locally oriented noncrossing trees. The paper established a bijection between k -plane trees and locally oriented $(k - 1)$ -noncrossing trees, demonstrating the intricate relationships between these structures and providing a foundation for further combinatorial analysis.

In addition to the enumeration of k -plane trees, the paper discussed the construction of bijections with Dyck paths of length $2(n - 1)$ such that the up-steps in the Dyck paths are given labels which satisfy a certain property, i.e., these Dyck paths consist of unit $(n - 1)$ up-steps labelled with integers from the set $\{1, 2, \dots, k\}$ and unit $(n - 1)$ down-steps, starting at the point $(0, 0)$ and ending at $(2n - 2, 0)$ where any two consecutive labels i and j of up-steps satisfies the condition $i + j \leq k + 1$. In [(Okoth, 2022a, Theorem 3)], the number of these Dyck paths is given by the formula

$$\frac{1}{n} \binom{(k+1)n-2}{n-1}.$$

Dyck paths are lattice paths that have applications in various areas of mathematics, including combinatorics and algebra. The author outlined a method for translating k -plane trees into Dyck paths, adhering to specific coherence conditions that

govern the labeling of steps in the path. This connection not only enhanced the understanding of k -plane trees but also highlighted their relevance in broader mathematical contexts, showcasing the interplay between different combinatorial objects and their applications in theoretical and practical scenarios.

Okoth and Wagner further refined the formula for k -plane trees in (Okoth & Wagner, 2024) allowing for the enumeration of these trees based on their label distributions. The main result was the establishment of a system of functional equations that could be solved explicitly, leading to a formula derived from the Lagrange-Bürmann formula. i.e. k -plane trees on n vertices were enumerated by

$$\frac{k}{n} \binom{(k+1)(n-1)}{n-1}.$$

This formula not only counts the total number of k -plane trees but also provided insights into the average number of occurrences of specific labels within these trees.

One of the significant contributions of the paper was the connection between k -plane trees and lattice paths. The authors demonstrated that k -plane trees can be bijectively related to lattice paths characterized by up-steps and down-steps. This relationship allowed for a deeper understanding of the combinatorial properties of k -plane trees and facilitated the enumeration process. The paper also explored the concept of k -noncrossing trees, which were defined as noncrossing trees whose vertices were labelled from the set $\{1, 2, \dots, k\}$ such that the sum of any adjacent vertices is not more than $k+1$. The authors drew parallels between k -plane trees and k -noncrossing trees, highlighting the similarities in their enumeration and structural properties.

In addition to the main results, the paper included various corollaries derived from the primary formula, particularly concerning the average number of occurrences of specific labels in k -plane trees. The authors also provided detailed proofs and examples to illustrate their findings, making the results accessible and applicable

to further research in combinatorial enumeration. Overall, the study enhanced the understanding of k -plane trees and their relationships with related combinatorial structures, paving the way for future explorations in this area of mathematics.

The set of 2-binary trees was introduced and enumerated by Gu, Li and Mansour in (Gu et al., 2008). Their paper discussed the concept of 2-binary trees, which are binary rooted trees where the root is colored black and other vertices can be either black or white. It highlighted the importance of trees in combinatorics and their applications across various fields such as mathematics, physics, and biology. The authors defined a 2-binary tree and explore its properties, including the enumeration of such trees based on their structure. They also introduced the concept of hybrid binary trees, a special class of 2-binary trees, and provided formulas for counting these trees. The paper emphasized the role of preorder traversal in analyzing the structure of these trees.

Additionally, the authors established bijections between 2-binary trees and related combinatorial structures, such as ternary trees, noncrossing trees, and Schröder paths. They specifically focused on 2-binary trees satisfying certain conditions and their correspondence to little Schröder paths, detailing the construction of these paths through a preorder traversal of the trees. The paper presented various enumeration results and statistical properties of 2-binary trees, showcasing their relationships with other combinatorial entities. Overall, the study contributed to the understanding of 2-binary trees and their significance in combinatorial mathematics.

Parallel work has examined classes of trees related to plane trees, including non-crossing trees. Noy showed that the number of noncrossing trees with n edges is enumerated by the formula of generalized Catalan numbers (Noy, 1998). Philippe Flajolet and Marc Noy later investigated various non-crossing configurations using analytic combinatorial techniques, establishing that distributions of parameters like the number of leaves in trees, edges in connected graphs, and components in forests exhibit asymptotic normality (Flajolet & Noy, 1999). They derived precise asymp-

otic formulas using singularity analysis of generating functions, which were shown to have unique dominant singularities of the square-root type.

The authors provided explicit formulas for the number of connected graphs and the distribution of components in graphs, closely related to classical combinatorial structures such as the Catalan numbers. Their methods offered a more algebraic perspective compared to earlier approaches that combined algebraic and differential techniques. They derived the asymptotic behaviour of the number of noncrossing partitions, expressed in terms of generating functions, and presented a generating function $P(z, y) = 0$ that showed relationships among various configurations, leading to mean and variance estimates for parameters of interest.

The butterfly decomposition, introduced in "Analytic combinatorics of non-crossing configurations," (Flajolet & Noy, 1999), was utilized to systematically analyze non-crossing graphs and related structures. This approach allowed the authors to derive generating functions capturing the essence of the configurations, which were then subjected to singularity analysis. This method revealed deeper combinatorial properties of non-crossing configurations, enabling the establishment of precise asymptotic estimates and limit laws for various parameters. The butterfly decomposition served as a crucial tool in the analytic framework, providing a clearer understanding of relationships between different types of non-crossing configurations, and has since been used in various other enumerations of noncrossing trees and their generalizations as in (Okoth, 2021a; Okoth & Wagner, 2015; Okoth, 2022b).

The paper by Hough, (Hough, 2003), explored the combinatorial properties of non-crossing trees, which are trees drawn on points arranged in a circle such that their edges do not intersect. A key focus of this paper was on the concept of descents, defined as edges that connect nodes with higher labels to those with lower labels along a path from the root. The author derived a generating function for descents in these noncrossing trees and demonstrated that this function, when evaluated at 2, corresponds to the generating function for connected noncrossing graphs. This

established a significant link between the structure of noncrossing trees and the broader category of noncrossing graphs.

A pivotal result presented in the paper was the bijection between noncrossing trees and connected noncrossing graphs, which proved how descents in trees could be transformed into additional edges in graphs. Specifically, for each descent in a noncrossing tree, a new edge was created that maintained the noncrossing property, thereby forming a connected noncrossing graph. The paper also raised several open questions regarding the relationships between trees with varying numbers of descents and related combinatorial structures, indicating potential avenues for further research in this area. Overall, the findings contributed to a deeper understanding of the combinatorial dynamics of noncrossing configurations.

In 2009, Yan and Liu (Yan & Liu, 2009) generalized noncrossing trees. They introduced 2-noncrossing tree and defined them as rooted trees where no two edges cross when drawn in a certain way, and could have vertices labeled as either black or white. The authors established a bijection between 2-noncrossing trees with a black root and 5-ary trees, which are rooted plane trees where each vertex has either 0 or 5 children. This bijection was significant as it allowed for the translation of properties and counts between these two types of trees, facilitating a deeper understanding of their combinatorial structures.

The authors presented several corollaries that highlighted important relationships, such as the equivalence between the number of children of the root in a 2-noncrossing tree and the number of internal vertices on the longest rightmost path in the corresponding 5-ary tree. Additionally, they discussed the implications of these relationships for counting specific types of trees, including 2-plane trees and ternary trees, thereby contributing to the broader field of combinatorial mathematics. They proved that the set of 2-noncrossing trees with root labelled 2 is enumerated by

$$\frac{1}{5n-4} \binom{5n-4}{n-1}.$$

This formula also enumerated the set of 5-ary trees (Yan & Liu, 2009). A bijection between the set of these two combinatorial objects was constructed by the said author. When the root is labelled 1, the formula for such trees with n vertices is

$$\frac{2}{5n-3} \binom{5n-3}{n-1}.$$

Panholzer and Prodinger introduced a prominent approach in the enumeration of noncrossing trees which is by first obtaining the (l, r) -representation of these trees (Panholzer & Prodinger, 2002). The (l, r) -representation was introduced as a specific visualization technique used for noncrossing trees, which are a unique class of combinatorial structures. In this representation, each vertex in the tree was assigned a label of either ' l ' or ' r ', indicating its position and relationship within the overall structure. This labeling was crucial as it helped maintain the properties of the tree, particularly the noncrossing condition, which is a defining characteristic of these trees. In the (l, r) -representation, the vertices were categorized based on their labels. This method not only clarified the relationships between the vertices but also facilitated the analysis of the tree's properties. A bijection between the set of noncrossing trees on n vertices and the set of ternary trees with n internal nodes was later created in (Panholzer & Prodinger, 2002).

In (Panholzer & Prodinger, 2009), Panholzer and Prodinger further studied the set of d -ary particularly focusing on vertices colored in a specific manner. The authors explored the relationships between labelled and unlabelled trees, providing both generating function proofs and direct bijective methods to establish these connections. The study was structured around two main theorems, one addressing the generating functions for non-empty d -ary trees and the other focusing on the bijective proof of the relationships between different tree structures. The methodology employed in the paper included the use of generating functions to derive functional equations that characterize the number of trees in various configurations. For instance, the authors derived a functional equation for the generating function of non-empty d -ary

trees, which was expressed as

$$T_d(z) = z(1 + T_d(z))^d.$$

This equation was foundational for enumerating the trees and lead to further results regarding the number of trees with specific properties. For instance, they found that unlabelled d -ary trees with n edges are enumerated by the generalized Catalan number,

$$\frac{1}{(d-1)n+1} \binom{dn}{n}.$$

Additionally, the authors utilized a recursive approach to establish bijections, where they considered the structure of trees and systematically broke down the components into smaller subtrees. This recursive splitting allowed for a clear mapping between labelled and unlabelled trees, ultimately demonstrating the equivalence of their enumerative properties. The paper concluded with corollaries that extended the results to broader classes of trees. The authors also found that the number of d -ary trees on n vertices whose vertices are labelled from the set $\{1, 2, \dots, k\}$ such that the label of the d^{th} child from the left of any internal vertex is either greater than or equal to that of the parent is given by

$$\frac{1}{k(d-1)n+1} \binom{(k(d-1)+1)n}{n}.$$

From this formula, when the labelling constraint is retained and $d = 2$, then k -binary trees are obtained. The formula for the number of k -binary trees on n vertices with root labelled k is

$$\frac{1}{kn+1} \binom{(k+1)n}{n}.$$

(Panholzer & Prodinger, 2009)

Pang and Lv found this formula to count weakly labelled k -trees in (Pang & Lv, 2010). In this paper, they discussed two classes of k -labeled trees: k -noncrossing trees and k -proper trees, both of which are counted by k -Catalan numbers. A k -labeled tree was defined as a tree where vertices, except for the root, are labeled from the set $\{1, 2, \dots, k\}$. The k -noncrossing tree was characterized by the condition that the children of the root are all labeled by a fixed integer, while the k -proper tree required that adjacent vertices have distinct labels. The authors established generating functions for both types of trees, providing a combinatorial framework for their enumeration.

The main results of the paper included theorems that enumerated k -noncrossing trees and k -proper trees using generating functions. Specifically, [(Pang & Lv, 2010, Theorem 2.1)] states that the k -noncrossing trees on n vertices, with children of the root labeled by a specific integer, are counted by the $(k + 1)$ -Catalan number of order $n - 1$, i.e.

$$\frac{1}{n} \binom{kn - 2}{n - 1}.$$

[(Pang & Lv, 2010, Theorem 2.2)] asserts that the number of k -proper trees on n vertices is equal to the k -Catalan number of order n . Additionally, the paper presented bijections between k -noncrossing trees and k -ary trees, as well as between k -proper trees and k -ary trees, highlighting the structural relationships among these tree types. The authors emphasized the potential applications of k -noncrossing trees and k -proper trees in computer and information science, suggesting that these structures can serve as efficient data representations.

Various bijections between the set of noncrossing trees and related combinatorial structures have similarly been obtained. Noncrossing trees and complete ternary trees are also known to be in bijection with lattice paths (consisting of unit vertical and unit horizontal steps) which lie weakly below the line $y = 2x$. These lattice paths were introduced by Yang and Jiang in (S.-L. Yang & Jiang, 2021). Other

bijections were studied by Dulucq and Penaud in (Dulucq & Penaud, 1993).

In the paper "On two subclasses of Motzkin paths and their relation to ternary trees," (Prodinger et al., 2020), the authors focused significantly on establishing bijections between S -Motzkin paths, T -Motzkin paths, and ternary trees. They first defined S -Motzkin paths as specific sequences of steps that adhere to certain conditions, and they demonstrated a bijection between the set of S -Motzkin paths of length $3n$ and the set of ternary trees on n vertices. This bijection was constructed by mapping the structure of S -Motzkin paths to the branching patterns of ternary trees, allowing for a clear correspondence between the two combinatorial objects. The authors provided a canonical decomposition method that facilitated this mapping, ensuring that each S -Motzkin path could be uniquely represented as a ternary tree.

The authors also explored bijections between S -Motzkin paths and noncrossing trees. They established a systematic method to convert an S -Motzkin path into a noncrossing tree by analyzing the structure of the path and its components. Each S -Motzkin path was broken down into "pieces," which were maximal subpaths that followed a specific pattern of steps. The characteristics of these pieces were then used to construct the corresponding noncrossing tree, where the number of children and the arrangement of subtrees were determined by the properties of the S -Motzkin path. The bijection was achieved by associating each piece of the S -Motzkin path with a node in the noncrossing tree, ensuring that the resulting tree maintained the noncrossing property. The authors provided detailed explanations and examples to illustrate this mapping, demonstrating how the combinatorial structures of S -Motzkin paths and noncrossing trees are intricately linked. In this study, we have extended the linkage between the S -Motzkin paths and related combinatorial structures.

Furthermore, the same paper, (Prodinger et al., 2020) introduced T -Motzkin paths and established a bijection between these paths and ordered pairs of S -Motzkin paths. The authors detailed a process for decomposing a T -Motzkin path into

two S -Motzkin paths, highlighting how the rightmost step of the T -Motzkin path plays a crucial role in this decomposition. This relationship not only enriched the understanding of T -Motzkin paths but also emphasized the interconnectedness of these combinatorial structures.

Okoth in (Okoth, 2021c) further refined the enumeration of 2-noncrossing trees by considering them with a given number of vertices labelled 1 or 2. He found the number of 2-noncrossing trees on n vertices with root labelled 2 and k vertices labelled 2 is

$$\frac{1}{n-1} \binom{2n-2}{k-1} \binom{3n-3}{n-k-1}.$$

Nyariaro and Okoth, in (Nyariaro & Okoth, 2024), delved into the combinatorial structures of labeled trees, focusing on various classes such as k -plane trees and k -binary trees, an extension of work done by Panholzer and Prodinger in (Panholzer & Prodinger, 2009). The authors presented various techniques for constructing labeled trees, including relabeling procedures that transform one type of tree into another while preserving certain properties. These transformations were crucial for establishing bijections and understanding the combinatorial properties of labeled trees. They explored the concept of bijections, which are one-to-one correspondences between different combinatorial objects, to facilitate the enumeration of these trees. By establishing these bijections, the authors aimed to simplify the counting process and provide deeper insights into the relationships between different classes of trees.

One of the key highlights of the paper is the discussion on noncrossing trees and their representation as plane trees. Additionally, the paper emphasized the role of Catalan numbers in counting plane trees, showcasing their significance in combinatorial mathematics. The authors referenced the work of Panholzer and Prodinger, who introduced a method for representing noncrossing trees based on the relative values of parent and child vertices. This representation helped in understanding

the structure of these trees and allowed for the development of bijective proofs that connect various tree classes. For instance, they studied the bijections between 2-binary trees with a root labeled by 2 and noncrossing trees on a different number of vertices and between the set of noncrossing trees on $n + 1$ vertices and the set of k -binary trees with root labelled 2. They also constructed a bijection between the set of k -binary trees with root labelled k and the set of weakly labelled k -trees and showed that weakly labelled k -trees on $n + 1$ vertices are enumerated by

$$\frac{1}{k(n-1)+1} \binom{(k+1)(n-1)}{n-1}.$$

When $k = 2$, they obtained the set of weakly labelled 2-trees on $n + 1$ vertices which is enumerated by the same formula enumerating the set of S -Motzkin paths of length n and 2-plane trees among many related combinatorial structures as i.e. the generalised Catalan numbers formula,

$$\frac{1}{2n+1} \binom{3n}{n}$$

listed in A001764 of (Sloane et al., 2018).

enumerated k -noncrossing trees and k -proper trees using generating functions. Specifically, [(Pang & Lv, 2010, Theorem 2.1)] Their results were later refined by Okoth et.al. (Okoth & Wagner, 2024) to take into consideration the label of the root i.e. the set of k -noncrossing trees on n vertices with a root labelled i is enumerated by

$$\frac{k+1-i}{2kn-k-i+1} \binom{(2k+1)n-k-i-1}{n-1}.$$

Okoth and Wagner also obtained a formula for the number of k -noncrossing trees with a given number of vertices of a certain label, generalizing a previous result by Okoth (Okoth & Wagner, 2024).

In (Okoth, 2022b), Okoth found a formula for the number of forests of k -noncrossing

trees and number of k -noncrossing trees with root labelled k such that the root is of degree d . The author also created a bijection between the set of k -noncrossing trees of order n with root labelled k and the set of complete $(2k + 1)$ -ary trees on $n - 1$ internal vertices.

Noncrossing increasing trees were introduced in (Asinowski & Mansour, 2008). Okoth extended the research into 2-noncrossing increasing trees in (Okoth, 2021a). He enumerated them by label of the vertices and root degree. One of the main results found in this paper is that the set of 2-noncrossing increasing trees on n vertices whose root is labelled 2 such that there are k vertices labelled 2 is given by the Catalan-like numbers,

$$\frac{1}{n-1} \binom{2n-2}{n-k-1} \binom{n-1}{k-1}.$$

The same author also created bijections with the sets of restricted lattice paths, Dyck paths, locally oriented noncrossing trees and 2-plane trees. Okoth and Wagner in (Okoth & Wagner, 2015) introduced the set of locally oriented noncrossing trees which are noncrossing trees whose edges are oriented from a vertex of a lower label to that of a larger label. The authors of the paper enumerated them by number of sources and sinks and showed that the number of locally oriented noncrossing trees on n vertices with k sources are ℓ sinks is (Okoth & Wagner, 2015)

$$\frac{1}{n-1} \binom{n-1}{k-1} \binom{n-1}{\ell-1} \binom{n-1}{k+\ell-1}.$$

The authors obtained a formula for these trees with a given outdegree and indegree sequences. They also created a bijection between the set of locally oriented noncrossing trees on n vertices with k sources and ℓ sinks and the set of ternary trees on $n - 1$ vertices such that there are $k - 1$ left-edges and $\ell - 1$ middle-edges. Their bijection was a modification of the bijection between noncrossing trees with $n - 1$ edges and ternary trees on $n - 1$ vertices that was obtained independently by Simion

and Postnikov as stated by Stanley in (Stanley, 1999, Solution 5.46).

2.3 Tree-like structures

In this section, we review available literature on three tree-like combinatorial structures, i.e. Husimi graphs, cacti, and oriented cacti, with a particular focus on Husimi structures.

Husimi graphs, named after the Japanese physicist Kōji Husimi, were first introduced and enumerated in his seminal work in 1950 (Husimi, 1950). These structures have since become an essential topic in graph theory and combinatorics, with applications ranging from statistical physics to theoretical chemistry. A Husimi graph, also known as a cactus graph, is a connected graph in which every edge belongs to at most one simple cycle. This unique property makes Husimi graphs a fascinating intermediate structure between trees (which have no cycles) and general graphs (which can have multiple overlapping cycles).

The study of Husimi graphs has been extensive since their introduction. In 1953, Harary and Norman, (Harary & Uhlenbeck, 1953), provided a detailed analysis of these structures, offering insights into their properties and enumerations. They investigated the combinatorial properties of Husimi trees, which are connected graphs characterized by the absence of multiple cycles along any line. The authors defined pure Husimi trees as those composed of a single type of polygon, while mixed Husimi trees consisted of various polygon types. The study employed combinatorial techniques, particularly Pólya's enumeration theorem, to derive functional equations that counted the number of these trees based on their structural components, such as lines, triangles, and quadrilaterals. The results highlighted the relationship between the number of main leaves and the overall structure of the trees, leading to a deeper understanding of their combinatorial nature.

The paper also discussed the symmetry properties of pure Husimi trees, particularly

those made up of quadrilaterals, and presented equations that accounted for the different symmetry types within these structures. The authors provided a framework for extending their findings to more complex configurations, including trees with mixed polygon types. Their work laid the groundwork for future research into the asymptotic behaviour and generalizations of Husimi trees, contributing to the broader field of combinatorial mathematics and its applications in statistical mechanics and graph theory.

One of the key aspects of Husimi graphs is their block structure. A block in a Husimi graph is either a single edge or a simple cycle. This block decomposition allows for a recursive approach to studying these graphs, which has been proven invaluable in their enumeration and analysis. Leroux et al. made significant contributions to this field in their 1992 paper, where they enumerated both labelled and unlabelled Husimi graphs, cacti, and oriented cacti (Leroux, 2004). Their work provided a comprehensive framework for understanding these structures. They found the labelled Husimi graphs on $[n]$ for $n \geq 1$ and k blocks to be enumerated by

$$\sum_{k \geq 0} S(n-1, k) n^{k-1}.$$

in [(Leroux, 2004, Proposition 3.1)], where $S(n, k)$ is the Stirling number of the second kind. This formula elegantly captured the relationship between the number of vertices, blocks, and possible configurations of Husimi graphs.

The connection between Husimi graphs and related combinatorial structures has been a fruitful area of research. For instance, the relationship between Husimi graphs and rooted trees has been explored, revealing interesting bijections and structural similarities. In recent years, colored Husimi graphs have gained attention. These are Husimi graphs where the blocks (cycles or edges) are assigned colors from a given set. The enumeration of colored Husimi graphs by block-color distributions has opened up new avenues for research and applications.

Okoth's PhD thesis in 2021 provided a comprehensive study of various tree-like structures, including Husimi graphs. The thesis explored the interconnectedness of labelled trees with the symmetric group and vertex-degree distributions. It also established connections between cycle-free connected families of set partitions and free cacti with specific vertex-degree distributions. One of the significant contributions of Okoth's work was the study of noncrossing Husimi graphs. These are Husimi graphs that can be drawn in the plane such that no edges intersect except at their endpoints. The enumeration of noncrossing Husimi graphs by block size, vertex degree, and block colors was achieved through the application of the butterfly decomposition technique. One of the results obtained was on the generalized Narayana number;

Proposition 2.1 ((Okoth, 2015, Corollary 6.2.2)). *The number of noncrossing Husimi graphs on $n \geq 2$ vertices is*

$$\frac{1}{n-1} \sum_k^{n-1} \binom{2n+k-2}{k-1} \binom{n-1}{k}$$

This formula not only provided a precise count of these structures but also revealed deep connections to other combinatorial objects, such as noncrossing partitions and certain types of lattice paths.

In (Okoth, 2021b), Okoth explored the enumeration of related combinatorial structures, particularly focusing on noncrossing Husimi graphs, plane cacti, and their bicoloured variants. The research employed generating functions and bijections to derive significant results regarding these structures. The author discussed the properties of Husimi graphs, which are connected graphs composed of complete subgraphs known as blocks. By establishing relationships between different types of tree-like structures, including cacti and oriented cacti, the paper provided a comprehensive overview of how these graphs can be counted based on their block sizes and configurations. The findings highlighted the intricate connections between these structures

and their combinatorial properties, offering insights into their enumeration.

Furthermore, the study delved into the concept of bicoloured noncrossing structures, presenting formulas that counted these graphs under specific conditions. The author emphasized the importance of vertex degrees and the arrangement of blocks in determining the overall structure of the graphs. By extending the results to include bicoloured noncrossing trees and cacti, the paper showcased the versatility of the methods used in the analysis. The research not only contributed to the theoretical understanding of these combinatorial structures but also opened avenues for further exploration in the field of graph theory.

The concept of plane Husimi graphs, introduced by Okoth in subsequent work, further expanded the field (Okoth, 2021b). In addition to the exploration of noncrossing and bicoloured structures, the paper provided a detailed examination of plane Husimi graphs, which are a specific type of graph characterized by their planar representation and ordered blocks.

Plane Husimi graphs are drawn in such a way that their blocks do not intersect, maintaining a clear and organized structure that allows for easier visualization and analysis. The author discussed how the properties of these graphs relate to the number of leaves and blocks, revealing important combinatorial relationships. By employing generating functions, the paper derived formulas that enumerate plane Husimi graphs based on the number of vertices and the configuration of their blocks. This analysis not only enhanced the understanding of plane structures but also illustrates the broader implications of these findings in the study of graph theory, particularly in how they relate to other tree-like structures. The results underscore the significance of plane Husimi graphs in combinatorial mathematics, providing a foundation for future research in this area.

Overall, Okoth's work stands as a significant contribution to the enumeration of noncrossing and plane tree-like structures, enriching the existing literature with

rigorous mathematical formulations and proofs. The enumeration of plane Husimi graphs according to the number of vertices, number of blocks, block types, and number of leaves provided new insights into the combinatorial properties of these structures. A particularly interesting result is the formula for the number of plane Husimi graphs on $n + 1$ vertices with k blocks:

$$\frac{1}{n+1} \binom{n+k}{k} \binom{n-1}{k-1}.$$

This formula showcased the intricate relationship between the number of vertices and blocks in plane Husimi graphs, and has led to further investigations into their structural properties. The study of Husimi graphs has also extended to their applications in various scientific fields. In statistical physics, Husimi graphs have been used to model complex systems with localized cycles. In theoretical chemistry, these structures have found applications in modeling certain types of molecular structures. Recent research has also explored the connection between Husimi graphs and random graph models. The study of random Husimi graphs has provided insights into the average properties of these structures, such as expected block sizes and distribution of cycles.

The concept of d -ary Husimi graphs, a generalization of plane Husimi graphs, has opened up new avenues for research. These structures, where each vertex can have up to d children, provide a bridge between Husimi graphs and d -ary trees and were discussed in (Onyango et al., 2023). The enumeration of d -ary Husimi graphs by number of blocks, block sizes, outdegree sequences, and outdegree of a given vertex revealed fascinating combinatorial patterns. One of the main results obtained from this paper was the number of d -ary Husimi graphs on n vertices and k blocks which was given by

$$\frac{1}{n} \binom{dn}{k} \binom{n-2}{k-1}.$$

In conclusion, the study of Husimi graphs and related structures continues to be

an active area of research in combinatorics. From their initial introduction in statistical physics to their current applications in various fields of mathematics and computer science, Husimi graphs have proven to be a rich source of combinatorial insights. The ongoing research in this area promises to uncover further connections and applications of these fascinating structures. Particularly, this study has focused on extending the research on plane Husimi graphs introduced in (Onyango et al., 2023) by creating various bijections with various related combinatorial structures enumerated by the same formula.

2.4 Lattice paths

Lattice path enumeration can be traced back to the Bertrand's classic ballot problem (Takács, 1962) where he posed a question on an election between two candidates A and B , enquiring the probability that candidate A would always be ahead of candidate B at each stage of counting.

This problem was solved combinatorially by sketching its lattice paths (consisting horizontal and vertical steps) in the $x - y$ plane. In this problem, they took a case of two candidates A and B competing during an election and enquired the probability of a candidate winning if the number of votes of A were m and those of B were n letting each vote of A be represented by a unit horizontal line and each vote of B be represented by a unit vertical line. The lattice path for this problem started at the origin $(0, 0)$, lying on or below the line $x = y$ and end at the point (m, n) . This representation gave a lattice path with the step set $\{(1, 0), (0, 1)\}$. Consider a case where candidate A got 6 votes and candidate B got 4 votes. One way this could be represented on a lattice path is as in figure 2.4.

Following the solution of this problem, a lot of attention has been put into the study of lattice paths under various restrictions (Mohanty, 2014) and the relationship of lattice paths to combinatorial structures which are counted by various formulas. Such structures include plane trees, k -plane trees, d -ary trees, Dyck paths, Motzkin

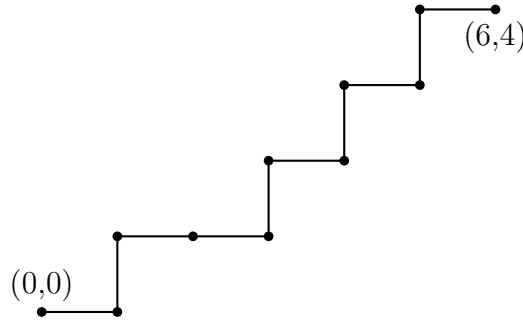


Figure 2.1: Path starts from (0,0) to (6,4)

paths and Deutsch paths among others. There are various lattice paths which we cannot exhaustively handle in this document, so in the sequel we only mention those relevant to our study.

In 1999, combinatorialist Emeric Deutsch (Deutsch, 1999) enumerated Dyck paths by means of generating functions. The statistics of his enumeration were path lengths, number of peaks and sum of peak heights in which the structures are counted by Catalan, Narayana and Fine numbers respectively. For example, in (Deutsch, 1999), Deutsch enumerated the set of Dyck paths by solving their generating function

$$D(x) = 1 + x^2 D(x)^2$$

and applying the Lagrange Inversion Formula upon setting $D(x) = \frac{F(x)}{x}$. He also found that the number of Dyck paths of semi length n with no returns of even length is given by the Catalan number.

When horizontal steps are introduced to Dyck paths, Motzkin paths are created. The generating function of the Motzkin paths is given as

$$M(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

In (Deutsch, 1999), the author explained the extension of Dyck paths to Motzkin paths and gave the explicit counting formula for the number of Motzkin paths.

A class of restricted Dyck paths whose peaks are only allowed on level 1 and on even-numbered levels was introduced in (Ekhad & Zeilberger, 2020). It was proved in the paper that these restricted paths are enumerated by Motzkin numbers and their generating function is $M = 1 + zM + z^2M^2$ where z marks a step. In (Prodinger, 2021c), the average height of these restricted Dyck paths and their relation to Motzkin paths was investigated. The authors computed the average height using singularity analysis of the generating functions and their associated plane trees. He also studied the amplitude of Motzkin paths, which is basically twice the height, i.e. if there exists a horizontal step on level h , which is the height, the amplitude is $2h + 1$, otherwise it is $2h$ (Prodinger, 2021c). In this paper, the author showed that the average amplitude is about twice the average height and there are about as many Motzkin paths with/without horizontal steps on the maximal level.

Deutsch paths were discussed in (Prodinger, 2021a). They are like Dyck paths in which extra down-steps of the form $(1, -j)$, for $j = 2, 3, 4, \dots$, are also allowed. The up-steps and down-steps are not symmetrical, which allows investigation of the paths from right to left. In the paper, a temporary upper boundary h is introduced where the paths are considered to be on a strip. Introducing the strips makes it possible for the generating functions to be described by a system of linear equations solvable by Cramer's rule. The generating functions for bounded Deutsch paths with open ends is the same as that of Motzkin paths. It was showed that Deutsch paths are generated by a transpose of the matrix used to generate Motzkin paths. The average height of both open and closed Deutsch paths was found to be the same value $2\sqrt{\frac{\pi n}{3}}$, around twice that of Motzkin paths (Prodinger, 2021c). A bijection between the set of open Deutsch paths and the set of Motzkin paths of the same length was also investigated in (Prodinger, 2021a).

A variation of Deutsch paths with down-steps of length $(1, 2, 3, \dots)$ was studied by (Prodinger, 2021b). In the said paper, the Deutsch paths were enumerated by the method of adding a new slice combined with the kernel method. The Deutsch paths

were decomposed into paths of maximal runs of up and down steps under various restrictions such as; paths ending with down-steps, paths ending with up-steps, each maximal run of down-steps to the x -axis starting at an odd level and by counting the number of maximal sequences of up-steps consisting of only one up-step in the Deutsch path.

The concept of skewness was first introduced to lattice paths by Deutsch and his co-authors by coming up with the set of skew Dyck paths in (Deutsch et al., 2010a). The authors used symbolic method to enumerate skew Dyck paths and found them to be enumerated by numbers closely related to Catalan numbers, i.e., the number S_n of skew Dyck paths of half-length n is

$$S_n = \sum_{k=1}^n \binom{n-1}{k-1} C_k$$

where C_k is the k^{th} Catalan number. These numbers are given in sequence A002212 of the online encyclopedia of integer sequences, OEIS, (Sloane et al., 2018). Enumerative results on skew Dyck paths with respect to length and the type of last step of the path, their bijections with various combinatorial objects and to super diagonal bar graphs were obtained in (Deutsch, Munarini, & Rinaldi, 2010b). In 2021, Baril et.al. enumerated skew Dyck paths according to the type of prefixes. He defined the prefixes as skew meanders $S(n, k)$, where n was the length of the path and k the y -coordinate of the last step. They discussed the recursive relations of the skew meanders, their generating functions and closed formulas with respect to the height and the number of left-steps (Baril, Ramirez, & Simbaqueba, 2021).

Skew Motzkin paths were introduced by Lu in the paper (Lu, 2017) as a generalization of skew Dyck paths. Moreover, the paper recognized skew Motzkin paths as a generalisation of Motzkin paths instead of a special category of Motzkin paths since a skew Motzkin path without the $(-1, -1)$ step is just a Motzkin path and one without the $(1, 0)$ step is a skew Dyck path. The sequence counting skew Motzkin paths is recorded as A086581 in OEIS (Sloane et al., 2018). In the same paper,

(Lu, 2017), the author derived a counting formula, a recurrence and a convolution formula that enumerates skew Motzkin paths according to path length and number of up steps. He found the number of skew Motzkin paths of length n as

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{(-1)^i}{n+1-i} \binom{n+1-i}{i} \binom{2n-3i+2}{n-i}.$$

Several other studies on Motzkin paths have also been done. The authors of (Prodinger et al., 2020) studied two subclasses of Motzkin paths and their bijections with other combinatorial objects. They introduced the set of S -Motzkin paths and proved that the number of these paths of length $3n$ is the generalized Catalan number,

$$\frac{1}{2n+1} \binom{3n}{n}.$$

The authors also introduced the set of T -Motzkin paths and showed that

$$\frac{1}{n+1} \binom{3n+1}{n}$$

is the formula for these paths of length $3n$. In addition, bijections between the set of S -Motzkin paths of length $3n$ and the set of ternary trees with n vertices as well as the set of noncrossing trees on $n+1$ vertices were also constructed. For T -Motzkin paths, the authors established bijections with the set of ordered pairs of ternary trees. The same authors also introduced a new class of paths, the U -paths, which are S -Motzkin paths without the initial horizontal step and thus the generating function of the S -Motzkin paths was generated from the generating function of U -paths. Their parameters of enumeration included number of returns, peaks, valleys and valleys on the x -axis in both S -Motzkin and T -Motzkin paths.

In 2019, Selkirk introduced a class of k_t -Dyck paths that generalises k -Dyck paths in her Msc thesis (Selkirk, 2019). The author created a bijection on the set of

$(k + 1)$ -ary trees and the set of k -Dyck paths to prove that there are

$$\frac{1}{kn + 1} \binom{(k + 1)n}{n}$$

k -Dyck paths of length $(k + 1)n$. She then used symbolic method and kernel method to prove that the number of k_t -Dyck paths of length $(k + 1)n$ is given by

$$\frac{t + 1}{(k + 1)n + t + 1} \binom{(k + 1)n + t + 1}{n}.$$

Also in 2019, Prodinger enumerated the set of k -Dyck paths with a negative boundary in (Prodinger, 2019). This paper extended the previous research done by Selkirk in (Selkirk, 2019). The author, using symbolic method, found the generating function of k -Dyck paths with negative boundary is

$$y = 1 + z^{k+1}y^{k+1}$$

where z marks the length of the path. By setting $y = 1 + w$ and applying the Lagrange inversion formula, the author found the counting formula for the k -Dyck paths with negative boundary. He also found the generating function of k_t -Dyck paths by path length. This paper also investigated what happens when the restriction $0 \leq t \leq k$ is negated. The k_t -Dyck paths are lifted by t units to create a normal k -Dyck path that ends at level t instead of the x -axis. To obtain a generating function for these paths, a temporary upper bound h was introduced which lead to a system of linear equations solved using the Cramer's rule. The arguments were made bijectively and analytically.

In 2022, Heuberger, Selkirk and Wagner (Heuberger et al., 2023) enumerated k_t -Dyck paths according to the number of down steps at height i modulo k . They used generating functions and bijections to obtain their results and constructed a bijection between the sets of k -plane trees and k_t -Dyck paths.

A study on partial skew Dyck paths has also been done in (Prodinger, 2022) where generating function for these paths was obtained using the kernel method. The skew Dyck paths are adjusted to the regular Dyck paths by replacing the $(-1, -1)$ step by $(1, -1)$ step and giving it a different color, creating partially decorated Dyck paths where the focus of study is these paths ending at the level $j \geq 0$. The paper also studied the dual skew paths which are a mirror version of the skew Dyck paths with steps counted from right to left. Similarly, the dual skew Dyck paths were reassembled into the traditional Dyck paths by replacing each $(-1, -1)$ step by $(1, -1)$ and assigning a different color. The author also studied skew Dyck paths that are allowed to go below the x -axis.

The study of k -box paths was introduced by Zhang and Zhuang in (Zhang & Zhuang, 2024). They defined these paths as a generalization of skew Dyck paths. Specifically, the authors represented $(1, -1)$ step by D , $(1, 1)$ step by U and $(-1, -1)$ step by L . For any k consecutive steps, they were expressed as a step to power k . A k -box path of size n with k peaks was defined as a skew Dyck path of half-length $(k + 2)n - 1$ having n factors of UD^kL . These paths were proved to be counted by

$$\frac{1}{(k + 2)n - 1} \binom{(k + 2)n - 1}{n}.$$

In the same paper, the authors created a bijection of the set of these paths and the set of $(k + 2)$ -ary trees having a total of $n - 1$ vertices. Moreover, they established a bijection between the set of k -box paths and the set of k_t -Dyck paths. Lastly, they enumerated k -box paths by number of returns to the x -axis and maximum number of sequence of ascents (sequence of steps moving from a lower level to a higher level) which they found to be enumerated by the Narayana numbers. This thesis studies the set of labelled 2-box paths which are an extension of the k -box paths where the boxes are labelled either 1 or 2 and proves that they are in bijection with the set of skew 2-Dyck paths.

David Callan introduced the set of plane trees with n non-leaf vertices and n leaves

where each non-leaf vertex has a single leaf and enumerated them by

$$\frac{1}{n+1} \binom{3n+1}{n}$$

, as recorded in sequence A006013 of (Sloane et al., 2018). This thesis shows that the set of plane trees with $k(n-1)$ vertices where each internal vertex has $k-1$ leaves is enumerated by Formula

$$\frac{1}{n} \binom{(k+1)n-2}{n-1}.$$

2.5 Various combinatorial sequences

2.5.1 Catalan numbers

The Catalan numbers, named after Belgian mathematician Eugène Charles Catalan, form one of the most fundamental and widely studied sequences in combinatorics (Larcombe, 1999). Their history, however, pre-dates Catalan himself. The sequence was first discovered by Leonard Euler in the 18th century when he was studying the problem of polygon triangulation (Pak, 1761). Catalan later provided a more systematic study of these numbers in the mid-19th century, leading to their current name. The Catalan numbers, denoted as C_n , have a simple closed form:

$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

As noted by Deutsch in his seminal 1999 paper, (Deutsch, 1999), an alternative representation of the Catalan number is:

$$\frac{1}{n} \binom{2n-2}{n-1}.$$

This formulation, while less common, is particularly useful in certain combinatorial

contexts, especially when dealing with tree structures. The importance of Catalan numbers in combinatorics is remarkable. They appear in a vast array of counting problems, often in seemingly unrelated contexts. Some of the most well-known structures enumerated by Catalan numbers and their generalizations include: the number of binary trees with n internal nodes, (Knuth, 1997), number of ways to triangulate a convex $(n+2)$ -gon, (Pak, 1761), number of Dyck words of length $2n$, (Deutsch, 1999), number of S -Motzkin paths, (Prodinger et al., 2020) and the number of skew Dyck paths of half-length n , (Deutsch et al., 2010a) among others.

One of the most significant works showcasing the versatility of Catalan numbers is Richard Stanley's book "Enumerative Combinatorics" (Stanley, 1999). In this work, Stanley lists over 200 different interpretations of Catalan numbers, highlighting their pervasive nature in discrete mathematics. The simplicity of their generating function simplifies the complex nature of many problems enumerated by Catalan numbers. Their recurrence relation has led to efficient algorithms for computing Catalan numbers and has connections to dynamic programming in computer science.

In the context of tree enumeration, as mentioned in the previous section, Catalan numbers enumerate plane trees with n vertices. This connection between Catalan numbers and tree structures is fundamental in combinatorics and has led to numerous insights in graph theory and computer science. The bijection between Catalan numbers and plane trees is typically established through a recursive construction in (Dershowitz & Zaks, 1980). Given that C_n counts the number of plane trees with n vertices, the authors constructed a tree with $n + 1$ vertices by taking a root node and attaching to it a sequence of subtrees, each of which is itself a plane tree. This recursive structure directly mirrors the recurrence relation for Catalan numbers.

In recent years, research on Catalan numbers has expanded into various directions. For instance, Aval et al., (Aval, Boussicault, & Dasse-Hartaut, 2011) studied q -analogs of Catalan numbers, extending the concept to quantum combinatorics. Flajolet and Sedgewick's work on analytic combinatorics, (Flajolet & Sedgewick, 2009)

provided deep insights into the asymptotic behavior of Catalan numbers and related sequences.

2.5.2 Motzkin numbers

Motzkin numbers, closely related to Catalan numbers, form another important sequence in combinatorics. Named after Theodore Motzkin (Aigner, 1998), these numbers have garnered significant attention due to their connections to various combinatorial objects and their relationship with Catalan numbers. As noted in the works of Aigner (Aigner, 1998) and Wang and Zhao (Wang & Zhang, 2015), Motzkin numbers M_n are intimately related to Catalan numbers C_k through the formula:

$$M_n = \sum_k \binom{n}{2k} C_k$$

This relationship highlights the deeper connection between the combinatorial objects counted by these two sequences. The combinatorial interpretation of Motzkin numbers, as discussed by Aigner and later proved by Deutsch (Deutsch, 1999), shows that they enumerate Motzkin paths. The simplicity of their generating function similarly simplifies the complex nature of many problems enumerated by Motzkin numbers.

Motzkin numbers have found applications in various areas of mathematics and computer science. In graph theory, they enumerate certain types of trees and matchings. In computer science, they appear in the analysis of certain algorithms and data structures. Recent research has expanded our understanding of Motzkin numbers and their generalizations. For instance, in (Woan, 2005), Woan studied weighted Motzkin numbers, providing a generalization that encompasses both Catalan and Motzkin numbers. Sapounakis et al., (Sapounakis, Tasoulas, & Tsikouras, 2007) explored colored Motzkin paths, extending the concept to more complex path structures.

The connection between Motzkin numbers and other combinatorial sequences has also been a fruitful area of research. For example, Sulanke in (Shapiro & Sulanke, 2000) established bijections between certain families of Motzkin paths and other combinatorial objects, including certain types of permutations and lattice paths. In the realm of algebraic combinatorics, Motzkin numbers have connections to representation theory such as in the study of certain algebras related to path enumeration.

The asymptotic behavior of Motzkin numbers has been studied using techniques from analytic combinatorics. Flajolet and Sedgewick's work, (Flajolet & Sedgewick, 2009) provides tools for analyzing the growth rate of Motzkin numbers and related sequences. In computer science, Motzkin numbers have found applications in the analysis of algorithms, particularly those involving tree structures and path-finding problems. They also appear in the study of formal languages and automata theory. Motzkin numbers have been extended to quantum combinatorics and providing connections to q -series and special functions.

The interplay between Catalan numbers, Motzkin numbers, and other combinatorial sequences continues to be an active area of research. These numbers provide a rich framework for understanding related combinatorial structures and have found applications in diverse areas such as computer science, biology, and physics. As research in this field progresses, new connections and generalizations of these numbers are continually being discovered, cementing their place as fundamental building blocks in the edifice of discrete mathematics. The ongoing exploration of Catalan and Motzkin numbers promises to yield further insights into the structure of discrete objects and the algorithms that manipulate them.

2.5.3 Narayana numbers

Narayana numbers, an important sequence in enumerative combinatorics, were first introduced by Indian mathematician T.V. Narayana in his paper "On lattices formed

by partitions of an integer and their applications to probability theory” (Banderier & Schwer, 2004). These numbers, denoted as $N(n, k)$, count various combinatorial objects, including certain types of lattice paths and parentheses structures.

Over the years, researchers have explored different combinatorial interpretations of Narayana numbers. Kreweras showed that they enumerate non-crossing partitions with a fixed number of blocks (Kreweras, 1972). Deutsch demonstrated that Narayana numbers count the number of Dyck paths with a specified number of peaks (Deutsch, 1999). In 2000, Sulanke provided a comprehensive survey of over 30 different combinatorial interpretations of Narayana numbers (Banderier & Schwer, 2002).

Narayana numbers are closely related to other important combinatorial sequences. For instance, the sum of Narayana numbers $N(n, k)$ for $k \in 1, \dots, n$ gives the n_{th} Catalan number. Björner and Wachs in (Björner & Wachs, 1983) explored connections between Narayana numbers and the Eulerian numbers in their work on q -analogues of Eulerian numbers. Several mathematicians have also worked on generalizations of Narayana numbers. A q -analogue of Narayana numbers and study their properties was done in 2008 and later in (Guo & Lian, 2022) they were explored further and their combinatorial interpretations found.

Narayana numbers have found applications in various areas. As originally explored by Narayana himself, they have applications in probability theory. They also appear in computer science, particularly in the analysis of certain algorithms and data structures, and in physics, specifically in some models of statistical mechanics. Recent work has continued to expand our understanding of Narayana numbers. Mansour and Sun in (Mansour & Sun, 2009) studied generating functions and recurrence relations for Narayana polynomials. More recently, Soykan in (Soykan, 2020) explored connections between Narayana numbers and certain graph colorings.

The study of Narayana numbers remains an active field in combinatorics, with new connections and applications continually being discovered. This ongoing research

continues to deepen our understanding of these fascinating numbers and their role in various mathematical and scientific contexts.

2.5.4 Schröder numbers

Schröder numbers (also referred to as *super-Catalan numbers*) represent a fascinating sequence in combinatorial mathematics. Named after Ernst Schröder, a prominent German mathematician of the 19th century, these numbers have captured the attention of mathematicians and computer scientists alike (Gessel, 2009; Ardila, 2015). Schröder's work in mathematical logic and algebra laid the groundwork for what would later become a cornerstone in enumerative combinatorics.

The Schröder numbers are divided into two distinct but related sequences: the large Schröder numbers and the little Schröder numbers. This categorization reflects fundamental differences in the combinatorial objects they enumerate. The relationship between these two sequences is interesting, with the large Schröder numbers being twice the corresponding little Schröder numbers, except for the first term.

Chen and Yuan in (Chen & Xiang, 2020) made significant contributions to our understanding of Schröder numbers through their work on Schröder paths. They demonstrated that large Schröder numbers enumerate what are known as large Schröder paths, while little Schröder numbers count little Schröder paths. These paths are specific types of lattice paths in the plane, subject to certain constraints. Large Schröder paths allow for diagonal steps in addition to the usual up and right steps, while little Schröder paths have additional restrictions on these diagonal steps (Ardila, 2015). This connection between numbers and paths provides a geometric interpretation of these abstract sequences, making them more tangible and easier to visualize.

The versatility of large and little Schröder numbers becomes apparent when we consider the wide array of combinatorial structures they enumerate. Standard Young

tableaux, which play a crucial role in representation theory and algebraic combinatorics, are one such structure. These tableaux are arrangements of numbers in a specific diagram shape, subject to certain ordering constraints.

Pechenik, in (Pechenik, 2014) explored the deep connections between increasing tableaux and little Schröder paths, particularly focusing on the combinatorial structures represented by these two concepts. The author established a bijection between rectangular 2-row increasing tableaux, $Inc_k(2 \times n)$ and small Schröder paths with n up-steps and k horizontal steps, which are counted by the well-known Schröder numbers. He also extended the known result that Catalan numbers enumerate the set of rectangular shaped standard Young tableau $SYT(2 \times n)$ with two rows when he showed that there is a bijection between the set of standard Young tableau $SYT(k, k, 1^{n-k})$ and the set of little Schröder paths with $n - k$ horizontal steps. This relationship highlighted how the enumeration of increasing tableaux could be interpreted through path combinatorics, providing a new perspective on both structures by giving the interpretations of representation theory through mappings between the sets $Inc_k(2 \times n)$ and $SYT(k, k, 1^{n-k})$.

Standard Young tableaux have made a wide appearance in fields such as geometry, representation theory of Lie groups and various complex symmetric groups (Fuchs & Schweigert, 2003). They have also found applications in Statistical and Quantum Mechanics in analysing particle behaviour, in computer science and in coding theory on error-correcting codes among other applications (Huffman & Pless, 2010; Stanley, 1999).

Pechenik, in (Pechenik, 2014), also showed that little Schröder numbers count various combinatorial objects, including the number of ways to traverse a grid using certain movement rules, which could be visualized as paths that do not cross a diagonal. By linking these paths to increasing tableaux, the paper demonstrated that each increasing tableau could be associated with a unique little Schröder path. This bijection not only facilitated the enumeration of tableaux but also revealed under-

lying symmetries and properties shared between these two combinatorial entities. The cyclic sieving phenomenon used further enriched this connection, as it provided a framework for understanding how these structures behave under specific group actions, leading to deeper insights into their combinatorial nature.

In the paper (Du, Fan, & Zhao, 2019), Du *et.al* obtained a refinement for the formula counting increasing tableau. He introduced the set of row-increasing tableaux denoted as $RInc_k(2 \times n)$ and showed that they are enumerated by

$$\frac{1}{n-k+1} \binom{2n-k}{k} \binom{2n-2k}{n-k},$$

where k entries are repeated. This formula also enumerates the set of large Schröder paths with k horizontal paths and n up-steps.

Polygon dissections represent another area where Schröder numbers appear. This involves counting the number of ways to divide a polygon into smaller polygons by drawing non-intersecting diagonals. Such problems have applications in computational geometry and computer graphics. In (Shapiro & Sulanke, 2000), the authors discussed the relationship between bracketings, planted trees, and dissections of convex polygons, establishing a bijective correspondence among these configurations. A dissection of a convex polygon involved dividing the polygon into non-overlapping regions using line segments that connect non-adjacent vertices. This process has become crucial in combinatorial geometry and has historical significance, dating back to the work of mathematicians like Euler and Segner, who studied triangulations of polygons (Pak, 1761).

The number of bracketings on a string of $n + 1$ letters was found to be equal to the number of dissections of a convex polygon with $n + 2$ sides [(Shapiro & Sulanke, 2000, Theorem 1)]. This bijection implied that both configurations had the same cardinality, which was foundational in combinatorial enumeration. The paper further explored how these dissections could be represented through planted full

binary trees, where each tree corresponded to a unique dissection of the polygon. Additionally, the authors introduced a method to double the set of dissections by coloring the base of the polygon, leading to a new set of configurations that could be mapped to Schröder paths.

This connection to Schröder numbers provided efficient ways to calculate these dissections for polygons of various sizes and has been extended in various papers since then. For instance, in Stanley's paper (Stanley, 1996), the author obtained a bijection between the set of dissections of a regular convex $(n + 2)$ -gon and the set of standard Young tableaux of shape $(k, k, 1^{n-k})$. Both structures were enumerated by little Schröder numbers.

Restricted lattice paths, which are paths on a grid subject to certain movement constraints, form yet another category of objects counted by Schröder numbers. These paths have applications in probability theory, statistical physics, and even in biological sequence analysis. Aguiar *et.al*, (Aguiar, 2006), created a bijection between the set of rooted planar trees with $n + 1$ leaves and m internal vertices where the internal vertices have at least two children and the set of lattice paths of length $2n$ from $(0, 0)$ to (n, n) with $n - m$ diagonal steps and comprising of $(1, 0)$, $(0, 1)$ and $(1, 1)$ steps which never go above the line $y = x$. The authors then showed that this set of lattice paths is enumerated by little Schröder numbers,

$$\frac{1}{n} \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j+1} \binom{n+j-1}{j} 2^j.$$

Similarly, Deutsch paths which start at the origin and use up-steps of length greater than or equal to one and unit down-steps, are also enumerated by the little Schröder numbers represented as sequence A001003 in (Sloane et al., 2018). The enumeration of these paths using Schröder numbers provides insights into random walk problems and diffusion processes.

In sequence A006013 of (Sloane et al., 2018), combinatorialist David Callan recorded

that

$$\frac{1}{n} \binom{3n+1}{n}$$

gives the number of ways to connect $2n$ points, in a line, labelled $1, 2, \dots, 2n$ with either zero or more non-crossing arcs such that the maximal sequence of isolated points have an even length. For example; when $n = 2$, there are 4 ways to connect 4 points which are: 12, 23, 14, 34 (note that we cannot connect points 13 or 24 since the sequences of the isolated points are not even).

Valid sequences, another structure enumerated by Schröder numbers, often arise in computer science, particularly in the study of programming language syntax and database query languages. These sequences must satisfy certain validity criteria, and the fact that they are counted by Schröder numbers allows for efficient algorithms in parsing and validation tasks.

Generalized Motzkin paths, which are extensions of the well-known Motzkin paths, also find their enumeration tied to Schröder numbers. These paths have connections to various problems in computer science and combinatorial optimization. The link to Schröder numbers allows for the analysis of more complex path structures and their properties. The connection between Motzkin and Schröder numbers is a significant aspect of combinatorial enumeration, particularly in the study of lattice paths. Both types of paths are defined on a grid and involve specific step types, but they differ in their constraints and characteristics. Motzkin paths are lattice paths that can include up steps, down steps, and level steps, allowing the path to touch the x -axis at multiple points. In contrast, Schröder paths, particularly small Schröder paths, are constrained to avoid horizontal steps on the x -axis, which leads to a different counting mechanism.

In the paper, (L. Yang & Yang, 2020), the authors established a bijection between small q -Schröder paths of semilength $n + 1$ and $(q + 2, q + 1)$ -Motzkin paths of

length n . This relationship highlighted how the enumeration of these paths could be interrelated through combinatorial techniques. The bijection not only provided a way to count the paths but also revealed deeper structural similarities between the two types of paths. Furthermore, the authors explored additional correspondences, such as between small 3-Schröder paths and Catalan rook paths, as well as between small 4-Schröder paths and Catalan queen paths, further illustrating the rich interplay between these combinatorial objects.

The relationship between Motzkin paths and Schröder numbers has piqued interest in various authors which is shown in the work by Gessel, (Gessel, 2009) . He defined a Riordan path as a Motzkin path that excludes flat steps on the x -axis making Riordan paths analogous to small Schröder paths, which were defined similarly but also did not include flat steps. As a result, small Schröder paths could be viewed as a subset of Motzkin paths, highlighting a direct connection between the two concepts. In terms of counting, the large Schröder numbers represent the total number of Schröder paths of semilength n , while the small Schröder numbers count only those paths that do not contain flat steps. On the other hand, Motzkin paths include flat steps, leading to a different enumeration approach. The interplay between these path types has been significant in combinatorial enumeration, as both have been employed to address various combinatorial problems, and their respective counts related through generating functions and other combinatorial techniques. Thus, while Motzkin and Schröder paths share foundational principles, their differences in step types and counting methods have created a rich area of study within combinatorial mathematics.

The combinatorics of Schröder numbers and their role in enumerating a variety of structures continues to be investigated. This thesis, has explored the interplay between these diverse combinatorial structures and several tree-like structures: non-decreasing 2-plane trees, plane Husimi graphs, and weakly labelled k -plane trees. This research direction was significant as it sought to establish connections between

seemingly disparate mathematical objects. Non-decreasing 2-plane trees are a specific type of rooted tree with ordering constraints, while plane Husimi graphs are graph structures with particular cyclic arrangements. Weakly labelled k -plane trees introduce labeling constraints to the already complex structure of plane trees.

The study of Schröder numbers and their relationship to related combinatorial structures represented a rich and active area of mathematical research. By exploring the connections between these numbers and diverse objects like Young tableaux, polygon dissections, and lattice paths, and further extending these connections to tree-like structures, this research not only advanced our understanding of combinatorics but also potentially impacting fields ranging from computer science to theoretical physics. The ongoing exploration of these relationships continues to reveal the deep and often surprising connections that exist within the world of discrete mathematics.

A closer analysis revealed specific gaps that this thesis sought to address. First, while Dyck paths and their variations have been extensively studied, the enumeration of skew 2-Dyck paths—particularly under specific labelling constraints remained largely unexplored. Second, although non-decreasing labelled trees naturally arise in restricted growth models and other combinatorial contexts, their systematic study and unified enumeration framework was still lacking in the literature. Finally, Husimi graphs, which had been primarily investigated in the context of theoretical physics, had received little attention from a combinatorial perspective, especially in establishing bijective links to other well-known structures such as plane trees. Addressing these gaps not only deepened the theoretical understanding of these combinatorial structures but also opened new avenues for applications in computer science, probability theory, and mathematical physics.

CHAPTER THREE: METHODOLOGY

In this chapter, the methods of enumeration applied in the research were reviewed.

3.1 Symbolic method

The symbolic method served as the foundational framework for all enumeration results presented in this thesis. This method provided a powerful framework for translating combinatorial structures into generating functions through systematic construction rules. The method begins with the fundamental concept of an atom, denoted by z , which represents the basic building block of size 1 in any combinatorial structure, with generating function x .

For any combinatorial family $\mathcal{A}$ containing a_n objects of size n , the associated ordinary generating function is constructed, i.e.

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

The symbolic method then provides systematic rules for combining these generating functions through natural combinatorial operations:

Union: If A and B are combinatorial classes with generating functions $A(x)$ and $B(x)$, then $(A \cup B)(x) = A(x) + B(x)$.

Cartesian Product: The class of ordered pairs (a, b) where $a \in A$ and $b \in B$ has

generating function $(A \times B)(x) = A(x)B(x)$.

Sequence Construction: The class of all finite sequences of elements from A has generating function $\text{Seq}(A(x)) = \frac{1}{1-A(x)}$.

Set Construction: The power set of A has generating function $P(A(x)) = \exp(A(x))$.

These transformations allows complex combinatorial constructions to be expressed as algebraic equations involving generating functions, simplifying many counting problems and providing a systematic approach to enumerative combinatorics.

This work applied the symbolic method systematically to enumerate several classes of trees: It established the symbolic equation for these trees by decomposing them according to their root structure where a tree was either a single vertex (atom z) or consisted of a root vertex connected to a sequence of subtrees. It was also extended to lattice path enumeration by modifying the basic symbolic construction to account for different step restrictions and ending conditions.

The symbolic method has been explained in details in (Flajolet & Sedgewick, 2009). An illustration to enumerate binary trees by obtaining their generating function showed that the number of binary trees with n vertices is enumerated by the Catalan numbers. The number of Dyck paths and k -Dyck paths has also been generated using this method (Selkirk, 2019).

3.1.1 Generating functions

In 2015, Ardila (Ardila, 2015) provided a comprehensive survey of algebraic methods in enumeration. He studied the theory of generating functions and how combinatorics problems are rephrased into linear equations and various methods of extracting coefficients applied. A major conclusion made was that generating functions are one of the most powerful tools that are used to determine explicit formulas of combinatorial structures.

This study concentrated on implicit functional equations, that are expressed in terms of themselves. Such an example is the functional generating function of the Catalan number in Equation 1.1.

The Lagrange inversion formula helps in extracting the coefficient of x^n in the generating function of the form $T(x) = x\Phi(T(x))$. It's stated as:

Theorem 3.1 (Lagrange Inversion Formula, (Wilf, 2005)). *The coefficient of x^n in an implicitly defined function $T(x) = x\Phi(T(x))$ is given by*

$$[x^n]T(x) = \frac{1}{n}[t^{n-1}]\Phi(t)^n$$

if $\Phi(t)$ has a power series expansion $\Phi(t) = \sum_{k=0}^{\infty} a_k t^k$ with $a_0 \neq 0$. More generally,

$$[x^n]T(x)^k = \frac{k}{n}[t^{n-k}]\Phi(t)^n.$$

This formula was used to extract coefficients from the generating functions of structures studied in this thesis that were obtained by the symbolic method.

Theorem 3.2 (Lagrange-Bürmann Theorem, (Comtet, 1974)). *Let $q(x)$ be a power series with complex coefficients and β be a complex number, satisfying an equation*

$$q(x) = \beta + x \cdot g(q(x)).$$

Then

$$[x^n]f(q(x)) = f(\beta) + \frac{1}{n!} \frac{d^{n-1}}{ds^{n-1}} \left(\frac{d}{ds}(f(s)) \cdot (g(s))^n \right)_{s=\beta}.$$

The Lagrange's inversion formula and Lagrange-Bürmann Theorem, are especially designed to solve implicit functional equations. Where the generating functions cannot not be expressed in a form applicable to Lagrange inversion formulas, manipulations on the functional equations are performed.

3.1.2 Path decomposition

The decomposition of paths is a fundamental concept in combinatorics and graph theory, particularly useful for analyzing lattice paths. This technique involves breaking down complex paths into simpler, more manageable components. It includes identifying the start and end points of the path, recognizing recurring patterns or substructures within the path, breaking the path into segments based on certain criteria (e.g., direction changes or visited vertices) and then expressing the overall path in terms of these smaller segments. Path decomposition often leads to recurrence relations or generating functions that can be used to count or characterize the paths. The power of path decomposition lies in its ability to transform complex path-counting problems into simpler, often recursive, structures that are more amenable to analysis and enumeration. For example, this method has been used to find the generating functions of skew Dyck paths in (Deutsch et al., 2010a).

3.1.3 Butterfly Decomposition

Tree decomposition methods provide powerful approaches for enumerating complex tree structures by breaking them into simpler components. One particularly elegant example is the butterfly decomposition, a specialized technique developed specifically for analyzing noncrossing trees.

The butterfly decomposition was introduced by Flajolet and Noy in (Flajolet & Noy, 1999) as a combinatorial method for counting and analyzing noncrossing configurations in planar graphs. This decomposition technique systematically breaks down complex noncrossing tree structures into fundamental building blocks called "butterflies," which resemble the wing structure of a butterfly when visualized.

The key insight of butterfly decomposition lies in its ability to transform the enumeration problem of complex noncrossing trees into a more tractable analysis of

their constituent butterfly components.

Definition 3.1. A *butterfly* is an ordered pair of noncrossing trees that share a common root vertex. The two trees in this pair form the "wings" of the butterfly structure.

The construction of a butterfly involved combining two noncrossing trees by identifying their root vertices, creating a unified structure where each original tree serves as one wing of the resulting butterfly. This gluing operation preserved the noncrossing property while creating a new combinatorial object with well-defined structural properties.

The decomposition process operated on individual vertices within noncrossing trees. Consider a vertex v in a noncrossing tree with outdegree d , meaning v has d children in the (l, r) -representation of the tree. Such a vertex could be systematically decomposed into exactly d distinct butterflies, each rooted at vertex v . This decomposition is illustrated in Figure 3.1, where the geometric relationship between the original vertex structure and its butterfly components is clearly demonstrated.

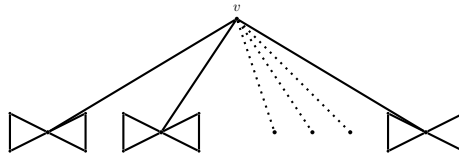


Figure 3.1: Decomposition of noncrossing trees into butterflies.

This method was used to achieve objective *iv* of this study.

3.2 Bijections

Two combinatorial classes $\mathcal{A}$ and $\mathcal{B}$ are combinatorially isomorphic if and only if their counting sequences are identical (Flajolet & Noy, 1999). This is similar to a bijection between the set of objects in $\mathcal{A}$ and the set of objects in $\mathcal{B}$. Suppose a counting formula for the number of objects in $\mathcal{A}$ is known but that of $\mathcal{B}$ is not

known. We can show that the known formula also counts the number of objects in $\mathcal{B}$ if we can construct a bijection between $\mathcal{A}$ and $\mathcal{B}$. This method is considered the most powerful method in enumerative combinatorics. The method involves giving a procedure of how to obtain a distinct element in a set B from a distinct element in A (injective step) and the reverse procedure (surjective step). Selkirk in (Selkirk, 2019) used this approach to enumerate the k -Dyck paths through a bijection with $(k + 1)$ -ary trees and also enumerated k_t -Dyck paths via ordered tuples of k -Dyck paths. We constructed bijections between the sets of combinatorial structures.

We used both symbolic and bijections methods described above to obtain our five objectives. Enumeration of lattice paths, trees and graphs is a problem that has been tackled using several methods as has been explained in details by (Krattenthaler, 2015). In this work, we used symbolic method to come up with generating functions and applied Lagrange Inversion Formula to extract the coefficients.

The following is a summary of the methods used to achieve the five objectives of this thesis. These are; decomposition of paths, generating functions, Lagrange inversion and a known theorem from (Liu & Wang, 2007) by Liu and Wang for objective (i). Symbolic method, decomposition of trees, bijections, generating functions and Lagrange inversion for objective (ii). Symbolic method, generating functions, Lagrange inversion formula and a known theorem due to Stanley in Theorem 4.23 for objective (iii). Butterfly decomposition, symbolic method, Lagrange-Burmman theorem, generating functions, Lagrange inversion formula and decomposition of trees for objective (iv) and bijections for objective (v).

CHAPTER FOUR: RESULTS AND DISCUSSIONS

4.1 Introduction

Chapter FOUR presents findings and discussions of the research. In Section 4.2, the set of skew 2-Dyck paths by length was enumerated and some of their statistics such as the bounds and limits of their enumerating sequence are discussed. Various bijections to skew 2-Dyck paths were also created. In Section 4.3, the set non-decreasing 2-plane trees was enumerated by number of vertices, root degree, label of eldest child and bijections with combinatorial structures such as large and little Schröder paths are constructed. In Section 4.4, the set of weakly labelled k -plane trees was introduced and enumerated and bijections with other sets of combinatorial structures were created. The set of weakly 2-noncrossing trees was enumerated in Section 4.5. Moreover, bijections with other structures of similar cardinality were obtained. Lastly, in Section 4.6, bijections between the set of plane Husimi graphs and the sets of plane trees, polygon dissections, standard Young tableaux, valid sequences, Deutsch paths and restricted lattice paths were constructed.

4.2 Skew 2-Dyck paths

In this section, skew 2-Dyck paths were studied. Recall from Chapter ONE that a *skew k -Dyck path* was a lattice path starting at $(0, 0)$, consisting of $(1, 1)$, $(k, -k)$

and $(-k, -k)$ steps and ending on the x -axis. When $k = 2$, This study obtain the *skew 2-Dyck paths*. Retaining the definition of length of the skew Dyck paths as stated in (Deutsch et al., 2010a), i.e. length referred to sum of lengths of all steps in the path. Skew 2-Dyck paths were of length $3n$ where n was the total number of $(2, 2)$ and $(-2, -2)$ steps. Each of these steps contributed 2 to the length of the path and the $2n$ unit $(1, 1)$ steps each contributed n . For our enumeration, This study referred to the skew 2-Dyck paths of length $3n$ as paths of half-length n .

Two examples of skew 2-Dyck paths of half-length 3 were given in Figure 4.1.

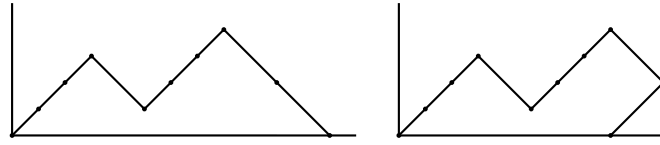


Figure 4.1: Skew 2-Dyck paths when $n = 3$

4.2.1 Enumeration of Skew 2-Dyck paths

Let $S(z)$ be the generating function for skew 2-Dyck paths, where z marks a down-step or a left-step. The paths were decomposed through path decomposition method specifically according to the first down-step or left-step that touched the x -axis. This was depicted in Figure 4.2.

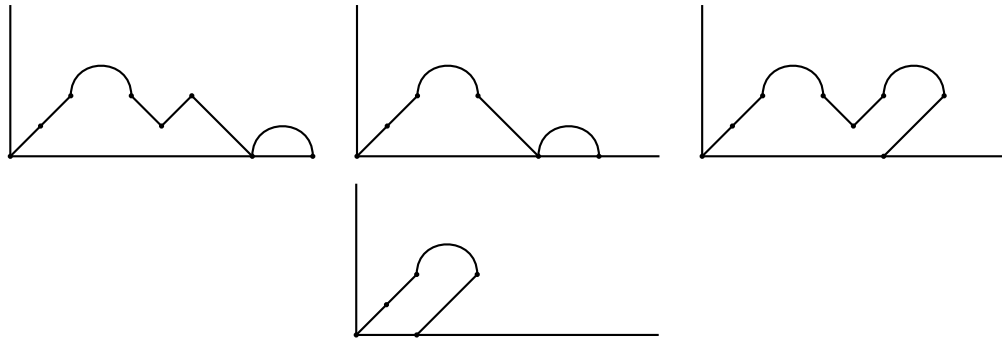


Figure 4.2: Decomposition of skew 2-Dyck paths

The generating function thus satisfied the equation

$$S(z) = z + 3zS(z)^2 + 4zS(z), \quad (4.1)$$

which could be written as:

$$S(z) = z(1 + 3S(z)^2 + 4S(z)).$$

Applying the Lagrange Inversion formula to extract the coefficients of z^n from this generating function;

$$\begin{aligned} [z^n]S(z) &= \frac{1}{n}[s^{n-1}](3s^2 + 4s + 1)^n \\ &= \frac{1}{n}[s^{n-1}](3s + 1)^n(s + 1)^n \\ &= \frac{1}{n}[s^{n-1}] \sum_{i=0}^{\infty} \binom{n}{i} 3^i s^i \sum_{j=0}^{\infty} \binom{n}{j} s^j \\ &= \frac{1}{n}[s^{n-1}] \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \binom{n}{i} \binom{n}{j} 3^i s^{i+j} \\ &= \frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i} \binom{n}{n-i-1} 3^i \end{aligned} \quad (4.2)$$

The Narayana number was given by $N(n, k) = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$ (Soykan, 2020). So

$$[z^n]S(z) = \sum 3^i N(n, k).$$

This study formally wrote the result as;

Theorem 4.1. *The number of skew 2-Dyck paths of half-length n was given by*

$$\sum_{i=0}^{n-1} 3^i N(n, i) \quad (4.3)$$

Consequently,

$$\begin{aligned}
[z^n]S(z) &= \frac{1}{n}[s^{n-1}](3s^2 + (4s + 1))^n = \frac{1}{n}[s^{n-1}] \sum_{i \geq 0} \binom{n}{i} 3^i a s^{2i} (4s + 1)^{n-i} \\
&= \frac{1}{n}[s^{n-1}] \sum_{i \geq 0} \sum_{j \geq 0} \binom{n}{i} 3^i \binom{n-i}{j} 4^j s^{2i+j} \\
&= \frac{1}{n} \sum_{i=0}^{\infty} \binom{n}{i} 3^i \binom{n-i}{n-2i-1} 4^{n-2i-1} \\
&= \frac{1}{n} \sum_{i \geq 0} \binom{n}{i} \binom{n-i}{i+1} 3^i 4^{n-2i-1}
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
[z^n]S(z) &= \frac{1}{n}[s^{n-1}]((1 + 2s)^2 - s^2)^n \\
&= \frac{1}{n}[s^{n-1}] \sum_{i \geq 0} \binom{n}{i} (1 + 2s)^{2i} (-1)^{n-i} s^{2(n-i)} \\
&= \frac{1}{n}[s^{n-1}] \sum_{i \geq 0} \binom{n}{i} \sum_{j \geq 0} \binom{2i}{j} 2^j s^{2n-2i+j} (-1)^{n-i} \\
&= \frac{1}{n} \sum_{i \geq 0} \binom{n}{i} \binom{2i}{2i-n-1} (-1)^{n-i} \\
&= \frac{1}{n} \sum_{i=0}^n \binom{n}{i} \binom{2i}{n+1} (-1)^{n-i} \\
&= \frac{1}{n} \sum_{i=0}^n \binom{n}{i} \binom{2(n-i)}{n+1} (-1)^i
\end{aligned} \tag{4.5}$$

So the number of skew 2-Dyck paths of half-length n was also given by (4.4) and (4.5). From (4.1), the quadratic equation (4.6)

$$3zS(z)^2 + (4z - 1)S(z) + z = 0 \tag{4.6}$$

was obtained which simplified to

$$S(z) = \frac{(1 - 4z) - \sqrt{4z^2 - 8z + 1}}{6z} \tag{4.7}$$

The correct root that gave an analytic generating function was chosen since the root needed to converge in the neighbourhood of $z = 0$. The first few terms of the

coefficients of $S(z)$ were 1, 4, 19, 100, 562, 3304, ..., which was sequence A007564 in (Sloane et al., 2018). From (4.7), This study obtained

$$1 - 6zS(z) - 4z = \sqrt{4z^2 - 8z + 1}. \quad (4.8)$$

Differentiating (4.8) with respect to z , and simplifying further gave

$$-6zS'(z) - 6S(z) - 4 = \frac{8z - 8}{2\sqrt{4z^2 - 8z + 1}}.$$

So,

$$(6zS'(z) + 6S(z) + 4)\sqrt{4z^2 - 8z + 1} = 4 - 4z.$$

Multiplying through by $\sqrt{4z^2 - 8z + 1}$ resulted in

$$(6zS'(z) + 6S(z) + 4)(4z^2 - 8z + 1) = (4 - 4z)\sqrt{4z^2 - 8z + 1}. \quad (4.9)$$

Substituting (4.8) into (4.9), resulted in

$$z(4z^2 - 8z - 1)S'(z) + (1 - 4z)S(z) - 2z = 0. \quad (4.10)$$

To obtain the recurrence equation, let $S(z) = \sum_{n=0}^{\infty} s_n z^n$ and $S'(z) = \sum_{n=1}^{\infty} n s_n z^{n-1}$ and substituting in (4.10) the study obtained

$$\sum_{n=1}^{\infty} n s_n z^{n-1} (4z^2 - 8z - 1) + \sum_{n=1}^{\infty} s_n z^n (1 - 4z) - 2z = 0$$

which was the same as

$$\sum_{n=1}^{\infty} 4n s_n z^{n+2} - \sum_{n=1}^{\infty} 8n s_n z^{n+1} + \sum_{n=1}^{\infty} n s_n z^n + \sum_{n=0}^{\infty} s_n z^n - \sum_{n=0}^{\infty} 4s_n z^{n+1} - 2z = 0.$$

This resulted in the recurrence

$$(n + 3)s_{n+2} = (8n + 12)s_{n+1} - 4n s_n \quad (4.11)$$

for the number of skew 2-Dyck paths with initial condition $s_1 = 1, \in s_2 = 4$.

Definition 4.1. (Liu & Wang, 2007) A sequence a_k was *log convex* if and only if the sequence $\frac{a_{k+1}}{a_k}$ was increasing.

For example $1, 4, 19, \dots$ was log convex since $\frac{4}{1}, \frac{19}{4}, \dots$ was increasing.

Theorem 4.2 ((Liu & Wang, 2007, Theorem 3.10)). *Let $z_n, n \geq 0$, be a sequence of positive numbers and satisfies the three-term recurrence $(\alpha_1 n + \alpha_0)z_{n+1} = (\beta_1 n + \beta_0)z_n - (\gamma_1 n + \gamma_0)z_{n-1}$ for $n \geq 1$, where $\alpha_1 + \alpha_0, \beta_1 + \beta_0, \gamma_1 + \gamma_0$ are positive for $n \geq 1$. Suppose that z_0, z_1, z_2 was log-convex, then the sequence $z_n (n \geq 0)$ was log-convex if one of the following conditions holds:*

- (i) $B, C \geq 0$,
- (ii) $B < 0, C > 0, AC \geq B^2$ and $z_0 B + z_1 C \geq 0$,
- (iii) $B > 0, C < 0, AC \leq B^2$ and $z_0 B + z_1 C \geq 0$

where

$$A = \begin{bmatrix} \beta_0 & \beta_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} B = \begin{bmatrix} \gamma_0 & \gamma_1 \\ \alpha_0 & \alpha_1 \end{bmatrix} \text{ and } C = \begin{bmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{bmatrix}.$$

For (4.11), $\alpha_0 = 3, \alpha_1 = 1, \beta_0 = 12, \beta_1 = 8, \gamma_0 = 0$ and $\gamma_1 = 4$ so that $A = 48, B = -12$ and $C = 12$. Since $B < 0, C > 12, AC = 576 > 144 = B^2$ and $1(-12) + 4(12) = 36 > 0$, then by *ii* in Theorem 4.2, S_n in (4.11) was log convex. To obtain the bounds of S_n given by the recurrence relation

$$4ns_n - (8n + 12)s_{n+1} + (n + 3)s_{n+2} = 0,$$

rewriting it as

$$(n + 2)s_{n+1} - (8n + 4)s_n + 4(n - 1)s_{n-1} = 0.$$

Since $4(n - 1)s_{n-1}$ was non-negative, then

$$(n + 2)s_{n+1} - (8n + 4)s_n \leq 0,$$

so that

$$(n+2)s_{n+1} \leq (8n+4)s_n.$$

This meant that

$$\frac{s_{n+1}}{s_n} \leq \frac{8n+4}{n+2} \leq \frac{8n}{n} = 8. \quad (4.12)$$

The sequence $\frac{a_{k+1}}{a_k}$ was increasing and bounded from above by 8.

We obtained its limit. According to [(Woan, 2005, Theorems 9 and 10)], a three-term recursive equation

$$(n+1)m_n = (2dn-d)m_{n-1} + (4c-d^2)(n-2)m_{n-2} \quad (4.13)$$

has the limit,

$$\lim_{n \rightarrow \infty} \frac{m_n}{m_{n-1}} = d + 2\sqrt{c}. \quad (4.14)$$

The recurrence under investigation was

$$(n+1)s_n = (8n-4)s_{n-1} - 4(n-2)s_{n-2}.$$

So,

$$\frac{s_n}{s_{n-1}} = \frac{8n-4}{n+1} - \frac{4(n-2)s_{n-2}}{(n+1)s_{n-1}}.$$

Let

$$a_n = \frac{2dn-d}{n+1} = \frac{8n-4}{n+1} \text{ and } b_n = \frac{(4c-d^2)(n-2)}{n+1} = -\frac{4(n-2)}{(n+1)}.$$

This implied that $d = 4$ and $c = 3$. Also, letting

$$x_n = a_n + \frac{b_n}{x_{n-1}}.$$

Then $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \frac{s_n}{s_{n-1}}$. From (4.13) and (4.14), gave

$$\lim_{x \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \left(a_n + \frac{b_n}{x_{n-1}} \right) = 4 + 2\sqrt{3}.$$

4.2.2 Bijections of skew 2-Dyck paths

From the k -box paths introduced by Zhang and Zhuang in (Zhang & Zhuang, 2024), the total number of box paths of size n was

$$\frac{1}{3n-1} \binom{3n-1}{n}.$$

This study introduced a new class of box paths where the factors were of the form $UDLD$. It called a path of the form $DUDLD$ a *box*.

Definition 4.2. A *labelled 2-box path* was a box path whose boxes are labelled either 1 or 2 except those that touch the x -axis with another box path on its right which receive label 1.

Figure 4.3 depicts a labelled 2-box path of size 2.

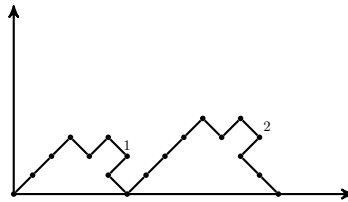


Figure 4.3: A labelled 2-box path with 2 factors.

This study established a bijection between the set of labelled 2-box paths and skew 2-Dyck paths in the following theorem.

Theorem 4.3. *There exists a bijection between the set of skew 2-Dyck paths of half-length n and the set of labelled 2-box paths with n factors.*

Proof. Given a skew 2-Dyck path, this study obtained the corresponding labelled 2-box path by performing the following transformations; For every left-step $(-2, -2)$ or down-step $(2, -2)$ encountered, delete it and draw a unit up-step, followed by a unit down-step, unit left-step then a final unit down-step (*UDLD* factor). For any box corresponding to a down-step, label it 1 and one corresponding to a left-step, label it 2. See Figure 4.4 for an example.

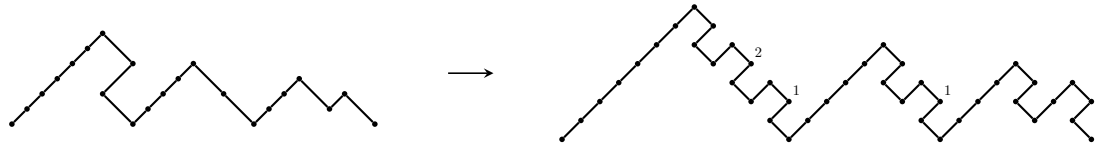


Figure 4.4: A procedure of obtaining a labelled 2-box path on 21 up-steps from a skew 2-Dyck path of half-length 7.

Conversely, given a labelled 2-box path with n factors, i.e. with $3n$ up-steps, for each factor (*UDLD*), replace it with a down-step of length 2. If the box was labelled 2, then label the down-step as 2. Now, replace each down-step labelled 2 with a left-step of length 2. This procedure was depicted in Figure 4.5 $\square$

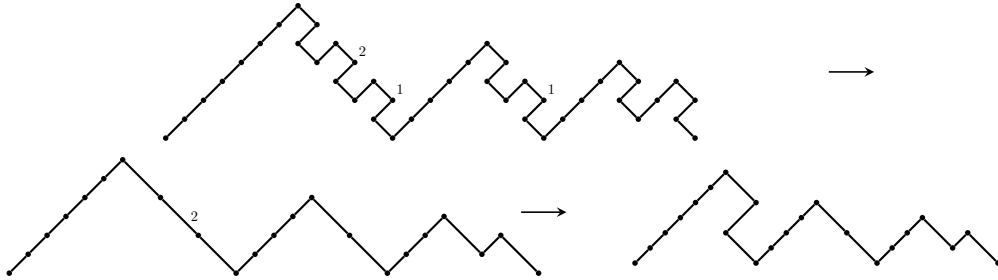


Figure 4.5: A procedure of obtaining a skew 2-Dyck path of half-length 7 from a labelled 2-box path on 21 up-steps.

4.2.2.1 3-leaf labelled plane trees

This study introduced the set of leaf labelled plane trees which was formally defined.

Definition 4.3. A *3-leaf labelled plane tree* was a plane tree such that all leaves are at a length multiple of 3 from the root and those leaves at length 1 from their

immediate branching point are labelled as follows: If the rightmost subtree rooted at its root was a leaf or was empty, then that leaf was labelled either 1 or 2 and if that subtree has atleast three vertices, then the leaf was strictly labelled 1.

Consider an example of a 3-leaf labelled plane tree in Figure 4.6.

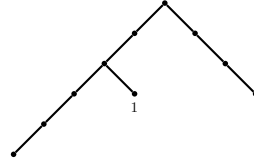


Figure 4.6: A 3-leaf labelled plane tree on 10 vertices.

Theorem 4.4. *There exists a bijection between the set of labelled 2-box paths with n factors and the set of 3-leaf labelled plane trees on $3n + 1$ vertices.*

Proof. Given a labelled 2-box path with n up-steps, starting at the origin of the path and drawing an initial vertex which will be the root, for each up-step encountered, draw a new vertex. If a down-step was encountered, move up the tree on the left hand side towards the root. If the down-step was labelled 1 or 2, then name the vertex encountered as one moves up the tree as 1 or 2 respectively. Since each factor (*UDLD*) corresponds to three down-steps then each branching point was a multiple of 3 from the last vertex drawn. Since each up-step has a corresponding vertex in the tree and an initial vertex was introduced, then the tree has $3n + 1$ vertices i.e. each factor has an up-step and each left and final down-step in a factor also has a corresponding up-step. The tree obtained was a 3-leaf labelled plane tree.

Conversely, given a 3-leaf labelled plane tree, traverse the tree in pre-order. For each vertex encountered as one moves away from the root, draw an up-step. Now, for three consecutive vertices encountered as one moves towards the root, draw a down-step, a left-step and a down-step. If a leaf labelled 1 or 2 was encountered, label the corresponding down-step 1 or 2 respectively. These are the labels of the boxes. The Dyck path obtained was a labelled 2-box path. The labelling of the boxes

agrees with the definition of labelled 2-box paths. An illustration of this bijection was shown in figure 4.7. □

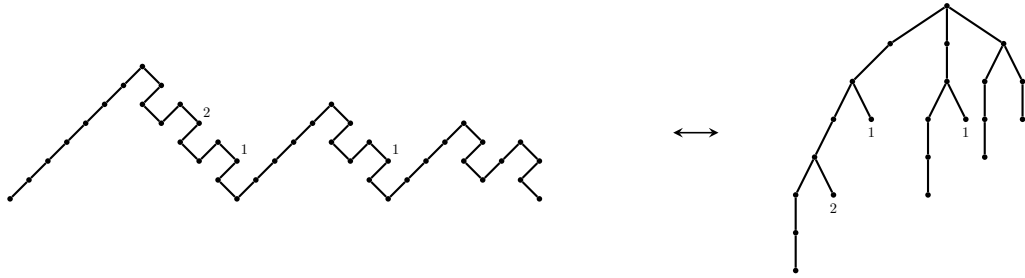


Figure 4.7: A labelled 2-box path with 7 factors and its corresponding 3-leaf labelled plane tree on 22 vertices.

Corollary 4.1. *There was a bijection between the set of skew 2-Dyck paths of half-length n and the set of 3-leaf labelled plane trees on $3n + 1$ vertices.*

4.2.2.2 2-edge labelled plane trees

This study defined 2-edge labelled plane trees as follows:

Definition 4.4. A 2-edge labelled plane tree was a plane tree whose rightmost subtree was of even length from the root to the leaf while all other subtrees are of even length from their branching point such that any two consecutive edges of the rightmost subtree are labelled either 1 or 2 and any of other subtrees are labelled 1, excluding the pair farthest from the root (branching point respectively).

Consider an example of a 2-edge labelled plane tree in Figure 4.8.

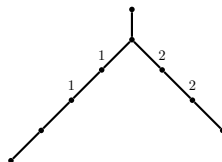


Figure 4.8: A 3-leaf labelled plane tree on 9 vertices.

Theorem 4.5. *There exists a bijection between the set of skew 2-Dyck paths of half-length n and the set of 2-edge labelled plane trees on $2n + 1$ vertices.*

Proof. Given a skew 2-Dyck path, this study obtained its corresponding 2-edge labelled plane tree by performing the following transformations; First, convert the $(2, -2)$ down-steps and $(-2, -2)$ left-steps of the path into pairs of $(1, -1)$ and $(-1, -1)$ steps respectively. Starting with an initial vertex which will be the root of the tree: For every up-step encountered, draw a vertex. For every two consecutive down-steps encountered, move up two steps in the constructed tree and label the 2 edges encountered as 1 (do not label the first two). For every two consecutive left-steps encountered, move one two steps towards the root and label the two consecutive edges as 2. The study obtained a 2-edge labelled plane tree on $2n + 1$ vertices since each up-step ($2n$ in total) and the initial root make up the vertices of the tree.

Conversely, given a 2-edge labelled plane tree, traverse it in pre-order to obtain the corresponding skew 2-Dyck path as follows: Moving away from the root, for any vertex encountered, draw an up-step, Moreover, moving towards the root for any unlabelled edges or edges labelled 1 encountered, draw a down-step and for any edge labelled 2 encountered, draw a left-step. From the skew path obtained, transform it into a skew 2-Dyck path by combining any pair of consecutive down-steps or left-steps into a new step of length 2. The resulting structure was a skew 2-Dyck path of half-length n . □

This bijection was illustrated in Table 4.1

Table 4.1: The four skew 2-Dyck paths of half-length 2 and their corresponding 2-edge labelled plane trees on 5 vertices.

Skew 2-Dyck paths	2-edge labelled plane trees

4.3 Non-decreasing 2-plane trees

4.3.1 Introduction

This study remarked that the set of plane trees with labels in the set $\{1, 2\}$ such that the sum of labels of any two adjacent vertices does not exceed 3 was introduced in (Gu & Prodinger, 2009). They were commonly referred to as *2-plane trees*. These trees have been studied extensively in (Lumumba et al., 2024; Mlotkowski & Penson, 2013; Okoth, 2021a). This study introduced and studied a new class of labelled plane trees which was baptised as *non-decreasing 2-plane trees*. The set of these trees was a subset of the set of 2-plane trees but first it was formally defined.

Definition 4.5. A *non-decreasing 2-plane tree* was a plane tree whose vertices are labelled from the set $\{1, 2\}$ with any two vertices i and j connected by an edge having the sum of the labels $i + j \leq 3$ and the labels of any siblings are non-decreasing from left to right.

See Figure 4.9 for an example of a non-decreasing 2-plane tree.

This additional ordering condition distinguished non-decreasing 2-plane trees from the general class of 2-plane trees. This study derived enumerative formulas for non-

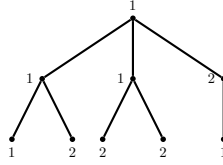


Figure 4.9: A non-decreasing 2-plane tree on 9 vertices with root labelled 1.

decreasing 2-plane trees according to three parameters; number of vertices, root degree and label of eldest child of the root by means of symbolic method, generating functions and the Lagrange inversion Formula.

Surprisingly, the generating functions for these trees with root labelled 1 or 2 were found to be the same as those for the well-known large or little Schröder numbers respectively. This study started off with enumeration using various parameters. It then obtain bijections with structures counted by little and large Schröder numbers.

4.3.2 Generating functions of non-decreasing 2-plane trees

There are two classes of non-decreasing 2-plane trees: those with roots labelled 1 and those with roots labelled 2. Let $W(x)$ and $B(x)$ be the generating function for non-decreasing 2-plane trees with roots labelled 1 and 2 respectively, where x marks a vertex. Due to the condition that any two adjacent vertices were given labels whose sum was at most 3, then when the root was labelled 2, the children of the root had to be labelled 1. This was depicted in Figure 4.10. The generating function $B(x)$ thus satisfies the equation

$$B(x) = x \cdot \frac{1}{1 - W(x)}. \quad (4.15)$$

For the non-decreasing 2-plane trees with root labelled 1, the children of the root could either be labelled 1 or 2 where all the children labelled 2 came at the right of the ones labelled 1. This was shown in Figure 4.11. The generating function $W(x)$

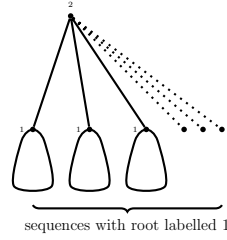


Figure 4.10: Non-decreasing 2-plane tree with root labelled 2.

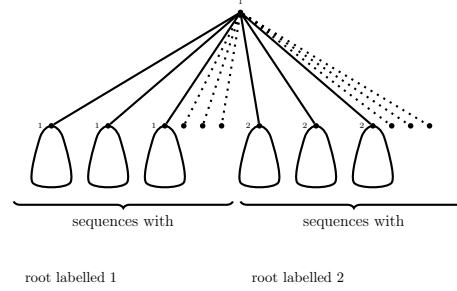


Figure 4.11: Non-decreasing 2-plane tree with root labelled 1.

was thus given by

$$W(x) = x \cdot \frac{1}{1 - W(x)} \cdot \frac{1}{1 - B(x)}. \quad (4.16)$$

Substituting (4.15) in (4.16), this study had

$$W(x) = \frac{B(x)}{1 - B(x)} \quad (4.17)$$

and from (4.15) this study got

$$W(x) = \frac{B(x) - x}{B(x)}. \quad (4.18)$$

By equating (4.17) and (4.18), this study had $2B(x)^2 - (1 + x)B(x) + x = 0$ which simplified to

$$B(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4}.$$

This was the generating function for the little Schröder numbers as discussed in Chapter ONE, Section 2.5.

Substituting (4.15) in (4.16) and simplifying, it got $W(x)^2 + (x - 1)W(x) + x = 0$.

This quadratic equation was solved to get

$$W(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2},$$

which was the generating function for large Schröder numbers.

4.3.2.1 Number of vertices

The number of non-decreasing 2-plane trees of n vertices with root labelled 1 created the sequence 1, 2, 6, 22, 90, $\dots$ which was the sequence of the Large Schröder numbers while those with root labelled 2 create the sequence 1, 1, 3, 11, 45, $\dots$ which was the sequence of the Little Schröder numbers.

Theorem 4.6. *The number of non-decreasing 2-plane trees with roots labelled 2 on n vertices is given by*

$$\frac{1}{n} \sum_{j=0}^{n-1} (-1)^{n-j-1} 2^j \binom{n}{j+1} \binom{n+j-1}{j}. \quad (4.19)$$

Proof. Since $2B(x)^2 - (1+x)B(x) + x = 0$ then

$$B(x) = \frac{x(1 - B(x))}{1 - 2B(x)}.$$

This was in a form this study could apply Lagrange inversion to extract the coefficient of x^n as shown below:

$$\begin{aligned} [x^n]B(x) &= \frac{1}{n} [b^{n-1}] (1-b)^n (1-2b)^{-n} \\ &= \frac{1}{n} [b^{n-1}] \sum_{i=0}^n \binom{n}{i} (-1)^i b^i \sum_{j=0}^{n-1} \binom{-n}{j} (-2)^j b^j \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{n-j-1} (-1)^{n-j-1} \binom{n+j-1}{j} 2^j. \end{aligned}$$

□

In the following theorem, this study obtained an equivalent result to Theorem 4.6, when the roots of the trees are labelled 1.

Theorem 4.7. *There are*

$$\frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j+1} \binom{n+j-1}{j} \quad (4.20)$$

non-decreasing 2-plane trees on n vertices with roots labelled 1.

Proof. From (4.17), this study had

$$B(x) = \frac{W(x)}{1 + W(x)}.$$

By (4.15), it got

$$B(x) = \frac{x}{1 - W(x)} = \frac{W(x)}{1 + W(x)}.$$

It followed that

$$W(x) = \frac{x(1 + W(x))}{1 - W(x)}. \quad (4.21)$$

Applying Lagrange inversion formula, the study got

$$\begin{aligned} [x^n]W(x) &= \frac{1}{n} [w^{n-1}] (1 + w)^n (1 - w)^{-n} \\ &= \frac{1}{n} [w^{n-1}] \sum_{i=0}^n \binom{n}{i} w^i \sum_{j=0}^{n-1} \binom{-n}{j} (-1)^j w^j \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{n-j-1} \binom{n+j-1}{j}. \end{aligned}$$

This completed the proof. □

4.3.2.2 Root degree

The total number of non-decreasing 2-plane trees on n vertices could be broken down to reflect the various degrees of the root.

Theorem 4.8. *The number of non-decreasing 2-plane trees on n vertices with roots of degree k labelled 2 was given by*

$$\frac{k}{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{k+j} \binom{n+j-2}{j}. \quad (4.22)$$

Proof. Since the subtrees rooted at a vertex of label 2 were always rooted at vertices of label 1 then the required result was:

$$\begin{aligned} [x^n]xW^k &= \frac{k}{n-1} [w^{n-k-1}] (1+w)^{n-1} (1-w)^{-(n-1)} \\ &= \frac{k}{n-1} [w^{n-k-1}] \sum_{i=0}^{n-1} \binom{n-1}{i} w^{i+j} \sum_{j=0}^{n-k-1} \binom{-(n-1)}{j} (-1)^j \\ &= \frac{k}{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{n-k-j-1} \binom{n+j-2}{j}. \end{aligned}$$

□

Setting $k = 1$ in (4.22), the study found that there were

$$\frac{1}{n-1} \sum_{j=0}^{n-2} \binom{n-1}{j+1} \binom{n+j-2}{j}$$

planted non-decreasing 2-plane trees on n vertices if the root was labelled 2. This study remarked that this formula was easily determined by attaching root vertices labelled 2 to non-decreasing 2-plane trees on $n-1$ vertices whose roots are labelled 1, i.e., This study replace n with $n-1$ in (4.20).

Corollary 4.2. *The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 and of degree k such that all the children of the root are labelled*

1 was given by

$$\frac{k}{n-1} \sum_{j=0}^{n-k-1} \binom{n-1}{k+j} \binom{n+j-2}{j}. \quad (4.23)$$

Proof. The proof followed easily from Theorem 4.7 after changing the root labels in trees under consideration from 1 to 2. $\square$

Theorem 4.9. *The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 and of degree ℓ such that all the children of the root are labelled 2 was given by*

$$\frac{\ell}{n-1} \sum_{j=0}^{n-\ell-1} \binom{n-1}{\ell+j} (-1)^{n-\ell-j-1} \binom{n+j-2}{j} 2^j. \quad (4.24)$$

Proof. Since all the subtrees rooted at the root are of label 2 then the desired result was:

$$\begin{aligned} [x^n]xB(x)^\ell &= \frac{\ell}{n-1} [b^{n-\ell-1}] (1-b)^{n-1} (1-2b)^{-(n-1)} \\ &= \frac{\ell}{n-1} [b^{n-\ell-1}] \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i b^i \sum_{j=0}^{n-\ell-1} \binom{-(n-1)}{j} (-2)^j b^j \\ &= \frac{\ell}{n-1} \sum_{j=0}^{n-\ell-1} \binom{n-1}{n-\ell-j-1} (-1)^{n-\ell-j-1} \binom{n+j-2}{j} 2^j. \end{aligned}$$

$\square$

This study now proved the following theorem:

Theorem 4.10. *The number of non-decreasing 2-plane trees on n vertices with roots of degree d labelled 1 such that there are k children labelled 1 was given by*

$$\frac{1}{n-1} \sum_{j=0}^{n-d-2} \frac{k(n-1) + j(d-k)}{n-d-j-1} \binom{n+k-d-2}{k+j} \binom{n+j-2}{j}. \quad (4.25)$$

Proof. This study extracted the coefficient of x^n in $xW(x)^k B(x)^{d-k}$ where $W(x)$ and $B(x)$ are the generating functions for non-decreasing 2-plane trees with roots

labelled 1 and 2 respectively. Here again, x marks a vertex:

$$\begin{aligned}
[x^n]xW(x)^k B(x)^{d-k} &= [x^{n-1}]W(x)^k W(x)^{d-k} (1+W(x))^{k-d} \\
&= [x^{n-1}]W(x)^d (1+W(x))^{k-d} \\
&= [x^{n-1}]W(x)^d \sum_{i \geq 0} \binom{k-d}{i} W(x)^i \\
&= \sum_{i \geq 0} \binom{k-d}{i} [x^{n-1}]W(x)^{d+i}.
\end{aligned}$$

By Lagrange Inversion Formula, this study got

$$\begin{aligned}
&[x^n]xW(x)^k B(x)^{d-k} \\
&= \sum_{i \geq 0} \binom{k-d}{i} \frac{d+i}{n-1} [w^{n-d-i-1}] (1+w)^{n-1} (1-w)^{-(n-1)} \\
&= \frac{1}{n-1} \sum_{i \geq 0} \binom{k-d}{i} (d+i) [w^{n-d-i-1}] \sum_{j \geq 0} \sum_{l \geq 0} \binom{n-1}{l} \binom{-(n-1)}{j} (-1)^j w^{l+j} \\
&= \frac{1}{n-1} \sum_{i \geq 0} \left[d \binom{k-d}{i} + (k-d) \binom{k-d-1}{i-1} \right] \sum_{j \geq 0} \binom{n-1}{n-d-i-j-1} \binom{n+j-2}{j}.
\end{aligned}$$

By Vandermonde Convolution, this study obtained

$$\begin{aligned}
&[x^n]xW(x)^k B(x)^{d-k} \\
&= \frac{1}{n-1} \sum_{j \geq 0} \left[d \binom{n+k-d-1}{n-d-j-1} + (k-d) \binom{n+k-d-2}{n-d-j-2} \right] \binom{n+j-2}{j} \\
&= \frac{1}{n-1} \sum_{j \geq 0} \frac{k(n-1) + j(d-k)}{n-d-j-1} \binom{n+k-d-2}{k+j} \binom{n+j-2}{j}.
\end{aligned}$$

This completed the proof. □

By summing over all k in (4.25), this study obtained the following corollary:

Corollary 4.3. *There are*

$$\frac{1}{n-1} \sum_{k=0}^d \sum_{j=0}^{n-d-2} \frac{k(n-1) + j(d-k)}{n-d-j-1} \binom{n+k-d-2}{k+j} \binom{n+j-2}{j}$$

non-decreasing 2-plane trees on n vertices with roots of degree d labelled 1.

4.3.2.3 Label of eldest child of the root

Since the eldest child of the root labelled 2 was always labelled 1, it was of less significance to enumerate them by label of the eldest child. This study then concentrated on non-decreasing 2-plane trees with root labelled 1.

Theorem 4.11. *The number of non-decreasing 2-plane trees on n vertices with root labelled 1 such that the eldest child of the root was labelled 1 was given by*

$$\frac{2}{n} \sum_{j=0}^{n-2} \binom{n}{j+2} \binom{n+j-1}{j}. \quad (4.26)$$

Proof. The decomposition of the trees in question was as shown in Figure 4.12.

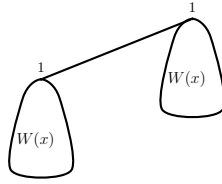


Figure 4.12: Decomposition of trees with root labelled 1 and eldest child of the root was labelled 1.

The generating function for these trees was $W(x)^2$ where $W(x)$ was the generating function for trees with root labelled 1. This study thus had that the number of non-decreasing 2-plane trees on n vertices with root labelled 1 such that the eldest child of the root was labelled 1 was

$$\begin{aligned} [x^n]W(x)^2 &= \frac{2}{n} [w^{n-2}] (1+w)^n (1-w)^{-n} \\ &= \frac{2}{n} [w^{n-2}] \sum_{i \geq 0} \sum_{j \geq 0} \binom{n}{i} \binom{-n}{j} (-1)^j w^{i+j} \\ &= \frac{2}{n} \sum_{j=0}^{n-2} \binom{n}{n-j-2} \binom{n+j-1}{j}. \end{aligned}$$

Thus the formula. □

Bijjective proof of Theorem 4.11. Label the root as r . Let the eldest child of r be i . Delete the edge connecting r to i . Introduce a new root j labelled 2 whose children are i and r in this order from left to right. The tree has root degree 2 labelled 2 on $n + 1$ vertices. The number of these trees was given by (4.22) if This study set $k = 2$. This was illustrated in Figure 4.13. □

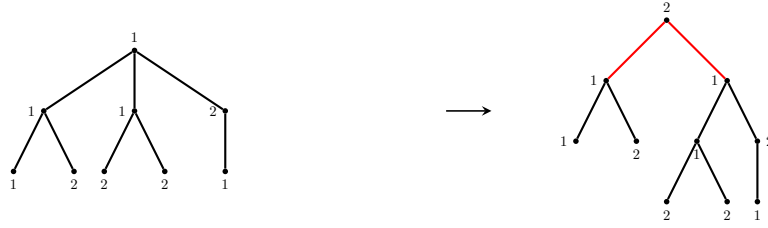


Figure 4.13: A non-decreasing 2-plane tree on 9 vertices with root labelled 1 and eldest child labelled 1 with its corresponding non-decreasing 2-plane tree on 10 vertices with root labelled 2.

Corollary 4.4. *The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 such that the eldest children of the roots are labelled 2 was given by*

$$\frac{1}{n} \sum_{j=0}^{n-2} \frac{3j+4-2n}{n-j-1} \binom{n}{j+2} \binom{n+j-1}{j}.$$

Proof. This was the number of non-decreasing 2-plane trees with roots labelled 1 such that the eldest child of the root was labelled 2 as obtained by subtracting the number in (4.26) from (4.20), i.e.,

$$\frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j+1} \binom{n+j-1}{j} - \frac{2}{n} \sum_{j=0}^{n-2} \binom{n}{j+2} \binom{n+j-1}{j}$$

which simplifies to the formula in the statement of the corollary. Alternatively, this could also be achieved by summing over ℓ in (4.24), i.e.,

$$\sum_{\ell=1}^{n-1} \sum_{j=0}^{n-\ell-1} \frac{\ell}{n-1} \binom{n-1}{\ell+j} (-1)^{n-\ell-j-1} \binom{n+j-2}{j} 2^j.$$

□

4.3.3 Bijections of non-decreasing 2-plane trees with root labelled 2

4.3.3.1 Leaf-labelled plane trees

Plane trees were one of the many structures counted by the Catalan numbers. The number of plane trees on $n \geq 2$ vertices and having k leaves was given by the Narayana number,

$$\frac{1}{n-1} \binom{n-1}{k} \binom{n-1}{k-1}.$$

So, if the leaves are given labels 1 and 2 then the number of such trees was

$$\frac{2^k}{n-1} \binom{n-1}{k} \binom{n-1}{k-1}.$$

This study had the total number of plane trees on $n \geq 2$ vertices whose leaves were labelled with integers 1 and 2 as

$$\frac{1}{n-1} \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{k-1} 2^k$$

which were little Schröder numbers. This study therefore proved the following result:

Theorem 4.12. *There was a bijection between the set of non-decreasing 2-plane trees on n vertices with roots labelled 2 and the set of plane trees on n vertices whose leaves are labelled with integers 1 and 2.*

Proof. Let T be a non-decreasing 2-plane tree on n vertices with root labelled by 2. Now, let v be a non-root internal vertex of T labelled 2. Detach the subtrees rooted at the children of v and attach them to the parent of v in order from left to right so that the children of v appear on the immediate right of v . Do this for all non-root internal vertices labelled 2. The labels of all internal vertices (inclusive of the root)

are dropped and the resultant structure was a plane tree on n vertices whose leaves are labelled 1 or 2.

For the reverse procedure, label the root of the plane tree as 2 and all the internal vertices as 1. Now, let ℓ_1 and ℓ_2 (if any) be leaves labelled 2 that share a parent such that ℓ_1 was older than ℓ_2 and that there are younger siblings of ℓ_1 labelled 1. Detach all the vertices between ℓ_1 and ℓ_2 or all on the right of ℓ_1 if there was no such ℓ_2 . Attach all the vertices between ℓ_1 and ℓ_2 or on the right of ℓ_1 if there was no such ℓ_2 and the subtrees rooted at them and attach the subtrees at ℓ_1 such that these vertices which were sharing a parent with ℓ_1 are now children of ℓ_1 . The attachment was done in order from left to right and the labels are retained. Do this to all vertices with the same properties as ℓ_1 and ℓ_2 . The structure obtained was a non-decreasing 2-plane tree on n vertices whose root was labelled 2.

A depiction of the bijection was given in Figure 4.14.

□

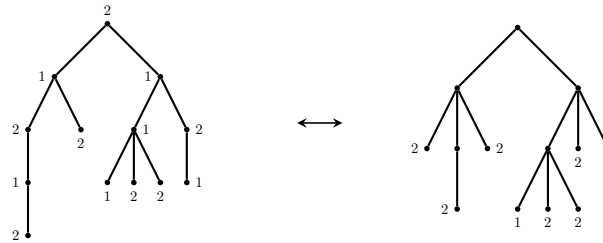


Figure 4.14: A non-decreasing 2-plane tree on n vertices and its corresponding plane tree on n vertices such that the leaves are labelled 1 or 2.

4.3.3.2 Little Schröder paths

Theorem 4.13. *There was a bijection between the set of non-decreasing 2-plane trees on $n + 1$ vertices with roots labelled 2 and the set of little Schröder paths of length n with no horizontal steps on the x -axis.*

Proof. Consider a non-decreasing 2-plane tree T on $n + 1$ vertices with root labelled

2. This study obtained the corresponding little Schröder path of length n with no horizontal step at level zero by the following procedure:

- (i) This study traversed T using preorder traversal and for every vertex labelled 1 (resp. labelled 2) encountered as one moves away from the root, draw a unit up-step (resp. horizontal step of length 2).
- (ii) Moreover, for every edge encountered as one moves towards the root, if the initial vertex of the edge was labelled 1, then draw a unit down-step. Otherwise, do not draw any step.

Let the number of non-root vertices labelled 2 be k . Then the number of vertices labelled 1 was $n - k$. Since for each vertex labelled 1, one draws a unit up-step as you move away from the root and unit down-step as you move towards the root, and for each non-root vertex labelled 2, draw horizontal step of length 2 as you move away from the root and no step as you move towards the root, then the total length of the path was $2n$. Since the children of the root are all labelled 1, then you are assured of having no horizontal path on the x -axis. So, by this procedure, this study obtained little Schröder paths from $(0, 0)$ to $(2n, 0)$ with no horizontal paths on the x -axis.

Conversely, a non-decreasing 2-plane tree on $n + 1$ vertices was obtained from a little Schröder path of length $2n$ with no horizontal step on the x -axis as follows:

- (i) Draw a vertex of the tree which will root of the tree. Label it 2. This study then build the tree recursively.
- (ii) Starting at the origin of the path, for each up-step encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 1. Note that there are no horizontal steps on the x -axis. So, the initial step was an up-step and the vertex corresponding to this vertex was labelled 1. Also, the condition

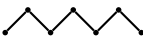
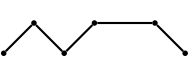
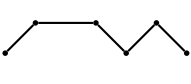
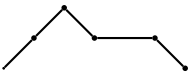
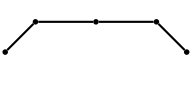
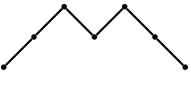
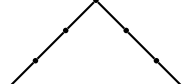
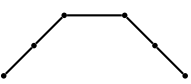
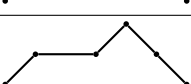
that there are no horizontal steps on the x -axis guarantees that all the children of the root are labelled 1.

- (iii) When a horizontal-step was encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 2. Moreover, move up the tree back to the vertex in which a new vertex has been attached.
- (iv) For every down-step encountered, move up the tree to the parent of the vertex last visited. This vertex was labelled 1.

Let the number of up-steps be k . Then the number of down-steps was k . Moreover, there are $n - k$ horizontal steps since each horizontal step was of length 2. Since each up-step corresponds to a non-root vertex labelled 1 and each horizontal step corresponds to a non-root vertex labelled 1, then there were a total of $k + n - k = n$ non-root vertices. Thus there were $n + 1$ vertices in the corresponding non-decreasing 2-plane tree. The coherence condition that there were no edges whose endpoints are labelled 2 was satisfied since for each horizontal step encountered a vertex labelled 2 was drawn but it was attached to a vertex which corresponds to a down-step which was labelled 1. Moving up the tree to the parent of the vertex last visited ensures this fact. This procedure stopped with the root and the corresponding structure was a non-decreasing 2-plane tree. □

Table 4.3.3.2 provides a list of all non-decreasing 2-plane trees on 4 vertices with roots labelled 2 and their corresponding little Schröder paths of length 6 such that there are no horizontal step on the x -axis.

Table 4.2: All non-decreasing 2-plane trees on 4 vertices with roots labelled 2 and their corresponding little Schröder paths of length 6 such that there are no horizontal step on the x -axis.

Non-decreasing 2-plane tree	Little Schröder path	Non-decreasing 2-plane tree	Little Schröder path
			
			
			
			
			
			

4.3.3.3 Restricted lattice paths

Consider lattice paths in which only horizontal steps $(1, 0)$, vertical steps $(0, 1)$ and diagonal steps $(1, 1)$ are allowed. Moreover, let the steps lie entirely above the line $y = x$. This study coined the name *restricted lattice paths* for these paths. In (Aguiar, 2006), it was showed the set of these paths from $(0, 0)$ and (n, n) was enumerated by little Schröder numbers,

$$\frac{1}{n} \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j+1} \binom{n+j-1}{j} 2^j.$$

This formula also counted non-decreasing 2-plane trees on $n + 1$ vertices with root labelled 2 as proved in the previous section. It was thus of combinatorial importance to construct a bijection between the set of restricted lattice paths and the set of non-decreasing 2-plane trees which was obtained in the next theorem.

Theorem 4.14. *There exists a bijection between the set of non-decreasing 2-plane trees on $n + 1$ vertices with root labelled 2 and the set of restricted lattice paths from*

$(0, 0)$ to (n, n) .

Proof. Consider a non-decreasing 2-plane tree T on $n + 1$ vertices with root labelled 2. This study obtained the corresponding restricted lattice path from $(0, 0)$ to (n, n) as follows:

- (i) This study used preorder traversal to visit the vertices of T and for each vertex labelled 1 (resp. labelled 2) encountered as one moved away from the root, draw a $(0, 1)$ step (resp. $(1, 1)$ step).
- (ii) Moreover, for every edge encountered as one moved towards the root, if the initial vertex of the edge was labelled 1, then draw a $(1, 0)$ step. Otherwise, do not draw any step.

Let the number of non-root vertices labelled 2 be k . Then, the number of vertices labelled 1 was $n - k$. Since for each vertex labelled 1, this study drew a $(0, 1)$ as one moved away from the root and a $(1, 0)$ as one moved towards the root, and for each non-root vertex labelled 2, it drew $(1, 1)$ as one moved away from the root and no step as one moved towards the root, then the path started at $(0, 0)$ and ended at (n, n) . Since the children of the root were all labelled 1, then this study assured the path lay entirely above the line $y = x$. So, by this procedure, this study obtained restricted lattice paths from $(0, 0)$ to (n, n) .

Conversely, a non-decreasing 2-plane tree on $n + 1$ vertices was obtained from a restricted lattice path from $(0, 0)$ to (n, n) by the following procedure:

- (i) Draw a vertex of the tree which was the root of the tree. Label it 2. This study then build the tree recursively.
- (ii) Starting at the origin of the path, for each $(0, 1)$ encountered, attach a vertex to the tree being constructed at the immediate vertex previously drawn and label it 1. Note that the path lies entirely above the line $y = x$. So, the initial

step was a $(0, 1)$ step and its corresponding vertex was labelled 1. Also, the condition that the path lies entirely above the line $y = x$ guaranteed that all the children of the root were labelled 1.

- (iii) When a $(1, 1)$ step was encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 2. Moreover, move up the tree back to the vertex in which a new vertex has been attached.
- (iv) For every $(1, 0)$ step encountered, move up the tree to the parent of the vertex last visited. This vertex was labelled 1.

Let the number of $(0, 1)$ steps be k . Then the number of $(1, 0)$ steps was k as well. Moreover, there were $n - k$ steps of type $(1, 1)$. Since each $(0, 1)$ step corresponded to a non-root vertex labelled 1, each $(1, 0)$ step corresponded to a non-root vertex labelled 1 and each $(1, 1)$ step corresponded to a non-root vertex labelled 2, then there were $k + (n - k)$ non-root vertices in the tree. Thus there were $n + 1$ vertices in the corresponding non-decreasing 2-plane tree. The coherence condition that there were no edges whose endpoints were labelled 2 was satisfied since for each $(1, 1)$ step encountered, a vertex labelled 2 was drawn but it was attached to a vertex which corresponds to a $(0, 1)$ step which was labelled 1. Moving up the tree to the parent of the vertex last visited ensured this condition was satisfied. Figures 4.15 and 4.16 show an illustration of the bijection. □

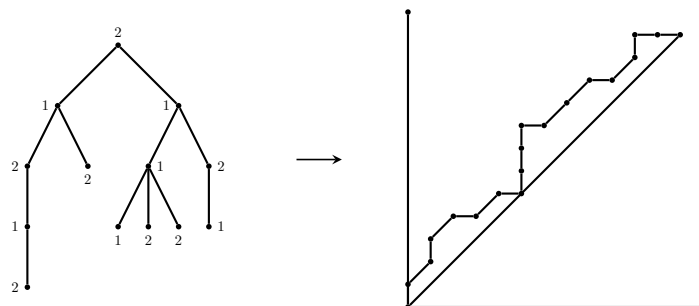


Figure 4.15: A procedure from a non-decreasing 2-plane tree on 13 vertices to a lattice path from $(0, 0)$ to $(12, 12)$.

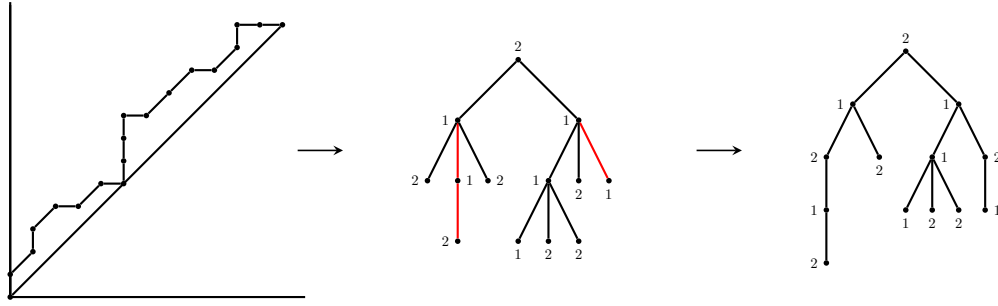


Figure 4.16: A procedure from a lattice path from $(0,0)$ to $(12,12)$ to a non-decreasing 2-plane tree on 13 vertices.

4.3.3.4 Increasing tableaux

A bijection between the set of standard Young tableaux and the set of increasing tableaux was demonstrated in (Pechenik, 2014). An increasing tableau, $Inc_k(2 \times n)$, was defined in Chapter TWO. This was a table comprising of 2 rows and n columns where both rows and columns have strictly increasing values, such that for each value i contained in the table, all the smaller positive integer values less than i also appear in the table. Every value in the table can be repeated at most twice and k represents the number of repeated terms. A bijection was constructed between the set of these structures and the set of non-decreasing 2-plane trees with root labelled 2 as below.

Theorem 4.15. *There was a bijection between the set of non-decreasing 2-plane trees with root labelled 2 on $n + 1$ vertices such that the number of vertices labelled 2 was $k + 1$ and the set of increasing Young tableaux, $Inc_k(2 \times n)$.*

Proof. Given a non-decreasing 2-plane tree on $n + 1$ vertices, with root labelled 2 and $k + 1$ vertices labelled 2, draw an empty table of 2 rows and n columns and fill the values by the below procedure.

- (i) Traverse the non-decreasing 2-plane tree by preorder traversal and draw arrows along the edges of the tree as one moves away from the root and towards the root. Each entry in the tableau corresponded to exactly one arrow.

- (ii) The first arrow led to the first child of the root and for this vertex, row 1 column 1 of the tableau was filled with the positive integer value 1 and one moved to the next arrow.
- (iii) Let i^{th} arrow point towards vertex v_i as one moved away from the root for each $i \geq 2$.
 - (a) If v_i was labelled 1, then one filled in the next positive integer value into the table in row 1.
 - (b) If v_i was labelled 2, then one filled in the next positive integer value into the table in both row 1 and row 2.
- (iv) Let i^{th} arrow point away from vertex v_i as one moved towards the root for each $i \geq 2$.
 - (a) If v_i was labelled 1, then one filled in the next positive integer value into the table in row 2.
 - (b) If v_i was labelled 2, then one did not fill in any value.
- (v) Steps (iii) and (iv) were performed until the whole tree was traversed.

The vertices labelled 2 were repeated in the tableau as in step (iii)(b). So the repeated values were k . Since all children of the root were labelled 1, the entry in row 1 column 1 of the tableau was always 1. In the table filled, all the entries of the rows and columns were strictly increasing and no entry was repeated more than twice, which satisfied the properties of the set of $Inc_k(2 \times n)$. An illustration of this procedure was given in Figure 4.17.

Conversely, consider increasing tableau of type $n \times 2$ where k positive integer values were repeated. The other values were thus $1, 2, \dots, 2n - k$. This study obtained a corresponding non-decreasing 2-plane tree on n vertices with root labelled 2 and having $k + 1$ vertices labelled 2 by the following procedure:

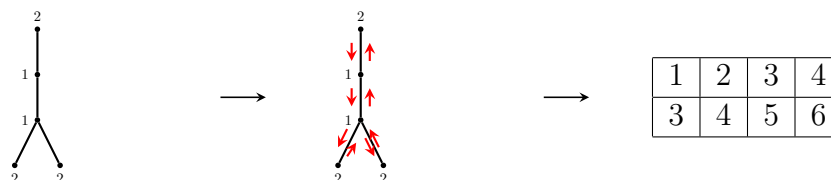


Figure 4.17: An increasing tableau, $Inc_2(2 \times 4)$, obtained from a non-decreasing 2-plane tree on 5 vertices.

- (i) Draw a root of the tree and label it 2. Then draw a child of the root and label it 1. This corresponds to entry 1 of row 1 column 1 of the tableau.
- (ii) Consider entries $2, 3, \dots$ in the tableau. Let the least of the values be i .
 - (a) If i was in row 1 but not in row 2 then draw a child attached to the last vertex drawn. Label it 1.
 - (b) If i was not in row 1 but in row 2 then move back to the parent of the last vertex drawn.
 - (c) If i was in row 1 and row 2 then draw a child attached to the last vertex drawn. Label it 2 and move back to the parent of the child.
- (iii) Perform step (ii) above for all values of i building the tree from left to right. The process ends at the root when the last integer in the tableau was considered.
- (iv) In the tree constructed, for any vertex i labelled 2, let the label of its sibling j on the immediate right be labelled 1. Detach the subtree rooted at j and attach it as a child of i , on the right of the existing children of i (if any).

The tree obtained had all vertices labelled either 1 or 2 and the labels of any vertices sharing an edge had a sum of not more than 3. Further, the labels of any siblings were non-decreasing from left to right forming a non-decreasing 2-plane tree on $n + 1$ vertices. Figure 4.18 represented this procedure. $\square$

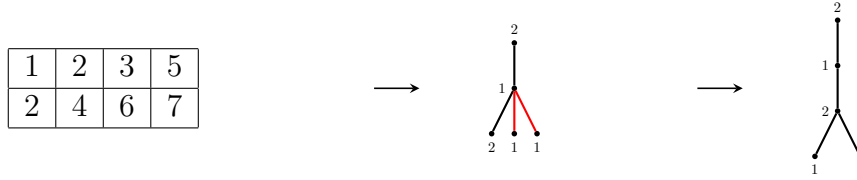


Figure 4.18: A procedure from an increasing tableau $Inc_1(2 \times 4)$ to a non-decreasing 2-plane tree on 5 vertices with 1 repeated value.

4.3.4 Bijections of trees with root labelled 1

In this section, this study constructed bijections between the set of non-decreasing 2-plane trees and the sets of large Schröder numbers and row-increasing tableaux.

4.3.4.1 Large Schröder paths

Theorem 4.16. *There exists a bijection between the set of non-decreasing 2-plane trees on $n + 1$ vertices with roots labelled 1 and the set of large Schröder paths of length $2n$.*

Proof. Given a non-decreasing 2-plane tree on $n + 1$ vertices, with root labelled 1, this study obtained a large Schröder path of length $2n$ by the procedure below: This study traversed the tree by preorder traversal and let the root to correspond with the starting point of the large Schröder path.

- (i) When a vertex labelled 1 (resp. labelled 1) was encountered as one moves away from the root, an up-step $(1, 1)$ (resp. horizontal step $(2, 0)$) was drawn.
- (ii) As one moved towards the root, if the initial point of the edge was a vertex labelled 1 (resp. labelled 2) then a down-step $(1, -1)$ (resp. do not draw any step) was drawn.

The corresponding structure obtained was a large Schröder path from $(0, 0)$ to $(2n, 0)$, comprised of the step $(0, 1)$, $(1, 0)$ and $(2, 0)$ on any of the n levels.

Conversely, a non-decreasing 2-plane tree on $n + 1$ vertices was obtained from a large Schröder path of length $2n$ as below:

- (i) Draw a root of the tree labelled 1. This study walked on the lattice path and built up the tree.
- (ii) When an up-step was encountered, a child (on the right) of the last vertex visited was drawn and labelled 1.
- (iii) For every down-step encountered, move up one step to the vertex labelled 1.
- (iv) When a horizontal-step was encountered, a child (on the right) of the last vertex visited was drawn and labelled 2. Move up one step to the parent of the child drawn.
- (v) In the tree constructed, for any vertex i labelled 2, let the label of its sibling j on the immediate right be labelled 1. Detach the subtree rooted at j and attach it as a child of i , on the right of the existing children of i (if any).
- (v) In the tree obtained, for any vertex labelled i , let the label of its sibling on the immediate right be j . If $j < i$, the subtree rooted at vertex j was cut and attached as a child of vertex i .

The resulting structure was a non-decreasing 2-plane tree on $n + 1$ vertices with root labelled 1 and the labels of siblings were non-decreasing from left to right. The bijection was illustrated in Figure 4.19. □

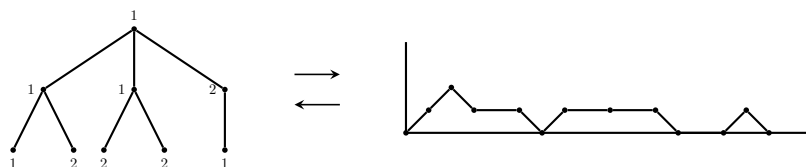


Figure 4.19: A non-decreasing 2-plane tree on 9 vertices with root labelled 1 and its corresponding large Schröder path of length 16.

4.3.4.2 Row-increasing tableaux

The set $RInc_k(2 \times n)$ of row-increasing tableaux with $2n$ entries and k repeated values was introduced and enumerated in (Du et al., 2019). They are an extension of the increasing tableaux with the entries of the rows being strictly increasing and those of the columns being weakly increasing.

Theorem 4.17. *There exists a bijection between the set of non-decreasing 2-plane trees on $n + 1$ vertices with roots labelled 1 such that k vertices are labelled 2 and the set of row-increasing tableaux, $RInc_k(2 \times n)$.*

Proof. The proof of this theorem was similar to that of Theorem 4.15 with the condition that the root in this case was labelled 1 and not 2. Since the children of the root can be labelled either 1 or 2 then it was possible to get two equal values on the same column in the tableau. This satisfied the condition that the values of the columns were weakly increasing. An illustration of this bijection was given in Table 4.3.4.2. □

Table 4.3: Non-decreasing 2-plane trees on 3 vertices with roots labelled 1 and corresponding row-increasing tableaux when $n = 2$.

Non-decreasing 2-plane tree	Row-increasing tableau				
	<table border="1" data-bbox="877 1400 981 1489"> <tr><td>1</td><td>3</td></tr> <tr><td>2</td><td>4</td></tr> </table>	1	3	2	4
1	3				
2	4				
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4.4 Weakly labelled k -plane trees

4.4.1 Introduction

Plane trees have been generalized by considering their block graphs (Okoth, 2021b; Onyango et al., 2023) and assigning labels to the vertices (Gu & Prodinger, 2009; Gu et al., 2010). This study introduced another generalization based on the latter.

Definition 4.6. A *Weakly labelled k -plane tree* is a plane tree whose root is unlabelled and all children of all internal vertices are given labels from the set $\{1, 2, \dots, k\}$ such that the labels are non-decreasing from left to right.

The following theorem, which this study proved in the next subsection, gave a formula counting these combinatorial structures.

Theorem 4.18. *There are*

$$\frac{1}{n} \binom{(k+1)n-2}{n-1} \quad (4.27)$$

weakly labelled k -plane trees on n vertices.

This study enumerated weakly labelled k -plane trees by number of vertices of a certain type, root degree, label of the eldest and youngest child of the root, number of leaves, forests and outdegree sequence in subsubsection 4.4.2 then constructed bijections between the set of weakly labelled k -plane trees and the sets of related combinatorial structures discussed in Chapter ONE.

4.4.2 Enumeration of weakly labelled k -plane trees

4.4.2.1 Occurrences of vertices of certain labels

Let $P_i(x, u_1, \dots, u_k) = P_i$ be the multivariate generating function for weakly labelled k -plane trees whose root was labelled i , where x marked a vertex and u_j marked a vertex labelled j for $j = 1, \dots, k$. Note that if the root was not labelled then any label could be assigned to it. Since $P_i(x, u_1, \dots, u_k)$ consisted of a root vertex and a sequence of subtrees rooted at vertices labelled 1, then those labelled 2, and so on, until this study got a sequence of subtrees rooted at vertices labelled k , therefore;

$$P_i = x \cdot \frac{1}{1 - u_1 P_1(x)} \cdot \frac{1}{1 - u_2 P_2(x)} \cdots \frac{1}{1 - u_k P_k(x)}. \quad (4.28)$$

Since $P_1 = P_2 = \cdots = P_k$, then from (4.28), this study obtained

$$P_i(x) = x \prod_{j=1}^k (1 - u_j P_i(x))^{-1}. \quad (4.29)$$

By Lagrange inversion formula [(Stanley, 1999, Theorem 5.4.2)], this study got

$$\begin{aligned} [x^n u_1^{a_1} \cdots u_k^{a_k}] P_i &= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^k (1 - u_j p)^{-n} \\ &= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^k \sum_{i \geq 0} \binom{-n}{i} (-u_j p)^i \\ &= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \prod_{j=1}^k \sum_{i \geq 0} \binom{n+i-1}{i} p^i u_j^i \\ &= \frac{1}{n} [p^{n-1}] p^{a_1 + \cdots + a_k} \prod_{j=1}^k \binom{n+a_j-1}{a_j} \\ &= \frac{1}{n} \prod_{j=1}^k \binom{n+a_j-1}{a_j} \end{aligned} \quad (4.30)$$

The last equality followed since $a_1 + \cdots + a_k = n - 1$, i.e., all non-root vertices were labelled with an integer in the set $\{1, 2, \dots, k\}$. Thus, this study had the following

theorem.

Theorem 4.19. *The number of weakly labelled k -plane trees on n vertices such that there are a_j vertices labelled j is given by*

$$\frac{1}{n} \prod_{j=1}^k \binom{n + a_j - 1}{a_j}. \quad (4.31)$$

If in this study set $k = 1$ in (4.31), then $a_1 = n - 1$ and it was found that there were

$$\frac{1}{n} \binom{2n - 2}{n - 1}$$

plane trees on n vertices, a Catalan number. To prove Theorem 4.18, it was summed over all a_j in (4.31), i.e.,

$$\begin{aligned} [x^n]P_i &= \sum_{a_1 + \dots + a_k = n-1} \frac{1}{n} \prod_{j=1}^k \binom{n + a_j - 1}{a_j} \\ &= \frac{1}{n} \sum_{a_1 + \dots + a_k = n-1} \prod_{j=1}^k \binom{-n}{a_j} (-1)^{a_j} \\ &= \frac{1}{n} \sum_{a_1 + \dots + a_k = n-1} (-1)^{a_1 + \dots + a_k} \prod_{j=1}^k \binom{-n}{a_j} \\ &= \frac{1}{n} (-1)^{n-1} \binom{-kn}{n-1} \\ &= \frac{1}{n} \binom{kn + (n-1) - 1}{n-1} \\ &= \frac{1}{n} \binom{(k+1)n - 2}{n-1}. \end{aligned} \quad (4.32)$$

Alternative proof of Theorem 4.18. Let $P_i(x)$ be the generating function for weakly labelled k -plane trees where x marked a vertex. Then

$$P_i = x \cdot \frac{1}{1 - P_1(x)} \cdot \frac{1}{1 - P_2(x)} \cdots \frac{1}{1 - P_k(x)}. \quad (4.33)$$

Since $P_1 = P_2 = \dots = P_k$, then

$$P_i = x \frac{1}{(1 - P_i(x))^k}.$$

By Lagrange inversion formula, this study obtained

$$\begin{aligned} [x^n]P_i &= \frac{1}{n}[p^{n-1}](1-p)^{-kn} \\ &= \frac{1}{n}[p^{n-1}] \sum_{i \geq 0} \binom{-kn}{i} (-p)^i \\ &= \frac{1}{n}[p^{n-1}] \sum_{i \geq 0} \binom{kn+i-1}{i} p^i \\ &= \frac{1}{n} \binom{kn+(n-1)-1}{n-1} = \frac{1}{n} \binom{(k+1)n-2}{n-1} \end{aligned} \quad (4.34)$$

□

4.4.2.2 Root degree and label of the eldest child of the root

This study proved the following result.

Lemma 4.1. *There are*

$$\frac{r}{n-1} \binom{(k+1)(n-1)-r-1}{n-r-1} \quad (4.35)$$

weakly labelled k -plane trees on n vertices with root of degree r such that all r children of the root are all labelled by i .

Proof. This study extracted the coefficient of x^n in xP_i^r as follows.

$$\begin{aligned}
[x^n]xP_i^r &= [x^{n-1}]P_i^r = \frac{r}{n-1}[p^{n-r-1}](1-p)^{-k(n-1)} \\
&= \frac{r}{n-1}[p^{n-r-1}] \sum_{j \geq 0} \binom{-k(n-1)}{j} (-p)^j \\
&= \frac{r}{n-1} \binom{k(n-1) + (n-r-1) - 1}{n-r-1} \\
&= \frac{r}{n-1} \binom{(k+1)(n-1) - r - 1}{n-r-1}. \tag{4.36}
\end{aligned}$$

□

Upon setting $k = 1$ in (4.35), this study obtained the formula for the number of plane trees with n such that the root was of degree r . Summing over all r in (4.35), this study found that there were

$$\frac{1}{n-1} \binom{(k-1)(n-1)}{n-2} \tag{4.37}$$

weakly labelled k -trees on n vertices. Bijections of these structures were considered in Subsection 4.4.3.4.

Theorem 4.20. *The number of weakly labelled k -plane trees on n vertices such that the root is of degree r and the eldest child of the root was labelled i is given by*

$$\frac{r}{n-1} \binom{(k+1)(n-1) - r - 1}{n-r-1} \binom{k-i+r-1}{r-1}. \tag{4.38}$$

Proof. By Lemma 4.1, there were

$$\frac{r}{n-1} \binom{(k+1)(n-1) - r - 1}{n-r-1}$$

weakly labelled k -plane trees on n vertices such that all the r children of the root were labelled j . If the eldest child of the root was to be relabelled i then the siblings of the eldest child of the root must be relabelled as siblings labelled i , followed by

a sequence of siblings labelled $i + 1$, so on until a sequence of siblings labelled k , on the far right. This was equivalent choosing $r - 1$ objects from a set with $k - i + 1$ elements such that repetitions was allowed. The number of such ways was

$$\binom{k - i + r - 1}{r - 1}. \quad (4.39)$$

The result thus followed by the product rule of counting. $\square$

This study obtained the following corollary upon summing over all i in (4.38), making use of hockey stick identity.

Corollary 4.5. *There are*

$$\frac{r}{n - 1} \binom{(k + 1)(n - 1) - r - 1}{n - r - 1} \binom{k + r - 1}{r} \quad (4.40)$$

weakly labelled k -plane trees on n vertices with root degree r .

The following version of Vandermonde type convolution was obtained in [(Kucukoglu & Simsek, 2017, Corollary 5.5)].

Identity 4.1. Let m_1 and m_2 be positive integers and n a non-negative integer then,

$$\sum_{j=0}^n \binom{m_1 + j - 1}{j} \binom{m_2 + n - j - 1}{n - j} = \binom{m_1 + m_2 + n - 1}{n}.$$

This study summed over all r in (4.40), making of using Identity 4.1, and obtained (4.27) as the number of weakly labelled k -plane trees on n vertices. Setting $i = k - i + 1$ in (4.38) this study obtained:

Corollary 4.6. *The number of weakly labelled k -plane trees on n vertices whose*

root is of degree r such that the youngest child of the root is labelled i is given by

$$\frac{r}{n-1} \binom{(k+1)(n-1)-r-1}{n-r-1} \binom{i+r-2}{r-1}. \quad (4.41)$$

Summing over all r in (4.38) and making use of Identity (4.1), this study arrived at the following result.

Corollary 4.7. *The number of weakly labelled k -plane trees on n vertices in which the eldest child of the root is labelled i is*

$$\frac{2k-i+1}{kn-i+1} \binom{(k+1)n-i-2}{n-2}. \quad (4.42)$$

By setting $i = 1$ in (4.42), this study found that:

Corollary 4.8. *There are*

$$\frac{2}{n} \binom{(k+1)n-3}{n-2} \quad (4.43)$$

weakly labelled k -plane trees on n vertices in which the eldest child of the root was labelled 1.

When this study deleted an edge connecting the root vertex to its eldest child and introduced a new vertex as a root so that the eldest child of the initial root became the eldest child of the new root and the initial root (now labelled 1) became the second child (youngest child) of the new root then, Corollary 4.8 followed easily by setting $n = n + 1$ and $r = 2$ in (4.35). The same arguments could be used to show that (4.43) also counted weakly labelled k -plane trees on n vertices such that the youngest child of the root was labelled k .

Corollary 4.9. *The number of weakly labelled k -plane trees on n vertices in which*

the youngest child of the root is labelled i is

$$\frac{k+i}{k(n-1)+i} \binom{(k+1)(n-1)+i-2}{n-2}. \quad (4.44)$$

Proof. Sum over all r in (4.41). □

Setting $i = k$ in (4.44), this study found that (4.42) also gave the number of weakly labelled k -plane trees on n vertices such that the youngest child of the root was labelled k .

4.4.2.3 Leaves

This study also enumerated weakly labelled k -plane trees according to number of leaves.

Theorem 4.21. *There are*

$$\frac{1}{n} \binom{n}{\ell} \sum_{q \geq 0} \binom{n-\ell}{q} (-1)^q \prod_{j=1}^k \binom{n-\ell-q+a_j-1}{a_j} \quad (4.45)$$

weakly labelled k -plane trees on n vertices with ℓ leaves such that a_j vertices are labelled j .

Proof. Let $P_i(x, v, u_1, \dots, u_k)$ be the multivariate generating function for weakly labelled k -plane trees with roots labelled i where x marked a vertex, v marked a leaf and u_j marked a vertex labelled j for $j = 1, 2, \dots, k$. Then the study got,

$$P_i(x, v) = xv - x + x \cdot \frac{1}{1 - u_1 P_1(x)} \cdot \frac{1}{1 - u_2 P_2(x)} \cdots \frac{1}{1 - u_k P_k(x)}. \quad (4.46)$$

Since $P_1 = P_2 = \dots = P_k$, it followed that

$$P_i = xv - x + x \frac{1}{\prod_{j=1}^k (1 - u_j P_i)} = x \left(v + \frac{1 - \prod_{j=1}^k (1 - u_j P_i)}{\prod_{j=1}^k (1 - u_j P_i)} \right). \quad (4.47)$$

Application of Lagrange inversion gave,

$$\begin{aligned}
[x^n v^\ell u_1^{a_1} \cdots u_k^{a_k}] P_i &= \frac{1}{n} [p^{n-1} v^\ell u_1^{a_1} \cdots u_k^{a_k}] \left(v + \frac{1 - \prod_{j=1}^k (1 - u_j p)}{\prod_{j=1}^k (1 - u_j p)} \right)^n \\
&= \frac{1}{n} [p^{n-1} v^\ell u_1^{a_1} \cdots u_k^{a_k}] \sum_{j \geq 0} \binom{n}{j} v^j \left(\frac{1 - \prod_{j=1}^k (1 - u_j p)}{\prod_{j=1}^k (1 - u_j p)} \right)^{n-j} \\
&= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \binom{n}{\ell} \left(1 - \prod_{j=1}^k (1 - u_j p) \right)^{n-\ell} \prod_{j=1}^k (1 - u_j p)^{-(n-\ell)} \\
&= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \binom{n}{\ell} \sum_{q \geq 0} \binom{n-\ell}{q} (-1)^q \prod_{j=1}^k (1 - u_j p)^{q-(n-\ell)} \\
&= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \binom{n}{\ell} \sum_{q \geq 0} \binom{n-\ell}{q} (-1)^q \prod_{j=1}^k \sum_{b \geq 0} \binom{-(n-\ell-q)}{b} (-p)^b u_j^b \\
&= \frac{1}{n} [p^{n-1} u_1^{a_1} \cdots u_k^{a_k}] \binom{n}{\ell} \sum_{q \geq 0} \binom{n-\ell}{q} (-1)^q \prod_{j=1}^k \sum_{b \geq 0} \binom{n-\ell-q+b-1}{b} p^b u_j^b \\
&= \frac{1}{n} \binom{n}{\ell} \sum_{q \geq 0} \binom{n-\ell}{q} (-1)^q \prod_{j=1}^k \binom{n-\ell-q+a_j-1}{a_j}. \tag{4.48}
\end{aligned}$$

The last equality followed since $u_1 + \cdots + u_k = n - 1$. Thus the proof. $\square$

This following corollary was then obtained by summing over all a_j in (4.45).

Corollary 4.10. *There are*

$$\frac{1}{n} \binom{n}{\ell} \sum_{a \geq 0} \binom{n-\ell}{a} \binom{(k+1)n - k\ell - ka - 2}{n-1} (-1)^a \tag{4.49}$$

weakly labelled k -plane trees on n vertices with ℓ leaves.

Proof. Let $P_i(x, v)$ be the bivariate generating function for weakly labelled k -plane trees with roots labelled i where x and v marked a vertex and a leaf respectively. Then this study got,

$$P_i(x, v) = xv - x + x \cdot \frac{1}{1 - P_1(x)} \cdot \frac{1}{1 - P_2(x)} \cdots \frac{1}{1 - P_k(x)}. \tag{4.50}$$

Since $P_1 = P_2 = \dots = P_k$, then

$$P_i = xv - x + x \frac{1}{(1 - P_i)^k} = x \left(v + \frac{1 - (1 - P_i)^k}{(1 - P_i)^k} \right). \quad (4.51)$$

By Lagrange inversion, this study got

$$\begin{aligned} [x^n v^\ell] P_i &= \frac{1}{n} [p^{n-1} v^\ell] \left(v + \frac{1 - (1 - p)^k}{(1 - p)^k} \right)^n = \frac{1}{n} [p^{n-1} v^\ell] \sum_{j \geq 0} \binom{n}{j} v^j \left(\frac{1 - (1 - p)^k}{(1 - p)^k} \right)^{n-j} \\ &= \frac{1}{n} [p^{n-1}] \binom{n}{\ell} (1 - (1 - p)^k)^{n-\ell} (1 - p)^{-k(n-\ell)} \\ &= \frac{1}{n} [p^{n-1}] \binom{n}{\ell} \sum_{a \geq 0} \binom{n-\ell}{a} (-1)^a (1 - p)^{ak - k(n-\ell)} \\ &= \frac{1}{n} [p^{n-1}] \binom{n}{\ell} \sum_{a \geq 0} \binom{n-\ell}{a} (-1)^a \sum_{b \geq 0} \binom{-k(n-\ell-a)}{b} (-p)^b \\ &= \frac{1}{n} [p^{n-1}] \binom{n}{\ell} \sum_{a \geq 0} \binom{n-\ell}{a} (-1)^a \sum_{b \geq 0} \binom{k(n-\ell-a) + b - 1}{b} p^b \\ &= \frac{1}{n} \binom{n}{\ell} \sum_{a \geq 0} \binom{n-\ell}{a} (-1)^a \binom{(k+1)n - k\ell - ka - 2}{n-1}. \end{aligned} \quad (4.52)$$

□

Equation (4.49) could be rewritten as

$$\frac{1}{n} \binom{n}{\ell} \sum_{a \geq 0} \binom{\ell - n + a - 1}{a} \binom{(k+1)n - k\ell - ka - 2}{kn - k\ell - ka - 1}. \quad (4.53)$$

Setting $k = 1$ and summing over all a , this study obtained the Nayarana number

$$\frac{1}{n} \binom{n}{\ell} \binom{n-2}{n-\ell-1} = \frac{1}{n-1} \binom{n-1}{\ell} \binom{n-1}{\ell-1} \quad (4.54)$$

which gave the number of plane trees on n vertices with ℓ leaves.

4.4.2.4 Forests

In this subsection, this study was interested in obtaining a formula for weakly labelled k -plane trees with a given number of components.

Theorem 4.22. *The number of forests of weakly labelled k -plane trees on n vertices and c components such that the roots of the components are labelled $1, 2, \dots, c$ from left to right is*

$$\frac{c}{n} \binom{(k+1)n - c - 1}{n - c}. \quad (4.55)$$

Proof. Let $P(x) = P$ be the generating function for weakly labelled k -plane trees on n vertices. Then,

$$P = \frac{x}{(1 - P)^k}.$$

A forest is a sequence of trees, so this study extracted the coefficient of x^n in P^c as follows.

$$\begin{aligned} [x^n]P^c &= \frac{c}{n} [p^{n-c}] (1 - p)^{-kn} = \frac{c}{n} [p^{n-c}] \sum_{j \geq 0} \binom{-kn}{j} (-p)^j \\ &= \frac{c}{n} \binom{kn + (n - c) - 1}{n - c} = \frac{c}{n} \binom{(k+1)n - c - 1}{n - c}. \end{aligned} \quad (4.56)$$

The roots of the components were then labelled $1, 2, \dots, c$ to avoid redundancies. $\square$

This study remarked that (4.55) could be obtained by considering a weakly labelled k -plane tree on $n + 1$ vertices such that the root was of degree c and all the children of the root were labelled i . The forest was attained by deleting the root vertex and all the edges incident to it. The roots of the components were then labelled $1, 2, \dots, c$ from left to right. The formula was thus obtained by setting $n = n + 1$ and $r = c$ in (4.35).

Setting $c = 1$ in (4.55), this study obtained the number of weakly labelled k -plane trees on n vertices. Upon setting $k = 1$ in (4.55), this study obtained a formula for the number of forests of plane trees on n vertices and c components.

4.4.2.5 Outdegree sequence

Theorem 4.23. *The number of weakly labelled k -plane trees on n vertices such that there are d_i vertices with i children where $i = 0, 1, \dots$ is*

$$\frac{1}{n} \binom{n}{d_0, d_1, \dots} \prod_{r \geq 0} \binom{k+r-1}{r}^{d_r}. \quad (4.57)$$

Proof. There were

$$\frac{1}{n} \binom{n}{d_0, d_1, \dots}$$

plane trees on n vertices with outdegree sequence $(d_0, d_1, \dots)$ (Rote, 1997). By the proof of Theorem 4.20, there were $\binom{r+k-1}{r}$ ways to assign labels to r children of a vertex. By the product rule of counting the result thus followed. $\square$

This study obtained the following identity by summing over all d_i in (4.57) and using (4.28).

Identity 4.2. Let $d_0, d_1, \dots$ be non-negative integers. Also let k and n be positive integers. Then

$$\sum_{d_0+d_1+\dots=n} \binom{n}{d_0, d_1, \dots} \prod_{r \geq 0} \binom{k+r-1}{r}^{d_r} = \binom{(k+1)n-2}{n-1}.$$

4.4.3 Bijections

This subsection was devoted to construction of bijections of related combinatorial structures which were related to weakly labelled k -plane trees.

4.4.3.1 k -ary Husimi graphs

In the sequel, this study relate weakly labelled k -plane trees to k -ary Husimi graphs.

Theorem 4.24. *There is a bijection between the set of k -ary Husimi graphs on n vertices and the set of weakly labelled k -plane trees on n vertices.*

Proof. Consider a k -ary Husimi graph on n vertices. This study obtained a weakly labelled k -plane tree with n vertices using the following procedure:

- (i) By Breadth First Search (BFS), it traversed the k -ary Husimi graph and gave label i to all the vertices on the block (which appeared at a lower level) if the block was the i^{th} block child of its parent. A block child j of a vertex i was a block attached to i such that all other vertices of b are at a lower level.
- (ii) Let $x, x_1, x_2, \dots, x_m$ be the vertices of a block of size m where x was the root and x_i was a vertex which appears on the left of x_{i+1} for all $i = 1, \dots, m - 1$. Delete all edges connecting x_i 's and create edges between x and all x_i 's where $i = 1, \dots, m$. Deletion of edges and creation of new edges ensured that the resultant structure was a plane tree.
- (iii) Repeat steps (i) and (ii) until all the vertices of the graph were labelled. Note that no vertex was deleted nor introduced, so the tree obtained had the same number of vertices as the Husimi graph.

Since the labels of all the vertices sharing a parent were dependent on their block child position in the k -ary Husimi graph, then these labels were strictly non-decreasing from left to right, creating a weakly labelled k -plane tree on n vertices.

This study obtained a reverse procedure as follows:

- (i) Starting with a weakly labelled k -plane tree, create a block consisting of vertices on the same level but with the same label, together with their parent.

- (iii) Position the block children according to the labels on the vertices and delete the labels, i.e, if the vertices were labelled i , then the block took position i in the k -ary Husimi graph. The structure obtained was a k -ary Husimi graph.

Figure 4.20 was a depiction of the bijection. □

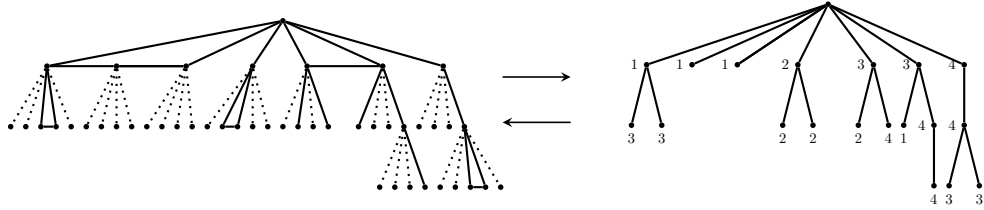


Figure 4.20: A 4-ary Husimi graph on 18 vertices with its corresponding weakly labelled 4-plane tree.

4.4.3.2 Dyck paths

In this subsection, this study obtained a bijection between the set of weakly labelled k -plane trees on n vertices and the set of Dyck paths with $n - 1$ up-steps labelled with integers from the set $\{1, 2, \dots, k\}$ which were non-decreasing from left to right.

Theorem 4.25. *There is a bijection between the set of weakly labelled k -plane trees on n vertices and the set of Dyck paths of length $2(n - 1)$ consisting of $n - 1$ unit up-steps labelled with integers from the set $\{1, 2, \dots, k\}$ such that labels of the up-steps starting at the origin are non-decreasing from left to right.*

Proof. This study proceeded as follows to obtain a Dyck path of length $2n - 2$ described in the statement of the theorem from a weakly labelled k -plane tree on n vertices: Using preorder traversal, this study visited the vertices of the tree and drew an up-step of the Dyck path for each vertex visited and a down-step if one visited the vertex for the last time as one moved towards the root. Label each up-step with the label of the corresponding vertex and don't draw any step for the root.

The Dyck path constructed was of length $2(n - 1)$ since each vertex (except the root) accounts for an up-step and a down-step. Since the labels of the children of each internal vertex were non-decreasing, then the labels of the up-steps starting at the origin were non-decreasing from left to right.

For the converse, this study started with a vertex which would be the root of the weakly labelled k -plane tree. Now, one moved through the Dyck path from left to right and recursively built the tree as follows: For each up-step encountered, draw a child of the last vertex visited and give it the label of the up-step. For each down-step encountered move up (on the right) to the parent of the last vertex visited.

Note that the number of up-steps of the Dyck path starting at the origin corresponded to the number of children of the root and the labels of up-steps at the origin were non-decreasing from left to right, thus this study obtained a weakly labelled k -plane tree. This procedure was illustrated in Figure 4.21. $\square$

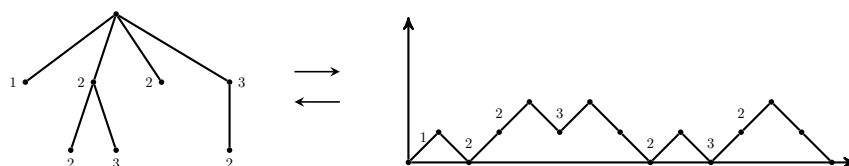


Figure 4.21: A weakly labelled 3-plane tree on 8 vertices and its corresponding Dyck path of length 14.

4.4.3.3 Plane trees with given number of leaves

This study proved the following result:

Theorem 4.26. *There is a bijection between the set of weakly labelled k -plane trees on n vertices and the set of plane trees with kn vertices where each internal vertex has exactly $k - 1$ leaves.*

Proof. This study constructed a plane tree satisfying the stated conditions from a given a weakly labelled k -plane tree on n vertices by the following procedure: For

each vertex of the weakly labelled k -plane tree, if there was no child of the vertex labelled $1 \leq i \leq k$, draw an imaginary leaf labelled i . Now, draw a leaf separating vertices of labels j and $j + 1$ where $1 \leq j \leq (k - 1)$. Delete the imaginary leaves and the labels of the vertex and its children.

Each internal vertex (inclusive of the root) had $k - 1$ leaves, since this study drew leaves to separate vertices of labels j and $j + 1$ where $1 \leq j \leq (k - 1)$. The resulting structure was a plane tree with kn vertices (each internal vertex contributes $k - 1$ leaves, giving a total of $n(k - 1)$ leaves which together with the initial n internal vertices give a total of kn vertices).

Conversely, This study obtained a weakly labelled k -plane tree as follows:

- (i) Consider all non-leaf children of each vertex x and let $\ell_1, \ell_2, \dots, \ell_{k-1}$ be the leaf children of x from left to right. Label all non-leaf children of x on the left of ℓ_1 as 1, all non-leaf children between ℓ_i and ℓ_{i+1} as $i + 1$ and those on the right of ℓ_{k-1} (if any) as k .
- (ii) Delete all the leaves.

The resulting structure had all vertices on the same level labelled with non-decreasing values and the vertices had labels from the set $\{1, 2, \dots, k\}$ from left to right hence it was a weakly labelled k -plane tree.

See Figure 4.22 for an example. □

4.4.3.4 Weakly labelled k -trees and k -plane trees

This study constructed two bijections regarding weakly labelled k -plane trees and k -plane trees as well as weakly labelled k -trees and k -plane trees.

Theorem 4.27. *There is a bijection between the set of weakly labelled k -plane trees on n vertices and the set of k -plane trees on n vertices whose roots are labelled 1.*

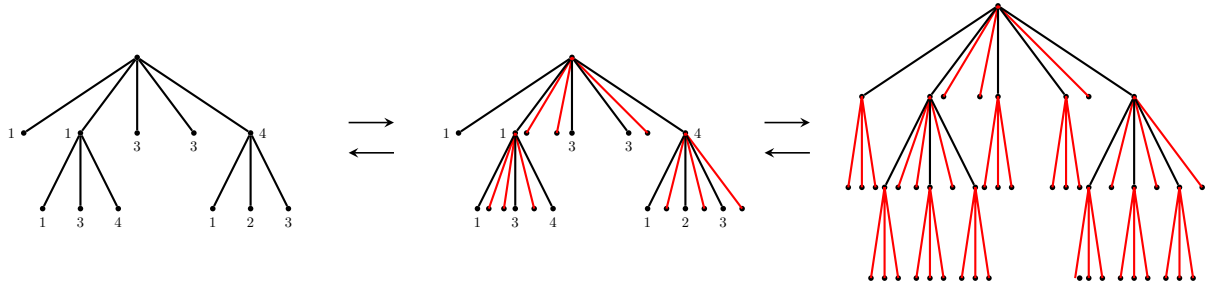


Figure 4.22: A weakly labelled 4-plane tree on 12 vertices and its corresponding plane tree on 48 vertices where each internal vertex having exactly 3 leaves.

Proof. Consider a weakly labelled k -plane tree T with n vertices. This study obtained its corresponding k -plane tree on n vertices by the following procedure: This study traversed T using BFS.

- (i) Label the root of the tree as 1.
- (ii) Let x be a non-root vertex and $x_1, x_2, \dots, x_m$ be its children arranged in this order from left to right. Also let y be the label of the parent of x . Let the labels of $x, x_1, x_2, \dots, x_m$ be $s, t_1, t_2, \dots, t_m$ respectively. Moreover, let x_r be the first vertex visited with the property that $s + t_r > k + 1$. This means that $s + t_i > k + 1$ for all $i \geq r$. Let S be a subtree of T rooted at x with $x_r, x_{r+1}, \dots, x_m$ as its children. Adjoin S to the parent of x such that $x_r, x_{r+1}, \dots, x_m$ are on the immediate right of x in this order.
- (iii) Assign the label $s + t_i - (k + 1)$ to the vertices initially labelled x_i for $i = r, r + 1, \dots, m$.
- (iv) Repeat steps (ii) and (iii) until all the vertices are traversed.

Since for each $i = 1, 2, \dots, m$, this study had $t_i \leq k$ and $y + s \leq k + 1$ then $y + (s + t_i - (k + 1)) \leq t_i \leq k < k + 1$ which satisfied the property that the sum of the labels of any adjacent vertices was at most $k + 1$. Hence the structure obtained was a k -plane tree.

Conversely, This study obtain a weakly labelled k -plane tree from a k -plane tree P , by the following procedure: This study traversed P using BFS.

- (i) Delete the label of the root.
- (ii) Let vertex x be a vertex of P with the property that its label was s and it has at least one sibling on its right with a label less than s . Moreover, let y be the first sibling of x , on its right, with a label greater than or equal to s . Note that This study may not have such a vertex. Let $x_1, x_2, \dots, x_m$ be the children of the root of x which appear in this order from left to right betThis studyen x and y or all of them to the right of x if there was no such a vertex y . Let the labels of $x_1, x_2, \dots, x_m$ be $t_1, t_2, \dots, t_m$ respectively. Detach the subtrees rooted at $x_1, x_2, \dots, x_m$ and attach them, in this order from left to right, to x on the right of the existing children.
- (iii) Relabel vertex x_i as $t_i - s + k + 1$ for $i = 1, 2, \dots, m$.
- (iv) Repeat steps (ii) and (iii) until all the vertices are traversed.

Since for each $i = 1, 2, \dots, m$, this study had $t_i < s$, then $t_i - s + k + 1 < k + 1$. So, the vertices were labelled with integers less than $k + 1$. Also, $s + (t_i - s + k + 1) \geq k + 1$. So the vertices $x_1, x_2, \dots, x_m$ were relabelled with integers greater than their initial labels. Hence the structure obtained was a weakly labelled k -plane tree. This bijection was illustrated in Figure 4.23. $\square$

Remark 4.1. In (Okoth, 2022a), a bijection between the set of k -plane trees on n vertices with root labelled by 1 and the set of Dyck paths from $(0, 0)$ to $(2n - 2, 0)$ with up-steps labelled from the set $\{1, 2, 3, \dots, k\}$ such that the sum of any two consecutive up-steps was at most $k + 1$ was constructed. From Theorem 4.25, it followed that there was a bijection between the set of these Dyck paths of length $2n - 2$ and the set of weakly labelled k -plane trees on n vertices, i.e., the correspondence was shown in Figure 4.24.

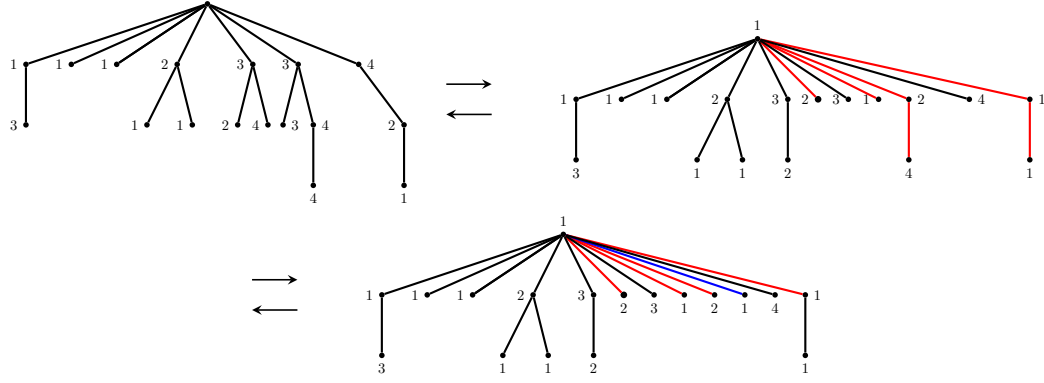


Figure 4.23: A weakly labelled 4-plane tree and its corresponding 4-plane tree with root labelled 1.

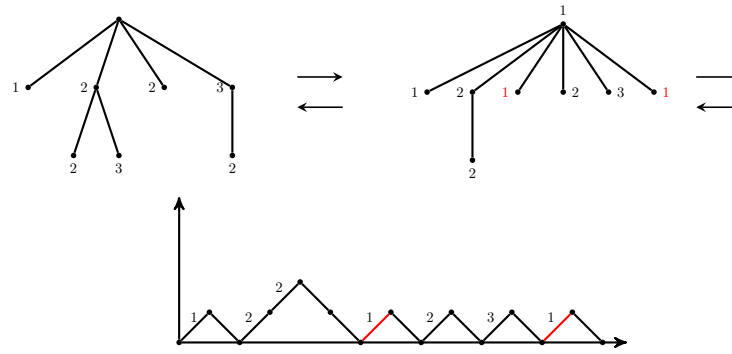


Figure 4.24: A weakly labelled 3-plane tree on 8 vertices with its corresponding a Dyck path from $(0, 0)$ to $(14, 0)$ where the sum of any two consecutive up-steps was at most 4 constructed via a 3-plane tree with root labelled 1.

By imitating the procedure used in the proof of Theorem 4.25, this study obtain a corresponding k -plane tree given a weakly labelled k -tree. The procedure was easily reversed. The result was stated in the following theorem.

Theorem 4.28. *There is a bijection between the set of k -plane trees with root labelled k on n vertices and the set of weakly labelled k -trees on n vertices.*

The bijection of Theorem 4.26 was with illustrated in the Figure 4.25.

4.4.3.5 k -Binary trees

This study used the rotation correspondence (Flajolet & Sedgewick, 2009), which gave a bijection between the set of plane trees and the set of binary trees to come

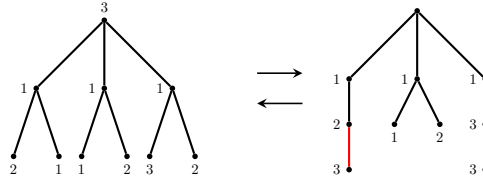


Figure 4.25: A 3-plane tree with root labelled 3 and its corresponding weakly labelled 3-tree on 10 vertices.

up with a bijection between the set of weakly labelled k -plane trees on n vertices with root labelled 1 and the set of k -binary trees on $n - 1$ vertices whose roots are labelled with integers in the set $i = \{1, 2, \dots, k\}$.

Theorem 4.29. *There is a bijection between the set of weakly labelled k -plane trees with n vertices and the set of k -binary trees on $n - 1$ vertices whose roots are labelled with integers in the set $i = \{1, 2, \dots, k\}$.*

Proof. Given a weakly labelled k -plane tree, this study obtained its corresponding k -binary tree by the following procedure:

- (i) Let x be the root vertex and $x_1, x_2, \dots, x_m$ be its children. Draw edges connecting vertices x_i and x_{i+1} where $i = 1, 2, \dots, m - 1$ if $m \geq 2$. Otherwise, do not draw any edge. Delete x and the edges incident to it.
- (ii) Let y be a non root vertex and $y_1, y_2, \dots, y_m$ be its children. Draw edges connecting vertices y_i and y_{i+1} where $i = 1, 2, \dots, m - 1$ if $m \geq 2$. Otherwise, do not draw any edge. Delete edges connecting y and y_i where $i = 2, 3, \dots, m$, there are such edges. Otherwise, do not delete any edge.
- (iii) Rotate the resulting structure in a clockwise direction.

The structure obtained was a binary tree with vertices labelled with integers from the set $\{1, 2, \dots, k\}$ such that for any vertex i with a right child j , the label of j was always greater than or equal to that of i . This was a k -binary tree. The root of the resulting k -binary tree was labelled with the label of the leftmost child of the root of

the weakly labelled k -plane tree. Since the root of the weakly labelled k -plane tree on n vertices was deleted then its corresponding k -binary tree has $n - 1$ vertices.

Conversely, to obtain a weakly labelled k -plane tree on n vertices given a k -binary tree on $n - 1$ vertices, This study follow the following procedure;

- (i) Rotate the k -binary tree in an anti-clockwise direction.
- (ii) Consider all the vertices $x_1, x_2, \dots, x_m$ on the first level connected by a path. Introduce a vertex p as the root of the tree and draw edges from p to all the vertices $x_1, x_2, \dots, x_m$. Delete edges from x_i to x_{i+1} where $i = 1, 2, \dots, m - 1$.
- (iii) For the subsequent levels, consider all the vertices $y_1, y_2, \dots, y_m$ on the same level connected by a path with the left most vertex y_1 connected to its parent y . Draw edges from y to $y_2, \dots, y_m$ and delete edges from y_i to y_{i+1} where $i = 1, 2, \dots, m - 1$.

Since this study introduced an extra vertex p which was the root of our resulting structure and the vertices of the resulting tree had labels from $1, 2, \dots, k$, the structure obtained was a weakly labelled k -plane tree n vertices.

This bijection was illustrated in Figure 4.26. □

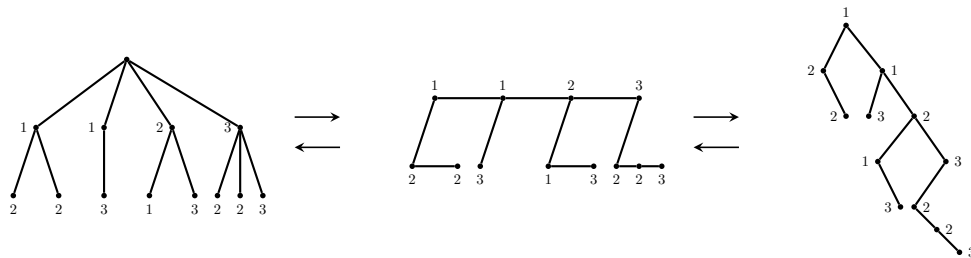


Figure 4.26: A weakly labelled 3-plane tree on 13 vertices and its corresponding 3-binary tree on 12 vertices.

4.4.3.6 S -Motzkin paths

This study constructed a bijection between the set of S -Motzkin paths and the set of weakly labelled 2-trees.

Theorem 4.30. *There was a bijection between the set of S -Motzkin paths of length $3n$ and the set of weakly labelled 2-trees on $n + 1$ vertices.*

Proof. Consider a weakly labelled 2-tree on $n + 1$ vertices. Now, this study assigned a pair (a, b) to each vertex (excluding the root), where a denoted the number of children of the vertex and b was the number of such children labelled 1. This study obtained the S -Motzkin path of length $3n$ as follows:

- (i) Identify the left most leaf, and in its place, draw an initial horizontal step of the S -Motzkin path.
- (ii) Using postorder traversal, This study visit the vertices of the tree. Let vertex i be assigned the pair (a, b) . For each vertex i (except the initial leaf and root), draw an $\nearrow$ step folloThis studyd by $b \searrow$ steps then a horizontal step folloThis studyd by $a - b \searrow$ steps.
- (iii) For the root, add an $\nearrow$ step folloThis studyd by $c \searrow$ steps where c was the degree of the root.

The structure obtained started with a horizontal step (step (i)) and the up-steps and horizontal steps were always alternating (step (ii)). Since the left most leaf did not contribute an up-step and a down-step, there were n up-steps and down-steps each. Moreover, the root did not contribute a horizontal step (step (iii)), so there were n such steps. Thus, the structure was an S -Motzkin path of length $3n$.

An example of how the procedure worked was given in Figure 4.27.

Conversely, to obtain a weakly labelled 2-tree from an S -Motzkin path, This study broke down the S -Motzkin path into pieces as in (Prodinger et al., 2020). That was, each piece contained an $\nearrow$ step followed by a series of $\searrow$ steps (if available) and a horizontal step followed by a series of $\searrow$ steps (if available) with the initial horizontal step as an independent piece and then followed the procedure below:

- Since this study introduced a root in step (i), the structure obtained had $n + 1$ vertices (if the S -Motzkin has n up steps, n down steps and n horizontal steps). Also all the children of the root were labelled 1 and other the vertices (except the root) were labelled either 1 or 2. This was a weakly labelled 2-tree. This procedure was described in Figure 4.28.

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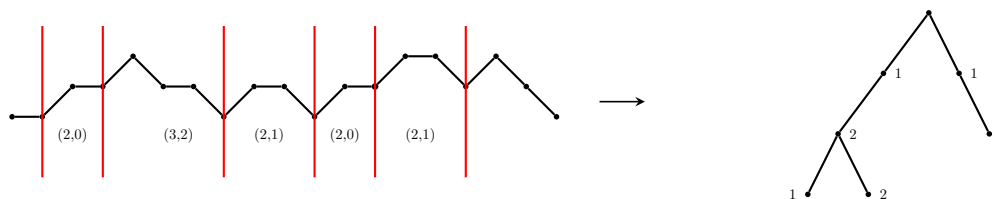


Figure 4.28: From an S -Motzkin path of length $n = 6$ to a weakly labelled 2-tree on 7 vertices.

Remark 4.2. By Theorems 4.26 and 4.27, it followed that there was a bijection between the set of S -Motzkin paths of length $3n$ and the set of 2-plane trees on $n + 1$ vertices with root labelled 2.

4.4.3.7 T -Motzkin paths

This study was interested in getting a bijection between the set of T -Motzkin paths and the set of weakly labelled 2-plane trees.

Theorem 4.31. *There is a bijection between the set of T -Motzkin paths of length $3n$ and the set of weakly labelled 2-plane trees on $n + 1$ vertices.*

Proof. To obtain a T -Motzkin path from a weakly labelled 2-plane tree, this study followed the following procedure:

- (i) Assign each vertex i of the weakly labelled 2-plane tree (including the root), a pair (a, b) where a denotes the total number of children of i and b denotes the number of children of i labelled 1.
- (ii) Delete the left most leaf of the tree.
- (iii) This study traverse the tree using postorder traversal and for each vertex i , draw an $/$ step, followed by $b \setminus$ steps then a horizontal step, followed by $a - b \setminus$ steps.

Since this study deleted the left most leaf from the tree (step (ii)), the structure obtained had n up-steps, n down-steps and n horizontal steps and it started with

an up-step. The structure was thus a T -Motzkin path of length $3n$. Figure 4.29 was an illustration of the procedure.

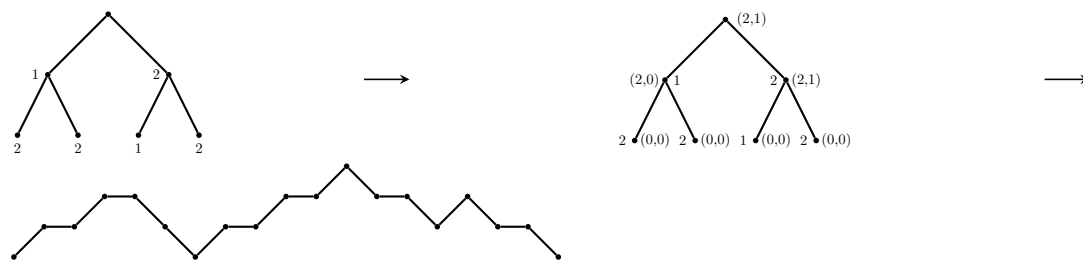


Figure 4.29: A T -Motzkin path of length 18 obtained from a weakly labelled 2-plane tree on 7 vertices.

Conversely, given a T -Motzkin path, this study constructed a weakly labelled 2-plane tree as follows:

- (i) Break down the T -Motzkin path into pieces, starting from the left; each piece consisting of an up-step, followed by a series of $\searrow$ steps (if any) and a horizontal step, followed by a series of $\searrow$ (if any).
- (ii) Consider the piece on the far right. Let the total number of its $\searrow$ steps be c . Draw a vertex with c children. This vertex was the root of the tree. Further, consider the position of horizontal step in this piece and let the number of $\searrow$ steps on the right of the horizontal step be s . In the tree this study constructed, label the right most s children of the root with 2. Similarly, let the number of $\searrow$ steps appearing on the left of the horizontal step be t and label the t left most children of the root with 1.
- (iii) For every other piece of the T -Motzkin path working from right to left, let i be the position of the horizontal step and j be the number of $\searrow$ steps and assign to each piece the pair (i, j) . Traversing the tree in preorder, working from right to left, assign $i - 2$ children labelled 1 and $j + 2 - i$ children labelled 2 (non-decreasing from left to right) on every available leaf.

The left most leaf of the tree does not correspond to any piece from the T -Motzkin path, hence the tree obtained has $n + 1$ vertices if the T -Motzkin path has n up-steps.

All vertices of the tree are labelled by either 1 or 2 and the labels of the vertices are non-decreasing from left to right thus the structure obtained was a weakly labelled 2-plane tree on $n + 1$ vertices.

An illustration of this procedure was provided in Figure 4.30. □

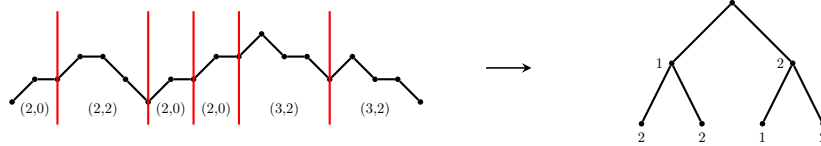


Figure 4.30: A weakly labelled 2-plane tree on 7 vertices obtained from a T -Motzkin path of length 18.

Corollary 4.11. *There exists a bijection between the set of T -Motzkin paths of length $3n$ and the set of binary Husimi graphs on $n + 1$ vertices enumerated by*

$$\frac{1}{n+1} \binom{3n+1}{n}. \quad (4.58)$$

Proof. In Theorem 4.24, this study obtained a bijection between the set of k -ary Husimi graphs on n vertices and the set of weakly labelled k -plane trees on n vertices. Since Theorem 4.28 gave a bijection between the set of T -Motzkin paths of length $3n$ and the set of weakly labelled 2-plane trees on $n + 1$ vertices, this study concluded that the set of T -Motzkin paths of length $3n$ and the set of binary Husimi graphs on $n + 1$ vertices were in bijection and hence both structures were enumerated using the same Formula (4.58). □

4.4.3.8 Ways of connecting points

This study described a bijection between the set of 2-plane trees on $n + 1$ vertices with roots labelled 1 and the set of ways of connecting $2n$ points, in a line, labelled $1, 2, \dots, 2n$ with either zero or more non-crossing arcs such that the maximal sequence of isolated points has even length.

Theorem 4.32. *There was a bijection between the set of 2-plane trees on $n + 1$ vertices with root labelled 1 and the set of ways of connecting $2n$ points, in a line, labelled $1, 2, \dots, 2n$ with either zero or more non-crossing arcs such that the maximal sequence of isolated points has an even length.*

Proof. Starting with a 2-plane tree with n edges, the corresponding way of connecting $2n$ points labelled $1, 2, \dots, 2n$ satisfying the conditions in the statement of the theorem was as follows:

- (i) Draw $2n$ points in a line ordered from left to right.
- (ii) If the root of the tree has degree m , split the tree into m ordered sub-trees $T_1, T_2, \dots, T_m$ rooted at the children of the root in order from left to right. Consider the $2n$ points and partition them into m partitions $s_1, s_2, \dots, s_m$ where each partition s_i has twice the number of vertices of the respective subtree T_i .
- (iii) Consider a subtree T_i with j vertices, where $i = 1, 2, \dots, m$. Let $i_1, i_2, \dots, i_{2j}$ be the points in its corresponding partition. If the root of T_i was labelled 1 then draw an arc from i_1 to i_{2j} . Otherwise, do not draw any arc.
- (iv) Let $T_{i_1}, T_{i_2}, \dots, T_{i_r}$ be subtrees of T_i . So, T_i has root degree r . This subtree partitioned $i_2, i_3, \dots, i_{2j-1}$ into r parts $t_1, t_2, \dots, t_r$ such that the number of points in t_p was twice the number of vertices of T_{i_p} where $p = 1, 2, \dots, r$. For each subtree, perform step (iii).
- (v) Continue with steps (iii) and (iv) until a single vertex remains as a subtree. If the vertex was labelled 1 then draw an arc connecting the corresponding two points in the partition. Otherwise, do not draw any arc.

Note that the root of the tree was not in any partition T_i . Moreover, since for each vertex labelled 1, this study drew a curve connecting 2 points and for each vertex labelled 2, it left 2 points not connected, the resulting structure had $2n$ points.

Conversely, given a structure connecting $2n$ points in a line with either zero or more non-crossing arcs, this study obtained a 2-plane tree on $n + 1$ vertices with root labelled 1 through the following procedure:

- (i) Working from right to left, locate any isolated point and connect it to the next isolated point on its immediate left using a dotted line.
- (ii) Now, working from left to right, subdivide the obtained structure at the end of each outer curve to obtain m partitions $s_1, s_2, \dots, s_m$. Draw a root labelled 1 of degree m .
- (iii) If partition s_i , where $i = \{1, 2, \dots, m\}$, has an outer complete arc (resp. dotted arc) then label the $i - th$ child of the root 1 (resp. 2).
- (iv) Working recursively in all partitions, if there are m smaller curves within the outer curve in partition s_i , then the $i - th$ child has degree m and it's a leaf if there are no smaller arcs present.

Since each isolated point was connected to the next isolated point on its immediate left then in the tree constructed, the sum of the vertices of any 2 adjacent vertices was at most $2 + 1 = 3$. This study also introduced the root of the tree, making the tree constructed to have $n + 1$ vertices and was hence a 2-plane tree. This procedure was illustrated in Table 4.4.3.8. □

4.5 Non-decreasing 2-noncrossing trees

4.5.1 Introduction

One approach that was prominent in the enumeration of noncrossing trees was by butterfly decomposition. This concept was described in Chapter THREE. This study applied it to enumerate the non-decreasing 2-noncrossing trees.

Table 4.4: 2-plane trees on 4 vertices with roots labelled 1 and corresponding way of connecting 6 points.

2-plane tree	Connecting 6 points	2-plane tree	Connecting 6 points	2-plane tree	Connecting 6 points

$N(x)^2$ was divided by x to avoid over counting vertices. Substituting (4.60) in (4.59), this study obtained

$$N(x) = \frac{x}{1 - \frac{N(x)^2}{x}}, \quad (4.61)$$

as the generating function for noncrossing trees with x marking a vertex. Equation (4.61) was easily solved using the following form of Lagrange inversion formula upon letting $N(x) = \sqrt{x}M(x)$.

The (l, r) -representation on noncrossing trees could be extended to k -noncrossing trees. This study used the subscript of l and r to denote the label of the vertex in the k -noncrossing trees. This was illustrated in Figure 4.31.

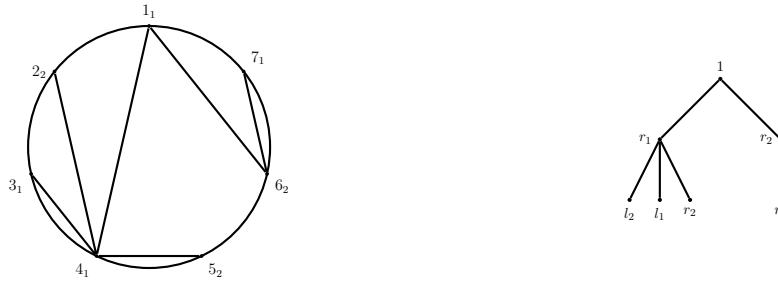


Figure 4.31: A 2-noncrossing tree on 7 vertices with root labelled 1 together with its (l, r) -representation.

This study now defined the non-decreasing 2-noncrossing tree as follows.

Definition 4.7. A *non-decreasing 2-noncrossing tree* is a 2-noncrossing tree whose (l, r) -representation is a plane tree in which for each internal vertex, all its children labelled l_1 , l_2 , r_1 and r_2 appear in this order from left to right.

In Figure 4.32, this study illustrates a non-decreasing 2-noncrossing tree on 7 vertices with root labelled 1. This study defined another new class of combinatorial structures:

Definition 4.8. A *non-decreasing 2-noncrossing increasing tree* is a non-decreasing 2-noncrossing tree whose edges are all ascents if paths from the root are considered

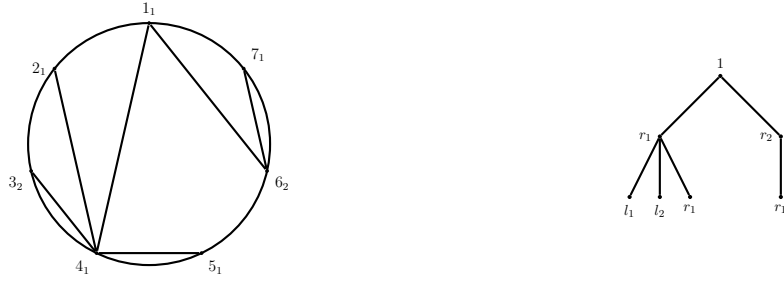


Figure 4.32: A non-decreasing 2-noncrossing tree on 7 vertices with root labelled 1 together with its (l, r) -representation.

in the (l, r) -representation of the 2-noncrossing tree.

So, in the (l, r) -representation of the non-decreasing 2-noncrossing increasing trees, there are no vertices labelled l_1 or l_2 .

Recall that from the paper by Okoth and Wagner in (Okoth & Wagner, 2015), the number of locally oriented noncrossing trees on n vertices with k sources are ℓ sinks was

$$\frac{1}{n-1} \binom{n-1}{k-1} \binom{n-1}{\ell-1} \binom{n-1}{k+\ell-1}. \quad (4.62)$$

If this study set $k + \ell = i + 2$ in (4.62), then;

$$\frac{1}{n-1} \binom{n-1}{k-1} \binom{n-1}{i-k+1} \binom{n-1}{i+1}. \quad (4.63)$$

Finally, summing over all k (making use of Vandermonde convolution) in (4.63), this study gets

$$\frac{1}{n-1} \binom{2n-1}{i} \binom{n-1}{i+1}. \quad (4.64)$$

Note that $i + 1$ was the number of non-root sources and sinks. So, if these vertices, i.e. non-root sources and sinks, come in two colours, then the total number of these

trees was

$$\frac{1}{n-1} \sum_{i=0}^{n-1} \binom{2n-1}{i} \binom{n-1}{i+1} 2^{i+1}. \quad (4.65)$$

This study first enumerated non-decreasing 2-noncrossing trees by number of vertices and label of the root.

4.5.2 Enumeration by number of vertices and vertices of a given type

Let $C(x, u)$ and $D(x, u)$ be the bivariate generating functions for non-decreasing 2-noncrossing trees with roots labelled 2 and 1 respectively, where x marked a vertex and u marked a vertex labelled 2. Since a butterfly rooted at a vertex labelled 2 could be considered to have one wing rooted at vertex labelled 2 and the other wing rooted at a vertex labelled 1 then this study found

$$C(x, u) = xu \cdot \frac{1}{1 - \frac{D^2}{x}}, \quad (4.66)$$

and

$$D(x, u) = x \cdot \frac{1}{\left(1 - \frac{D^2}{x}\right)} \cdot \frac{1}{\left(1 - \frac{CD}{x}\right)}. \quad (4.67)$$

Let $C = xu(1 + y)$. Then from (4.66), This study got

$$D^2 = \frac{xy}{1 + y}. \quad (4.68)$$

Substituting for $C = xu(1 + y)$ and (4.68) in (4.67), this study obtained

$$D^2 u(1 + y) - D + x(1 + y) = 0. \quad (4.69)$$

Further, substituting (4.68) in (4.69), it got

$$D = x(1 + (1 + u)y). \quad (4.70)$$

Now, substituting (4.70) in (4.68) to arrive at

$$y = x(1 + y)(1 + (1 + u)y)^2. \quad (4.71)$$

Equation (4.71) was in the right form to apply Lagrange inversion. So,

$$\begin{aligned} [x^n u^k]C &= [x^n u^k]uxy = [x^{n-1} u^{k-1}]y \\ &= \frac{1}{n-1} [t^{n-2} u^{k-1}]((1 + (1 + u)t)^2(1 + t))^{n-1} \\ &= \frac{1}{n-1} [t^{n-2} u^{k-1}] \sum_{i \geq 0} \binom{2n-2}{i} ((1 + u)t)^i \sum_{j \geq 0} \binom{n-1}{j} t^j \\ &= \frac{1}{n-1} [t^{n-2} u^{k-1}] \sum_{i \geq 0} \binom{2n-2}{i} (1 + u)^i \sum_{j \geq 0} \binom{n-1}{j} t^{i+j} \\ &= \frac{1}{n-1} [t^{n-2} u^{k-1}] \sum_{i \geq 0} \binom{2n-2}{i} \sum_{a \geq 0} \binom{i}{a} \sum_{j \geq 0} \binom{n-1}{j} t^{i+j} u^a \\ &= \frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{i}{k-1} \binom{n-1}{n-i-2}. \end{aligned} \quad (4.72)$$

Similarly,

$$\begin{aligned} [x^n u^k]D &= [x^n u^k](1 + u)xy = [x^{n-1} u^k](1 + u)y \\ &= \frac{1}{n-1} [t^{n-2} u^k](1 + u)((1 + (1 + u)t)^2(1 + t))^{n-1} \\ &= \frac{1}{n-1} [t^{n-2} u^k](1 + u) \sum_{i \geq 0} \binom{2n-2}{i} ((1 + u)t)^i \sum_{j \geq 0} \binom{n-1}{j} t^j \\ &= \frac{1}{n-1} [t^{n-2} u^k] \sum_{i \geq 0} \binom{2n-2}{i} (1 + u)^{i+1} \sum_{j \geq 0} \binom{n-1}{j} t^{i+j} \\ &= \frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{i+1}{k} \binom{n-1}{n-i-2}. \end{aligned} \quad (4.73)$$

Adding (4.72) and (4.73), this study obtained

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{k+i+1}{i+1} \binom{2n-2}{i} \binom{i+1}{k} \binom{n-1}{i+1}. \quad (4.74)$$

This study formally stated these results in the following theorem:

Theorem 4.33. *The number of non-decreasing 2-noncrossing trees on n vertices with root labelled 2 such that there are k vertices labelled 2 is*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{i}{k-1} \binom{n-1}{n-i-2}$$

and with root labelled 1 such that there are k vertices labelled 2 is

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{i+1}{k} \binom{n-1}{n-i-2}.$$

In total, there are

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{k+i+1}{i+1} \binom{2n-2}{i} \binom{i+1}{k} \binom{n-1}{i+1}$$

non-decreasing 2-noncrossing trees on n vertices such that there are k vertices labelled 2.

Corollary 4.12. *The number of non-decreasing 2-noncrossing trees on n vertices with root labelled 2 is*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{n-1}{i+1} 2^i \quad (4.75)$$

and with root labelled 1 is

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{n-1}{i+1} 2^{i+1}. \quad (4.76)$$

In total, there are

$$\frac{3}{n-1} \sum_{i=0}^{n-2} \binom{2n-2}{i} \binom{n-1}{i+1} 2^i$$

non-decreasing 2-noncrossing trees on n vertices.

Proof. This study summed over all k in Theorem 4.12 or set $u = 1$ in (4.71) and extracted the coefficient of x^n in C and D respectively. Finally, the respective equations were added together. $\square$

Setting $k = 1$ in (4.72) or $k = 0$ in (4.73) and summing over all i , this study rediscovered the formula for the number of noncrossing trees on n vertices, i.e.,

$$\frac{1}{n-1} \binom{3n-3}{n-2}.$$

4.5.3 Enumeration by root degree

Let $D(x) = \sum_{n=0}^{\infty} D_n x^n$ and $C(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating functions for non-decreasing 2-noncrossing trees with roots labelled 1 and 2 respectively. Let $d(n, r)$ be the number of non-decreasing 2-noncrossing trees on n vertices such that vertex 1 was labelled 2 and was of degree r . This study proved the following lemma, with the proof mimicked from the previous work of Noy (Noy, 1998).

Lemma 4.2. *The numbers $d(n, r)$ could be written in terms of the numbers D_n as follows:*

$$d(n, r) = \sum_{\substack{j_1+j_2+\dots+j_{2r}=n+r-1 \\ j_1, j_2, \dots, j_{2r} \geq 1}} D_{j_1} D_{j_2} \cdots D_{j_{2r}}. \quad (4.77)$$

Proof. Let N be a non-decreasing 2-noncrossing tree on n vertices with root labelled 2 such that the root has degree r . The root was adjacent to r vertices $v_1 < v_2 < \dots < v_r$ all labelled 1. For every $j = 1, 2, \dots, r-1$, the subgraph induced by N on the vertex set $\{v_j, \dots, v_{j+1}\}$ was the disjoint union of two non-decreasing 2-noncrossing trees with roots labelled 1. This procedure gave a total of $2r-2$ trees. Also, the subgraphs induced on $\{2, \dots, v_1\}$ and $\{v_r, \dots, n\}$ are both non-decreasing

2-noncrossing trees with roots labelled 1. This study thus had $2r$, non-decreasing 2-noncrossing trees with roots labelled 1. Let these trees have $j_1, j_2, \dots, j_{2r}$ vertices respectively. So $j_1 + j_2 + \dots + j_{2r} = n + r - 1$. In addition, a family of $2r$ non-decreasing 2-noncrossing trees with roots labelled 1 on the corresponding vertex set determined a unique non-decreasing 2-noncrossing tree N with root labelled 2. This completed the proof. $\square$

This study now proved the following;

Theorem 4.34. *The number of non-decreasing 2-noncrossing trees with root labelled 2 on n vertices such that the root has degree r and the total number of vertices labelled 2 is k is given by*

$$\frac{2r}{n-r-1} \binom{2n-2r-2}{k-1} \sum_{i \geq 0} \binom{k-1}{i} \binom{3n-r-i-4}{n-r-2} (-1)^i. \quad (4.78)$$

Proof. The generating function $C(x, u)$ of non-decreasing 2-noncrossing trees with root labelled 2 was given by $C(x, u) = xu(1+y)$ where x and u marks a vertex and a vertex labelled 2 respectively, and $y = x(1+y)(1+(1+u)y)^2$. Let $F = 1+y$ so that $C = x u F$, $y = F - 1$ and $F = 1 + xF(F + u(F - 1))^2$.

The convolution of Lemma 4.2 implied that

$$N_r = \frac{1}{x^{r-1}} N^{2r} = x^{r+1} F^{2r}$$

where the coefficients in the series N_r give the number of non-decreasing 2-noncrossing trees with a root labelled 2 and of degree r . Since

$$F = 1 + xF(F + u(F - 1))^2,$$

this study applied Lagrange-Bürmann Inversion Formula (Theorem 3.2) to the series

N_r with $\beta = 1, g(s) = s(s + u(s - 1))^2$ and $f(s) = s^{2r}$ to get

$$\begin{aligned}
[x^n u^{k-1}]F^{2r} &= \frac{2r}{n!} \frac{d^{n-1}}{ds^{n-1}} \left\{ s^{n+2r-1} (s + u(s - 1))^{2n} \right\}_{s=1} \\
&= \frac{2r}{n!} \frac{d^{n-1}}{ds^{n-1}} \left\{ \sum_{i \geq 0} \binom{2n}{i} (s - 1)^i u^w a s^{3n+2r-i-1} \right\}_{s=1} \\
&= \frac{2r}{n!} \binom{2n}{k-1} \frac{d^{n-1}}{ds^{n-1}} \left\{ (s - 1)^{k-1} s^{3n+2r-k} \right\}_{s=1} \\
&= \frac{2r}{n!} \binom{2n}{k-1} \sum_{i \geq 0} \binom{k-1}{i} (-1)^{k-i-1} \frac{d^{n-1}}{ds^{n-1}} \left\{ s^{3n+2r+i-k} \right\}_{s=1}. \quad (4.79)
\end{aligned}$$

It followed that,

$$\begin{aligned}
[x^n u^{k-1}]F^{2r} &= \frac{2r}{n!} \binom{2n}{k-1} \sum_{i \geq 0} \binom{k-1}{i} (-1)^{k-i-1} (3n + 2r + i - k) \\
&\quad (3n + 2r + i - k - 1) \cdots (2n + 2r + i - k + 2) \\
&= \frac{2r}{n!} \binom{2n}{k-1} \sum_{i \geq 0} \binom{k-1}{i} (-1)^{k-i-1} \frac{(3n + 2r + i - k)!}{(2n + 2r + i - k + 1)!} \\
&= \frac{2r}{n} \binom{2n}{k-1} \sum_{i \geq 0} \binom{k-1}{i} (-1)^{k-i-1} \binom{3n + 2r + i - k}{n - 1}. \quad (4.80)
\end{aligned}$$

Therefore,

$$\begin{aligned}
[x^n u^k]N_r &= [x^{n-r-1} u^{k-1}]F^{2r} \\
&= \frac{2r}{n - r - 1} \binom{2n - 2r - 2}{k-1} \sum_{i \geq 0} \binom{k-1}{i} (-1)^{k-i-1} \binom{3n - r + i - k - 3}{n - r - 2}. \quad (4.81)
\end{aligned}$$

This completed the proof. □

Setting $k = 1$ (4.78), this study rediscovered that:

Corollary 4.13. *There are*

$$\frac{2r}{n - r - 1} \binom{3n - r - 4}{n - r - 2}$$

noncrossing trees on n vertices with root of degree r .

Theorem 4.35. *The number of non-decreasing 2-noncrossing trees on n vertices whose root is labelled 2 and of degree r is given by*

$$\frac{2r}{n-r-1} \sum_{i=0}^{2n-2r-2} \binom{2n-2r-2}{i} \binom{n+r+i-2}{2r+i} (-2)^i. \quad (4.82)$$

Proof. The generating function $C(x)$ of non-decreasing 2-noncrossing trees with root labelled 2 was given by $C(x) = x(1+y)$ where x marks a vertex. This study had $y = x(1+y)(1+2y)^2$. Let $F = 1+y$ so that $C = xF$, $y = F-1$ and $F = 1 + xF(2F-1)^2$.

The convolution of Lemma 4.2 implied that

$$N_r = \frac{1}{x^{r-1}} N^{2r} = x^{r+1} F^{2r}$$

where the coefficients in the series N_r give the number of non-decreasing 2-noncrossing trees with a root labelled 2 and of degree r . Since

$$F = 1 + xF(2F-1)^2,$$

This study applied Lagrange-Bürmann Inversion Formula to the series N_r with $\beta = 1$, $g(s) = s(2s-1)^2$ and $f(s) = s^{2r}$ to get

$$\begin{aligned} [x^n] F^{2r} &= \frac{2r}{n!} \frac{d^{n-1}}{ds^{n-1}} \{s^{n+2r-1} (2s-1)^{2n}\}_{s=1} \\ &= \frac{2r}{n!} \sum_{i \geq 0} \binom{2n}{i} (-1)^i 2^{2n-i} \frac{d^{n-1}}{ds^{n-1}} \{s^{3n+2r-i-1}\}_{s=1} \\ &= \frac{2r}{n!} \sum_{i \geq 0} \binom{2n}{i} (-1)^i 2^{2n-i} (3n+2r-i-1)(3n+2r-i-2) \cdots (2n+2r-i+1) \\ &= \frac{2r}{n!} \sum_{i \geq 0} \binom{2n}{i} \frac{(3n+2r-i-1)!}{(2n+2r-i)!} (-1)^i 2^{2n-i} \\ &= \frac{2r}{n} \sum_{i \geq 0} \binom{2n}{i} \binom{3n+2r-i-1}{n-1} (-1)^i 2^{2n-i}. \end{aligned} \quad (4.83)$$

Therefore,

$$\begin{aligned}
 [x^n]N_r &= [x^{n-r-1}]F^{2r} \\
 &= \frac{2r}{n-r-1} \sum_{i=0}^{2n-2r-2} \binom{2n-2r-2}{i} \binom{3n-r-i-4}{n-r-2} (-1)^i 2^{2n-2r-i-2}.
 \end{aligned} \tag{4.84}$$

Hence the proof. $\square$

This study remarked that (4.82) can also be obtained by summing over all k in (4.78).

4.5.4 Forests of non-decreasing 2-noncrossing trees

In Theorem 4.36, this study considered forests of non-decreasing 2-noncrossing trees with the following two properties:

- each component was rooted at a vertex with the smallest label.
- the components are non-decreasing 2-noncrossing trees with the root labelled 2, and the components do not intersect each other.

The case in which the roots of the components are all labelled 1 was considered in Theorem 4.37.

This study used generating functions and also adopted the proof of Okoth in (Okoth, 2022b) so as to prove the result.

Theorem 4.36. *There are*

$$\frac{1}{n-k} \binom{n}{k-1} \sum_{i=0}^{n-k-1} \binom{n-1}{i+k} \binom{2n-2k}{i} 2^i \tag{4.85}$$

forests of non-decreasing 2-noncrossing trees on n vertices with k components such that the roots of the components are labelled 2.

Proof. Let $C(x)$ be the generating function for the components, i.e. non-decreasing 2-noncrossing trees satisfying the given conditions, where x marks a vertex. This study decompose the forests according to components containing vertex 1 as shown in Figure 4.33. If this component was on p vertices then there are p spaces that are to be filled by forests $T_1, T_2, \dots, T_p$. Some of these components may be empty.

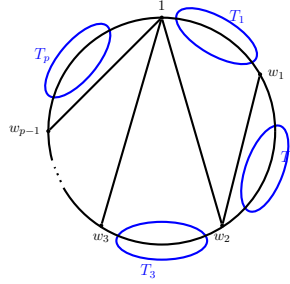


Figure 4.33: Decomposition of forests of non-decreasing 2-noncrossing trees according to vertex 1.

So if $C(x) = \sum_{n \geq 1} c_n x^n$, then the generating function $T(x, z)$ for forests where z marked the number of components would satisfy

$$T(x, z) = 1 + z \sum_{m \geq 1} c_m x^m T(x, z)^m = 1 + zC(xT(x, z)).$$

Let $C(x) = x(1 + y(x))$, then by (4.71) with $u = 1$, This study had $y(x) = x(1 + y)(1 + 2y)^2$. Now, let $T(x, z) = 1 + zs$. Then, $1 + zs = 1 + zC(x(1 + zs))$ or $s = C(x(1 + zs))$.

This study defined a as $a(t) = \frac{t}{C^{-1}(t)}$. Since $x(1 + zs) = C^{-1}(s)$, then $x(1 + zs) = \frac{s}{a(s)}$. This implied that $s = x(1 + zs)a(s)$. This was in a form this study could apply Lagrange inversion to obtain

$$[x^n z^k]T(x, z) = [x^n z^{k-1}]s = \frac{1}{n} [t^{n-1} z^{k-1}]((1 + zt)a(t))^n = \frac{1}{n} \binom{n}{k-1} [t^{n-k}]a(t)^n. \quad (4.86)$$

This study only needed to obtain $[t^{n-k}]a(t)^n$ so as to complete the proof. By definition, $t = C\left(\frac{t}{a(t)}\right)$.

This study wrote $C(x) = x(1 + y)$, to have

$$t = \frac{t}{a(t)} \left(1 + y \left(\frac{t}{a(t)} \right) \right) \quad (4.87)$$

or

$$a(t) = 1 + y \left(\frac{t}{a(t)} \right) \quad (4.88)$$

where $y(x)$ satisfied $y(x) = x(1 + y)(1 + 2y)^2$.

It followed that

$$a(t) - 1 = y \left(\frac{t}{a(t)} \right) = \frac{t}{a(t)} \left(1 + y \left(\frac{t}{a(t)} \right) \right) \left(1 + 2y \left(\frac{t}{a(t)} \right) \right)^2 \quad (4.89)$$

$$= \frac{t}{a(t)} \cdot a(t)(2a(t) - 1)^2. \quad (4.90)$$

Therefore, $a(t) = 1 + t(2a(t) - 1)^2$. This study let $b(t) = a(t) - 1$, then

$$b(t) = t(2(a(t) - 1) - 1)^2 = t(2b(t) + 1)^2.$$

This study used Lagrange inversion to get,

$$\begin{aligned} [t^{n-r}]a(t)^n &= [t^{n-r}](1 + b(t))^n = \sum_{i \geq 0} \binom{n}{i} [t^{n-k}]b(t)^i \\ &= \sum_{i \geq 0} \binom{n}{i} \frac{i}{n-k} [t^{n-r-i}](2t + 1)^{2n-2k} \\ &= \frac{n}{n-k} \sum_{i \geq 0} \binom{n-1}{i-1} [t^{n-k-i}] \sum_{j \geq 0} \binom{2n-2k}{j} 2^j t^j \\ &= \frac{n}{n-k} \sum_{i \geq 0} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-k-i}. \end{aligned} \quad (4.91)$$

So, the number of forests of non-decreasing 2-noncrossing trees with n vertices and

k components such that the components have roots labelled 2 was

$$\begin{aligned}
 [x^n z^k]T(x, z) &= \frac{1}{n} \binom{n}{k-1} [t^{n-k}]a(t)^n \\
 &= \frac{1}{n} \binom{n}{k-1} \frac{n}{n-k} \sum_{i \geq 0} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-k-i} \\
 &= \frac{1}{n-k} \binom{n}{k-1} \sum_{i=1}^{n-k} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-k-i}. \tag{4.92}
 \end{aligned}$$

□

Setting $k = 1$ in (4.85), This study obtain the formula for the number of non-decreasing 2-noncrossing trees with n vertices such that the root of each tree was labelled by 2. The formula was already obtained in (4.75).

Theorem 4.37. *There are*

$$\frac{1}{n-k} \binom{n}{k-1} \sum_{i=0}^{n-k-1} \binom{n-1}{i+k} \binom{2n-2k}{i} 2^{k+i} \tag{4.93}$$

forests of non-decreasing 2-noncrossing trees on n vertices with k components such that the roots of the components are labelled 1.

Proof. Let B be the generating function for the components, i.e. non-decreasing 2-noncrossing trees such that each component was rooted at the a vertex with the smallest label, the roots of the components are labelled 1 and the components do not intersect. This study decomposed the forests according to components containing vertex 1 as shown in Figure 4.33 above. If this component was on p vertices then there are p spaces that are to be filled by forests $T_1, T_2, \dots, T_p$. Some of these components may be empty. Let $D(x) = \sum_{n \geq 1} d_n x^n$ be the generating function for non-decreasing 2-noncrossing trees with root labelled 1. The generating function $T(x, z)$ for forests where z marked the number of components would satisfy

$$T(x, z) = 1 + z \sum_{m \geq 1} d_m x^m T(x, z)^m = 1 + z D(x T(x, z)). \tag{4.94}$$

Let $D(x) = x(1 + 2y(x))$, then by (4.71) with $u = 1$, this study had $y(x) = x(1 + y)(1 + 2y)^2$. Now, let $T(x, z) = 1 + 2zs$. Then, $1 + 2zs = 1 + zD(x(1 + 2zs))$ or $2s = D(x(1 + 2zs))$.

This study defined a as $a(t) = \frac{t}{D^{-1}(t)}$. Since $x(1 + 2zs) = D^{-1}(s)$, then $x(1 + 2zs) = \frac{s}{a(s)}$. This implies that $s = x(1 + 2zs)a(s)$. This was in a form this study can apply Lagrange inversion to obtain

$$\begin{aligned} [x^n z^k]T(x, z) &= [x^n z^{k-1}]2s = 2 \cdot \frac{1}{n} [t^{n-1} z^{k-1}]((1 + 2zt)a(t))^n \\ &= \frac{2}{n} \binom{n}{k-1} 2^{k-1} [t^{n-k}]a(t)^n. \end{aligned}$$

By the proof of Theorem 4.36,

$$[t^{n-k}]a(t)^n = \frac{n}{n-k} \sum_{i \geq 0} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-k-i}.$$

So, the number of forests of non-decreasing 2-noncrossing trees with n vertices and k components whose roots are labelled 1 was

$$\begin{aligned} [x^n z^k]T(x, z) &= \frac{2^k}{n} \binom{n}{k-1} [t^{n-k}]a(t)^n \\ &= \frac{2^k}{n} \binom{n}{k-1} \frac{n}{n-k} \sum_{i \geq 0} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-k-i} \\ &= \frac{1}{n-k} \binom{n}{k-1} \sum_{i=1}^{n-k} \binom{n-1}{i-1} \binom{2n-2k}{n-k-i} 2^{n-i}. \end{aligned} \quad (4.95)$$

□

Setting $k = 1$ (4.93), This study obtained (4.76) which was the number of non-decreasing 2-noncrossing trees on n vertices with root labelled 1.

4.5.5 Non-decreasing 2-noncrossing increasing trees

This study enumerated non-decreasing 2-noncrossing increasing trees according to number of vertices, root degree, label of the leftmost child of the root and number of forests. It started off with number of vertices.

4.5.5.1 Number of vertices

Let $I(x, u)$ and $J(x, u)$ be the bivariate generating functions for non-decreasing 2-noncrossing increasing trees with roots labelled 2 and 1 respectively, where x and u are marking vertices and vertices labelled 2 respectively. Since the left wings butterflies were missing in non-decreasing 2-noncrossing increasing trees then

$$I(x, u) = xu \cdot \frac{1}{1 - J}, \quad (4.96)$$

where u marked a vertex labelled 2, and

$$J(x, u) = x \cdot \frac{1}{1 - J} \cdot \frac{1}{1 - I}. \quad (4.97)$$

Let $I = xu(1 + y)$. Then from (4.96), this study got

$$J = \frac{y}{1 + y} = y(1 + y)^{-1}. \quad (4.98)$$

Substituting for $I = xu(1 + y)$ and (4.98) in (4.97), this study obtained

$$y = x(1 + y)(1 + (1 + u)y). \quad (4.99)$$

It noted that (4.99) was in the right form to apply Lagrange inversion formula and now extracted the coefficient of $x^n u^k$ in the generating functions $I(x, u)$ as follows:

$$\begin{aligned}
[x^n u^k]I &= [x^n u^k]u x y = [x^{n-1} u^{k-1}]y \\
&= \frac{1}{n-1} [t^{n-2} u^{k-1}]((1 + (1+u)t)(1+t))^{n-1} \\
&= \frac{1}{n-1} [t^{n-2} u^{k-1}] \sum_{i \geq 0} \binom{n-1}{i} ((1+u)t)^i \sum_{j \geq 0} \binom{n-1}{j} t^j \\
&= \frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{i}{k-1} \binom{n-1}{n-i-2}. \tag{4.100}
\end{aligned}$$

Similarly,

$$\begin{aligned}
[x^n]J &= [x^n u^k]y(1+y)^{-1} = [x^n u^k]y \sum_{a \geq 0} \binom{-1}{a} y^a = \sum_{a \geq 0} \binom{-1}{a} [x^n u^k]y^{a+1} \\
&= \sum_{a \geq 0} \binom{-1}{a} \frac{a+1}{n} [t^{n-a-1} u^k]((1 + (1+u)t)(1+t))^n \\
&= \sum_{a \geq 0} \binom{-1}{a} \frac{a+1}{n} [t^{n-a-1} u^k] \sum_{i \geq 0} \binom{n}{i} ((1+u)t)^i \sum_{j \geq 0} \binom{n}{j} t^j \\
&= \sum_{a \geq 0} \binom{-1}{a} \frac{a+1}{n} [t^{n-a-1}] \sum_{i \geq 0} \binom{n}{i} \binom{i}{k} \sum_{j \geq 0} \binom{n}{j} t^{i+j} \\
&= \frac{1}{n} \sum_{i \geq 0} \sum_{a \geq 0} \left[\binom{-1}{a} - \binom{-2}{a-1} \right] \binom{n}{i} \binom{i}{k} \binom{n}{n-i-a-1}
\end{aligned}$$

By Vandermonde convolution, this study had

$$\begin{aligned}
[x^n]J &= \frac{1}{n} \sum_{i \geq 0} \left[\binom{n-1}{n-i-1} - \binom{n-2}{n-i-2} \right] \binom{n}{i} \binom{i}{k} \\
&= \frac{1}{n} \sum_{i \geq 0} \frac{i}{n-1} \binom{n}{i} \binom{i}{k} \binom{n-1}{n-i-1} \\
&= \frac{1}{n-1} \sum_{i=1}^{n-1} \binom{n-1}{i-1} \binom{i}{k} \binom{n-1}{n-i-1} \\
&= \frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{i+1}{k} \binom{n-1}{n-i-2}. \tag{4.101}
\end{aligned}$$

Adding (4.100) and (4.101), it got

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{k+i+1}{k} \binom{n-1}{i} \binom{i}{k-1} \binom{n-1}{i+1}.$$

This study thus had the following theorem.

Theorem 4.38. *The number of non-decreasing 2-noncrossing increasing trees on n vertices with roots labelled 2 such that there are k vertices labelled 2 is given by*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{i}{k-1} \binom{n-1}{i+1} \quad (4.102)$$

and if the roots are labelled 1 such that there are k vertices labelled 2 is

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{i+1}{k} \binom{n-1}{i+1}. \quad (4.103)$$

The total number of non-decreasing 2-noncrossing increasing trees on n vertices such that there are k vertices labelled 2 is

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{k+i+1}{k} \binom{n-1}{i} \binom{i}{k-1} \binom{n-1}{i+1}.$$

Summing over all k in (4.102) and (4.103), This study get:

Corollary 4.14. *The number of non-decreasing 2-noncrossing increasing trees on n vertices with roots labelled 2 is*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{n-1}{i+1} 2^i \quad (4.104)$$

and if the roots are labelled 1 is

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{n-1}{i+1} 2^{i+1}. \quad (4.105)$$

In total, there are

$$\frac{3}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i} \binom{n-1}{i+1} 2^i.$$

non-decreasing 2-noncrossing increasing trees on n vertices.

Setting $k = 1$ in (4.102) or $k = 0$ in (4.103), this study obtained that there were

$$\frac{1}{n-1} \binom{2n-2}{n}$$

noncrossing increasing trees on n vertices, another manifestation of Catalan numbers as proved in (Asinowski & Mansour, 2008).

4.5.5.2 Label of leftmost child of the root

The leftmost child of the root of non-decreasing 2-noncrossing increasing tree whose root was labelled 2, in its (l, r) -representation, was always labelled 1. Thus, it was not interesting to study such trees. This study therefore enumerate trees in which the root was labelled 1.

Theorem 4.39. *The number of Non-decreasing 2-noncrossing trees on n vertices with root labelled 1 such that the leftmost child of the root is labelled 1 is given by*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{(2i-n+2)(i+1)}{n-2} \binom{n-1}{i+1} \binom{n-1}{i} 2^i. \quad (4.106)$$

Proof. Since the generating function for the number of non-decreasing 2-noncrossing increasing trees with root labelled 1 was $J(x) = y(1+y)^{-1}$, where $y = x(1+y)(1+2y)$

and x marks a vertex, the required formula was

$$\begin{aligned}
[x^n]J(x)^2 &= [x^n]y^2(1+y)^{-2} = \sum_{a \geq 0} \binom{-2}{a} [x^n]y^{a+2} \\
&= \sum_{a \geq 0} \binom{-2}{a} \frac{a+2}{n} [t^{n-a-2}] (1+t)^n (1+2t)^n \\
&= \frac{1}{n} \sum_{a \geq 0} \left[\binom{-2}{a} - 2 \binom{-3}{a-1} \right] [t^{n-a-2}] \sum_{j \geq 0} \binom{n}{j} \sum_{i \geq 0} \binom{n}{i} 2^i t^{i+j} \\
&= \frac{1}{n} \sum_{a \geq 0} \left[\binom{-2}{a} - 2 \binom{-3}{a-1} \right] \sum_{i \geq 0} \binom{n}{n-a-i-2} \binom{n}{i} 2^i \\
&= \frac{1}{n} \sum_{i \geq 0} \left[\binom{n-2}{i} - 2 \binom{n-3}{i} \right] \binom{n}{i} 2^i \\
&= \frac{1}{n-2} \sum_{i=0}^{n-2} \frac{2i-n+2}{n-i} \binom{n-2}{i} \binom{n-1}{i} 2^i. \tag{4.107}
\end{aligned}$$

The second last equality follows by Vandermonde convolution. $\square$

Corollary 4.15. *The number of Non-decreasing 2-noncrossing trees on n vertices with root labelled 1 such that the leftmost child of the root was labelled 2 was given by*

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \frac{(n-2i-2)(i+1) + 2n-4}{n-2} \binom{n-1}{i+1} \binom{n-1}{i} 2^i. \tag{4.108}$$

Proof. This study obtained (4.108) by subtracting (4.106) from (4.105). $\square$

Equation (4.108) therefore gave the number of non-decreasing 2-noncrossing increasing trees in which the root was labelled 1 and all the vertices attached to the root were all labelled 2.

4.5.5.3 Root degree

Theorem 4.40. *There are*

$$\frac{1}{n-q-1} \sum_{i=0}^{n-p-q-1} \frac{q(n-q-1) + i(p-q)}{n-p-1} \binom{n-p-1}{q+i} \binom{n-q-1}{i} 2^i \tag{4.109}$$

non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 1 such that there are p and q vertices adjacent to the root which are labelled by 1 and 2 respectively.

Proof. This study extracted the coefficient of x^n in the generating function $xJ(x)^pI(x)^q$.

$$\begin{aligned}
& [x^n]xJ(x)^pI(x)^q \\
&= [x^{n-1}] \left(\frac{y}{1+y} \right)^p (x(1+y))^q = [x^{n-q-1}] y^p (1+y)^{q-p} \\
&= \sum_{a \geq 0} \binom{q-p}{a} [x^{n-q-1}] y^{p+a} \\
&= \sum_{a \geq 0} \binom{q-p}{a} \frac{p+a}{n-q-1} [t^{n-q-p-a-1}] (1+t)^{n-q-1} (1+2t)^{n-q-1} \\
&= \frac{1}{n-q-1} \sum_{a \geq 0} \left[p \binom{q-p}{a} + (q-p) \binom{q-p-1}{a-1} \right] \\
& \quad [t^{n-p-q-a-1}] \sum_{j \geq 0} \binom{n-q-1}{j} \sum_{i \geq 0} \binom{n-q-1}{i} 2^i t^{i+j}. \tag{4.110}
\end{aligned}$$

So, this study had

$$\begin{aligned}
& [x^n]xJ(x)^pI(x)^q \\
&= \frac{1}{n-q-1} \sum_{a \geq 0} \left[p \binom{q-p}{a} + (q-p) \binom{q-p-1}{a-1} \right] \\
& \quad \sum_{i \geq 0} \binom{n-q-1}{n-p-q-a-i-1} \binom{n-q-1}{i} 2^i \\
&= \frac{1}{n-q-1} \sum_{i \geq 0} \left[p \binom{n-p-1}{q+i} + (q-p) \binom{n-p-2}{q+i} \right] \binom{n-q-1}{i} 2^i \\
&= \frac{1}{n-q-1} \sum_{i=0}^{n-p-q-1} \frac{q(n-q-1) + i(p-q)}{n-p-1} \binom{n-p-1}{q+i} \binom{n-q-1}{i} 2^i. \tag{4.111}
\end{aligned}$$

The second last equality followed by Vandermonde convolution. Thus the proof. $\square$

This study remarked that the label of root of the tree can be changed to 2, if all the vertices adjacent to the root are labelled 1. So on setting $q = 0$ in (4.109), this study obtained the following corollary.

Corollary 4.16. *The number of non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 2 such that the root has degree p is*

$$\frac{p}{n-1} \sum_{i=0}^{n-p-2} \binom{n-p-2}{i} \binom{n-1}{i+1} 2^{i+1}. \quad (4.112)$$

If this study substitute for $p = 1$ in (4.112), it gets the total number of planted non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 2. This formula was given as

$$\frac{1}{n-1} \sum_{i=0}^{n-3} \binom{n-3}{i} \binom{n-1}{i+1} 2^{i+1}. \quad (4.113)$$

Equation (4.113) was the same as (4.105) with $n = n - 1$. This study rediscovered the following result already proved at the beginning of this section.

Corollary 4.17. *The number of non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 2 is*

$$\frac{1}{n-1} \sum_{i=0}^{n-1} \binom{n-1}{i+1} \binom{n-1}{i} 2^i.$$

Proof. This study summed over all p in (4.112). □

By setting $p = 0$ in (4.109), This study obtained that:

Corollary 4.18. *The number of non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 1 of degree q if all the vertices adjacent to the root are labelled 2 is*

$$\frac{q}{n-1} \sum_{i=0}^{n-q-1} \binom{n-1}{q+i} \binom{n-q-2}{i} 2^i. \quad (4.114)$$

With $q = 1$ in (4.114), this study got the number of planted trees of non-decreasing

2-noncrossing increasing trees on n vertices such that the root was labelled 1 as

$$\frac{1}{n-1} \sum_{i=0}^{n-2} \binom{n-1}{i+1} \binom{n-3}{i} 2^i.$$

This equation was easily obtained from (4.104) by setting $n = n - 1$.

4.5.5.4 Forests

Theorem 4.41. *There are*

$$\frac{1}{n-q} \binom{p+q}{p} \sum_{i=0}^{n-p-q} \frac{q(n-q) + i(p-q)}{n-p} \binom{n-p}{q+i} \binom{n-q}{i} 2^i \quad (4.115)$$

forests of non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 1 such that there are p and q components whose roots are labelled by 1 and 2 respectively.

Proof. This study extracted the coefficient of x^n in the generating function $J(x)^p I(x)^q$.

$$\begin{aligned} & [x^n] J(x)^p I(x)^q \\ &= [x^n] \left(\frac{y}{1+y} \right)^p (x(1+y))^q = [x^{n-q}] y^p (1+y)^{q-p} = \sum_{a \geq 0} \binom{q-p}{a} [x^{n-q}] y^{p+a} \\ &= \sum_{a \geq 0} \binom{q-p}{a} \frac{p+a}{n-q} [t^{n-q-p-a}] (1+t)^{n-q} (1+2t)^{n-q} \\ &= \frac{1}{n-q} \sum_{a \geq 0} \left[p \binom{q-p}{a} + (q-p) \binom{q-p-1}{a-1} \right] [t^{n-p-q-a}] \sum_{j \geq 0} \binom{n-q}{j} \sum_{i \geq 0} \binom{n-q}{i} 2^i t^{i+j} \\ &= \frac{1}{n-q} \sum_{a \geq 0} \left[p \binom{q-p}{a} + (q-p) \binom{q-p-1}{a-1} \right] \sum_{i \geq 0} \binom{n-q}{n-p-q-a-i} \binom{n-q}{i} 2^i \\ &= \frac{1}{n-q} \sum_{i \geq 0} \left[p \binom{n-p}{q+i} + (q-p) \binom{n-p-1}{q+i} \right] \binom{n-q}{i} 2^i \\ &= \frac{1}{n-q} \sum_{i=0}^{n-p-q} \frac{q(n-q) + i(p-q)}{n-p} \binom{n-p}{q+i} \binom{n-q}{i} 2^i. \end{aligned} \quad (4.116)$$

Since there were $\binom{p+q}{p}$ different arrangements for the components This study have

the desired result by product rule of counting. $\square$

By setting $q = 0$ in (4.115), this study obtained:

Corollary 4.19. *The number of forests of non-decreasing 2-noncrossing increasing trees on n vertices with p components whose roots are all labelled 1 is*

$$\frac{p}{n} \sum_{i=0}^{n-p-1} \binom{n-p-1}{i} \binom{n}{i+1} 2^{i+1}. \quad (4.117)$$

When this study substituted for $p = 1$ in (4.117), it got the total number of non-decreasing 2-noncrossing increasing trees on n vertices with root labelled 1. This formula was given as

$$\frac{1}{n} \sum_{i=0}^{n-2} \binom{n-2}{i} \binom{n}{i+1} 2^{i+1}. \quad (4.118)$$

Equation (4.118) was given by (4.104). By setting $p = 0$ in (4.115), this study obtained that:

Corollary 4.20. *The number of forests of non-decreasing 2-noncrossing increasing trees on n vertices with q components whose root are labelled 2 is*

$$\frac{q}{n} \sum_{i=0}^{n-q} \binom{n}{q+i} \binom{n-q-1}{i} 2^i. \quad (4.119)$$

With $q = 1$ in (4.119), it got the number of non-decreasing 2-noncrossing increasing trees on n vertices such that the root was labelled 2 as

$$\frac{1}{n} \sum_{i=0}^{n-1} \binom{n}{i+1} \binom{n-2}{i} 2^i.$$

This equation was obtained in (4.105).

4.5.6 Bijections of non-decreasing 2-noncrossing trees

A bijection between the set of Non-decreasing 2-noncrossing trees and the set of locally oriented noncrossing trees with a given number of sources and sinks was given as follows:

4.5.6.1 Locally oriented noncrossing trees

Theorem 4.42. *There is a bijection between the set of non-decreasing 2-noncrossing trees on n vertices with root labelled 1 and the set of locally oriented noncrossing trees on n vertices with non-root sources and sinks coming in 2 colours.*

Proof. Let T be a non-decreasing 2-noncrossing tree on n vertices with root labelled by 1. Obtain its (l, r) -representation. It was worth noting that there are no (l_2, r_2) and (r_2, r_2) edges. Orient all the edges of T from a vertex of lower label towards a vertex of higher label. Note that the edges are labelled away from vertices labelled l at a lower level and also towards r if the other endpoint was labelled l . If both endpoints were labelled r , then the edges were oriented away from the root. Then do the following:

- (i) This study traversed the tree in preorder.
 - (a) If a vertex v which was neither a source nor a sink labelled l_2 was encountered then detach the subtrees rooted at the children of v labelled r_1 and attach them to the parent of v in order from left to right so that the detached children of v appear on the immediate right of v . Relabel these detached children of v as l_1 and orient the edge towards the parent of v . Also relabel the remaining children of v labelled l_1 (resp. l_2) as r_1 (resp. r_2) and orient the edges away from v . Vertex v becomes a source labelled l_2 by this procedure.

- (b) If a vertex v which was neither a source nor a sink labelled r_2 was encountered then detach the subtrees rooted at the children of v labelled r_1 and attach them to the parent of v in order from left to right so that the detached children of v appear on the immediate right of v . By this procedure, vertex v becomes a sink labelled r_2 .
- (ii) By step (i), this study found that vertices which are neither sources nor sinks are labelled r_1 or l_1 . Drop the label 1 so that they are labelled r or l . Finally, the vertices labelled r_1 and r_2 are sinks (resp. labelled l_1 and l_2 are sources), where the subscripts 1 and 2 are the colours.

The tree obtained was a locally oriented noncrossing tree in which non-root sources and sinks come in two colours.

For the reverse procedure, obtain the (l, r) -representation of the locally oriented noncrossing tree in which non-root sources and sinks come in two colours 1 and 2, i.e, r_1 and r_2 are sinks, and l_1 and l_2 are sources. Label root as 1 and all vertices which are neither sources nor sinks as l_1 and r_1 if they were initially labelled l and r respectively. Traverse the locally oriented noncrossing tree, in preorder and the following.

- (i) If a non-root source v , labelled l_2 , was encountered, detach the subtrees rooted at its siblings, labelled l_1 (if any), on its immediate right and attach them to v in order from left to right so that the detached siblings of v appear on the immediate right of the initial children of v . Relabel these detached siblings of v as r_1 . Moreover, relabel the initial children of v labelled r_1 (resp. r_2) as l_1 (resp. l_2). Vertex v becomes a non-source vertex labelled l_2 .
- (ii) If a sink v , labelled r_2 , was encountered, detach the subtrees rooted at its siblings, labelled r_1 (if any), on its immediate right and attach them to v in order from left to right so that the detached siblings of v appear on the

immediate right of the initial children of v retaining their label. By this procedure, vertex v becomes a non-sink vertex labelled r_2 .

This study removed the orientations of the edges. By this procedure, it obtained a non-decreasing 2-noncrossing trees on n vertices with root labelled 1. The bijection was illustrated in Figure 4.34.

□

4.5.6.2 Complete ternary trees

In (Okoth & Wagner, 2015), Okoth and Wagner obtained the following result:

Proposition 4.1 (Okoth and Wagner, 2015). *The was a bijection between the set of locally oriented noncrossing trees on n vertices with k sources and ℓ sinks and the set of ternary trees on $n - 1$ vertices with $k - 1$ middle-edges and $\ell - 1$ left-edges.*

In the bijection, vertex 1 (a source) and vertex n (a sink) are not used in determining the position of the edges. By Theorem 4.42, vertex n was a sink labelled 1 or 2. So this study used this label to assign a colour to the root of a ternary tree to obtain the following result.

Proposition 4.2. *There was a bijection between the set of non-decreasing n -noncrossing trees on n vertices with root labelled 1 and the set of ternary trees with $n - 1$ vertices such that all left-edges and middle-edges come in two colours and the root also comes in two colours.*

In Drake's thesis [(Drake, 2008, Example 1.6.9)], he showed that the number of complete ternary trees with $2n - 1$ leaves (i.e. the tree has $n - 1$ internal vertices) such that all internal vertices come in two colours and the rightmost child of each

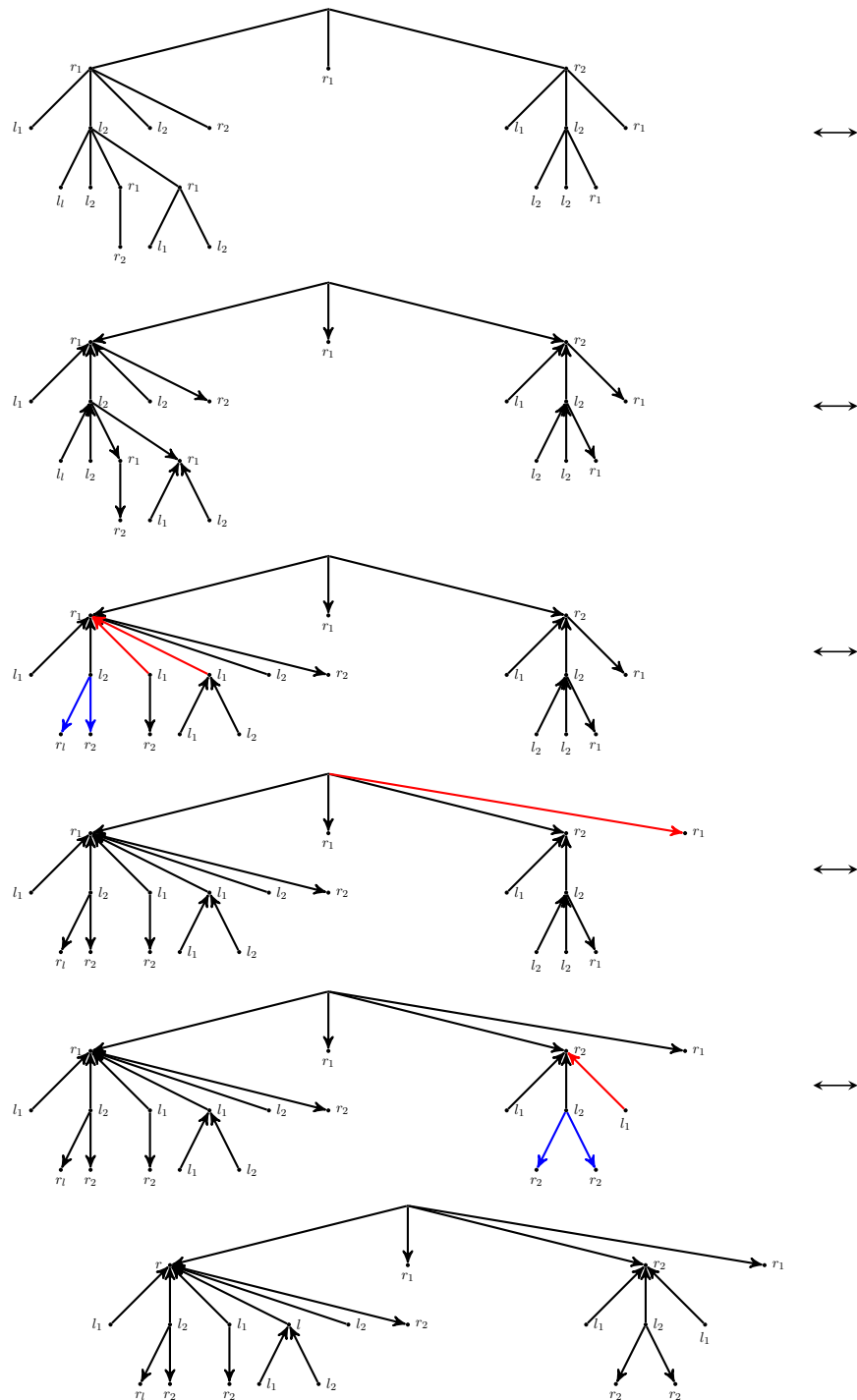


Figure 4.34: Obtaining a locally oriented noncrossing tree in which non-root sources and sinks come in two colours from a non-decreasing 2-noncrossing trees with root labelled 1 and vice versa.

internal vertex was assigned a different colour from that of its parent was given by

$$\frac{1}{n-1} \binom{2n-1}{i} \binom{n-1}{i+1} 2^{i+1}.$$

This study therefore prove the following theorem.

Theorem 4.43. *There exists a bijection between the set of non-decreasing 2-noncrossing trees on n vertices with root labelled 1 and the set of complete ternary trees with $2n-1$ leaves such that all internal vertices are coloured 1 or 2 and the right child of each internal vertex was assigned a different colour from that of its parent.*

Proof. From Proposition 4.2, this study had a bijection between the set of non-decreasing 2-noncrossing trees on n vertices with root labelled 1 and the set of ternary trees with $n-1$ vertices such that left- and middle-edges are coloured 1 or 2 and the root was also coloured 1 or 2. So, the required bijection was achieved via the corresponding locally oriented noncrossing tree and root labelled ternary tree. From a ternary tree on $n-1$ vertices in which the root was coloured 1 or 2 and the left- and middle-edges are coloured 1 or 2, assign the colour of the left- and middle-edge to the vertex just below it. This study obtained a ternary tree in which the first two children of each non-root vertex are coloured 1 or 2 and the root was also coloured 1 or 2. This study completed the tree by attaching $2n-1$ vertices (which will be leaves) so that each internal vertex has 3 children, i.e. This study obtained a complete ternary tree then remained to assign colours to the rightmost child of each internal vertex.

- (i) Since a right child of each internal vertex, which was also an internal vertex, was assigned a different colour from that of its parent, then there was one way of doing that, i.e. if a non-root internal vertex was coloured 1 (resp. 2), then its right child must be coloured 2 (resp. 1).
- (ii) If the colour of the root was chosen to be 1 (resp. 2) then an internal vertex which was a right child of the root (if any) was assigned colour 2 (resp. 1).

Proof. The proof follows from Theorem 4.43 by assigning one colour (instead of two) to the root of the complete ternary tree in which all internal vertices come in two colours and the rightmost child of each internal vertex was assigned a different colour from that of its parent. $\square$

4.5.6.3 3-Schröder paths

Theorem 4.44. *There was a bijection between the set of complete ternary trees with $2n + 1$ leaves such that the root comes in two colours, the first two children of all internal vertices come in two colours and the rightmost child of each internal vertex has a colour different from that of its parent and the set 3-Schröder paths of length n .*

Proof. This study constructed the bijection via Dyck paths in which the upsteps are of type $(1, 2)$ and come in two colours.

Let T be complete ternary tree with $2n + 1$ leaves such that the root comes in two colours, the first two children of all internal vertices come in two colours and the rightmost child of each internal vertex has colour different from that of its parent. Let the colours be 1 and 2. This study first obtained the corresponding Dyck path with n upsteps: Traversing the ternary tree in preorder, draw a $(1, 2)$ step labelled 1 (resp. labelled 2) for each internal vertex coloured 1 (resp. coloured 2) and a $(1, -1)$ step for each leaf, except the last one. This resulted in a Dyck path in which the upsteps are labelled 1 or 2. Since the upsteps correspond to internal vertices, then the number of upsteps was n and the downsteps are $2n$, i.e., one less than the number of leaves in the complete ternary tree.

This study remarked that for each upstep, there are two corresponding downsteps, i.e., the downsteps on the same level with the upstep, if one moves from left to right, without crossing any steps. Now, this study obtained the 3-Schröder paths of length n from the Dyck path as follows: For each upstep labelled 2, create a $(2, 1)$ step

and delete a corresponding downstep which was at a higher level. Finally remove all labels in the Dyck path. When a $(2, 1)$ step was created, the length of the path increases by 1. Also as a downstep was deleted the length of the path decreases by 1. So, the initial length of the Dyck path was retained. The resultant structure was thus a 3-Schröder paths of length n . Note that if the root of the complete ternary tree was coloured 1 (resp. 2) then the first step in the 3-Schröder path was of type $(1, 2)$ (resp. $(2, 1)$). This procedure was illustrated in Figure 4.36.

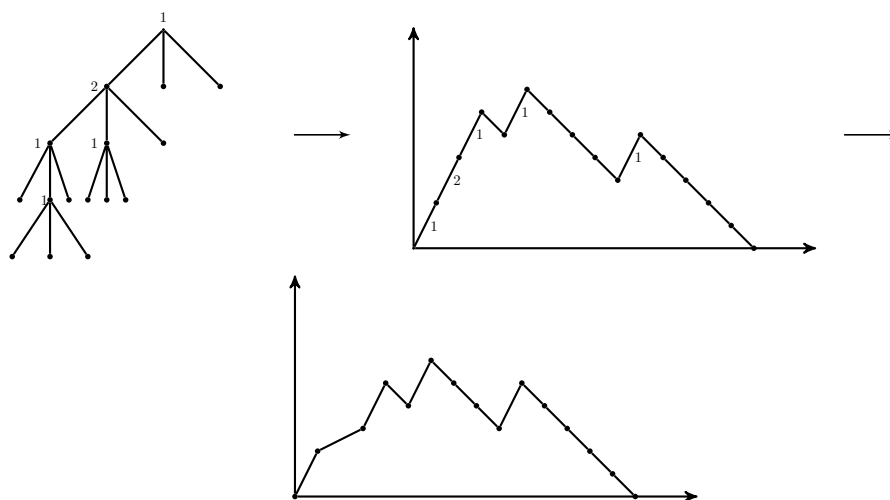


Figure 4.36: Obtaining a 3-Schröder path of length 5 from a complete ternary tree with 5 internal vertices.

Conversely, let S be a 3-Schröder paths of length n . Then label all the $(1, 2)$ steps as 1 and for each $(2, 1)$ step, draw a $(1, 2)$ step labelled 2 and introduce a corresponding downstep. The resulting figure was a Dyck path in which the upsteps are of type $(1, 2)$ labelled 1 or 2. This study then built a complete ternary tree from left to right as follows: Starting at the origin of the Dyck path, draw a root coloured 1 (resp. 2), of the corresponding tree if the first upstep was labelled 1 (resp. 2). Then move to the next step of the Dyck path. If an upstep labelled 1 (resp. 2) was encountered then introduce a new vertex attached to the previous vertex in the tree and colour it 1 (resp. 2). If a downstep was encountered, then draw a leaf and move back to the parent of the leaf. Continue until all the steps in the Dyck path are considered. Finally, introduce a last leaf so as to complete the tree. The resultant combinatorial object was a complete ternary tree in which the root was given the label of the initial

step in the Dyck path (or type of the initial upstep in the 3-Schröder path), the first two children of each internal vertex comes in two colours and the rightmost child of each internal vertex has a colour different from that of its parent. A depiction of this process was given in Figure 4.37.

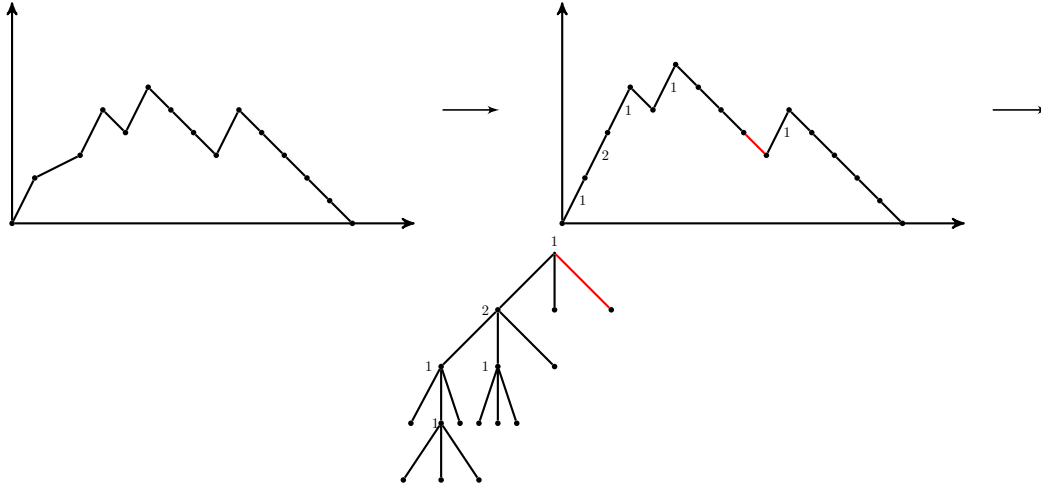


Figure 4.37: Obtaining a complete ternary tree with 5 internal vertices from a 3-Schröder path of length 5.

□

Based on the bijection, this study obtained the following corollary.

Corollary 4.22. *The set of complete ternary trees with $2n - 1$ leaves with root coloured 2 was enumerated by*

$$\frac{1}{n-1} \sum_{i=0}^{n-1} \binom{2n-2}{i} \binom{n-1}{i+1} 2^i.$$

By Theorems 4.43 and 4.44, this study found:

Corollary 4.23. *There is a bijection between the set of non-decreasing 2-noncrossing trees on $n + 1$ vertices with root labelled 1 and the set of 3-Schröder paths of length n .*

Corollary 4.24. *There is a bijection between the set of non-decreasing 2-noncrossing trees on $n + 1$ vertices with root labelled 2 and the set of small 3-Schröder paths of*

length n .

4.6 Plane Husimi graphs

Recall, from (Okoth, 2021b), that the number of plane Husimi graphs on $n + 1$ vertices with k blocks was given by

$$\frac{1}{n+1} \binom{n+k}{k} \binom{n-1}{k-1}. \quad (4.120)$$

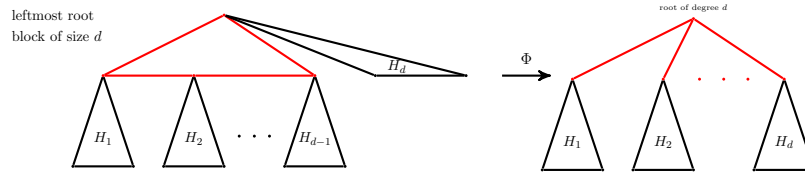
This study noted that when it summed over all k in Equation (4.120), it obtained little Schröder numbers. In this section, this study constructed bijections between the set plane Husimi graphs and related combinatorial structures enumerated by a formula whose sum was the little Schröder numbers.

4.6.1 Plane trees

Richard Stanley, in his paper [(Stanley, 1999, Exercise 6.39,(b))], included plane trees with a given number of endpoints and no vertices of outdegree 1 as one of the structures enumerated by the little Schröder numbers. This study constructed a bijection between the set of plane Husimi graphs and the set of these trees which was a generalization of the known bijection between the sets of plane trees and binary trees.

Theorem 4.45. *There is a bijection between the set of plane trees with $n+1$ vertices and k internal vertices such that each internal vertex has at least one child and the set of plane Husimi graphs on $n+1$ vertices with k blocks.*

Proof. Let H be a plane Husimi graph on $n + 1$ vertices with k blocks. Let the leftmost block child of the root be of size d and let the plane Husimi graphs attached at the vertices of leftmost block child be $H_1, H_2, \dots, H_d$ as shown in the left of Figure 4.38.

Figure 4.38: The map Φ .

This study defined a map Φ as follows: The image $\Phi(H)$ was a graph with an identified vertex (root) such that $H_1, H_2, \dots, H_d$ are its subgraphs attached in this order from left to right to the children of the root. Refer to Figure 4.38. The corresponding plane tree on $n + 1$ vertices was thus obtained by recursively applying Φ to transform $H_1, H_2, \dots, H_d$ to trees. Each block contributes an internal vertex. In Figure 4.39, this study gave a depiction of the procedure.

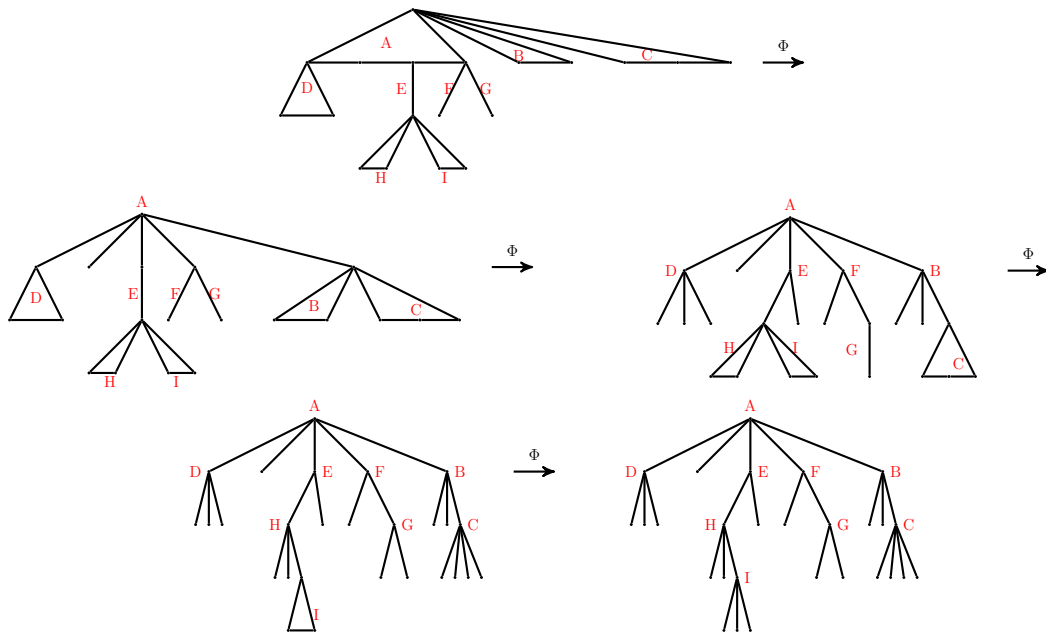


Figure 4.39: Procedure of obtaining a plane tree from a plane Husimi graph.

This study obtained the reverse procedure Φ^{-1} : Consider a plane tree on $n + 1$ vertices with k internal vertices such that each internal vertex has at least two children. For each internal vertex with i children, This study create a block of size i consisting of the vertex and its children except the rightmost child. Delete the rightmost edge and adjoin the rightmost child to its root. The structure obtained was a plane Husimi graph on n vertices with k blocks since each internal vertex

contributes a block. This procedure was illustrated in Figure 4.40.

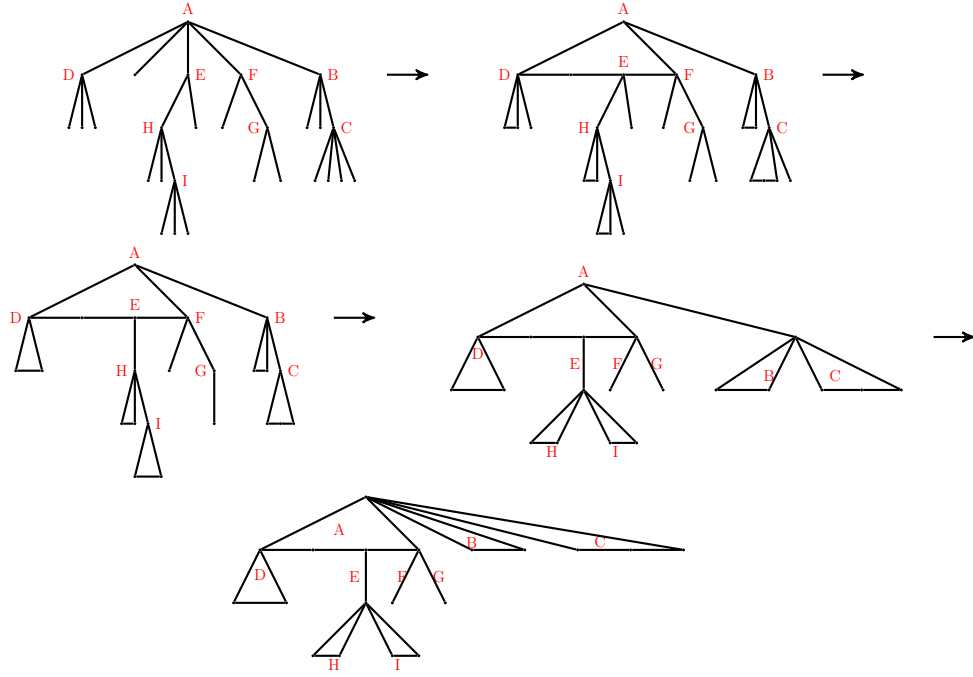


Figure 4.40: Procedure of obtaining a plane Husimi graph from a plane tree.

□

This study noted that in the bijection Φ described in the proof of Theorem 4.45, block size responds to outdegree in the plane tree. Okoth in [(Okoth, 2021b, Theorem 4.1)] showed that the number of plane Husimi graphs on n vertices with k blocks such that n_i blocks are of size $i \geq 2$ was given by

$$\frac{1}{n} \binom{n+k-1}{k} \binom{k}{n_2, n_3, \dots}.$$

From the bijection in Theorem 4.45, this study thus obtained the following corollary:

Corollary 4.25. *There are*

$$\frac{1}{n} \binom{n+k-1}{k} \binom{k}{n_2, n_3, \dots}$$

plane trees on n vertices with k internal vertices such that there are n_i vertices with outdegree i for every $i \geq 2$ and no vertices of outdegree 1.

This study also noted that if the blocks are all of size 2, then the bijection described in this section was the rotation correspondence between plane trees and binary trees obtained in (Flajolet & Sedgewick, 2009).

4.6.2 Polygon Dissections

Among the listed structures in the OEIS enumerated by little Schröder numbers was the set of dissections of a regular polygon.

Theorem 4.46. *There is a bijection between the set of dissections of regular polygons on $n + 2$ vertices with k regions and the set of plane Husimi graphs on $n + 1$ vertices with k blocks.*

Proof. Consider a regular $(n + 2)$ -gon with $k - 1$ diagonals. The diagonals divide the polygon into k regions. Label the upper left vertex as 1. Label the remaining vertices as 2 through to $n + 2$ in the anticlockwise direction. Let $v_1, v_2, \dots, v_m$, where $v_i \leq v_{i+1}$ and $m \geq 3$, be vertices of a region. Now, create edges $v_1v_2, v_2v_3, \dots, v_{m-2}v_{m-1}, v_{m-1}v_1$. Do this for all the regions. The graph obtained was a plane Husimi graph whose vertices have been labelled $1, 2, \dots, n + 1$ in preorder, i.e., visit the root, leftmost subgraph, second leftmost subgraph and so on. The labels can thus be dropped. Since a region in the polygon corresponds to a block in the plane Husimi graph, the graph was on $n + 1$ vertices with k blocks. In Figure 4.41, this study gave an illustration of this procedure.

This study obtained the reverse procedure: Consider a plane Husimi graph H on $n + 1$ vertices with k blocks. Traverse H in preorder and label a vertex i if it was the i^{th} vertex to be traversed. This study obtained the corresponding regular $(n + 2)$ -gon with k regions as follows: Draw a regular polygon with $n + 2$ vertices labelled $1, 2, \dots, n + 2$ in anticlockwise direction such that vertex 1 was the upper left vertex. Inscribe the plane Husimi graph in the polygon such that an edge ij in the Husimi graph was a straight line in the polygon. Note that vertex $n + 2$ was isolated.

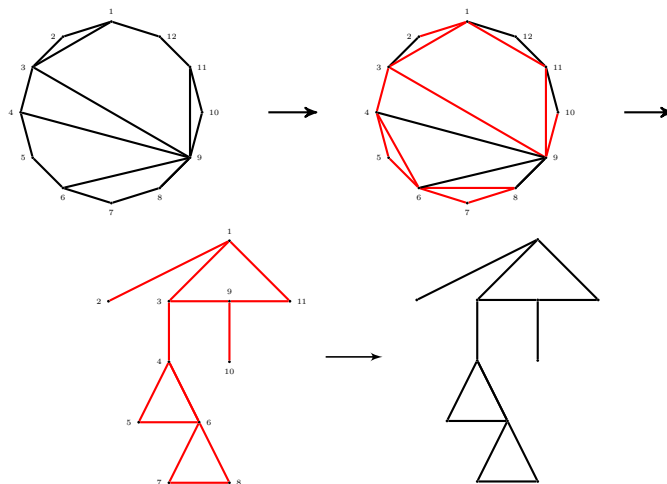


Figure 4.41: Obtaining a plane Husimi graph from a polygon dissection.

Let m be the vertex incident to vertex 1 that has the largest label. Delete this edge and draw straight lines from 1 and m to $n + 2$. This study now moved around the polygon, starting at vertex 1, in anticlockwise direction and add straight lines inside the polygon recursively as follows:

- (i) Let u and v be vertices with the smallest and largest label in a block respectively. Moreover, let w be a vertex with the largest label that was incident to u . The vertex w belongs to another block and uw was a diagonal line dissecting the polygon. Delete edge uw and draw a new edge vw .
- (ii) For $i = 1, 2, \dots, n$, draw the lines from i to $i + 1$ if they are not already there.

This study described the reverse procedure in Figure 4.42. □

This study noted that the root degree in the plane Husimi graph corresponded to the number of regions which are adjacent to vertex 1 in the polygon dissection thus the following result:

Corollary 4.26. *There are*

$$\frac{d}{k} \binom{n+k-d-1}{k} \binom{n-1}{k-1}$$

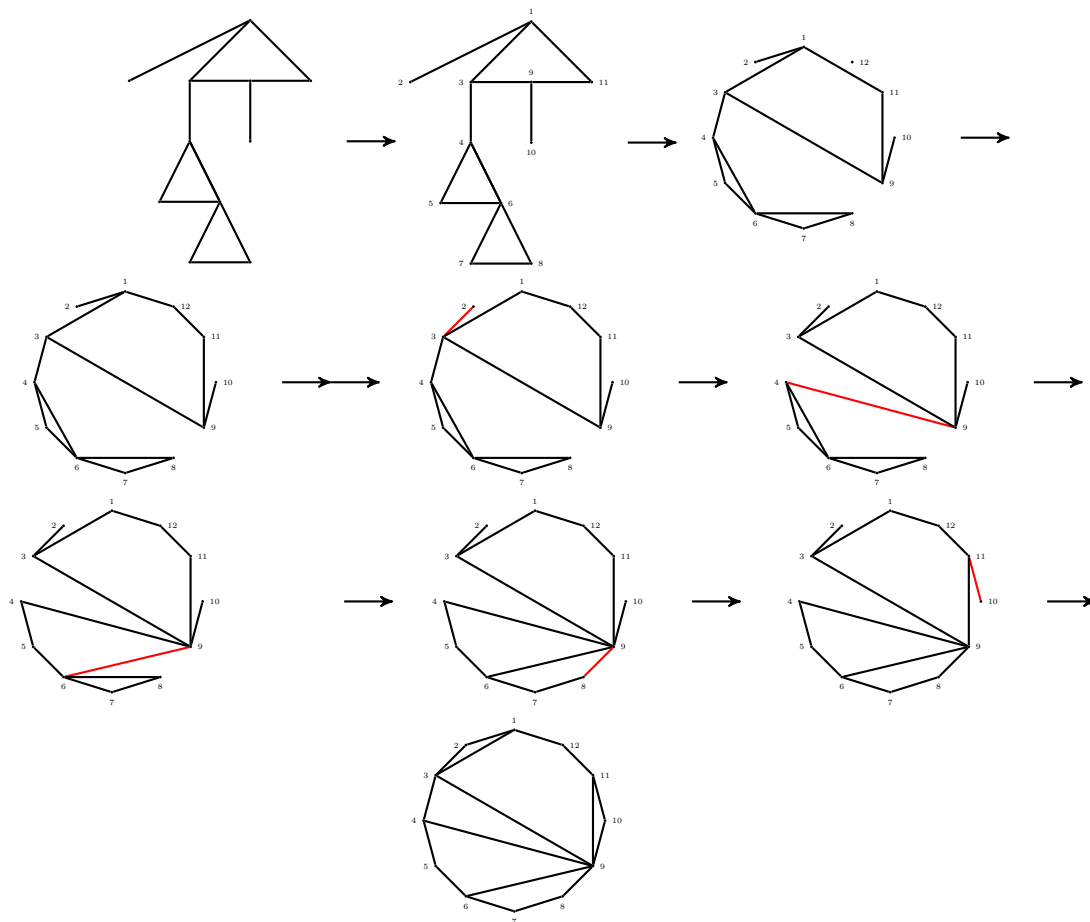


Figure 4.42: Obtaining a polygon dissection from plane Husimi graph.

different dissections of a regular $(n+2)$ -gon with k diagonals such that a fixed vertex was incident to d regions.

4.6.3 Standard Young tableaux

The number of standard Young tableaux of shape $(k, k, 1^{n-k})$ (where the first two entries represent the size of rows and the third entry the size of the column) was among the structures given by the little Schröder numbers. In (Stanley, 1996), Stanley constructed a bijection between the set of the polygon dissections previously discussed and Standard Young tableaux. This study therefore obtained the bijection;

Theorem 4.47. *There was a bijection between the set of plane Husimi graphs on $n+1$ vertices with k blocks and the set of standard Young tableaux of type $(k, k, 1^{n-k})$.*

Proof. Consider a plane Husimi graph H on $n+1$ vertices with k blocks. This study obtained a standard Young tableau by the following procedure:

- (i) Traverse H by preorder and obtain a sequence $s = (s_1, s_2, \dots, s_k)$ where s_i was one less than the size of the i^{th} block to be traversed. The sequence has length k since each block contributes an entry in the sequence.
- (ii) Get the partial sum sequence $p = (p_1, p_1 + 1, p_2 + 1, \dots, p_{k-1} + 1)$ where $p_j = s_1 + s_2 + \dots + s_j$ for $j = 1, 2, \dots, k$.
- (iii) Traverse H again but this time, label the vertices of H (except the root) as $1, 2, \dots, n$ starting at the vertex where this study first encounter a leaf. Note that a vertex was given a label if it was encountered for the last time.
- (iv) Mark the vertices with labels in the sequence p with an asterisk, $*$ and remove the labels leaving the asterisks.
- (v) Traverse the plane Husimi graph again and record an entry i in the:
 - (a) first row of the Young tableau if the i^{th} item to be visited was a block.
 - (b) second row, if the i^{th} item to be visited was a vertex with an asterisk.
 - (c) first column, if the i^{th} item to be visited was a vertex without an asterisk.

In Figure 4.43, this study got a standard Young tableau from a plane Husimi graph using the procedure described above.

This study provided the reverse procedure: Consider a sequence $(1, 2, \dots, n+k)$ from a standard Young tableau. Perform the following procedure to obtain a plane Husimi graph:

- (i) Change all the entries in the sequence that are in the second row to x_i^* where $i = 1, 2, \dots, k$ and all the entries of first column (except of row 1 and row 2) to x_j where $j = 1, 2, \dots, n-k$. Rename the remaining integers as $1, 2, \dots, k$ in this order.

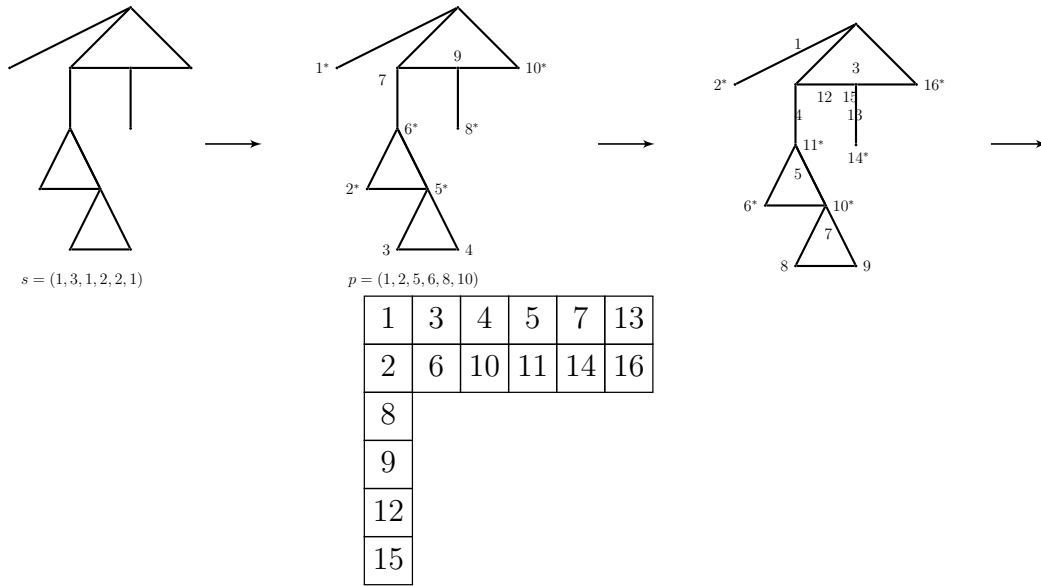


Figure 4.43: Obtaining a Standard Young tableau from plane Husimi graph.

- (ii) Change entry 1 of the sequence to the number of x_i 's and x_i^* that are appear before x_2^* .
- (iii) Change entry $2 \leq j \leq k - 1$ of the sequence to the number of x_i^* 's and x_i 's that appear from x_j^* to x_{j+1}^* but not inclusive of x_{j+1}^* .
- (iv) Change entry k of the sequence to the number of x_i 's and x_i^* 's that are encountered after x_k^* and inclusive of x_k^* .
- (v) Change all the x_j 's and x_j^* 's to x .
- (vi) Starting at the first entry in the sequence, draw a block size of i if the entry was $i - 1$ such that one vertex was identified as the root. Then move to the child in the block that appears on the far left. If the second entry was $j \geq 1$ then attach a block of size $j + 1$. Otherwise, if the entry was x then move to the second child in the block. If there was no available vertex to attach a block then move back to the root until you find a vertex to attach a block. Continue until are the entries in the sequence are encountered.

The graph obtained was a plane Husimi graph on $n + 1$ vertices with k blocks since each entry greater than 0 contributes a block and x contributed a vertex. An

example to explain this procedure was given in Figure 4.44.

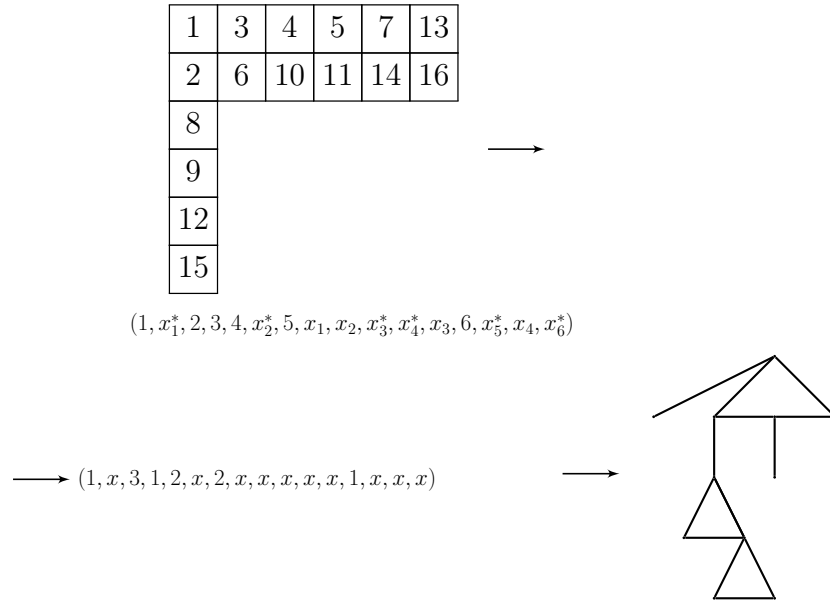


Figure 4.44: Obtaining a plane Husimi graph from a Standard Young tableau.

□

4.6.4 Valid Sequences

Consider a sequence $(s_1, s_2, \dots, s_{n+k})$ of integers which satisfy the following properties:

- (i) Either $s_i \geq 1$ or $s_i = -1$, for all $i = 1, 2, \dots, n+k$
- (ii) The number of $s_i \geq 1$ was k and the remaining ones are all equal to -1.
- (iii) The sum $s_1 + s_2 + \dots + s_{n+k} = 0$.
- (iv) The partial sum $p_i = s_1 + s_2 + \dots + s_i \geq 0$ for all $i = 1, 2, \dots, n+k$.

This study referred to such sequences as *valid sequences*. Stanley (Stanley, 1999) constructed a bijection between the set of these sequences and standard Young tableaux of type $(k, k, 1^{n-k})$. This showed that Equation (4.120) counted valid sequences of length $n+k$. This study therefore constructed the following bijection;

Theorem 4.48. *There is a bijection between the set of plane Husimi graphs on $n+1$ vertices with k blocks and the set of valid sequences with k entries greater than 0 and n entries which are equal to -1 .*

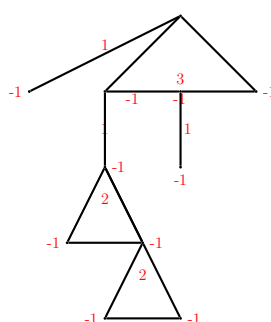
Proof. Let H be a plane Husimi graph on $n+1$ vertices with k blocks. This study obtained a sequence of length $n+k$ as follows: This study traverse H using preorder traversal or depth first search by visiting a block before its vertices. This study wrote $i-1$ if a block of size $i > 1$ was visited. Otherwise, record -1 if a vertex visited was a leaf or it was a non-leaf visited for the last time on the way back to the root. This study didn't record a value for the root. This study noted the following:

- (i) The length of the sequence was $n+k$ since each vertex in the graph (except the root) and each block contributes an entry in the sequence. Let the sequence be $(s_1, s_2, \dots, s_{n+k})$.
- (ii) Either $s_i \geq 1$ or $s_i = -1$.
- (iii) Exactly k entries are greater than 0 and exactly n entries are equal to $s_i = -1$.
- (iv) The sum $s_1 + s_2 + \dots + s_{n+k} = 0$ since each vertex (except the root) contributes 1 as This study record a contribution from the block and -1 if This study record a contribution of the vertex.
- (v) The partial sum $p_i = s_1 + s_2 + \dots + s_i \geq 0$ for all i since a record of -1 was done when a vertex was visited was leaf or was a non-leaf encountered at the last time on the way back to root. This was done after the recording of the size of the block to which the vertex belongs.

The sequence obtained was thus a valid sequence.

This study now obtained the reverse procedure: Consider a valid sequence of length $n+k$ with k entries greater than 0 and n entries equal to -1 . This study obtained a plane Husimi graph by the following procedure. Starting at the first entry in the

sequence, draw a block of size i if the entry was $i - 1$ such that one vertex was identified as the root. Then move to the child in the block on the far left. If the second entry was $j \geq 1$ then attach a block of size $j + 1$. Otherwise, if the entry was -1 then move to the second child in the block. If there was no available vertex to attach a block then move back to the root until you find a vertex to attach a block. The graph obtained was a plane Husimi graph on $n + 1$ vertices with k blocks since each entry greater than 0 contributes a block and -1 contributes a vertex. This bijection was described in Figure 4.45. $\square$



$$\Leftrightarrow (1, -1, 3, 1, 2, -1, 2, -1, -1, -1, -1, 1, -1, -1, -1)$$

Figure 4.45: Bijection plane Husimi graph and valid sequences

4.6.5 Deutsch paths

This study related Deutsch paths to plane Husimi graphs and extended them to the valid sequences discussed in Theorem 4.48.

Theorem 4.49. *There is a bijection between the set plane Husimi graphs with $n + 1$ vertices with k blocks and the set of Deutsch paths with n down steps and k up steps.*

Proof. Let H be a plane Husimi graph on $n + 1$ vertices with k blocks. This study obtained a Deutsch path with k up steps and n down steps as follows: This study traverse H using preorder traversal. A block was visited before the subtrees attached to its vertices. Draw an up step of length $i - 1$ if a block of size $i > 1$ was visited. Otherwise, draw a unit down step if the vertex visited was a leaf or was a non-leaf

vertex traversed for the last time on the way back to the root.

For the reverse procedure, start at the origin of the Deutsch path and draw a block of size i if the length of the up step was $i - 1$, such that one vertex was identified as the root. Then move to the child in the block that appears on the far left. If the next path was an up step of length $j \geq 1$ then attach a block of size $j + 1$. Otherwise, if the next path was a unit down step then move to the second child in the block. If there was no available vertex to attach a block then move back to the root until you find a vertex to attach a block. The graph obtained was a plane Husimi graph on $n + 1$ vertices with k blocks since each up step contributed a block and each down step contributed a vertex. The bijection was described in Figure 4.46.

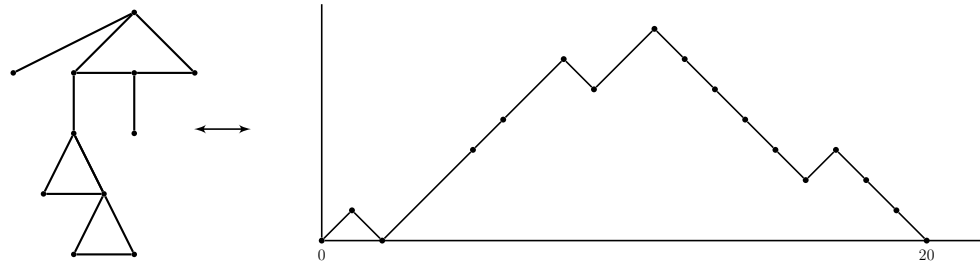


Figure 4.46: Bijection between plane Husimi graphs and Deutsch paths.

□

From the bijection constructed, the degree of the root of plane Husimi graphs corresponded to number of hills (number of sections starting and ending at the x-axis) in the Deutsch path thus Equation (4.120),

$$\frac{d}{k} \binom{n+k-d-1}{k} \binom{n-1}{k-1}$$

gave the number of Deutsch path with n down steps, k up steps and d hills.

This study then proved the following result with valid sequences.

Theorem 4.50. *There is a bijection between the set of valid sequences of length*

$n + k$ and the set of Deutsch paths with n down steps and k up steps.

Proof. Consider a valid sequence of length $n + k$. There are n entries which are equal to -1 and k entries which are greater than 0 . Let the sequence be $(s_1, s_2, \dots, s_{n+k})$. Starting at the beginning of the sequence, for each entry $s_i > 0$, draw up step of length s_i and for each entry $s_i = -1$, draw a unit down step. This study obtained a Deutsch path with n down steps and k up steps.

To reverse the procedure, start at the origin of the Deutsch path and record the length of each up step and -1 for each down step. This study now obtained a valid sequence of length $n + k$:

- (i) The length of the sequence was $n + k$ since each up step (k in number) contributes an entry greater than 0 and each down step (n in number) contributes an entry -1 to the sequence. Let the sequence be $(s_1, s_2, \dots, s_{n+k})$.
- (ii) The sum $s_1 + s_2 + \dots + s_{n+k} = 0$ since the sum of the lengths of up steps was n and that of the down steps was n . Since each down step was given an entry -1 , then the total sum was 0 .
- (iii) The partial sum $p_i = s_1 + s_2 + \dots + s_i \geq 0$ for all i since a record of -1 was done when a down step was visited. Since the path does not go below the x -axis, the sum will always be non-negative.

See Figure 4.47 for an example. □

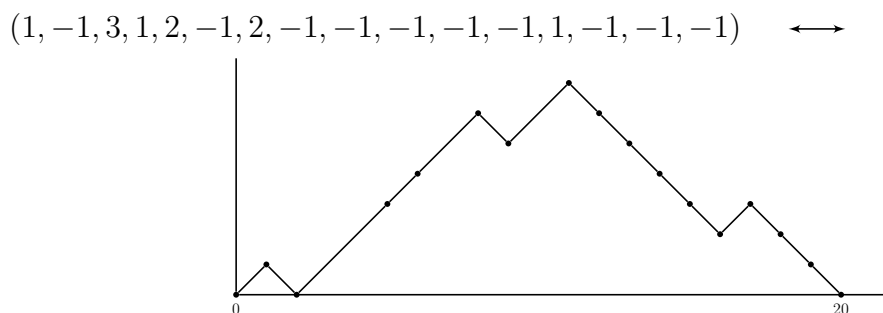


Figure 4.47: Bijection between valid sequences and Deutsch paths.

4.6.6 Restricted lattice paths

Consider lattice paths that use unit horizontal step and unit vertical steps that begin at the origin and lie weakly below the line $y = mx$. These lattice paths have been studied by various authors, for example (Krattenthaler, 2015; Humphreys, 2010). This study referred to these lattice paths as *restricted lattice paths* and constructed a bijection between the set of plane Husimi graphs and the set of restricted lattice paths in which horizontal steps were allowed to have length greater than or equal to 1. This study similarly extended the results to the set of valid sequences.

Theorem 4.51. *There was a bijection between the set of plane Husimi graphs on $n + 1$ vertices with k blocks and the set of restricted lattice paths with k horizontal steps and n vertical steps.*

Proof. Let H be a plane Husimi graph on $n + 1$ vertices with k blocks. This study obtained a restricted lattice path with k horizontal steps and n vertical steps as follows: This study traversed H using preorder traversal. Again a block was visited before its vertices. Draw a horizontal step of length $i - 1$ if the block traversed was of size $i > 1$. Otherwise, draw a unit vertical step if the vertex visited was a leaf or was a non-leaf visited for the last time on the way back to the root. The path does not cross the line $y = x$.

For the reverse procedure, start at the origin of the restricted lattice path and draw a block of size i if the length of the horizontal step was $i - 1$, such that one vertex was identified as the root. Then move to the child of the block on the far left. If the next path was a horizontal step of length $j \geq 1$ then attach a block of size $j + 1$. Otherwise, if the next path was a unit vertical step then move to the second child in the block. If there was no available vertex to attach a block then move back towards the root until you find a vertex to attach a block. The graph obtained was a plane Husimi graph on $n + 1$ vertices with k blocks since each horizontal step contributes a block and each vertical step contributes a vertex. Figure 4.48 was an illustration

of the bijection.

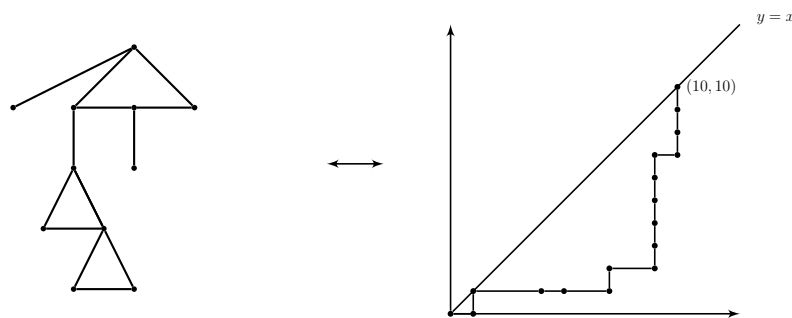


Figure 4.48: Bijection between plane Husimi graphs and restricted lattice paths.

□

This study also had that;

Theorem 4.52. *There was a bijection between the set of valid sequences of length $n + k$ and the set of restricted lattice paths with k horizontal steps and n vertical steps.*

Proof. This study constructed a bijection between the set of valid sequences of length $n + k$ and restricted lattice paths with n vertical steps and k horizontal steps as follows: Consider a valid sequence of length $n + k$. There are n entries which are equal to -1 and k entries which are greater than 0 . Let the sequence be $(s_1, s_2, \dots, s_{n+k})$. Starting at the beginning of the sequence, for each entry $s_i > 0$, draw a horizontal step of length s_i and for each entry $s_i = -1$, draw a unit vertical step. This study obtained a restricted path with n vertical steps and k horizontal steps. Since $s_1 + s_2 + \dots + s_i$ for $i \geq 1$, then the path lie weakly below the line $y = x$.

To reverse the procedure, start at the origin of the restricted lattice path and record the length of each horizontal step and -1 for each vertical step. This study show that sequence obtained was a valid sequence of length $n + k$:

- (i) The length of the sequence was $n + k$ since each horizontal step (k in num-

ber) contributes an entry greater than 0 and each vertical step (n in number) contributes an entry -1 to the sequence. Let the sequence be $(s_1, s_2, \dots, s_{n+k})$.

- (ii) The sum $s_1 + s_2 + \dots + s_{n+k} = 0$ since the sum of the lengths of horizontal steps was n and that of the vertical steps was n . Since each vertical step was recorded as -1 in the sequence, then the total sum was 0.
- (iii) The partial sum $p_i = s_1 + s_2 + \dots + s_i \geq 0$ for all i since a record of -1 was done when a vertical step was visited. Since the path stayed weakly below the line $y = x$, the sum would always be non-negative.

See Figure 4.49 for an example.

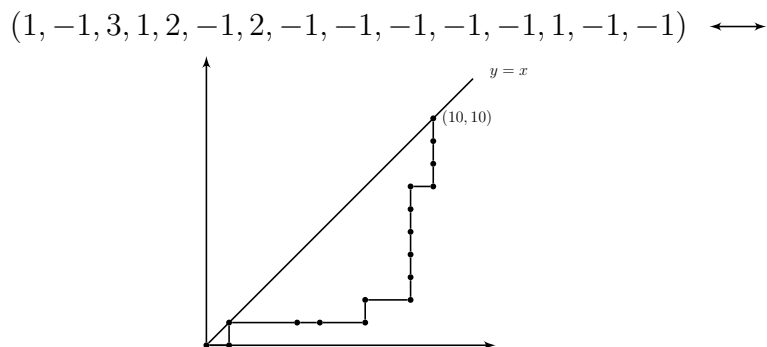


Figure 4.49: Bijection between valid sequences and restricted lattice paths.

□

CHAPTER FIVE: CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter provides a conclusion of this thesis as well as recommendations for future research, with emphasis on practical applications that address real-world societal problems.

5.2 Conclusion

This thesis achieved five main objectives, each contributing to both theoretical mathematics and practical problem-solving across multiple disciplines. First, it obtained a formula counting skew 2-Dyck paths by path length (Theorem 4.1), demonstrating log-concavity and establishing that consecutive term ratios were bounded above by 8 with limit approaching $4 + 2\sqrt{3}$. It established bijections with labelled 2-box paths (Theorem 4.3), 3-leaf labelled plane trees (Theorem 4.4), and 2-edge labelled plane trees (Theorem 4.5). These results have direct applications in computational biology for modeling RNA secondary structures and protein folding pathways, where the bounded growth properties help predict stable molecular configurations.

Second, this study introduced non-decreasing 2-plane trees which were plane trees with integer labels from $\{1, 2\}$ where adjacent vertex label sums were at most 3 and sibling labels increased left to right. Their generating functions corresponded to

little and large Schröder numbers (Equation (1.2)). Using Lagrange inversion, the study enumerated these trees by various parameters (Theorems 4.8, 4.10, 4.6, 4.7 and 4.11) and constructed bijections with multiple combinatorial structures including Schröder paths, lattice paths, and Young tableaux (Theorems 4.12, 4.13, 4.14, 4.15, 4.16 and 4.17). This work directly addressed network reliability problems in telecommunications, where these trees model hierarchical network structures with capacity constraints.

Third, the set of weakly labelled k -plane trees have been introduced. These are plane trees with an unlabelled root and children of all internal vertices are labelled from the set $\{1, 2, \dots, k\}$ in non-decreasing order from left to right. These trees have been enumerated by number of vertices of a certain label (Theorem 4.19), root degree (Lemma 4.1), label of the eldest child of the root (Theorem 4.20), number of leaves (Theorem 4.21), forests (Theorem 4.22) and outdegree sequence (Theorem 4.23). Bijections between the set of these trees and the set of k -ary Husimi graphs (Theorem 4.24), set of Dyck paths of a given labelling (Theorem 4.25), set of plane trees with some given number of leaves (Theorem 4.26), set of k -plane trees with root labelled k and 1 (Theorem 4.27 and Theorem 4.28), set of k -binary trees (Theorem 4.29), set of S -Motzkin paths (Theorem 4.30), set of T -Motzkin paths (Theorem 4.31) and ways of connecting a number of points in a line (Theorem 4.32) have also been created. These structures solve phylogenetic reconstruction problems in evolutionary biology, where researchers need to determine species relationships from genetic data while maintaining temporal ordering constraints.

Fourth, two new classes of combinatorial objects i.e. non-decreasing 2-noncrossing trees and non-decreasing 2-noncrossing increasing trees were introduced. The concept of butterfly decomposition and the (l, r) -representation to define and enumerate these trees was used. The non-decreasing 2-noncrossing trees have the (l, r) -representation as plane trees where each internal vertex has its children labelled l_1, l_2, r_1 and r_2 in this order from left to right while the non-decreasing 2-noncrossing

increasing trees are non-decreasing 2-noncrossing trees whose edges are all ascents if paths from the root are considered in the (l, r) -representation of the 2-noncrossing trees. non-decreasing 2-noncrossing trees have been enumerated by parameters such as number of vertices (Theorem 4.33), label of the root (Corollary 4.12), root degree (Theorem 4.34) and number of forests (Theorem 4.36). For non-decreasing 2-noncrossing increasing trees, the statistics of enumerations were; number of vertices (Theorem 4.38), label of left child of the root (Theorem 4.39), root degree (Theorem 4.40) and number of forests (Theorem 4.41). The enumerative formulas were found to be another generalization of the little and large Schröder numbers. Bijections have also been obtained with the sets of related combinatorial structures such as; locally oriented noncrossing trees (Theorem 4.42), complete ternary trees (Theorem 4.43), and 3-Schröder paths (Theorem 4.44). These results address urban planning optimization, particularly in designing non-intersecting transportation networks where traffic flow directions has to satisfy hierarchy constraints.

Last, in section 4.6, bijections between the set of plane Husimi graphs and various sets of combinatorial structures were established. i.e. the set of plane trees (Theorem 4.45), set of number of polygon dissections (Theorem 4.46), set of standard Young tableaux (Theorem 4.47), set of valid sequences (Theorem 4.48), set of Deutsch paths (Theorem 4.49) and set of restricted lattice paths (Theorem 4.51). This work has immediate applications in statistical physics for modelling polymer networks. This research mainly addressed Richard Stanley's challenge for combinatorial interpretations of Schröder number structures while solving concrete problems across multiple domains.

5.3 Recommendations

From the enumeration of skew 2-Dyck paths, further research on their applications and generalizations is recommended. In particular, extending the analysis to a generalized set of skew k -Dyck paths may create new insights. Study of potential

application into areas such as physics and computer science is also recommended.

From the study of non-decreasing 2-plane trees, a study on generalization of non-decreasing 2-plane trees to non-decreasing k -plane trees such that the sum of labels of adjacent vertices is at most $k+1$ is recommended. Their generating functions can be obtained for parameters such as forests and bijections of non-decreasing k -plane trees also established. From the bijections created for the set of non-decreasing 2-plane trees, there are related combinatorial structures yet to be established. These include set partitions and permutations among others. Further research on constructing such bijections is recommended. Research on potential applications of the generalized class of weakly labelled k -plane trees in areas such as computer science, phylogenetics, or data structures, where ordered tree-like structures play a role is also recommended.

Similarly, a study can be done to generalize the results on non-decreasing 2-noncrossing trees to k -noncrossing increasing trees. Various bijections have been constructed in relation to non-decreasing 2-noncrossing trees via complete ternary trees. It would be interesting to obtain direct bijections. In addition, there are combinatorial structures which stem from Hopf algebras and in phylogenetics. Connections between these structures and non-decreasing 2-noncrossing trees could also be investigated. Lastly, the labelling concept of non-decreasing applied in our objectives can also be extended to k -ary trees and enumerative results obtained.

Interdisciplinary collaborations would be essential for validating and implementing the theoretical advances. Partnerships with medical researchers would enable epidemic modelling validation. Collaborations with telecommunications companies would allow network optimization testing in operational environments. Working with environmental agencies on pollution tracking systems would demonstrate the practical value of lattice path models for environmental monitoring and engaging with computer scientists on algorithm implementation and performance analysis would ensure the theoretical advances translated into efficient computational tools.

CHAPTER A: Appendix

1.1 Papers from the thesis

Results from this thesis have been published in the following papers:

- Yvonne Wakuthii Kariuki and Isaac Owino Okoth, **Bijections of plane Husimi graphs and certain combinatorial structures**, European Journal of Mathematics and Applications, Volume 3 (2023), Article ID 21.
<https://doi.org/10.28919/ejma.2023.3.21>
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