

**Asymptotic Behaviour of Solutions of Discretised
Unbounded Positive Symmetric and Dirac Operators on
Hilbert Spaces**

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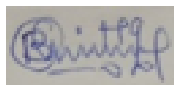
DECLARATION

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ABSTRACT

Stability analysis has been investigated for deficiency indices and spectrum of symmetric higher order differential operators. It has been established that deficiency indices and discrete spectrum are stable under bounded perturbations whereas singular continuous spectrum is not stable even under finite rank perturbations. However, little has been done to establish the stability of existence of positive self-adjoint operator extensions under unbounded perturbations. Similarly, asymptotic analysis for discretised Dirac operators with unbounded coefficients was limited. This study has added to existing knowledge on the related areas as well as analysed the stability of national income using asymptotics of the solutions. The main goal of this research was to analyse the stability of existence of positive self-adjoint operator extension of symmetric operators under unbounded perturbations and asymptotic behaviour of solutions of discrete Dirac operators. The specific objectives of this study were to: 1) Construct positive perturbed difference operators with unbounded potentials, 2) Analyse the necessary and sufficient conditions for the stability of existence of positive self-adjoint operator extension under unbounded perturbations, 3) Investigate the spectral properties of self-adjoint operator extensions of the perturbed difference operators with unbounded coefficients, and 4) Examine the asymptotic behavior of solutions of the discretised Dirac system with unbounded potentials. Inner products of Hilbert spaces was applied to obtain coefficients growth conditions for the construction of positive symmetric difference operators. Asymptotic summation based on discretised Levinson theorem, and von Neumann theorem for the existence of self-adjoint operator extensions were used to determine the existence of the positive self-adjoint operator extensions of the positive symmetric operators. The M-matrix of square summable eigensolutions was constructed and its spectral measures applied to determine the spectral properties of the positive self-adjoint operator extensions. Finally, the matrix asymptotic summation was applied to obtain the asymptotic properties of the Dirac difference operator. Suppose L_1 and L_2 are fourth and second-order symmetric difference operators with $L = L_1 - zL_2$, where z is a real constant, and if the leading coefficient of L_1 dominates the coefficients of L_2 , then L_1 and L are positive symmetric operators. Furthermore, the deficiency indices of L_1 and L are (n, n) , $2 \leq n \leq 4$, with each operator L_1 and L having positive self-adjoint operator extensions even if their coefficients are unbounded. The absolutely continuous spectra of H_1 and H , the positive self-adjoint operator extensions of L_1 and L respectively are subsets of $(0, \infty)$ of spectral multiplicity two whenever the leading coefficient of L_1 is unbounded. The solutions of the Dirac system were the product of their transforming matrices and the direct product of the eigenvalues matrix with asymptotic behavior of the solutions determined by the unbounded potentials. These results can be applied by economists to study the stability of national income via the Keynesian model. Due to the complexity in computations of roots of polynomials of higher degrees more than four, similar analysis could not be investigated for order six or more. It is thus recommended that order six or more could be analysed in future using numerical analysis techniques.

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DEDICATION

I dedicate this work to my family.

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LIST OF ABBREVIATIONS AND ACRONYMS

$\mathcal{L}$	symmetric difference map . . .	5
L	minimal difference operator . .	5
$\mathcal{H}$	Hilbert Space	13
$D(L)$	domain of L	13
L^*	maximal difference operator .	13
H	self-adjoint operator extension	16
O	Landau symbol ("big O ") . . .	17
σ_{ac}	absolutely continuous spectrum	121

CHAPTER ONE

INTRODUCTION

In this chapter, we have given the background of our study, definitions of terms used, statement of the problem, objectives, justification and significance of the research.

1.1 Mathematical Background

The study of difference operators plays a significant role in the analysis of discrete dynamical systems, numerical approximations of differential equations and various applications in mathematical physics. The study of the asymptotic behaviour of solutions of symmetric difference operators with unbounded coefficients in Hilbert spaces lies at the intersection of functional analysis, operator theory and discrete spectral theory. A central theme in this research was understanding the asymptotic behaviour of solutions of symmetric difference and discretised Dirac equations, particularly when the coefficients are unbounded. The asymptotic analysis of such equations provide crucial insights into stability, oscillation and the spectral properties of the associated operators.

The solutions to a difference equation can exhibit various growth behaviours including polynomial, exponential or oscillatory growth. A difference operator is a discrete analog of the differential operator. Given a sequence y_n , the forward and backward difference operators are respectively defined as:

$$\Delta y_n = y_{n+1} - y_n \text{ and } \nabla y_n = y_n - y_{n+1}.$$

Here, $y_n = y(n)$ for every $n \in \mathbb{N}$. The symmetric difference operator δ , is given by $\delta y_n = \frac{y_{n+1} - y_{n-1}}{2}$. For a sequence a_n , a second order symmetric difference equation takes the form

$$a_n \delta^2 y_n + b_n \delta y_n + c_n y_n = 0$$

where a_n and b_n are coefficient sequences and c_n represents a potential term. The asymptotic nature depends on the coefficients a_n, b_n and c_n particularly when they are unbounded. The classical results like Levinson-Benzaid Lutz's theorem help in understanding the stability of such solutions. An essential aspect of symmetric difference operators is their spectral properties. Self-adjointness conditions are necessary to ensure well defined spectral decompositions. In particular, whenever the coefficient a_n grow unbounded, the essential spectrum of the operator may extend beyond the classical setting.

The oscillation theory investigates whether the solutions oscillate infinitely often or extend to a limit. The classical Sturmian theory for difference equations provides tools to examine oscillatory behaviour even for the case of unbounded coefficients.

1.1.1 Differential and Difference Operators

The study of spectral problems of differential operators has played a vital role in both theoretical and practical aspects. Spectral problems of differential operators are classified into two. To begin with, those defined over finite intervals with well-behaved coefficients are referred to as regular. In such a case, fine spectral properties are expected. For instance, it is expected that the spectral set is discrete, infinite and unbounded and that the eigenfunction basis is complete for a corresponding space. On the other hand, spectral problems which are not regular are referred to as singular. It is difficult to discuss them since the spectral set may be much more complicated. The investigation of the singular spectrum is also not easy using perturbation theory due to their unstable nature.

In the study of spectral problems for linear ordinary differential equations, the the-

ory of Weyl-Titchmarsh is very important. The theory began with the renowned work by Weyl (Weyl, 1910). Weyl gave a dichotomy of the limit point and limit circle cases for singular spectral problems of second order, originally self-adjoint linear differential equations. He was followed by Titchmarsh (Titchmarsh & Weiss, 1962) and many others. Since 1910 until 1945, these mathematicians developed and improved the spectral theory of self-adjoint differential operators of second order and other higher order operators. Their work was furthered by Coddington and Levinson (Coddington, Levinson, & Teichmann, 1956) and others in the late 1940s and 1950s. Additional results were found for operators of second order. Operators of order more than two were also examined.

Further study continued in the 1960s and 1970s with the work of Atkinson (Atkinson & Weiss, 1964) on regular Hamiltonian systems and Everitt and Kumar [(W. Everitt & Kumar, 1976), (W. N. Everitt & Kumar, 1976)] on higher order scalar problems. The work for this period was summarised by Atkinson, Everitt and Kumar. There were many other contributors. One contributor that perhaps deserves special attention is Walker (Walker, 1974) who showed that every scalar self-adjoint problem of an arbitrary order can be reformulated as an equivalent self-adjoint Hamiltonian system. This eliminated the need to discuss scalar problems and systems separately.

Hinton and Shaw (Hinton & Shaw, 1983), Krall [(Krall, 1989), (Krall, 2002)] and Remling (Remling, 1996) made great progress in the 1980s and 1990s. They considered singular spectral problems in the Hamiltonian form, following the lead of Atkinson. In 2000s, Clark and Gesztesy (S. Clark & Gesztesy, 2000), Qi and Chen (Qi & Chen, 2004), Qi (Qi, 2005), Brown and Evans (Brown, Evans, & Plum, 2003), Remling (Remling, 1996), Shi (Y. Shi, 2004), Sun et al. (Sun, Shi, & Chen, 2007), Zheng and Chen (Zheng & Chen, 2006) have made significant progress by considering spectral problems for Hamiltonian differential systems.

Another key result in this area was the Glazman-Krein-Naimark (GKN) theorem. This theory was originally established by Glazman, Krein and Naimark in the 1950s. The GKN theorem is equivalent to the Stone-von Neumann theory which provides the necessary and sufficient conditions for the existence of self-adjoint extensions of closed symmetric operators in Hilbert space. However, it has particular advan-

tage over the Stone-von Neumann theory when it comes to consideration of related boundary problems. The GKN theory was developed from the view of complex symplectic geometry by Everitt and Markus (W. Everitt & Markus, 1997). They proved that there exists one-to-one correspondence between the set of all self-adjoint extensions of the minimal operator generated by quasi-differential expressions and the set of all Lagrangian subspaces of related boundary space.

Zheng (Zheng, n.d.) established the GKN theory for continuous linear Hamiltonian systems while Remling (Remling, 1996) did geometric characterisation of singular self-adjoint boundary conditions for Hamiltonian systems. The geometric aspects of self-adjointness for Sturm-Liouville problems were investigated in terms of complex boundary conditions by Wu and Cao (Cao & Wu, 2004) and Wu, Kong and Zettl (Kong, Wu, & Zettl, 2000).

Sun, Shi and Chen (Sun et al., 2007) developed GKN theory for a class of discrete Hamiltonian systems. They introduced minimal and maximal operators, GKN-sets and boundary space for the system. Algebraic characterisations were made for the domains of self-adjoint extensions of the minimal operator. They established a very close relationship between the domains of self-adjoint extensions and the GKN sets. It was shown that one-to-one correspondence existed among the set of all self-adjoint extensions, the set of all d -dimensional Lagrangian subspaces of the boundary space.

The study of the theory of difference equations has received a lot of attention from researchers. Today it occupies a pivotal position in applicable analysis and will for sure continue to play a crucial role in mathematics as a whole. It is a fact that difference equations emerged much earlier than differential equations and hence played an instrumental role in laying the foundation for the development of differential equations. Studies have been done to investigate the spectra of even higher order symmetric differential and difference operators and elegant results have been obtained. Earlier research show that deficiency indices and discrete spectrum are stable under bounded perturbations whereas absolutely continuous spectrum is stable under compact perturbations. On the other hand, singular continuous spectrum is unstable (highly fluid) even under finite rank perturbations. Similar results are lacking for existence of positive self-adjoint operator extensions, particularly when

an even higher order symmetric difference operator is perturbed by a constant multiple of an even lower order symmetric difference operator.

In this study, we focussed on the stability of the existence of positive self-adjoint extensions of the minimal perturbed difference operator and their spectral stability analysis.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be even symmetric difference maps of order four and two respectively. We have therefore, considered the following difference equations:

$$\mathcal{L}_1 y(t) = (\omega(t))^{-1} \{ \Delta^2 \{ p_2(t) \Delta^2 y(t-2) \} - \Delta \{ p_1(t) \Delta y(t-1) \} + p_0(t) y(t) \} \quad (1.1)$$

and

$$\mathcal{L}_2 y(t) = (\omega(t))^{-1} \{ -\Delta \{ q_1(t) \Delta y(t-1) \} + q_0(t) y(t) \} \quad (1.2)$$

being fourth and second order difference equations respectively defined on the weighted Hilbert space, $\ell_\omega^2(\mathbb{N})$, $\omega = \omega(t)$. However, for simplicity we have assumed that $\omega(t) = 1$ for all $t \in \mathbb{N}$. This implies we have worked on the Hilbert space $\ell^2(\mathbb{N})$ which is isomorphic to $\ell_\omega^2(\mathbb{N})$ for all $\omega(t) = 1$. We assume that the coefficients $p_2(t), p_1(t), p_0(t), q_1(t)$ and $q_0(t)$ are real valued functions and are allowed to be unbounded, continuously differentiable such that $\Delta^2 y(t)$ and $(\Delta^2 y(t))^2$ tend to zero as $t \rightarrow \infty$. In this case, $\Delta y(t)$ is a forward difference operator defined by

$$\Delta y(t) = y(t+1) - y(t).$$

We have considered difference operators generated by the difference maps above,

L_1 and

$$L = L_1 - z L_2, \quad (1.3)$$

where L is obtained by perturbing L_1 by a scalar multiple of L_2 , $z \in \mathbb{R}$. The main interest was in the stability of the existence of positive self-adjoint operator extensions of the perturbed difference operator.

1.1.2 Dirac Operators

The study of Dirac operators is an essential matter in modern mathematics as well as theoretical physics. It has a deep relationship with geometry, topology and quantum mechanics. The background of Dirac operators is embedded in the interaction between differential geometry, functional analysis and the concepts of partial differential equations. In order to understand the starting point of Dirac operators, one needs to explore the concepts and structures like Clifford algebras, spin structures and the connection between Dirac operators and geometry of manifolds.

Clifford algebras and spin structures are algebraic structures that generalise the properties of complex numbers and quaternions. They are usually constructed from vector spaces with quadratic form and therefore they are central in defining Dirac operators. For a real vector space V with the quadratic form Q , the Clifford algebra, $Cl(V, Q)$, is an associative algebra defined by V based on the relation $v^2 = Q(v)$ for every $v \in V$. This relationship encodes the geometric structure for the quadratic form and enables construction of algebraic structures which reflect the underlying geometry of the space.

Clifford algebras are very useful in spin structures; geometric structures which allow definition of spinors on a given manifold. Spinors are mathematical objects which transform in a particular way subject to rotations. They are very important for the study of Dirac operators. Spinors are natural objects on which Dirac operators act. As such, the ability to define them on a given manifold is key in construction of Dirac operators.

The Dirac operator is defined as a first-order differential operator acting on sections of spinor bundle over Riemannian or Lorentzian manifold. It is best defined in relation to Clifford algebras and spin structures that give the algebraic and geometric foundations enabling its construction. Suppose M is a Riemannian manifold having a spin structure, the Dirac operator D acting on a spinor field ϕ is expressed as

$$D = \sum_{i=1}^n e_i \cdot \nabla_{e_i},$$

for $\{e_i\}$ is an orthonormal frame of the tangent bundle denoting Clifford multiplication.

tion and ∇ is a covariant derivative on the spinor bundle (Lawson Jr & Michelsohn, 1989).

Spectral Theory and Stability Analysis of Dirac Operators

Spectral theory of Dirac operators is an important tool in the analysis of stability. The spectrum of Dirac operator provides an insight about the geometry and topology of the fundamental manifold and therefore provides a systematic way of analysing stability of solutions to differential equations. Friedrich (Friedrich, 2000) showed that the Dirac operator has discrete spectrum consisting of real eigenvalues which accumulate at infinity on a compact Riemannian manifold. Friedrich proved that the eigenvalues of the Dirac operator has close relationship to the geometry of the manifold and provide a powerful tool for analysing the global properties of the manifold.

The Weyl law is one of the key results in the spectral theory of Dirac operators. It describes the asymptotic behaviour of the eigenvalues of the Dirac operator. It says that the number of eigenvalues of the Dirac operator less than a given value λ does grow asymptotically as

$$N(\lambda) \sim \frac{vol(M)}{(4\pi)^{\frac{1}{2}} \Gamma(\frac{n}{2}+1)} \lambda^n,$$

where $vol(M)$ refers to the volume of the manifold M , n is the dimension of M and Γ is gamma function. The Weyl law usually links the local geometry of the manifold encoded by the volume and the global properties for the Dirac operator (Gilkey, 2018).

The spectrum of the Dirac operator plays a key role in understanding the stability of solutions of differential equations. In a case where the Dirac operator has a gap in its spectrum; there are no eigenvalues tending to zero, then a small perturbation to the system may not result to a large change in the behaviour of the solutions. Booss and Wojciechowski (Booss & Wojciechowski, 1993) established that this was true given that the absence of small eigenvalues permits that the system is robust to any perturbations and hence the solutions remain stable over time. However, if the Dirac operator had small or zero eigenvalues, then the system would be unstable such that

a small perturbation would lead to significantly large changes in the behaviour of the solutions.

Dirac Operators in Quantum Field Theory and Stability

In this case, Dirac operators are crucial in formulating fermionic fields and analysis of their stability. The Dirac equation, a relativistic wave equation of fermions is usually derived from the Dirac operator. It describes how particles like electrons and quarks behave.

The Dirac equation gives a basis for understanding the stability of fermionic fields in quantum field theory. Witten (Witten, 1981) presented a new proof of the positive energy theorem in classical general relativity, a crucial result that says that the total energy of a gravitating system is always non-negative and zero only for flat Minkowski space. This theorem is fundamental for the stability of Minkowski space as the ground state of general relativity. This ensures that no state of a lower energy exists to which it could decay. It was noted that the proof by Edward Witten was simple and elegant compared to the one provided earlier by Schoen and Yau which relied mostly on complex geometric methods.

Witten's approach strengthens spinors and the Dirac equation, which is kind of unexpected in the context of classical general relativity since spinors are typically associated with quantum field theory. The proof involved construction of solutions to the Dirac equation on an asymptotically Euclidean initial value hypersurface and demonstrated that the existence of such solutions implied the positivity of the total energy. The major insight is that when the Dirac operator is squared, it yields terms related to the energy-momentum tensor through Einstein's equations thus enabling Witten to connect the behaviour of spinors to the gravitational field's energy.

The semiclassical stability of Minkowski space was addressed, showing that it could not decay via quantum tunneling into a negative energy state. This result was significant as it rules out a potential instability mechanism that could arise if negative energy states were allowed in general relativity. The proof relied on the absence of asymptotically Euclidean gravitational instantons, the solutions to the Euclidean Einstein equations that could mediate such decays in principle.

The study has broader influence on the relationship between classical general relativity and quantum field theory, especially in the context of supergravity. Witten suggested that his proof could be understood as the classical limit of a supergravity argument, in which the total energy would be expressed as a sum of squares of supersymmetry charges. This connection hints at a deeper link between the stability of classical spacetime and the structure of supersymmetric theories.

The work provided a new and elegant proof of the positive energy theorem besides opening up new avenues for exploring the connection between classical and quantum aspects of gravity, more so in the context of supergravity and higher-dimensional theories. The results reinforce the stability of Minkowski space and underscore the role of spinors in understanding the fundamental properties of spacetime.

Dirac Operators on Curved Spacetimes and Stability in General Relativity

Dirac operators are usually used in describing how fermions behave in the presence of gravitational fields. Normally, the curved spacetime Dirac operator is defined on a spinor bundle over Lorentzian manifold and comprises of the effects of the curvature of spacetime on the fermionic fields. Parker and Taubes (Parker & Taubes, 1982) showed that Dirac operators on a curved spacetime are crucial in understanding the connection between quantum mechanics and general relativity. They noted that the Dirac operators were essential in the study of black holes, cosmology and the early universe.

In stability analysis, Dirac operators on curved spacetimes provide a basis for analysing stability of fermionic fields in the presence of gravitational fields. For instance, stability of solutions of the Dirac equation on curved spacetime enables one to fully understand the behaviour of particles near black holes or in the early universe. Friedrich (Friedrich, 2000) provided a detailed introduction of the theory of Dirac operators and spinors in Riemannian geometry which focussed on their algebraic, topological and analytical properties. The research was deeply rooted in the connection between differential geometry, topology and mathematical physics in the context of spin structures and Dirac operators on Riemannian manifolds.

A major theme of the study was the analysis of Clifford algebras and their representations, which are key to understanding spinors. Clifford algebras provide usually provide a framework for defining spinors, that are essential objects in the study of Dirac operators. The construction of Clifford algebras associated with quadratic forms and their role in defining spin groups and spin representations were outlined. These algebraic structures are crucial when it comes to generalisation of the concept of spinors from flat Euclidean space to curved Riemannian manifolds.

Besides, the analytical properties of Dirac operators, including their essential self-adjointness, Fredholm properties and spectral theory were discussed. The study discussed the eigenvalue estimates for the Dirac operator, especially in the context of compact Riemannian manifolds. The Schrödinger-Lichnerowicz formula, that relates the square of the Dirac operator to the Laplacian and the scalar curvature, was a crucial tool in deriving the estimates. The research also explored the concept of Killing spinors, which are spinor fields satisfying a specific differential equation related to the geometry of the manifold. The existence of Killing spinors imposes strong geometric constraints on the manifold, such as being an Einstein space with constant scalar curvature.

In general, the study bridged between algebraic, topological and analytical methods to explore the rich connection between spinors, Dirac operators and the geometry of Riemannian manifolds. It emphasizes the importance of spin structures in modern differential geometry and their applications in mathematical physics.

The Atiyah-Singer Index Theorem

This theorem is one of the most powerful results in modern mathematics. It usually relates the index of the Dirac operator to the topological invariants of the manifold like the Euler characteristic and signature. In (M. F. Atiyah & Singer, 1963), the index of the Dirac operator is defined as the difference between the dimensions of the kernel and of the cokernel of the operator. They showed that the index of the Dirac operator provides a bridge between the analytical properties of the operator and the topological properties of the manifold.

In stability analysis, the index theorem provides a profound tool that enables the

understanding of the stability of solutions of differential equations. Booss and Wojciechowski (Booss & Wojciechowski, 1993) showed that if the index of the Dirac operator was zero, then the system would be stable and a small perturbation to the system would not lead to large changes in the behaviour of the solutions. On the contrary, if the index of the Dirac operator was not zero, then the system would be unstable and as a result, any small perturbations would lead to significant changes in the behaviour of the solutions.

Consider the continuous matrix Cauchy problem for the Dirac system:

$$D'(x) + J(x)D(x) = A_\mu D(x), \quad D(0) = I \quad (1.4)$$

where $x \in [0, 1]$, $A_\mu = i\mu J_0$ and

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J(x) = \begin{bmatrix} 0 & \delta_1(x) \\ \delta_2(x) & 0 \end{bmatrix}, \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\mu \in \mathbb{C}$ is a spectral parameter, $\delta_j(x) \in L_1[0, 1]$, $j = 1, 2$ are complex valued functions known as potentials. In this case, the potentials are integrable. Rzepnicki (Rzepnicki, 2021) focussed on the asymptotic behaviour of solutions to the Dirac system (1.4). The main goal was to derive sharp asymptotic formulas for the eigenvalues and eigenfunctions of Sturm-Liouville operators associated with the Dirac system. The study was mainly concerned with the behaviour of solutions in a strip $|Im\mu| \leq d$ as μ tends to infinity. The paper presented technical lemmas and estimates that were needed to derive the main results. These included bounds on integral operators and behaviour of solutions in the strip P_d . Rzepnicki used these estimates to control the remainder terms in the asymptotic expansions. The solutions of (1.4) is a matrix D with entries from the space of absolutely continuous functions on $[0, 1]$.

The findings significantly advanced the understanding of the asymptotic behaviour of solutions to the Dirac system with integrable potentials. By providing sharp asymptotic formulas for eigenvalues and eigenfunctions, the work contributed to the spectral theory of differential operators. These results have implications for the study of Sturm-Liouville operators and their associated spectral problems, which are very key in mathematical physics and applied mathematics. The results are not

are applicable to the specific spectral problem considered and to a broader class of boundary conditions including strongly regular and regular conditions in the sense of Birkhoff. It was noted that the method developed in the study could be used to study inverse spectral problems and other related issues.

Rzepnicki extended and refined previous results by handling more general potentials and providing a more comprehensive asymptotic expansions. The methods developed in the research were applicable to other problems involving differential operators with integrable potentials, making it a valuable contribution to the field.

In our study, we have considered the discrete counterpart of these Dirac operators and analysed the form and asymptotic behaviour of their solutions under various growth and decay conditions.

1.2 Basic Concepts

In this section, we have given definitions and basic concepts related to the topic of this study.

Definition 1.2.1. A Hilbert space, $\mathcal{H}$ is a complete inner product vector space. It generalizes the notion of Euclidean space and is key in functional analysis and operator theory. In the context of this research, the Hilbert space $\ell^2(\mathbb{N})$ is used, which consists of square-summable sequences. The inner product for two sequences $y_1(t)$ and $y_2(t)$ in $\ell^2(\mathbb{N})$ is defined as:

$$\langle y_1, y_2 \rangle = \sum_{t=1}^{\infty} y_1(t) \overline{y_2(t)}.$$

The completeness of the space ensures that Cauchy sequences converge within the space, which is important for the analysis of operators and their spectral properties (Kreyszig, 1991).

Definition 1.2.2. Spectral theory studies the spectrum of operators, which includes eigenvalues, continuous spectrum, and residual spectrum. (Simon, 1978).

In the context of this research, the spectral properties of self-adjoint extensions of

perturbed difference operators are analysed, specifically the absolutely continuous spectrum and its multiplicity.

Let L be a linear operator defined on the Hilbert space $\mathcal{H}$ and $D(L) \subseteq \mathcal{H}$ be the domain of L . We associate the operator

$$L_\lambda = L - \lambda I$$

where λ is a complex number and I is the identity operator on $D(L)$. Suppose L_λ has an inverse, we denote it by $R_\lambda(L)$ such that

$$R_\lambda(L) = L_\lambda^{-1} = (L - \lambda I)^{-1}$$

and refer to it as the resolvent of L .

Definition 1.2.3. A regular value λ of L is a complex number such that $(L - \lambda I)^{-1}$ exists, is bounded and is densely defined on $\mathcal{H}$.

Definition 1.2.4. The resolvent set $\rho(L)$ is the set of all regular values λ of L .

Definition 1.2.5. The spectrum of L , $\sigma(L)$, is the complement of its resolvent set, that is, $\sigma(L) = \mathbb{C} - \rho(L)$ in the complex plane $\mathbb{C}$. A $\lambda \in \sigma(L)$ is called the spectral value of L . The spectrum $\sigma(L)$ is partitioned into the following three disjoint sets (Kreyszig, 1991):

- i. The point spectrum or discrete spectrum $\sigma_p(L)$ is the set such that $(L - \lambda I)^{-1}$ does not exist. A $\lambda \in \sigma_p(L)$ is referred to as an eigenvalue of L .
- ii. The continuous spectrum $\sigma_c(L)$ is the set such that $(L - \lambda I)^{-1}$ exists, is densely defined but unbounded.
- iii. The residual spectrum $\sigma_r(L)$ is the set such that $(L - \lambda I)^{-1}$ exists, not densely defined but may be bounded or not.

Definition 1.2.6. The operator L is said to be densely defined if $D(L)$ is dense in $\mathcal{H}$, that is, $\overline{D(L)} = \mathcal{H}$, for a Hilbert space $\mathcal{H}$.

Definition 1.2.7. The operator L is said to be symmetric if L is densely defined and $L \subset L^*$, $D(L) \subset D(L^*)$, implying that $\langle Lx, y \rangle = \langle x, L^*y \rangle = \langle x, Ly \rangle$ for all $x \in D(L)$ and $y \in D(L^*)$ (Weidmann, 2012).

Symmetric operators are essential studying self-adjoint extensions, which are crucial for spectral theory. In the context of difference operators, the symmetry condition ensures that the operator has well-defined spectral properties.

Definition 1.2.8. The operator L is said to be positive if $\langle Ly(t), y(t) \rangle = \sum_{t=1}^n Ly(t) \cdot y(t) > 0$ for $y(t) \in \ell^2(\mathbb{N})$ (Kato, 1966).

Positive operators are crucial in the study of stability and spectral theory as they ensure that the spectrum lies in the positive real line. In this research, the positivity of the perturbed difference operators has been analysed under unbounded perturbations.

Definition 1.2.9. Let

$$Ly(t) = \sum_{k=0}^n (-1)^k \Delta^k [p_k(t) \Delta^k y(t-k)] \quad (1.5)$$

be a $2n^{th}$ – order difference equation which can be converted into its first order form using the quasi-differences given by Shi (Y. Shi, 2006)

$$\begin{aligned} x_i(t) &= \Delta^{i-1} y(t-i), & 1 \leq i \leq n \\ u_n(t) &= p_n(t) \Delta^n y(t-n) \\ u_k(t) &= \sum_{l=k}^n (-1)^{l-k} \Delta^{l-k} [p(t) \Delta^l y(t-l)], & 1 \leq k \leq n-1. \end{aligned}$$

Quasi-differences are essential in the formulation process of the system for higher-order difference operators.

The maximal operator L^* generated by the map L in (1.5) is defined on the largest possible domain in $\ell_\omega^2(\mathbb{N})$ that is mapped onto $\ell_\omega^2(\mathbb{N})$ where $\omega = \omega(t)$ and $\omega(t) = 1$ for every $t \in \mathbb{N}$. Define the Hamiltonian form of (1.5) by

$$\mathcal{J} \Delta Y(t) - P(t) R(Y(t)) = W(t) F(t) \quad (1.6)$$

Therefore

$$D(L^*) = \{y \in \ell^2(\mathbb{N}) : \exists f \in \ell_\omega^2(\mathbb{N}), \mathcal{J} \Delta Y(t) - P(t) R(Y(t)) = W(t) F(t), t \in \mathbb{N}\}.$$

Here, $L^* y(t) = f(t)$. $R(Y(t))$ is the partial shift operator defined by $R(Y)(t) = (x(t+1), u(t))^{tr}$ while $W(t) = \text{diag}(w(t), 0, \dots, 0)$, $w(t) > 0$, is the weighted function. $\mathcal{J}$ is a canonical symplectic matrix, that is,

$\mathcal{J} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$ while $P(t) = \begin{bmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{bmatrix}$ with the non-zero elements of $n \times n$ matrices A, B and C given by $A_{j,j+1} = 1, B_{n,n} = p_n^{-1}$ and $C_{j,j} = p_{j-1} - zq_{j-1}$ for $j = 1, 2, \dots, n$. The pre-minimal operator L' is defined by

$$D(L') = \{y \in D(L^*) : \exists n \in \mathbb{N}, y(a) = y(t) = 0\}$$

and $L'y(t) = L^*y$. The closure of the pre-minimal operator $L', \bar{L}' = L$, is the minimal operator.

Definition 1.2.10. (Weidmann, 2012) The deficiency index, $def L$, is the pair

$$def L = (dim N_{L^*-i}, dim N_{L^*+i}).$$

N_{L^*-i} is the null space of $L^* - iI$ and N_{L^*+i} is the null space of $L^* + iI$. Thus $L^* - iI$ is the set of all elements $y(t)$ such that $Ly(t) = iy(t)$. If we use a nonreal complex spectral parameter w , then for $Imw > 0$, we have $dim N_{L^*-i} = dim N_{L^*-w}$ and $dim N_{L^*+i} = dim N_{L^*-\bar{w}}$ with $N_+ = dim N_{L^*-w}$ and $N_- = dim N_{L^*-\bar{w}}$. Even though the definition of the deficiency indices depend on w , the dimension of the null spaces are independent of w provided that w remains in either of the half-planes. For $Imw > 0$, N_+ and N_- may be finite or infinite. Hence, $def L = (N_+, N_-)$.

Definition 1.2.11. The operator L has self-adjoint extensions if its deficiency indices are equal, that is, $def L = (n, n)$ and n is not equal to zero. Otherwise for $n = 0$, the operator L is self-adjoint and $L = L^*$ [(Naimark, 1968), (Schmüdgen, 2012)].

Self-adjoint operators have real spectra and are instrumental in quantum mechanics and other areas of mathematical physics. The existence of self-adjoint extensions of symmetric operators is a central theme in this research, particularly when dealing with unbounded perturbations.

Definition 1.2.12. Fix two $n \times n$ matrices α_1 and α_2 , that is, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{n \times n}$, with

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I_2, \alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0_2$$

and set the boundary condition for the left regular end point a as

$$(\alpha_1, \alpha_2)Y(a) = 0.$$

Given $a = 0$, the self-adjoint extension, H of L is defined by

$$D(H) = \{y \in D(L^*) : (\alpha_1, \alpha_2)Y(0) = 0\}, L^*y = Hy, \text{ for all } y \in D(H)$$

and $L \subset H = H^* \subset L^*$ in the operator sense.

Definition 1.2.13. Stability of a property of a system exists if the property remains invariant even after a reasonable perturbation of the system.

Consider the Sturm-Liouville operator on the interval $[a, b)$,

$$L = \frac{d^2}{dx^2} + q(x),$$

where $q(x)$ is a real-valued potential. The operator L is defined with boundary conditions at $x = a$ and $x = b$. The Weyl-Titchmarsh m -function is therefore defined with regards to the solutions to the differential equation

$$-\phi''(x) + q(x)\phi(x) = w\phi(x),$$

where $w \in \mathbb{C}$ is a spectral parameter. Let $\phi(x, w)$ and $\psi(x, w)$ be linearly independent solutions satisfying some boundary conditions at $x = a$.

Definition 1.2.14. (Hislop & Sigal, 2012) The Weyl-Titchmarsh m -function, $m(w)$ is the ratio

$$m(w) = \frac{\psi(b, w)}{\phi(b, w)}.$$

The m -function is crucial in the spectral theory of Sturm-Liouville operators and generally self-adjoint operators. It characterises the spectral properties of the operators in the context of boundary value problems on intervals. It is usually analytic both in the upper and lower half-planes of the w -plane. It has a close relationship with the spectral measure of the operator L . In particular, the boundary values of

$m(w)$ on the real axis determine the spectral measure. As such, is dependent on the boundary conditions imposed at $x = b$ hence different boundary conditions result to different m -functions.

Definition 1.2.15. The M-Matrix is a generalisation of the Weyl-Titchmarsh m -function and so it relates the asymptotics of the eigenfunction of higher order difference operators to the spectrum of their self-adjoint extensions. The M-matrix is defined from those solutions that stay absolutely square summable as the spectral parameter approaches the real line (Remling, 1998).

Definition 1.2.16. The "Big O " notation focuses on the asymptotic behaviour of a function particularly in the context of norms and operators. Let X and Y be normed vector spaces and let f and g be functions from X to Y . We say that $f(x) = O(g(x))$ as $x \rightarrow x_0$ if there exist constants $C > 0$ and $\delta > 0$ such that for all $x \in X$ with $0 < \|x - x_0\| < \delta$, the following inequality holds:

$$\|f(x)\| \leq C\|g(x)\|.$$

Therefore $f(x) = O(g(x))$ means that $f(x)$ grows at most as fast as $g(x)$, up to a constant factor as x approaches x_0 . In functional analysis, "Big O " notation is used to describe the growth or decay of functions, operators or norms in a precise, asymptotic manner. Hence it is a crucial tool for understanding the behaviour of functions and operators in varied contexts, including approximation theory, operator theory and the study of differential and difference equations (Kreyszig, 1991).

Definition 1.2.17. Let X and Y be normed vector spaces and let f and g be functions from X to Y . We say that $f(x) = o(g(x))$ as $x \rightarrow x_0$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$ with $0 < \|x - x_0\| < \delta$, the following inequality holds:

$$\|f(x)\| \leq \epsilon\|g(x)\|.$$

The "little o " notation is a precise way of describing asymptotic dominance of one function over another in functional analysis. It is commonly used in the study of limits, approximations and error terms in functional analysis and related fields (Kreyszig, 1991).

Definition 1.2.18. Dirac operators are first-order differential operators that arise in quantum mechanics and are used to describe fermions. They are closely related to the geometry and topology of manifolds (Thaller & Thaller, 1992). In this research, the discretised Dirac operator is studied and its asymptotic behavior is analysed for the case of unbounded potentials.

1.3 Statement of the Problem

Even higher order symmetric differential equations with almost constant coefficients have been constructed, analysed and spectral properties for their self-adjoint operator extensions obtained exhaustively. However, similar results have not been exhausted for the case when an even higher order symmetric difference equation with unbounded coefficients is perturbed by a constant multiple of an even lower order symmetric difference equation having unbounded coefficients. Analysis of the stability of the existence of positive self-adjoint operator extension of minimal difference operator generated by $L_1 - zL_2$ and spectral properties of self-adjoint operator extensions when perturbations are done with operators having unbounded coefficients does not yet exist. On the same note, the asymptotic behaviour of the fundamental solutions of Dirac operators as one example of positive operators with integrable potentials has been investigated and a variety of asymptotic formulae for the solutions to such Dirac system in different settings and under various assumptions on the potentials have been generated. However, similar study for the case of the discretised Dirac system is lacking. Moreover, when the potentials are unbounded, the analysis of how asymptotic behaviour of solutions of the discretised Dirac system is affected by the growth of the potentials is lacking. In this study, we have analysed the asymptotic behaviour of solutions of discretised Dirac system with bounded and unbounded potentials. This we have done using asymptotic summation based on Levinson-Benzaid-Lutz's theorem.

1.4 Main Objective of the Study

In this study we have considered the perturbed difference equation (1.3) where the order of L_1 is greater than that of L_2 and the discretised counterpart of the Dirac system (1.4). The main goal of this research was to analyse the stability of existence of positive self-adjoint operator extension of symmetric operators under unbounded perturbations and asymptotic behaviour of solutions of discrete Dirac operators.

1.5 Objectives of the Study

The objectives of this study were to:

- (i). Construct positive perturbed difference operators with unbounded potentials.
- (ii). Analyse the necessary and sufficient conditions for the stability of the existence of positive self-adjoint operator extensions under unbounded perturbations.
- (iii). Investigate the spectral properties of self-adjoint operator extensions of the perturbed difference operators with unbounded coefficients.
- (iv). Examine the asymptotic behavior of solutions of the discretised Dirac system with unbounded potentials.

1.6 Justification of the Study

This study was rooted in the need to address gaps in the existing literature and to extend the understanding of the stability and asymptotic behaviour of solutions for symmetric operators and Dirac operators especially in the context of unbounded perturbations and discretised systems. While the stability of deficiency indices and discrete spectrum under bounded perturbations has been very established, there is a lack of research on the stability of the existence of positive self-adjoint operator extensions under unbounded perturbations. This was very relevant for higher-order symmetric difference operators with unbounded coefficients. The study aimed to

fill this gap by analysing the necessary and sufficient conditions for the stability of such extensions. Again, even though the asymptotic behaviour of solutions for continuous Dirac operators has been extensively studied, similar analysis for discretised Dirac systems was lacking. This was particularly true when the potentials are unbounded. The study intended to address this by examining the asymptotic behaviour of solutions for discretised Dirac systems with both bounded and unbounded potentials.

The study has contributed to the spectral theory of symmetric and Dirac operators, which has broad applications in mathematical physics, quantum mechanics, and numerical analysis. Understanding the stability of these operators under perturbations was crucial for ensuring the robustness of solutions in various physical and mathematical models. The results of the study could be applied in the stability analysis of national income over time. This is a significant practical implication since the stability of national income is closely tied to economic security, employment opportunities and overall, the quality of life. By providing insights into the stability of solutions for this class of operators, the study could help in modeling and predicting economic stability.

The study employed advanced mathematical tools such as asymptotic summation based on the Levinson-Benzaid-Lutz theorem. This technique enables one to analyse the asymptotic behaviour of solutions for difference operators with unbounded coefficients, which has not been exhaustively explored before. Thus the study advances the methodological structure for analysing such systems. Moreover, the study bridges the gap between continuous and discrete models, which is essential for numerical approximations and simulations in various scientific and engineering applications.

1.7 Significance of the Study

This work is anchored on the stability analysis of perturbation theory of difference operators and the asymptotic behaviour of the solutions of the discretised Dirac System. Therefore, the spectral results may be of help to economists in stability

analysis of national income via the Keynesian economic model. The stability of national income underpins many aspects of quality of life, from economic security and employment opportunities to, access to essential services and psychological well-being. States having stable economies usually exhibit higher standards of living and greater overall satisfaction among their citizens.

1.8 Impact on Regional Development

The study suggests that the results could enable regional governments to gain a competitive economic advantage in the world market. By understanding the stability of national income and other economic indicators, governments are able make informed decisions which lead to reduced unemployment, increased income and higher standards of living for their citizens. Moreover, the study aligns with the broader goals of sustainable development by providing tools for analysing and ensuring economic stability which is a crucial component of sustainable growth.

1.9 Papers from the Thesis

The results from this thesis have been published in the following papers:

- (i) Owino, B., Nyamwala, F., and Ambogo, D. (2024). Stability of Krein-von Neumann self-adjoint operator extension under unbounded perturbations. *Annals of Mathematics and Computer Science*, 23, 29-47.
<https://doi.org/10.56947/amcs.v23.300>
- (ii) Owino, B., Nyamwala, F., and Ambogo, D. (2024). Solution forms of discretised dirac system with summable and unbounded potentials. *Gulf Journal of Mathematics*, 17(1), 44-57.
<https://doi.org/10.56947/gjom.v17i1.1723>

1.10 Structure of Thesis

This thesis has been organised as follows: Chapter ONE where we have given mathematical background of our study as well as the basic concepts which are key to presenting the subsequent chapters. The Statement of the problem, Objectives of the study, Justification of the study, Impact on regional governments, Significance of the study and a list of paper publications from this thesis are equally given. Chapter TWO provides literature review on stability, spectral theory and asymptotic behaviour of solutions of differential, difference and Dirac operators. In Chapter THREE we have explained the methods applied to achieve the objectives of our study. In Chapter FOUR we have presented the results and discussions of the study and finally, the Conclusion and recommendations from this thesis are given in Chapter FIVE.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Stability analysis has been done for solutions to differential and difference equations, deficiency indices and the spectrum of self-adjoint operator extensions as well as subspaces under bounded perturbations. Also, the asymptotic behaviour of solutions of the continuous Dirac system has been investigated. A number of results have been obtained in these studies. In this Chapter, we have done a review of related literature to spectrum of differential and difference operators, stability analysis of deficiency indices and spectrum and asymptotic behaviour of solutions of Dirac operators.

2.2 Differential Operators

Eastham (M. Eastham, 1982) formulated asymptotic theory for differential equations with large coefficients which are like the powers of the independent variable. Some critical case was identified which formed a borderline between scenarios where all the solutions have asymptotically an exponential character in terms of the coefficients and where only two solutions have that character. Asymptotic forms of the fundamental set of solutions were obtained for the critical case.

Behncke and Remling (Behncke & Remling, 1997) studied asymptotic integration of linear differential equations. They did establish some new result on the asymp-

otic integration of linear differential systems of the form $Y' = (\Lambda + R)Y$ where Λ is diagonal and R is very small. They particularly allowed for different scales of asymptotic behaviour of the individual matrix elements of Λ and R . Moreover, they discussed an application to self-adjoint fourth order differential equation.

It is known that the classical Titchmarsh-Weyl m-function for second order differential operators normally admit a generalisation to larger classes of differential equations. Despite that, the use of the general M-matrix in applications usually proves to be difficult. As such, Remling (Remling, 1998) interpreted the m-function in line with Hilbert space notions and proved that the classical m-function may be recovered as part of the more complicated one. This particular trick was applied to obtain results on the spectral multiplicity and stability properties of the spectrum.

In the year 1999, Remling (Remling, 1999) was interested in the location and the nature of the spectrum of self-adjoint differential equation of order four with some asymptotic assumptions made on the coefficients. The methods of asymptotic integration and Titchmarsh-Weyl M-matrix were applied. As a result, asymptotic formulae for the solutions of the differential equation were obtained and used to characterise the spectral properties of the M-matrix. The desired results were realized by characterisation of the spectral properties based on the boundary behaviour of the M-matrix, M .

Behncke, Hinton and Remling (Behncke, Hinton, & Remling, 2001) discussed the spectral properties of higher order ordinary differential operators. They observed that spectral properties were preserved when the coefficients vary from constants by small perturbations. Their approach was based on analysis of the associated differential equations by use of asymptotic techniques.

In the year 2002, Agarwal et al. (Agarwal et al., 2002) studied asymptotic behaviour of solutions of differential equations. They presented some recent contributions on asymptotic behaviour of solutions of second order differential equations besides the behavioural properties of positive solutions of singular Emden-Fowler-type equations. They showed that the existence and explicit asymptotic form of non-oscillatory solutions for a large class of differential equations can be proved and

provided respectively. They imposed further restrictions on the sign of integrable coefficient of the equation and obtained the necessary and sufficient conditions for the solutions to have a specified asymptotic behaviour as the independent parameter tends to infinity. The existence, uniqueness and asymptotic behaviour of positive solutions of singular Emden-Fowler-type equations were also studied. For the cases when the coefficients of the equation in consideration were of constant sign or of alternating sign were systematically discussed. As such, the existence and non-existence of results for the positive solutions of Emden-Fowler-type systems were proven.

Behncke (Behncke, 2002) determined the absolutely continuous spectrum of differential operator and established that the absolutely continuous part of any self-adjoint extension of the differential operator is unitarily equivalent to the operator multiplication if and only if the coefficients satisfy the decay and smoothness conditions.

Killip and Simon (Killip & Simon, 2003) did a survey of spectral theory of one-dimensional differential and finite difference operators (Jacobi matrices, Krein systems and Schrödinger operators). It was illustrated that the link between those results was the use of sum rules which related the coefficients and spectral data. The proof and systematic application of Case's sum-rules for Jacobi matrices were discussed. Special interest was in the linear combination of two of Case's sum-rules having strictly positive terms. Complete classification of the spectral measures of all Jacobi matrices J where $J - J_0$ is Hilbert Schmidt and the proof of Nevai's conjecture that Szego" condition holds if $J - J_0$ is a trace class was done.

The proof and systematic application of Case's sum rules for Jacobi matrices were further discussed by Abdelrahman (Abdelrahman, 2021). The study showed the bound of Jacobi matrix and completely described the point and absolutely continuous spectrum. For the case of singular continuous spectrum, it was concluded that additional assumptions were required. A proof of characterisation of the characteristic function of a row contraction operator was given and a verification of its defect operator shown. They also proved commutability of the operator of such row contraction.

In the year 2006, Behncke (Behncke, 2006a) studied spectral theory of differential operators by the method of asymptotic integration and was able to derive estimates for the eigenfunctions and the M-matrix. It was shown that the spectral properties of the differential operator are directly related to the asymptotics of the eigenfunctions given that the m-function is Stieltjes transform of the spectral measure. It was pointed out that asymptotic integration method excludes coefficients that are very oscillatory or that their derivatives decay rather too slowly. As such, for all the cases studied, there was no singular continuous spectrum. These results were extended by Behncke (Behncke, 2010b) in various directions. First, slower decay but higher order differentiability was allowed. This enabled the derivation of the expansion for the diagonalising matrices or transformations. In the second case, unbounded coefficients were allowed. This demanded further transformations so as to arrive at Levinson's form besides some modification. It was noted that standard results extend to weakly singular coefficients but very singular coefficients generally result to discrete spectra.

Behncke (Behncke, 2006b) studied spectral theory of higher order differential operators and determined the absolutely continuous spectrum of $2n^{th}$ order differential operators whose coefficients satisfied some smoothness and decay properties. This was achieved by use of asymptotic integration and analysis of the associated M-matrix. It was obtained that the form of solutions precludes the absence of singular continuous spectrum.

Dolgikh and Mirzoev (Dolgikh & Mirzoev, 2006) studied deficiency indices and spectrum of self-adjoint extensions of some classes of differential operators. In their work, asymptotic behaviour of a solution of a quasidifferential equation of arbitrary order, even or odd with complex valued coefficients are considered in the neighbourhood of the point $+\infty$ and in the neighbourhood of the origin. Their results enable one to establish defect indices of the corresponding minimal symmetric differential operators and the structure of the spectrum of self-adjoint operator extensions of such operators. Based on asymptotic formulae for solutions to the quasidifferential equation, the deficiency indices for a differential operator generated by the differential expression with the leading coefficient vanishing in the interval $[0, \infty)$ were obtained.

In the year 2007, Bodine and Lutz (Bodine & Lutz, 2007) considered the asymptotic equivalence of perturbed linear systems of both differential and difference equations. Specifically they studied perturbations with triangularly-induced structure and established that growth conditions may be weakened substantially. They also gave results for not necessarily triangular perturbations which interpolate between the theorems of Levinson and Hartman-Wintner. An analogous result for systems of linear difference equations were also given.

Ezzinbi and Ouhinou (Ezzinbi & Ouhinou, 2009) were concerned with stability of the solution semigroup for some linear partial functional differential equations having infinite delay in a Banach space. This was done for the particular case when exponential stability fails. They used characteristic equation to compute the order of every pole of the resolvent operator that is associated with the infinitesimal generator of the solution semigroup. The results enabled them to give sufficient conditions for stability of the solution semigroup.

Nyamwala (F. Nyamwala, 2010a) analysed the spectrum of a four term $2n^{th}$ order differential operator with unbounded coefficients and proved that it is non-limit-point. This was possible by strengthening of the dichotomy condition and weakening of the decay conditions. Using stringent conditions, it was further shown that deficiency index for this particular operator is determined by the behaviour of the coefficients. Besides, they proved the absence of singular continuous spectrum and that the absolutely continuous spectrum has multiplicity two.

Lakshmikantham (Lakshmikantham, 2011) considered stability and asymptotic behaviour of solutions of differential equations in a Banach space. Nonlinear variation of parameters formula was developed and used to obtain certain results on asymptotic behaviour of solutions of perturbed equations. The sufficient conditions for the differential equation in a Banach space to possess asymptotic equilibrium were given. They discussed stability and asymptotic behaviour of solutions using the norm and a Lyapunov-like function as candidates. The method of vector Lyapunov function was developed and used in studying the properties of solutions of differential equations. It was shown that this particular method was effective for qualitative study of interconnected systems whose multivariability and other properties make it difficult

for construction of a single Lyapunov function for these systems. Stability theory concerning delay differential equations in a Banach space were equally presented.

Weidmann (Weidmann, 2012) studied linear operators in Hilbert spaces and analysed the spectral theory of self-adjoint operators (for compact operators and for a general case) and characterisation of spectral properties. The theory of von Neumann's extension for symmetric operators was established and the results on perturbation theory for self-adjoint operators were obtained. Besides, applications of partial differential to Schrodinger and Dirac operators were illustrated.

Asymptotic integration has been a powerful tool to determine the deficiency indices and the spectra of higher order differential operators. Behncke and Hinton (Behncke & Hinton, 2013) outlined asymptotic integration and illustrated typical results and properties through examples. They calculated deficiency indices and found the location and multiplicity of absolutely continuous spectrum. Besides, they showed the absence of singular continuous spectrum. Finite singular points were considered and it was established that new results arose from competing terms of operators with unbounded coefficients.

Asymptotic behaviour and stability of second order neutral delay differential equations was considered by (G. Chen, van Gaans, & Lunel, 2014). This was done by the methods of spectral projection and ordinary differential equation. They discussed the relationship between these two methods and illustrated certain features by use of examples. Moreover, they introduced a third method, the fixed point method, to study asymptotic behaviour. They obtained that under similar assumptions, the ordinary differential equation approach was equivalent to the spectral approach. Despite that, the spectral approach had certain advantages following the fact that the conditions for the spectral method were weaker than those required for the ordinary differential equation. Besides, they noted that the spectral approach could also be used to study asymptotic behaviour of the neutral delay differential equations having matrix coefficients.

2.3 Difference Operators

In (Dijksma & de Snoo, 1974), the authors presented an extensive theory of self-adjoint extensions of symmetric linear manifolds (subspaces) in Hilbert spaces. The research built on previous work by Coddington, Phillips, McKelvey and others, and extended the understanding of symmetric and self-adjoint operators to the context of non-densely defined operators and their extensions. The study began with the introduction of the concept of symmetric and self-adjoint linear manifolds in a Hilbert space. The focus was on self-adjoint extensions of symmetric subspaces which play a crucial role in the study of differential operators and quantum mechanics, where self-adjointness ensures the existence of a well-defined spectral decomposition.

The main tool applied in the study was the Cayley transform which usually maps symmetric operators to isometric operators and self-adjoint operators to unitary operators. The authors did generalise the Cayley transform to linear manifolds hence enabling them to analyse both symmetric and self-adjoint subspaces in a unified framework. They showed that the resolvent of a self-adjoint subspace satisfies certain functional equations, which are used in characterising the spectral properties of the self-adjoint extension. The study also introduced the concept of generalised resolvents and spectral families, which are key for the understanding of the extensions of symmetric subspaces.

One of the major results was the characterisation of all self-adjoint extensions of a given symmetric subspace in terms of analytic contractions between the deficiency spaces of the symmetric subspaces. The deficiency spaces were defined as the orthogonal complements of the range of the symmetric space and they play a vital role in the determination of the possible extensions of the symmetric space. It was shown that each self-adjoint extension corresponds to a family of subspaces which are determined by analytic contractions between the deficiency spaces.

The study also considered the minimal self-adjoint extensions of symmetric subspaces and proved that all such extensions with the same spectral function were unitarily equivalent. The result was significant as it provided a way to classify self-adjoint extensions based on their spectral properties.

In general, the study provided a significant contribution to the theory of self-adjoint extensions especially in the context of non-densely defined operators. The use of the Cayley transform and the exhaustive analysis of deficiency spaces provided a fundamental framework for understanding the extensions of symmetric subspaces with potential applications in differential equations, quantum mechanics and operator theory.

Nyamwala (F. Nyamwala, 2010b) studied spectral theory for difference operators on Hilbert spaces and showed that the minimal difference operator whose coefficients are almost constant is limit point. This was done under given appropriate smoothness and decay conditions and by asymptotic summation. It was assumed that the self-adjoint operator extension of the minimal difference operator has no singular continuous spectrum. However, its absolutely continuous spectrum agrees with that of the limiting operator with constant coefficient. Precisely, if the characteristic limiting polynomial has $2k$ roots of absolute value one, then the absolutely continuous spectrum has spectral multiplicity k .

Shi and Chen (Y. Shi & Chen, 1999) studied spectral problems of second-order vector difference equations with two-point boundary value conditions where the matrix valued coefficient of the leading term may be singular. The study was remarkable in the context of discrete systems especially in the fields of numerical analysis, quantum mechanics, and control theory where difference equations always model discrete-time systems. The concept of self-adjointness of the boundary value conditions was introduced. This ensured that the corresponding difference operator was self-adjoint. This self-adjointness is always essential for the spectral theory since it usually guarantees that the eigenvalues are real and the eigenfunctions are orthogonal. The study was structured in a way that it first began by definition of the self-adjointness of the boundary conditions and then construction of an admissible function space in whereby the difference operator was self-adjoint. They showed that the eigenvalue problem had a finite number of real eigenvalues and corresponding eigenfunctions which formed a complete orthogonal set in the admissible space. This result is analogous to the spectral theory of self-adjoint operators in continuous systems but adapted to the discrete setting.

One of the major contributions of the study was the introduction of Rayleigh's principle and minimax theorems for the eigenvalues of the difference equation. These variational principles provided a way of characterising the eigenvalues by way of optimization problems which involved the Rayleigh quotient. Shi and Chen also proved a comparison theorem for the eigenvalues of two different Sturm-Liouville problems and showed that if the coefficients of one problem dominated those of the other, the corresponding eigenvalues were ordered appropriately.

The study extended the earlier work on scalar difference equations to the case of vectors thus it addressed the additional complexity introduced by matrix coefficients and boundary conditions. The results have potential applications in numerical analysis, where discrete systems are used to approximate continuous problems and in quantum mechanics where difference equations always model discrete energy levels. Generally, the study provided a robust framework for the spectral analysis of vector difference equations hence bridging the gap between discrete and continuous systems.

Shi (Y. Shi, 2006) established the Wey-Titchmarsh theory for some class of discrete linear Hamiltonian systems over a half-line. In this particular work, the properties of solutions, regular spectral problems and the corresponding maximal and minimal operators were studied. Matrix disks were constructed and consequently shown to be nested and converging to a limit set. The relationships among the rank of the matrix radius of the limiting set, the number of linearly independent square summable solutions and deficiency indices of the minimal operator were established. Following the above results, singular discrete linear Hamiltonian systems was classified based on the deficiency indices of the minimal operator. A number of equivalent conditions on cases of limit point and limit circle were obtained. Various problems in the limit point cases were considered. Amongst them were, the fundamental properties of square solutions, properties of Weyl function (which is the unique element in the set for this case) and the inhomogeneous boundary problems, self-adjointness of the corresponding Hamiltonian operator, the relation between the spectrum of the Hamiltonian operator, analyticity of the Weyl function and dependence on spectrum on the boundary data, in which some separable results for the spectrum were

obtained.

Bohner et al. (Bohner, Došlý, & Kratz, 2009) comprehensively studied the oscillation properties in symplectic difference systems, which are discrete counterparts of linear Hamiltonian differential systems. Their interest was on the oscillation theory of these systems, which was very crucial for understanding the behaviour of solutions in their discrete settings especially in numerical methods for solving Hamiltonian systems and in discrete calculus of variations and optimal control theory.

It was noted that the symplectic systems were closely related to discrete quadratic functionals, which arise in second-order nonlinear problems in discrete calculus of variations and optimal control. The study extended the already well developed oscillation theory of linear Hamiltonian differential systems to the discrete case by providing a discrete version of the Sturmian separation theorem. This theorem states that the difference in the number of focal points of any two conjoined bases of the system is at most equal to the dimension of the system. The study established a Sturmian comparison theorem for symplectic difference systems which was used to compare the number of focal points of solutions of two different systems. They derived the conditions under which the number of focal points of one system could be bounded by those of another and therefore it gave a discrete analogue of the continuous Sturmian comparison theorem.

The research concentrated on the spectral theory associated with these systems especially focusing on eigenvalue problems and the Rayleigh principle for the discrete quadratic functionals. The researchers proved the completeness of finite eigenvectors in the space of admissible sequences and gave technical statements used in the proofs of the main results. The results had significant contributions on the numerical methods in Hamiltonian dynamics. That was because symplectic systems preserved the Hamiltonian structure and hence ensuring accurate long-term numerical simulations. The discrete oscillation theory developed in the study provided a theoretical foundation for analysing the stability and convergence of numerical schemes for Hamiltonian systems.

The research was significant since it made great contributions to the understand-

ing of discrete symplectic systems by extending classical continuous results to the discrete setting and offering new insights and tools for analyse these systems in different applications, including numerical analysis, control theory and variational calculus. The robust mathematical approach and the derived theorems enhanced the theoretical foundation of discrete Hamiltonian systems thereby giving way for further research and applications in related fields.

Behncke and Hinton (Behncke & Hinton, 2011) did spectral analysis for general linear Hamiltonian systems. They assumed that the coefficients were asymptotically constants and satisfied certain smoothness and decay conditions. Their results apply to the case of Dirichelet index and spectrum. Behncke and Nyamwala [(Behncke & Nyamwala, 2011a), (Behncke & Nyamwala, 2011b)] studied spectral theory of higher even order difference operators with almost constant coefficients and established their spectrum. They showed that given the appropriate smoothness and decay conditions on the coefficients, the singular continuous spectrum is empty while the absolutely continuous spectrum agrees with that of the limiting operator with constant coefficients. Further, it is shown that the operators with the coefficients that are approximately constant in general exhibit an absolutely continuous spectrum that is equal to the one of the corresponding constant coefficient operator.

Behncke and Nyamwala (Behncke & Nyamwala, 2012) analysed the spectrum of higher order differential operators with unbounded coefficients and established that the eigenvalues of the characteristic polynomials for these operators fall into distinct classes or clusters. Moreover, the spectral properties, deficiency indices and spectra of the underlying operators are superpositions of the contributions from the individual clusters. These results were realised after quantitative improvement of Levinson's theorem. In the year 2013, they (Behncke & Nyamwala, 2013) applied asymptotic summation to extend a number of results of differential operators to the discrete setting. They investigated the spectra of fourth order degenerate operators and compared the results with those of corresponding differential operators. They established that even though there are many similarities between the two classes of operators, their spectral results may be quite distinct.

Shi and Sun (Y. Shi & Sun, 2011) did an investigation on the self-adjoint extensions

of second-order symmetric linear difference equations which are the discrete analogs of differential equations usually encountered in quantum mechanics and other areas of mathematical physics. The study was informed by the need to understand the spectral properties of such equations in the context of singular boundary conditions where traditional methods would not apply. Yuming Shi and Huaging Sun employed the Glazman-Krein-Naimark (GKN) theory for Hermitian subspaces to characterise these self-adjoint extensions both as subspace extensions and operator extensions, with regards to boundary conditions.

They began by introducing the second-order symmetric linear difference equation and the associated Hilbert space, ℓ_ω^2 , where I is set of integers. The maximal, pre-minimal and minimal subspaces corresponding to the difference equation were defined and their properties explored. It was shown that the minimal subspaces are non-densely defined Hermitian operators and as such, it complicated the application of classical self-adjoint extension theories. To address that difficulty, the authors extended the von Neumann theory to Hermitian subspaces. This allowed them to characterise self-adjoint extensions in a more general way.

The detailed analysis of the defect indices of the minimal subspaces which are very crucial in determining the existence and form of self-adjoint extensions was a significant contribution of the study. Shi and Sun established relationships between the defect indices of the minimal subspaces in the whole interval and those in the left-hand and right-hand half intervals. The analysis led to an exhaustive characterisation of self-adjoint subspace extensions with regards to the boundary conditions, which is a key result of the study.

The study also discussed self-adjoint operator extensions which are a subset of the subspace extensions. The necessary and sufficient conditions for a subspace extension to be an operator extension particularly in the cases where one or both endpoints of the interval are finite were provided. They also considered special cases like periodic and anti-periodic boundary conditions and showed how they fit into the overall framework. In conclusion, the research promoted the understanding of self-adjoint extensions for symmetric linear difference equations by providing a complete framework which accommodated both regular and singular endpoints.

Their results have implications for the spectral theory of discrete systems and may be applied to problems in quantum mechanics where discrete models are often used to approximate continuous systems.

Agure et al (Agure, Ambogo, & Nyamwala, 2013) studied the spectral theory of fourth-order difference equations with unbounded coefficients and particularly investigated the deficiency indices and the absolutely continuous spectrum of these equations. The study was done within the structure of subspace theory, which is usually key in the analysis of difference operators which may not be densely defined or single-valued as opposed to their continuous counterparts. They employed techniques from Hamiltonian systems and asymptotic analysis so as to derive the results which were then applied to bounded and unbounded coefficients. It was shown that for bounded coefficients, the spectrum could be purely discrete under certain conditions. For instance, when the weight function $w(t)$ tends to infinity as t tends to infinity. In the case of unbounded coefficients, they applied asymptotic summation and Levinson-Benzaid-Lutz theorem to approximate the roots of the characteristic polynomial and derived the asymptotic behaviour of the solutions. They demonstrated that the absolutely continuous spectrum could cover the whole real line depending on the growth rates of the coefficients.

The major contribution of the study was the determination of the deficiency indices and the absolutely continuous spectrum for fourth-order difference equations. They showed that under appropriate smoothness and decay conditions on the coefficients, the deficiency indices of the minimal subspace were (n, n) , where $2 \leq n \leq 4$ and the absolutely continuous spectrum of the self-adjoint extension subspace was the whole real line with a spectral multiplicity of one. It was established that the results were invariant under higher-order smoothness and decay conditions. The results extended earlier work on second-order difference equations to a more complex fourth-order case.

The authors addressed the case where one of the coefficients, q_1 to infinity as t tend to infinity. In this case, they showed that the deficiency indices were either $(3, 3)$ or $(2, 2)$. This result was dependent on whether q_1^{-1} was summable or not. The absolutely continuous spectrum was either empty or covered the whole real line

respectively.

In overall, the study provided a comprehensive investigation of the spectral properties of fourth-order difference equations and hence extending earlier results from second-order equations and continuous systems. The use of subspace theory and asymptotic techniques allowed them to handle the complexities that arise from unbounded coefficients, which are usually more challenging in the discrete setting compared to the continuous case. Their results have implications for the study of Hamiltonian systems and spectral theory in both discrete and continuous contexts, providing new intuition into the behaviour of higher-order difference operators.

Ren and Shi (Ren & Shi, 2011) studied defect indices of Linear Hermitian systems. They proved that the defect index of the minimal subspace and the number of linearly independent solutions that are square summable are equal if and only if the definiteness condition is satisfied.

Nyamwala (F. O. Nyamwala, 2015) used subspace theory to investigate absolutely continuous spectrum of fourth order difference equations. The results showed that the absolutely continuous spectrum exists outside some bounded interval. The spectral multiplicity and the location of absolutely continuous spectrum of self-adjoint subspace extension generated by three term fourth order difference equations whose coefficients are bounded were compared under some asymptotic conditions. Nyamwala (F. O. Nyamwala, 2017) studied the essential and continuous spectrum of symmetric difference equations. In the study, it is shown that the growth conditions of the coefficients are the determining factors of the deficiency indices and existence of the components of spectrum. Subspace theory and asymptotic summation were used to prove the necessary and sufficient conditions for the existence of essential spectrum of self-adjoint subspace extension.

Nyamwala and the other authors (F. O. Nyamwala, Ambogo, & Ngala, 2020) showed that self-adjoint operator extensions of minimal second order difference operators have only discrete spectrum when the odd order coefficient is unbounded but grows or decays according to specific conditions. However, self-adjoint operator extensions of minimal difference operators have absolutely continuous spectrum of multiplicity

one under similar growth and decay conditions on the coefficients.

De Monvel (de Monvel, Janas, & Zielinski, 2012) investigated the asymptotic behaviour of large eigenvalues for a certain class of finite difference self-adjoint operators having a compact resolvent in the Hilbert space. The study was particular on higher-order symmetric difference operators and hence generalising the earlier results on tridiagonal Jacobi matrices. The main interest is to derive simple remainder estimates in the asymptotics of eigenvalues for this class of operators that are applicable in quantum physics and other areas of mathematics.

A self-adjoint operator was introduced and the conditions under which it had compact resolvent were established. This ensured the existence of an orthonormal basis of eigenvectors with eigenvalues tending to infinity. The major assumptions involved the asymptotic behaviour of the diagonal entries and the off-diagonal entries of the operator. Specifically, the diagonal entries were assumed to grow polynomially while the off-diagonal entries were controlled by some power of the diagonals.

De Monvel presented a discussion of the application of the min-max principle to obtain eigenvalue estimates and gave a general approach for deriving the estimates using commutator techniques. The operators that were unitarily similar to the self-adjoint operator but have smaller off-diagonal entries were constructed. This made it possible to control the error terms in the asymptotics of the eigenvalues. The study extended the understanding of the asymptotics of eigenvalues of higher-order difference operators, giving a strong approach for analysing such operators under general conditions. The results have potential applications in quantum mechanics and other fields where spectral properties of discrete operators are of interest. The techniques developed in the study could be further extended to study non-smooth entries and other types of operators.

Katsunori(Iwasaki, 1997) presented a comprehensive analysis of the asymptotic behaviour of linear difference equations in locally convex spaces, particularly the behaviour of solutions as the index approached infinity. The concept of the profile operator which plays a key role in the understanding of the asymptotic behaviour of solutions to these equations was introduced. The profile operator is a bounded

linear operator that captures the dominant behaviour of the solution sequence as the independent variable grows large. The study provided a clear factorial asymptotic expansions for the solutions and established Gevrey estimates, which were essential for understanding the growth and decay rates of the solutions.

The study was structured about the analysis of asymptotic pairs (V, P) , where V was a subspace of the formal sequence space V_f and P was a sequence of linear operators. The conditions under which these pairs exhibited well-behaved asymptotic properties were defined with emphasis on rapidly decreasing pairs and Gevrey pairs. Rapidly decreasing pairs deal with sequences which decay faster than any polynomial while Gevrey pairs involves sequences with specific exponential decay rates, characterized by Gevrey indices (t, a) .

The results were applicable in confluent hypergeometric systems, the Humbert systems to be specific, that arise in the study of partial differential equations. Katsunori demonstrated how the theoretical approach could be used to analyse the asymptotic behaviour of solutions to these systems hence providing explicit formulae for the profile operator.

In summary, the study made huge contributions to the theory of difference equations by providing a rigorous approach for asymptotic analysis in locally convex spaces. The results have potential applications in the field of mathematics, including spectral theory, differential equations and mathematical physics where the understanding of the long-term behaviour of solutions is crucial. The explicit algorithms and estimates given in the study offer practical techniques for analysing complex systems thus making this research both theoretically and practically significant.

In the year 1987, Benzaid and Lutz (Benzaid & Lutz, 1987) asymptotically represented solutions of linear difference equations of the form

$$Y(t+1) = [A_o + A(t)]Y(t)$$

when $k \rightarrow +\infty$ and $A(k)$ is "small" and also established that the dichotomy condition on the diagonal part and the growth condition on the perturbation term are the properties responsible for the asymptotic behaviour of the solutions. They derived some known asymptotic results from a central point of view as well as a number of

new extensions and generalization of them. They constructed some examples that demonstrate the need for a dichotomy-type condition.

Shi and Wu (G. Shi & Wu, 2009) presented a number of classes of explicit self-adjoint Sturm-Liouville difference operators with either a non-Hamiltonian leading coefficient function, or a non-definite weight function, or a non self-adjoint boundary condition. These examples were obtained by use of a generalised procedure of constructing difference operators realizing discrete Sturm-Liouville problems and minimum conditions for difference operators to be self-adjoint with respect to a natural quadratic form. They showed that the discrete Sturm-Liouville problem do admit a difference operator realisation if and only if it does not have all complex numbers as eigenvalues. Further, spectral properties of self-adjoint Sturm-Liouville difference operators are studied and to be particular, several eigenvalue comparison results are proven.

Gingold and Xue (Gingold & Xue, 2007) proposed and studied a new method for asymptotic summation of linear systems of difference equations. This was based on the introduction of a given summation equation that pinpointed sufficient conditions for asymptotic summation. The conditions served as a framework from which new and old theorems follow. The analogues of the fundamental theorems of Levinson, Hartman and Wintner were shown to follow from one and the same framework.

Garkavenko et al (Garkavenko, Zgolich, & Uskova, 2018) studied spectral properties of second order difference operator with a growing potential, acting on the complex Hilbert space of square summable complex sequences that are indexed by integers. They explored the spectral properties of difference operator and the corresponding Sturm-Liouville operators. The method of similar operators was employed. This method allows one to reduce the study of the operator to one with a block-diagonal matrix hence it forms a basis for estimating the eigenvalues and eigenvectors of difference and differential operators with a growing potential. As such, they obtained asymptotic estimates of eigenvalues, eigenvectors and spectral projections of the difference operator with growing potential.

Spectral theory from the second-order q -difference operator was developed by Dhaouadi

(Dhaouadi, 2007). An integral representation of the inverse of the q -difference operator was given and the resolvent operator obtained. An analogue of the Poincare inequality was given as an application and the Zeta function for the q -difference operator introduced. Properties of such operator were formulated and spectral measure obtained.

Liu and Shi (Liu & Shi, 2016) looked at regular approximations of isolated eigenvalues of singular second-order symmetric linear difference equations. They proved that the k^{th} eigenvalues of a given self-adjoint subspace extension is absolutely the limit of the k^{th} eigenvalue of the induced regular self-adjoint subspace extension for the case that every end point is regular or in the limit circle case. The error estimates for the approximated eigenvalues were given. Furthermore, they showed that isolated eigenvalues in each gap of the essential spectrum of any self-adjoint subspace extension are precisely the limits of eigenvalues suitably chosen induced regular self-adjoint subspace extensions. This was done for the case that at least one end point is in the limit point case.

Reháková (Reháková, 2016) studied asymptotic formula for solutions of linear second order difference equations and established sufficient conditions in terms of regular variation for all eventually positive solutions to satisfy some asymptotic formulae. They obtained regular variations of all these solutions. Varied related problems were discussed and a number of examples given.

In the year 2017, Liu and Shi (Liu & Shi, 2017) approximated eigenvalues below the essential spectra of singular second-order symmetric linear difference equations with at least one end point in the limit point case. They gave sufficient condition so that the k^{th} eigenvalue of a self-adjoint subspace below its essential spectrum is purely the limit of the k^{th} eigenvalues of a sequence of self-adjoint subspaces. This was applied to singular second-order symmetric linear difference equations and approximations of eigenvalues below essential spectra obtained. Given any self-adjoint subspace extension of the corresponding minimal subspace, the k^{th} eigenvalue below its essential spectrum is exactly the limit of the k^{th} eigenvalues of any sequence of constructed induced regular self-adjoint subspace extensions.

Evamaria Russ (Russ, 2017) studied the structure and properties for the dichotomy spectrum of linear difference equations in Banach spaces. The assumption of existence of compact flow was avoided since any compact subset of the positive half-line would occur as a dichotomy spectrum. As such, the upper and lower dichotomy index was introduced by way of a measure of noncompactness. This enabled decomposition of the dichotomy spectrum into the upper, essential and lower dichotomy spectrum. The possible cases of the upper and lower dichotomy spectrum were described and explicit examples of difference equations were given to illustrate different cases of the Spectral Theorem. Applications to non-linear difference equations and the continuous time situation were shown.

Liu (Liu, 2019) was concerned with stability analysis of self-adjointness and deficiency indices of linear relations under perturbations in Hilbert spaces. Linear relations usually generalise the concept of linear operators and are very important in the study of multivalued and nondensely defined operators like those which arise in Hamiltonian systems. The study extended the classical perturbation theory for linear operators to linear relationships and therefore provided new insights and weakened some conditions of existing results.

The study reviewed the foundational concepts of linear relationships including their adjoints, deficiency indices and the notion of relative boundedness. Liu established several results that demonstrated the stability of deficiency indices and self-adjointness under bounded and relatively bounded perturbations. Particularly, the results show that if a closed Hermitian relation was perturbed by another relation with a relative bound less than one, the deficiency indices remained stable. This particular result was necessary for understanding how perturbations affect the spectral properties of linear relations.

Besides, the study highlighted the importance of the closedness of the perturbed relation which was preserved under some given conditions. This ensured that the spectral properties, including self-adjointness, were maintained under such perturbations. The results have practical implications especially in the study of differential equations and Hamiltonian systems where linear relations oftenly arise. In conclusion, the study was significant as it advanced the understanding of perturbation

theory for linear relations through extension of the classical results from linear operators to a more generalised setting. The findings provided a strong structure for analysing the stability of self-adjointness and deficiency indices under perturbations with applications in different areas of mathematical physics and differential equations. The work also opened avenues for future research into the spectral properties of linear relations and their perturbations.

The main result in the study states that if a relatively bounded perturbation is of a Hermitian relation with a bound less than one, then the perturbed relation is self-adjoint if and only if the Hermitian relation is self-adjoint. This result generalised the known Kato-Rellich theorem for linear operators to the context of linear relations. The result ensured that the deficiency indices of perturbed relation are the same as those of the Hermitian as long as the perturbed relation was relatively bounded with a bound less than one.

Anabby et al. (Annaby, Hamza, & Makharesh, 2018) studied Sturm-Liouville theory of the q, ω -Hahn difference operators in regular setting. They defined Hilbert space of q, ω -Hahn square summable functions in terms of Jackson-Nörlund integral. They discussed in detail how to formulate self-adjoint operator and the properties of the eigenvalues and eigenfunctions. In the same work, Green's function was developed and a study for q, ω -Fredholm integral operator established. As such eigenfunctions expansion theorem was developed and illustrative examples shown.

Later in the year 2020, Allahverdiev and Tuna (Allahverdiev & Tuna, 2020) considered spectral theory of singular Hahn difference equation of the Sturm-Liouville type. They defined a new Hilbert space of the solutions of these equations and constructed on it the Fourier transform and proved the Parseval equation. Hence, they proved the existence of a spectral function for the Hahn difference equations of the Sturm-Liouville type and established Parseval equality and an expansion formula in eigenfunctions in terms of the spectral function.

Mutlu (Mutlu, 2020) studied second order difference equation with self-adjoint operator coefficients defined on the whole axis. The focus was on the spectral analysis of a non-selfadjoint second-order difference equation with operator coefficients which is

the discrete analog of the Sturm-Liouville differential operator. The study was significant as difference equations are widely used in modeling various problems in physics, biology, economics and engineering. Mutlu considered the spectral properties of a discrete operator defined in a Hilbert space which consisted of vector sequences with values in a separable Hilbert space. The discrete operator was generated by a second-order difference equation having non-selfadjoint operator coefficients which were assumed to be completely continuous and invertible.

The study began by the introduction of the Jost solution, which is an operator solution of the eigenvalue equation associated with the discrete operator. Mutlu derived the asymptotic and analytical properties of the Jost solution and used it to determine the continuous spectrum of the discrete operator which was shown to be $[-2, 2]$. This result was obtained through decomposition of the discrete operator into a selfadjoint operator and a compact perturbation and application of the Weyl's theorem on compact perturbations.

They then analysed the discrete spectrum and spectral singularities of the discrete operator. The discrete spectrum consisted of eigenvalues while spectral singularities are points in the continuous spectrum where the Jost function (an operator function in this case) is not invertible. With the help of Keldysh's theory on the singular points of analytic operator functions, they demonstrated that the discrete spectrum was bounded and countable with limit points lying within the circle. They also showed that the set of spectral singularities was compact and had zero Lebesgue measure.

Finally, they proved that under an additional assumption on the decay rate of the coefficients, the discrete operator had a finite number of eigenvalues and spectral singularities. This result was significant since it provided a condition for which the spectral properties of the operator were well-behaved amidst the non-selfadjoint nature of the coefficients.

In general, the study extended the spectral analysis of difference equations to the case of infinite-dimensional operator coefficients by providing new insights and tools for studying non-selfadjoint operators. The results obtained in the study have po-

tential applications in quantum mechanics where non-Hermitian Hamiltonians and spectral singularities are of increasing interest.

Spectral theory on Hilbert spaces forms the basis of modern functional analysis as it provides an important tool that enables one to understand the behaviour of linear operators on infinite dimensional spaces. The study of P -operators in this framework is crucial with a wide range of applications in different fields like signal processing, quantum mechanics and partial differential equations. The theory of P -operators investigates the transformations which preserve positivity, which is key in modeling different physical phenomena and mathematical processes through complimentary problem. P -operators were closely linked to the conception of order, stability and growth. This gives the great intuition about the underlying structure of the space in consideration.

Rashid and Johnson (Johnson et al., 2025) investigated the spectral properties of positive operators and uncovered their eigenvalues, eigenvectors and spectral decompositions. They analysed the link between P -property and the spectral characteristics of P -operators hence giving light about their geometric, algebraic and analytical properties. Therefore, they provided a deep understanding of the spectral theory of P -operators thus enabling one to handle advanced topics in functional analysis and its applications.

2.4 Stability of Difference and Differential Equations, Deficiency Indices and Spectrum

The Stability of deficiency indices of symmetric operators under perturbations has received a lot of attention. Weyl (Weyl, 1910) studied the deficiency indices of second order self-adjoint differential equations. Weyl showed that the deficiency index is equal to the number of linearly independent square integrable solutions of the differential equation for every spectral parameter $\lambda \in \mathbb{R}$. Later, Atkinson (Atkinson & Weiss, 1964) studied the maximal deficiency index of Hamiltonian differential systems and established its invariance under bounded perturbation.

In order to determine the stability properties of functional differential equations, also referred to as hereditary equations, difficulties usually arise from the associated difference equation. Therefore, it is normally important to understand the problems of stability for hereditary equations. Melvin (Melvin, 1974) focussed on the stability properties of functional difference equations, which enables one to comprehensively understand the behaviour of neutral functional differential equations (hereditary equations). The stability of difference operators and their sensitivity to perturbations was exhaustively explored, particularly for the case of time lags, which are common in physical systems like electrical circuits, feedback control systems, and biological processes.

The study began by first discussing a special case of a linear autonomous difference equation whereby the stability of the difference operator was defined based on its response to perturbations. It was illustrated that certain classes of difference operators, even though are stable under specific conditions, they could become unstable under small perturbations in time lags. This kind of sensitivity was demonstrated by use of an example where a stable operator becomes unstable when the time lag was altered a little. A subclass of difference operators that maintained their stability under small perturbations was identified for the case where the sum of the absolute values of the coefficients was less than one. That condition ensured that the system stayed stable even when the time lags were perturbed.

The study generalised the results to functional difference equations on different function spaces. Theorems that established exponential bounds for solutions to these equations which are essential for the analysis of the stability of neutral functional differential equations were also provided. These theorems are very useful in applications where the stability of systems depend on the behaviour of the associated difference equations.

A number of examples which illustrated the application of the theoretical results were included. The examples covered a range of scenarios including linear and nonlinear difference operators and demonstrated how the stability conditions could be applied to specific problems. One particular example involved a neutral functional differential equation, where a double application of Liapunov's direct method was

used to prove global exponential asymptotic stability.

The research was significant enough as it contributed to the understanding of stability in functional difference equations and provided practical techniques for analysing the stability of systems with time delays. The results are very relevant for applications in control theory, biology, and engineering, where time lags and stability are key factors. The study also highlighted the need of carefully modeling physical systems to avoid instability caused by little perturbations in time lags.

Warming and Hyett (Warming & Hyett, 1974) analysed stability and accuracy of finite difference approximations for simple linear partial differential equations by way examining the modified partial differential equation. The modified equation, precluding round-off errors usually represents the actual partial differential equation being solved whenever a numerical solution is by use of finite difference scheme. The modified equation is normally derived by expanding each term of the difference scheme into Taylor series and removing time derivatives of order higher than the first through algebraic manipulation. The connection between Heuristic stability theory, which is normally based on the modified equation approach, and the von Neumann method was established. Beyond determining the necessary and sufficient conditions for computational stability, a truncated version of the modified version of the modified equation may provide an intuition into the nature of dissipative errors, always related amplitude damping and dispersive errors, usually related to phase distortions, in the numerical solution. Their analysis enables one to understand and improve the performance of finite difference methods.

Ortega (Ortega, 1973) studied stability of difference equations and convergence of iterate process. A number of connections between the theory of convergence of iterate processes and that theory of Lyapunov stability of ordinary difference equations were considered. Local convergence and asymptotic stability, Lyapunov functions, domain of attraction, global convergence and asymptotic stability, total stability and rounding errors as well as non-autonomous equations and variable operator iterations were equally discussed.

In the year 1977, Behncke and Focke (Behncke & Focke, 1977) investigated the sta-

bility of deficiency indices for closed symmetric operators on Hilbert spaces under perturbations. The main focus was on extending known results about the stability of selfadjointness to a more general case involving deficiency indices. They demonstrated that the deficiency index of the perturbed operator remained the same as that of the unperturbed operator if the perturbation was relatively bounded with a bound strictly less than one. This result extended the known Kato-Rellich theorem, which deals with the stability of selfadjointness.

Behncke and Focke formulated a number of theorems and corollaries which provided conditions under which the deficiency indices remained stable. For instance, they showed that if a dissipative operator was perturbed by another dissipative operator with a relative bound less than one, the deficiency index of the perturbed operator remained the same as that of the unperturbed operator. This result was extended to symmetric operators on Hilbert spaces hence providing a generalisation of the Kato-Rellich result. Besides, they addressed cases whereby the relative bound of the perturbation was equal to one, showing that the deficiency index would decrease in such scenarios. They also provided technical lemmas and additional theorems to handle more complex situations which deal with perturbations with a relative bound of one and showed that the codimension of the range of the perturbed operator is not less than that of the unperturbed operator.

Moreover, the authors applied the results to polynomials of symmetric operators and showed that the deficiency index of a polynomial in a symmetric operator could be determined by examining the leading term of the polynomial. This had implications for the theory of differential equations especially in cases of limit-point and limit-circle classifications.

The study was significant as it contributed to the understanding of how perturbations do affect the deficiency indices of symmetric operators. The results have broad applications in the study of unbounded operators, differential equations and quantum mechanics where the stability of operators under perturbations is a key concern. The paper provided a complete set of tools and conditions under which the deficiency indices remained stable hence extending and refining existing theories in the field.

The concepts of boundedness, relative boundedness and deficiency indices of linear relations was introduced by Cross (Cross, 1998) who showed that the deficiency of a subspace are stable under bounded perturbation under certain conditions [(Cross, 1998), Corollary III.7.6]. It has been established that the deficiency of a subspace is closely related to its deficiency indices.

Vidyasagar et al (Vidyasagar, Boyarsky, & Vannelli, 1982) presented a detailed analysis of the stability properties of perturbed linear non-stationary systems in the continuous-time and discrete-time settings. The main interest was on how to relate the stability of a perturbed system to that of an unperturbed system. They establish a number of results under some assumption on the perturbation. For the case of continuous-time systems, they demonstrated that if the unperturbed system was exponentially stable and $B(t)$ vanished at infinity, the perturbed system remained asymptotically stable. A similar result was shown for discrete-time systems, where geometric stability of the unperturbed system ensured asymptotic stability of the perturbed system. The authors also considered the concept of matrix measures and Lyapunov functions to derive tighter bounds on the stability of perturbed systems particularly when the unperturbed system was stationary.

The study further extended these results to cases where the perturbation $B(t)$ or B_k belonged to specific function spaces. This provided the necessary and sufficient conditions for uniform stability. The discrete-time analogues of these results were also extensively established with a focus on the geometric stability of the unperturbed system.

Aulbach et al (Aulbach, Minh, & Zabreiko, 1994) investigated the concept of spectral dichotomy for linear difference equations in Banach spaces by introducing a new structure for analysing the stability of such equations. They extended the known notion of exponential dichotomy through introduction of spectral dichotomy that was based on the spectral properties of the associated linear operators. The method allowed for a more generalised analysis of stability specifically in cases where the traditional exponential dichotomy would not apply.

Aulbach and the coauthors revisited earlier work on the stability of linear difference

equations especially the ones that focussed on the effects of perturbations on the stability of solutions. They stated that earlier results depended on conditions such as bounded growth and exponential stability, which were not always applicable. In order to address this kind of limitation, the authors invoked the concept of spectral dichotomy which was defined in terms of the spectrum of a linear operator associated with the difference equation. In particular, the equation was said to have a spectral dichotomy if the spectrum of the operator did not intersect the unit circle in the complex plane. This condition was shown to be closely related to the general exponents of the system that measures the growth and decay rates of the solutions.

Several major results were established, including the equivalence between spectral dichotomy and exponential dichotomy in the non-degenerate case where the operators were invertible. They also proved that spectral dichotomy was preserved under small perturbations. This provided a strong framework for analysing the stability of perturbed systems. This is specifically needful in the applications where the system was subject to external disturbances or modeling uncertainties. Moreover, they explored the reducibility of systems with exponential dichotomy in Hilbert spaces and showed that such systems could be transformed into a reduced form where the dynamics were decoupled. This particular finding was important for simplification of the analysis of complex systems.

The research contributed to the theory of linear difference equations by giving a more flexible and general method to stability analysis. The introduction of spectral dichotomy opened up new avenues for studying systems that did not satisfy the traditional conditions for exponential stability hence making the results applicable to a wider class of problems in dynamical systems and control theory. The research laid the groundwork for further research into the perturbation theory of spectral dichotomy and its applications in different fields including evolutionary processes and stochastic systems.

In the year 1996, Aulbach and Nguyen (Aulbach & Minh, 1996) studied the concept of spectral dichotomy for linear difference equation. In their study, the notion of dichotomy was essential as it enabled the application of the theory of linear operators to study dynamic properties of non-autonomous linear difference equations. They

extended their result on the equivalence of the spectral dichotomy and exponential dichotomy to the class of linear difference equations that were not necessarily invertible. They further investigated equations on the set of positive integers for which they established the necessary and sufficient conditions for exponential and uniform stability. They began with the study of geometry of the spectrum of the shift operator associated with the underlying difference equation. They drew certain conclusions from the structure of the spectrum of the shift operator. These included the results on equivalence of the spectral and exponential dichotomy, the Roughness Theorem and the results on the splitting of linear difference equations. They further derived a new criterion on the exponential dichotomy for difference equations on the integers and on the exponential or uniform stability of difference equation defined on natural numbers only. The different notions of dichotomy were designed mainly for equations with invertible solution operators.

Aleksandrov (Aleksandrov, 1996) considered stability of equilibrium of non-stationary systems. In the study, Aleksandrov investigated the effect of external non-stationary perturbations on the asymptotic properties of solutions of systems of non-linear differential equations. A formula for constructing Lyapunov functions was proposed. The formula was aimed at establishing conditions for which the perturbation does not destroy the asymptotic stability, instability or boundedness of solutions of a system. It was shown that it was possible for the order of the perturbations to be less than that of the right-hand sides of the unperturbed equations.

Clark (S. L. Clark, 1996) did spectral analysis for a class of second order difference equation and observed that real valued solutions do exhibit some type of stable asymptotic behaviour for certain real values of the spectral parameter. Such asymptotic behaviour do lead to characterisation of the limit circle and limit point cases of such equations. Using the same asymptotic behaviour, Clark established that strong non-subordinacy criterion is satisfied on subintervals of the real line for equation of limit point case. The subintervals are part of the absolutely continuous spectrum of self-adjoint realisation of such equations. The nature of the discrete spectrum for such self-adjoint realisations has also been discussed.

Later in the year 2000, Aleksandrov (Aleksandrov, 2000b) analysed the solutions of

a class of non-linear systems in the first approximation. The method of Lyapunov functions was used to obtain the sufficient conditions for stability with regards to all the variables and asymptotic stability with regards to part of the variables for the case when the first approximation system was primarily non-linear. In the same year, Aleksandrov (Aleksandrov, 2000a) studied stability conditions for non-linear systems with parameters that vary with time. Certain classes of non-linear oscillatory systems with time dependent coefficients were investigated. In order to obtain the sufficient conditions of asymptotic stability of equilibrium position for systems, the method of construction of non-autonomous Lyapunov's functions was proposed. The effects of external perturbations on stability of equilibrium was studied. It was shown that for some types of non-stationary perturbations, the asymptotic stability would be preserved particularly for the case when their order agreed with that of functions in the unperturbed equations.

Edwards and Ford (Edwards & Ford, 2002) considered the boundedness and stability of solutions to difference equations and provided a unified theoretical structure that applied to both autonomous and nonautonomous equations as well as linear and nonlinear systems. Their intention was to consolidate and extend existing results, offering a complete and easily applicable theory to analyse the qualitative behaviour of solutions to difference equations. The work was particularly relevant in the context of numerical analysis where difference equations usually arise as discrete approximations of continuous processes like differential equations. The study began by defining key concepts of stability like asymptotic stability and exponential stability, and distinguishes between boundedness and stability, which are often combined in the literature. They emphasized that while boundedness and stability coincide for linear equations, they could diverge for nonlinear systems. This distinction was necessary for one to comprehend the behaviour of more complex systems where solutions may be bounded but not stable or vice versa.

In the analysis, the Lipschitz condition was key and was used to derive sufficient conditions for stability and boundedness. Several theorems based on both uniform and local Lipschitz conditions were presented thus providing a flexible basis for analysing a broader range of problems. For instance, Theorem 4.1 which stated

that if a function in an autonomous difference equation satisfied a uniform Lipschitz condition with constant, then any solution was asymptotically stable. This result was extended to nonautonomous equations in Theorem 4.2 where the Lipschitz constants had to satisfy some given conditions. The authors addressed the limitations of linearization methods, which are commonly used in stability analysis but may fail to capture the true behavior of nonlinear systems. Instead, they advocate for a direct nonlinear analysis based on Lipschitz conditions, which avoids the pitfalls of linearization and therefore give more sophisticated results. This approach was particularly useful for analysing numerical methods applied to nonlinear differential equations where preservation of the qualitative properties of the original system was crucial.

A number of examples which demonstrated the application of the theory to both simple nonlinear difference equations and numerical methods for differential equations were presented. These examples showed how the Lipschitz-based method could be used to analyse stability and boundedness in a various contexts including systems with multiple equilibrium points. The study provided valuable contributions to the theory of difference equations by giving a unified and practical structure for the analysis of stability and boundedness. The emphasis on Lipschitz conditions and direct nonlinear analysis made the results broadly applicable especially in numerical analysis and in the study of dynamical systems.

Krause and Pituk (Krause & Pituk, 2004) focussed on the boundedness and stability of solutions to higher order difference equations. They established the sufficient conditions under which the difference equations generated order preserving dynamical systems with regards to discrete exponential ordering. They illustrated that the boundedness of solutions as well as the local and global stability of equilibrium could be determined by analysing some simple first order equation. The authors introduced the concept of discrete exponential ordering that generalises the standard partial ordering and allows for less restrictive monotonicity conditions on the function. They also provided an explicit criteria for the global asymptotic stability of equilibrium especially for systems where the considered function is not necessarily monotone in all its arguments.

Krause and Pituk (Krause & Pituk, 2004) applied the results to specific models, including second order non-linear difference equation from macroeconomics model. They derived explicit stability conditions to those models hence they demonstrated the practical utility of their theoretical results. The results highlighted the interplay between the structure of the difference equation and the properties of stability of its solutions thus giving a framework for analysing a wide range of dynamical systems. Their analysis provided a robust theoretical foundation for the understanding of the behaviour of higher order difference equations with application in economics and biology. The findings emphasized the importance of monotonicity and ordering in the determination of stability and boundedness of solutions hence offering tools for both theoretical analysis and practical modeling.

Gilbert (D. Gilbert, 2005) looked at the relationship between asymptotic behaviour of solutions of the Sturm-Liouville equation and the spectral properties of the corresponding self-adjoint operators. Specifically, they reviewed the salient features of subordinacy by examining two standard cases, the half-line operator and the full-line operator. They assumed the following conditions: that the coefficient function is locally integrable, 0 is a regular end point for the half-line case and Weyl's limit point case holds at the infinite end points. Consequences of the theory for informal characterisation of spectrum in terms of bounded solutions were observed. Extensions of this theory to related differential and difference operators were considered and its application together with other asymptotic methods, to various typical problems about spectral analysis were discussed.

Burenkov and Lamberti (Burenkov & Lamberti, 2007) studied the spectral stability of non-negative self-adjoint operators with compact resolvents, with a main focus on Neumann-type operators. They established a general framework for analysing the stability of eigenvalues under domain perturbations, invoking the concept of "transition operators" to measure the proximity of operators on different domains. They proved a general spectral stability theorem that provided necessary and sufficient conditions for the stability of eigenvalues hence giving a unified approach to the problem. This result was then applied to derive several results, especially for Neumann-type operators including semi-continuity of eigenvalues under inner

domain perturbations and sharp stability estimates for the variation of eigenvalues.

With regards to second-order uniformly elliptic operators with homogeneous Neumann boundary conditions, Burenkov and Lamberti (Burenkov & Lamberti, 2007) demonstrated that eigenvalues are semi-continuous under minimal assumptions on the domains. Therefore, the study provided a robust theoretical foundation for understanding the stability of eigenvalues under domain perturbations with significant implications for the analysis of elliptic operators and their spectral properties.

In the paper of (Wen, Wu, & Lu, 2008), stability theorem of non-linear fractional order differential equation was proved by use of Gronwall-Bellman lemma. A criterion for designing controller gains for stabilization was derived. Moreover, theoretical results were substantiated by simulation of results of the fractional-order chaotic Lorenz system with anticipated design requirements. Later, Wang, Yang and Zhang (Wang, Yang, & Zhang, 2016) studied stability of commensurate and incommensurate fractional-order systems. They suggested two comparison inequalities for incommensurate fractional-order systems. A stability criterion for a class of incommensurate fractional-order system and three stability criteria for a typical class of commensurate non-linear fractional-order systems were given. Lastly, the fractional-order Liu system and fractional-order Chen system were used as examples in showing how the proposed results could be applied to stabilization problems.

In the year 2009, Azizov et al. (Azizov, Behrndt, Jonas, & Trunk, 2009) invoked compact and finite rank perturbations of closed subspaces in $X \times Y$, where X and Y are Hilbert spaces, in terms of difference between the orthogonal projections of X and Y to the subspaces. It was shown that a closed subspace is of a finite or compact perturbation of another closed subspace if at all the difference between their resolvents is a compact or finite rank operator belonging to the intersection of their respective resolvent sets whenever this particular intersection is not empty and $X = Y$. They also worked on stability properties of the spectral points of positive and negative type and type π in the case of non-self-adjoint under various kinds of perturbations in the Krein spaces (Azizov, Behrndt, Jonas, & Trunk, 2011).

Ren and Shi (Ren & Shi, 2010) investigates the defect index of singular symmetric

linear difference equations of order $2n$ with real coefficients and one singular endpoint. The main focus was on determining the defect index d which characterizes the number of linearly independent square summable solutions of the equation. In the study, it was shown that the defect index d satisfied the inequality $n \leq d \leq 2n$, and all the values within this range were realized. This paralleled the well-known result of Glazman for differential equations.

The authors also established various criteria for the limit point case and the strong limit point case, which are crucial for understanding the self-adjoint extensions of the associated operators. The limit point case corresponded to $d = n$ while the limit circle case corresponded to $d = 2n$. The research provided the necessary and sufficient conditions for the equation to be in the limit point case, based on the asymptotic behavior of the coefficients and solutions of the difference equation.

Furthermore, the study explored the strong limit point case which was a stronger condition than the limit point case and ensured that some bilinear forms associated with the equation vanished at infinity. The results were derived using the Weyl-Titchmarsh theory for singular linear discrete Hamiltonian systems and asymptotic analysis of the solutions. The findings generalised earlier results for second-order difference equations to higher-order equations and provided a deeper understanding of the defect index and limit cases for singular symmetric linear difference equations. The research has significant implications for the spectral theory of difference equations and their applications in various fields including mathematical physics and engineering. The results also contributed to the wider theory of self-adjoint extensions of symmetric operators which is crucial in the study of boundary value problems.

Chen et al. (J. Chen, Fu, Niculescu, & Guan, 2010a) studied stability analysis of linear systems with parameter variations and in particular, linear systems with time-invariant delay. They characterised asymptotic behaviour of the characteristic zeros of the systems. They pointed out that the asymptotic behaviour determines whether the imaginary zeros cross one half plane into the other or not. As such, they noted that the asymptotic behaviour was responsible in determining the stability of the system. Mathematical tools of eigenvalue series of holomorphic operator

matrices were developed. Furthermore, it was noted that the eigenvalue perturbation analysis was crucial in analysis of stability as it provided efficient solutions of the computation of problems related to control systems analysis and design like time-delay systems. This work was extended in the paper (J. Chen, Fu, Niculescu, & Guan, 2010b) whereby an application of the eigenvalue series to the analysis of linear -invariant delay systems was presented. The focus was on asymptotic behaviour of the critical characteristic zeros on imaginary axis. Systems in state-space form and of quasi-polynomials were considered in developing eigenvalue analysis of perturbation method. The results showed that the zero asymptotic behaviour of time-delay systems could be generally characterised by solution of the eigenvalue problem as well as by computing the derived forms of the quasi-polynomials in the case that they were described using quasi-polynomials.

It has been established that the essential spectrum of the self-adjoint operator is invariant whenever the perturbing term is a densely defined and Hermitian, symmetric and relatively compact operator to the self-adjoint operator (Schmüdgen, 2012). On the other hand, the absolutely continuous spectrum is likely to disappear under such kind of perturbation even though the perturbing term is very small. When the perturbing term is finite rank, the absolutely continuous spectrum is invariant (Kato, 1966). Notably, these particular results have been widely applied in studying the stability of the spectra of symmetric linear differential operators as well as Jacobi operators with Schrödinger operators having a strong physical background included. It has been obtained in (Ren & Shi, 2011) and (Y. Shi & Sun, 2011) that the maximal and minimal operators generated by symmetric linear difference expressions are generally non-densely defined or multi-valued despite the fact that the corresponding definiteness condition being satisfied.

Discrete Hamiltonian systems arise from the discretisation of continuous Hamiltonian systems and discrete processes which act according to Hamiltonian principle like discrete physical problems, discrete control problems and variational problems for the sum of integrals. They have been in existence and widely studied since early 1960s. Spectral theory of singular discrete systems was initially studied by Atkinson (FV, 1964). In the year 2013, Zheng (Zheng, 2013) studied invariance of deficiency

indices under perturbation for discrete Hamiltonian systems with one singular end point. The invariance of the minimal and maximal deficiency indices under bounded perturbation was built. This was done by use of Hermitian linear relation as generalised by linear Hermitian system. The results paralleled those obtained earlier by Atkinson on linear Hamiltonian differential systems.

Mauroy and Mezić (Mauroy & Mezić, 2016) studied global stability of nonlinear systems. They employed an approach referred to as natural extension of classic linear stability analysis to non-linear systems. They established the necessary and sufficient relationship of the existence of particular eigenfunctions of Koopman operator and the global stability property of hyperbolic fixed points and limit cycles.

Shi (Y. Shi, 2016) analysed the stability of essential spectra of self-adjoint subspaces for the case of finite rank and compact perturbations in Hilbert spaces. This was possible by first establishing the relationships between compact perturbation of closed subspaces and relatively compact perturbation of their operator parts. A characterisation of compact perturbation based on the difference between the operator parts of the perturbed and unperturbed subspaces. It was shown that self-adjoint subspace remains self-adjoint under either relatively bounded perturbation with relative bound less than one or relatively compact perturbation or compact perturbation with an additional condition. The invariance of the essential spectra of self-adjoint subspaces was therefore proven separately under relatively compact and compact perturbations. It was noted that the results generalised those for self-adjoint operators to self-adjoint subspaces.

Ren (Ren, 2018) was concerned with stability index for linear relations in Banach and Hilbert spaces and its applications. Various results on the stability of closedness and the stability of indices of operators in Banach spaces were extended to the linear relations in Banach spaces. Besides, the stability of deficiency indices of dissipative linear relations in Hilbert spaces was equally studied. The stability of deficiency indices of dissipative linear relations generated by second order difference expressions was discussed and shown to be stable under relatively bounded perturbations with relative bounds less than 1 under some additional conditions.

Gümüş (Gümüş, 2018) investigated the global asymptotic stability and the rate of convergence of solutions to a system of difference equations. The study focussed on understanding the long term behaviour of solutions especially their convergence to unique positive equilibrium point. It was established that the equilibrium was asymptotically stable, all positive solutions converged regardless of initial conditions. This showed that semicycles (sequences of terms above and below the equilibrium) have at most m terms. The rate of convergence of solutions to the equilibrium was derived and linked to the eigenvalues of the systems's Jacobian matrix evaluated at the equilibrium.

Gümüş (Gümüş, 2018) provided examples to illustrate the theoretical results hence demonstrating the stability and convergence behaviour. The research offered a comprehensive analysis of the global behaviour of solutions to the system of difference equations, giving both theoretical and practical tools for understanding stability and convergence. The findings contribute to the broader field of dynamical systems and difference equations with the potential application to modeling real world phenomena.

Liu and Shi (Liu & Shi, 2019) were interested in stability of deficiency indices of Hermitian subspaces, linear relations under relatively bounded perturbations in Hilbert spaces. They established a number of results on invariance of deficiency indices of Hermitian subspaces under relatively bounded perturbations. Consequently, the invariance of self-adjointness of Hermitian subspaces under relatively bounded perturbations was obtained. Further, they showed that the deficiency indices would shrink in the special case when the relative bound is equal to one. Their results generalised the corresponding results for symmetric operators to more general Hermitian subspaces.

In the year 2020, Liu and Xu (Liu & Xu, 2020) also applied perturbation theory of Hermitian linear relations and established the invariance of deficiency indices for discrete Hamiltonian systems under bounded perturbations. They obtained the invariance of limit types for the systems under bounded perturbations; they built criteria for invariance of limit circle and limit point cases for the systems.

The theory of perturbation of linear relations has received a lot of attention. The following authors, (Wilcox, 2014), (Y. Shi, 2016) and (Y. Shi, Xu, & Ren, 2018) obtained results on stability of closedness, boundedness, self-adjointness and spectra of linear relations.

Shi (Y. Shi, 2016) investigated the stability of essential spectra of self-adjoint subspaces, also referred to as linear relations, in Hilbert spaces under varied types of perturbations, including finite rank, compact and relatively compact perturbations. The study extended classical perturbation theory for self-adjoint operators to self-adjoint subspaces, which are more general and may be multi-valued or non-densely defined. The main contributions included establishment of relationships between the operator parts of perturbed and unperturbed subspaces, and characterisation of compact and finite rank perturbations in terms of these operator parts.

Shi (Y. Shi, 2016) showed that the essential spectrum of a self-adjoint subspace remained invariant under compact perturbations, provided certain conditions were met. Particularly, it was shown that if a self-adjoint subspace was perturbed by a relatively compact or finite rank operator, the essential spectrum of the perturbed subspace coincided with that of the original subspace. The results generalised the known theorems from operator theory such as the Kato-Rellich theorem, to the context of subspaces. Moreover, the study provided sufficient conditions under which a perturbed self-adjoint subspace remained self-adjoint even when the perturbation was relatively bounded or relatively compact.

The results have significant impact in the study of differential and difference equations, specifically in cases where the operators involved are non-densely defined or multi-valued. The results are applicable in a wide range of problems in mathematical physics and applied mathematics especially in the analysis of singular Hamiltonian systems and other dynamical systems with complex boundary conditions.

As such, the research enhanced the understanding of spectral stability for self-adjoint subspaces under perturbation and provided a vibrant framework for analysing the effects of compact and finite rank perturbations. The results generalise classical operator theory as well as offering new tools for studying the spectral properties of

more complex systems in mathematical physics and applied mathematics.

Wilcox (Wilcox, 2014) explored the essential spectra of linear relations and extended the well-established theory for single-valued linear operators to a more general context of multi-valued linear operators. The research defined a number of essential spectra based on semi-Fredholm properties and the index, and established their fundamental properties. The perturbation theory for semi-Fredholm relations was applied to generalize Weyl's theorem thus demonstrating the stability of essential spectra under small and compact perturbations. Besides, a Möbius transform spectral mapping theorem was provided, enriching the spectral theory of linear relations even more.

Wilcox then introduced essential resolvent sets and spectra and explored their properties, proving that these sets were closed and interrelated through a series of inclusions. The research also investigated the behavior of nullity, deficiency and the indices within the essential spectra, giving a detailed analysis of their constancy and variation across different components of the essential resolvent sets. The perturbation results are particularly significant, as they affirm that the essential spectra remain invariant under relatively compact perturbations, generalising classical results for single-valued operators. The Möbius transform theorem further extended the applicability of these findings by provision of a spectral mapping relationship between a linear relation and its inverse.

Shi and the coauthors (Y. Shi et al., 2018) considered the boundedness, closedness, and closability of linear relations and generalised the concept of single-valued linear operators to include multi-valued operators. The study introduced a new linear operator induced by a linear relation and established its relationship with other induced operators thus providing a bridge between multi-valued and single-valued operators. This allowed for the application of classical operator theory to the study of linear relations.

Major contributions of the study included the development of equivalent characterisations for the closedness and closability of linear relations as well as conditions for boundedness and relative boundedness. The Shi and the partners also explored the

stability of closedness under bounded and relatively bounded perturbations hence they generalised classical results for single-valued operators to the multi-valued case. The closed graph theorem for linear relations in Banach spaces was completed and new intuitions into the behavior of Hermitian subspaces were provided including the conditions for boundedness from below and from above.

The study greatly extended existing results in the theory of linear operators and offered new tools and characterisations for analysing multi-valued operators. The results have potential applications in different fields like differential equations, control theory and functional analysis where multi-valued operators arise naturally. The research not only generalised classical operator theory but also improved and relaxed some assumptions in existing results hence provided a more comprehensive understanding of linear relations in normed and Banach spaces.

Liu and Shi (Liu & Shi, 2022) studied stability of deficiency indices of Hermitian subspaces under relatively bounded perturbations in Hilbert spaces. A number of results on invariance of deficiency indices of Hermitian subspaces under relatively bounded perturbations were obtained. Consequently, invariance of self-adjointness of Hermitian subspaces under relatively bounded perturbations were established. Moreover, they showed that the deficiency indices would shrink when the relative bound is equal to 1. Their results generalised the corresponding results for symmetric operators to more general Hermitian subspaces.

Liu (Liu, 2020) investigated the invariance of deficiency indices of second-order symmetric linear difference equations under perturbations. The invariance of the deficiency indices of the corresponding minimal subspaces under bounded and relatively bounded perturbation was built by application of perturbation theory of Hermitian linear relations. Consequently, the invariance of limit types of second-order symmetric linear difference equations under bounded and relatively bounded perturbations was obtained.

Carnicer et al. (Carnicer, Mainar, & Peña, 2023) analysed the stability of the representation of finite rank operators in terms of a basis. This was done by introduction of a conditioning as a measure of the stability properties. The conditioning, was that

closer to the Lebesgue function and as such it improved some other conditionings. They were able to obtain improved bounds for conditioning of Fourier sums regarding orthogonal basis particularly for Legendre, Chebyshev and disk polynomials. They also considered Lagrange and Newton formulae for interpolating polynomial.

Halanay (Halanay, 1966) proved asymptotic formula for differential inequality which involved maximum functional. The formula was then employed in stability theory of linear systems with some delay. The inequality was referred to as the Halanay inequality. Mohamed and Gopalsamy (Mohamad & Gopalsamy, 2000) studied discrete and continuous Halanay-type inequalities for non-autonomous scalar systems having discrete and distributed delays. They studied certain discretised versions of functional differential equations. They pointed out that the asymptotic behaviour of the continuous and discrete type of systems do not often coincide despite the fact there are a number of numerical schemes to approximate solutions of continuous type systems. Moreover, they showed that discrete time inequalities which are the analogues of continuous time inequalities conserve the stability conditions correlating with continuous time Halanay-type inequalities.

Liz and Ferreiro (Liz & Ferreiro, 2002) studied global stability of generalised difference equations. They proved the discrete analogue of the continuous Halanay. They provided a simple and effective discrete version of the Halanay inequality and applied it to derive the necessary and sufficient conditions for stability of equilibrium in a generalised difference equation. They highlighted the need for this particular discrete inequality in analysing the asymptotic behaviour of solutions in the context of numerical schemes for functional delay differential equations. They established a discrete Halanay inequality and showed that given particular conditions, the solutions to the inequality converge exponentially to zero. This result was extended so as to provide conditions for stability for global asymptotic stability of solutions to generalised difference equations. They demonstrated that whenever the nonlinear term in the difference equation was appropriately bounded, the solutions converged to zero hence enabling stability of the system.

Liz and Ferreiro (Liz & Ferreiro, 2002) bridged the gap between continuous and discrete systems, offering a fundamental framework necessary in the analysis of

stability of generalised difference equations. Their results are useful in numerical methods and application in fields like population dynamics and control theory, where discrete models are useful to approximate the continuous processes. The research advanced theoretical understanding as well as practical tools for ensuring stability of discrete systems which are derived from continuous models.

Again in the year 2011, Liz and Ferreiro (Liz, 2011) studied stability of non-autonomous difference equations. They explored stability properties of non-autonomous difference equations majoring onto two techniques that were motivated by the stability results for functional differential equations. The first technique involved discrete inequalities and monotonicity arguments similar to the Halanay inequality. On the other hand, the second technique was based on the known $3/2$ stability results for functional differential equations. The study provided a comprehensive overview of the two methods thereby giving new insights, proving new and interesting results as well as presenting applications and open problems.

The Halanay results are effective for establishing global exponential stability in difference equations. Liz and Ferreiro (Liz, 2011) demonstrated how the Halanay-type results could be generalised and applied in various contexts including Banach spaces and systems of difference equations. They highlighted the simplicity and applicability of the two methods, hence making them valuable tools for analysing stability in discrete processes.

The $3/2$ stability results were derived from the qualitative theory of functional differential equations and are adapted to difference equations by use of piecewise constant arguments. The approach enables one to directly apply stability theorems from functional differential equations to difference equations thus providing the necessary and sufficient conditions for uniform asymptotic stability. Liz and Ferreiro (Liz, 2011) discussed the sharpness of these conditions and suggested possible improvements and open problems for further investigations.

Besides, Liz and Ferreiro (Liz, 2011) bridged the gap between the stability theories of differential and difference equations and offered robust methods for analysing the stability of non-autonomous systems. Their results have major implications to

various fields like population dynamics, economics and neural networks where discrete models are common. They advanced theoretical understanding and provided practical tools that ensure stability of discrete processes.

In the year 2006, Stević (Stević, 2006) analysed global stability and asymptotics of certain classes of rational difference equations. The focus was proving that the positive equilibrium point of those equations was globally asymptotically stable. The study was built from the previous work by Xianyi and Deming (Li & Zhu, 2004) who had established that the global asymptotic stability of the positive equilibrium for two specific rational difference equations. Stević extended these results to six additional classes of rational difference equations and proved that the positive equilibrium was globally asymptotically stable for all of them. The proof strengthened global convergence result from Kruse and Nesemann (Kruse & Nesemann, 1999), which provided sufficient conditions for global asymptotic stability of a unique positive equilibrium in a general difference equation. Stević unified the six equations into a single form and showed that the linearised equation about the equilibrium implied local stability and that positive equilibrium was a global attractor for all positive solutions.

Stević (Stević, 2006) explored asymptotics of solutions of oscillatory and non-oscillatory difference equations. For non-oscillatory solutions, asymptotic expressions were derived by a way that was motivated by Berg's work (Berg, 2002) on nonlinear difference equations. The analysis involved making specific assumptions and application of a theorem that ensured existence of solutions within asymptotic stripes. For the case of oscillatory solutions, periodic sequences that describe oscillatory behaviour were identified and showed that the solutions equally exhibited asymptotic properties similar to the non-oscillatory ones.

Therefore, the research by Stević (Stević, 2006) significantly extended the understanding of global stability and asymptotic behaviour of rational difference equations and provided a unified approach to prove global asymptotic stability and derive a detailed expression for both oscillatory and non-oscillatory solutions. The findings contribute to the broader theory of difference equations and have got potential applications in different fields where such equations model dynamic systems. -prey

models hence enabling the achievement of satisfactory results.

Deng, Wu and Li (Deng, Wu, & Li, 2006) analysed the stability of differential equations with time dependent delay by introducing the concept of interval stability and applied it to a generalised Gauss-type predator-prey models. They established a number of theorems to help analyse the stability of such systems on the finite time intervals and in the sense of Liapunov. The introduction of interval stability extended the traditional notion of stability to finite time intervals and the application of the results to predator-prey models with time dependent delays.

Deng' et al. (Deng et al., 2006) demonstrated that the stability of equilibrium solutions in systems with time dependent delays could be analysed by use of part-metric and the concept of interval stability. They provided sufficient conditions for Liapunov stability and interval stability, illustrating that the stability of equilibrium solutions could be determined from the behaviour of the characteristic equation of the system. Besides, they explored the effects of time dependent delays on the stability of predator-prey models especially in cases where delays appeared in the prey's growth term or the predator's response function.

The findings showed that stability of the equilibrium solutions could change over time with respect to the nature of the delay and its derivative. They provided conditions under which the equilibrium remained stable or became unstable, giving insights into the dynamic behaviour of the systems. The research promoted the understanding of stability in differential equations with time dependent delays, offering a strong framework for analysing interval stability and Liapunov stability. The results were applicable in the study of dynamical systems in different fields including but not limited to biology, ecology and control theory where time dependent delays play an important role in the behaviour of the system.

In the paper of Matignon (Matignon, 1996), results for stability for control theory were provided for finite-dimensional linear fractional differential systems. The internal and external stabilities were considered for the fractional differential systems of state-space form. Equally, thorough examination was done on external stability of fractional differential systems of polynomial representation. The main result showed

that stabilities existed if and only if the roots of certain polynomial were outside the closed angular sector, $|\arg(\sigma)| \leq \frac{\alpha\pi}{2}$. This generalised in a special way the results for the integer $\alpha = 1$.

Fractional differential equations have been an area of interest to many researchers in the last decades. The necessity had been to model different stable physical phenomena that are not of exponential type. Matignon (Matignon, 1998) addressed stability properties and modelling issues by generalised fractional differential systems. The systems were considered in the time-domain and frequency-domain.

Mondal et. al (Mondal, Khan, Vishwakarma, & Saha, 2018) evaluated first order linear homogeneous difference equation in fuzzy environment. They illustrated the refinement of these equations using three different ways involving fuzzy numbers. Consequently, they discussed the existence and stability analysis of the obtained solutions of fuzzy difference equations under different cases of impreciseness. Triangular and generalised triangular fuzzy numbers were considered hence enabling illustration of numerical experiments that showed efficient solutions graphically and in tabular form.

The authors of (Kassim, Abdeljawad, Ali, & Abdo, 2021) considered stability of certain non-linear initial value functional differential problems within a generalised fractional derivative of different orders. They established enough conditions on the non-linear terms so as to analyse the solutions' decay to zero as a power function. Various versions of inequalities were combined and generalised via Bihari inequality. Certain properties of the generalised fractional derivative and regularisation techniques were applied. Examples affirming the validity of the results were presented.

In the paper (Bajodek, Seuret, & Gouaisbaut, 2022), Bajodek, Seuret and Gouaisbaut analysed the stability of the reaction-diffusion equation interconnected with finite-dimensional system. The main goal was to give a Lyapunov analysis resulting to an efficient and precise criteria for stability. The study was possible by use of dedicated Lyapunov functionals which contain information depending on the state of the augmented finite-dimensional system having the first Legendre coefficients and the remainder of the Legendre expansion. The Legendre development enabled

efficient formulation of Bessel and Wirtinger inequalities. This led to adequate stability conditions expressed as linear matrix inequalities that were scalable. Numerical examples were used to demonstrate the accuracy of the stability results and their potential.

Brandibur and Kaslik (Brandibur & Kaslik, 2023) reviewed the conditions for stability and instability and extended them to multi-term homogeneous linear functional differential equations of Caputo derivatives and constant coefficients. An extensive review of stability analysis of two and three terms fractional order differential equation was given and extended to four-term fractional differential equations. They characterised stability and instability properties in terms of coefficients of the multi-term fractional differential equations hence realising fractional-order-dependent and fractional-order-independent stability and instability properties were described for fractional-order differential equations having an arbitrary number of fractional derivatives.

Dassios, Liu and Milano (Dassios, Liu, & Milano, 2024) performed stability analysis for systems of differential equations with multiple and distributed delays. They defined the characteristic equation used to analyse stability equilibrium of systems of multiple delay differential equations. A matrix as a result of the coefficients of the characteristic matrix equation was constructed and its spectrum used to derive conditions for stability of the system. They compared the stability equivalency between systems delay differential equations and related their results to distributed delay systems.

A good number of applied problems from physics, biology, mechanics and other fields are best described by partial differential or delay differential equations. This always requires the construction and study of infinite dimensional dynamical systems. Rabah et al. (Rabah, Sklyar, & Rezounenko, 2005) considered stability theory of systems with delayed argument. They proved that most methods and approaches that have been used before were developed for retarded differential equations. However, the case of neutral type systems still remained more difficult and less studied. Their attention was necessitated by the fact that neutral type systems exhibited essentially two types of stability: exponential and strong asymptotic non-exponential

stability. It was noted that asymptotic non-exponential stability was impossible for retarded systems. However, they could occur for neutral type systems. To be precise enough, it was shown in (Brumley, 1970) that for a higher order differential equation of neutral type, the smooth solutions decay essentially slower than exponential. This was possible since neutral type equation could have infinite sequence of roots of the characteristic equation with negative real parts approaching zero. In such a case, the equation is not exponentially stable and one would require more subtle methods so as to characterise such type of asymptotic stability.

Rabah and the other authors (Rabah et al., 2005) considered an operator model of the system in Hilbert space and used results on the existence of Riesz basis of invariant finite dimensional subspaces so as to verify its dissipativity. Their main result was concerned with the conditions of asymptotic non-exponential stability. They showed that the property of asymptotic stability is not determined only by the spectrum of the system but primarily depends on the geometric spectral characteristic of its main neutral term but of which one of them is asymptotically stable while the other is unstable.

Stochastic differential equations with Markovian switchings have received much attention of researchers in the past decades. Results ranging from stability and structural analysis of scientific and engineering applications have been obtained. It has been established that Stochastic differential equations with infinite Markovian switchings possess distinct features. For instance, asymptotic mean square stability are equivalent for Stochastic differential equations with finite Markovian switchings. However, that link does not hold for the case of infinite Markovian switchings. Ma and Jia (Ma & Jia, 2016) focussed on the stability analysis of linear Itô stochastic differential equations with countably infinite Markovian switchings. They proposed a spectral criterion for exponential stability which they used to clarify the relationship between exponential stability and stochastic stability. They further showed that even if the stochastic differential equation with infinite Markovian switchings were stochastically stable, it would not be exponentially stable. Furthermore, for any finite-energy random disturbance, it was exponential stability and not stochastic stability that would guarantee the perturbed stochastic differential equations with

infinite Markovian switchings to be stable.

Kaslik and Sivasundaram (Kaslik & Sivasundaram, 2012) presented various analytical and numerical approaches for the stability analysis of linear fractional-order delay differential equations. They mainly focussed on asymptotic stability. However, they also discussed bounded-input bounded-output stability. The possibility of applying the Laplace transform method in analysis of stability was investigated jointly with the corresponding characteristic equation which is widely used in bounded-input bounded-output stability analysis. Moreover, they showed that different characteristic equation, having one parameter Mittag-Leffler function, could be obtained by use of known method steps which provide the necessary condition for asymptotic stability. Besides, stability criteria based on Argument principle were obtained. They numerically evaluated the stability regions by use of those methods and presented comparison results.

The investigation of qualitative properties of solutions, stability of solutions to be precise, is a very important problem in the theory and applications of the differential equations. The most efficient tool for the study of stability of non-linear systems was provided by Lyapunov's second method (Stamova & Stamov, 2013). Its application to systems having delay was developed in two different directions. The first method was by use of Lyapunov functional method and the Razumikhin technique (Hale, 2006). The second method was by use of Lyapunov-Krasovskii functional method (Kolmanovskii & Nosov, 1986). Differential equations with maximum is a special class of functional differential equations having automatic regulation (Popov, 1966).

Bainov and Voulov (Bainov & Voulov, 1992) formulated mathematical models of differential equations with maximum. Interesting results on stability by use of a modified Lyapunov's first method were obtained. The theorems for existence, uniqueness and continuability were established and some oscillation results obtained by Bainov and Zahariev (Bainov & Zahariev, n.d.). Stamova and Stamov (Stamova & Stamov, 2013) investigated the stability of the zero solution of differential equations with maximum by use of Lyapunov and Razumikhin techniques. They established the sufficient conditions for stability, uniform stability and asymptotic stability.

2.5 Dirac Operators

Dirac operators were first introduced by Dirac P.A.M [(Dirac, 1928a),(Dirac, 1928b)] in the year 1928. The aim was to treat electron quantum mechanically, an idea that was to make this treatment consistent with Lorentz transformations in special relativity, the Laplace operator that occurs in the Schrödinger equation should be written as the square of some first order differential operator D . This was done on four-dimensional space-time with the Minkowski metric. We note that other than their motivation from Physics, Dirac operators have become more relevant to various areas in Mathematics. These include existence of Riemannian metrics of positive scalar curvature (Lichnerowicz, 1963), representation theory [(M. Atiyah & Schmid, 1977),(Parthasarathy, 1972)], geometry and topology. Many applications involve the Atiyah-Singer index theorem [(M. F. Atiyah & Singer, 1968a),(M. F. Atiyah & Singer, 1968b)], which relates the space of solutions of the equation $Ds = 0$ to the geometry and topology of the space in consideration.

Peter Hochs (Hochs, 2023) considered the case of n -dimensional Euclidean space and introduced Dirac operators and their index theory. The special case of Spin-Dirac operators is discussed. Moreover, the Atiyah-Singer index theorem for such operators is stated and a deduction made on the consequence to existence of Riemannian metrics of positive scalar curvature. Kiselev (Kiselev, 1999) analysed the asymptotic behaviour of the solution of the two dimensional Dirac system with rapidly oscillating coefficients, that holds uniformly with respect to the complex variable and the two real variables.

Albeverio et. al (Albeverio, Hryniv, & Mykytyuk, 2007) studied spectral properties of Dirac operators on the interval $(0, 1)$ with potentials belonging to $L_p(0, 1)$ and for some $p \in [1, \infty)$. The aim was to solve the direct and inverse spectral problems for one-dimensional Dirac operators on a finite interval under some likely least restrictive assumptions on the potentials. They established the algorithm of reconstructing the potential from two spectra or from one spectrum and the corresponding norming constants. They provided a complete solution of the inverse spectral problem.

Coskun(Coskun & Yokus, 2020), considered the spectrum of a non-self-adjoint dis-

crete Dirac operator generated by a discrete Dirac system with a general boundary condition in the Hilbert space. The main interest was on the determination of the sets of eigenvalues and spectral singularities of the Dirac operator, and showing that, under certain conditions, the Dirac operator has a finite number of eigenvalues and spectral singularities with finite multiplicity.

The study invoked techniques from functional analysis and the theory of analytic functions to analyse the spectrum of the Dirac operator. The Jost solution of the operator was derived and the resolvent of the Dirac operator was constructed by the Green's function. The eigenvalues and spectral singularities were characterised based on the zeros of the Jost function. Under the assumption that the sequences satisfied certain summability conditions (Naimark's and Pavlov's conditions), it was shown that the Dirac operator had a finite number of eigenvalues and spectral singularities, each of finite multiplicity. The results generalised earlier findings by considering a more general boundary condition and providing a basis for future research in spectral theory, scattering theory, and applied physics. The study also laid the framework for further investigations into direct and inverse problems related to discrete Dirac operators.

The point spectrum of Dirac operators has been studied by (Berthier & Georgescu, 1987) and more particularly, the behaviour of eigenfunctions of Dirac operators at infinity. The aim was to obtain conditions for the absence of eigenvalues embedded in the continuous spectrum for Dirac Hamiltonians with long range and locally singular potential. The principal results were in the following directions: distribution of eigenvalues of the Dirac operator, exponential decay for the eigenfunctions of the Dirac operator at infinity, absence of eigenvalues of the Dirac operator and finally result about discreteness of the spectrum with long range and locally singular potentials.

Hughes and Schmidt (Hughes & Schmidt, 2014) showed that the absolutely continuous part of the spectral function of one dimensional Dirac operator on half-line with a constant mass term and a real square integrable potential is strictly increasing throughout the essential spectrum. The proof here was based on estimates for the transmission coefficient for full line scattering problem with truncated potential and

a subsequent limiting procedure for the spectral function. Moreover, they showed that the absolutely continuous spectrum persists when an angular momentum term is added hence establishing the results for spherically symmetric Dirac operators in higher dimensions.

Schmidt (Schmidt, 1997), investigated the location and nature of the spectrum of one-dimensional Dirac operator with a potential, q being infinite at infinity such that the positive variation of $\frac{1}{q}$ is bounded, does cover the whole real line and is purely absolutely continuous. The major result, Theorem 1, established that if the potential q satisfied some regularity conditions, particularly, if q tends to infinity and the positive variation of $\frac{1}{q}$ is bounded, then the spectrum of the Dirac operator would be purely absolutely continuous and covering the entire real line. This extended earlier results by relaxing the regularity requirements on q and allowing for a broader class of potentials including the non-decreasing functions.

The proof relied on the method of subordinacy, which reduced the problem to proving that all solutions of the eigenvalue equation were globally bounded. The main idea involved analysis of the function R , which captured the behaviour of the solutions and controlled its growth with the help of Gronwall-type lemma. The study also provided a counterexample to demonstrate that if the positive variation of $\frac{1}{q}$ was unbounded, the spectrum would not be purely absolutely continuous as the operator could have eigenvalues.

Furthermore, the study offered a direct proof of a special case of the subordinacy method that simplified the argument for the boundedness of the solutions. The results are crucial for understanding the spectral properties of Dirac operators in relativistic quantum mechanics, especially in cases where the potential grows indefinitely. The findings generalised previous work and provided a robust framework for analysing the spectrum of such operators under less restrictive conditions.

Shi and Chen (G. Shi & Chen, 2005) discussed the location and nature of the spectrum of the matrix Dirac system. It was illustrated that under some additional conditions on the potential $n \times n$ real-valued Hermitian matrix with entries that are locally integrable on the positive half-line, the spectrum of the Dirac operator is

purely absolutely continuous outside certain gaps containing eigenvalues of Hermitian matrix. These results extended those of Schmidt (Schmidt, 1997) to the higher dimensional cases.

Cuenin and Siegel (Cuenin & Siegl, 2018) analysed the spectral properties of one-dimensional non-self-adjoint Dirac operators perturbed by complex and nonsymmetric potentials. They established the existence and asymptotics of weakly coupled eigenvalues emerging from the thresholds of the essential spectrum by way of the Birman-Schwinger principle and complex analysis techniques. The Lieb-Thirring inequalities, which provide upper bounds for the number of eigenvalues in some subsets of the complex plane were derived, hence advancing previous results by offering tighter bounds and better weights in the eigenvalue sums. The analysis was also extended to physical applications, including the damped wave equation and armchair graphene nanoribbons.

For the damped wave equation, it was shown that the spectrum of the associated operator shifted due to damping. They provided the Lieb-Thirring type inequalities and bounds on the number of eigenvalues. However, in the case of armchair graphene waveguides, they analysed the eigenvalues emerging from the edges of the essential spectrum and derived their asymptotics in the weak coupling regime. The results were obtained through decomposition of the resolvent of the waveguide operator and employment of a unitary transformation to relate it to the one-dimensional Dirac operator.

The study highlighted the crucial nature of the massive Dirac operator whereby even small perturbations was able to create eigenvalues and compared it with the non-critical massless case. The implications of the findings for the number of eigenvalues in compact subsets of the complex plane were discussed. This provided explicit bounds that depend on the norms of the potential. Generally, the study presented a comprehensive analysis of non-self-adjoint Dirac operators, giving new insights into their spectral properties and applications in physical systems. The results have great significance for the understanding of the behavior of eigenvalues in the case of complex perturbations and have potential implications for the study of damped wave equations and graphene waveguides.

Amar (Amar, 2002) considered one-dimensional Dirac system on the semiaxis whose potentials are continuous and gave two results about spectral decomposition of such systems. The notion of subordinated solutions developed by D. Gilbert and D. Pearson (D. J. Gilbert & Pearson, 1987) for Schrödinger equation was adapted and an idea of Jitomirskaya-Last was used to establish a characterisation of the spectrum in terms of the existence and non-existence of subordinate solutions to the system. A characterisation of the absolutely continuous spectrum which relies on the transfer matrix was exhibited.

Clark (S. L. Clark, 1994) studied the absolutely continuous spectrum of a Dirac system with a matrix-valued long-range, short-range and oscillatory potentials. The system had one singular end point at infinity. Further conditions on the potentials were given that guaranteed particular asymptotic behaviour of an energy functional associated with a certain set of solutions. The asymptotic behaviour guaranteed the existence of a purely absolutely continuous spectrum outside a gap containing the origin.

Nenciu (Nenciu, 1976) investigated the self-adjointness and invariance of essential spectrum of Dirac operators perturbed by singular potentials. The interest was on the construction of a distinguished self-adjoint extension for such operators. The main criterion for this extension was that the wave functions in its domain possess finite mean kinetic energy thus ensuring physical relevance. The study generalised perturbation theory for non-semibounded self-adjoint operators by use of quadratic forms, providing an approach to define and show the uniqueness of the distinguished self-adjoint extension. This approach was more general than previous ones such as the pseudo-Friedrichs extension and covers a broader class of potentials, including those dominated by Coulomb-like potentials.

The key findings included the proof that the essential spectrum of the distinguished extension was contained within the spectrum of the free Hamiltonian and that the operator had a distinguished self-adjoint extension under some conditions on the potential. They also demonstrated that the resolvent of the perturbed operator could be expressed through a Born series and provided conditions under which the essential spectrum remained invariant.

The study covered general theory, the free-particle Dirac Hamiltonian, the perturbing potential and applications to Dirac operators. It concluded with a theorem which stated that for potentials dominated by Coulomb-like potentials or nonsingular potentials, there existed a unique self-adjoint extension with the essential spectrum contained within that of the free Hamiltonian. This findings extended earlier results and provided a more comprehensive basis for handling singular potentials in relativistic quantum mechanics.

Allahverdiev (Allahverdiev, 2003) constructed a space of boundary values for minimal symmetric Dirac operator with deficiency indices $(2, 2)$. All maximal dissipative (accretive), self-adjoint and other extensions of such symmetric operators are described in terms of boundary conditions at infinity. The maximal dissipative operators with separated boundary conditions were investigated. In particular, it was established that when separated boundary conditions are considered, at infinity the non self-adjoint boundary conditions are simultaneously described. The self-adjoint dilation and its incoming and outgoing spectral representations have been constructed hence making it possible to determine the scattering matrix of the dilation. Besides, a functional model of the dissipative operator was constructed and its characteristic function defined. The theorem on completeness of the system of eigenvectors and associated vectors of the dissipative Dirac operators have been proved.

Aygar et al (Aygar, Bairamov, & Yardımcı, 2015) investigated the spectral properties of a self-adjoint matrix-valued discrete Dirac system. They derived a polynomial-type Jost solution for the system and analysed its analytical properties and asymptotic behaviour. They illustrated that the continuous spectrum of the system filled the segment $[-2, 2]$ by use of the Weyl compact perturbation theorem. Moreover, they showed that the system had a finite number of simple real eigenvalues under certain conditions. The authors began with the introduction of the boundary value problem and definition of the Jost solution. The Jost solution plays an important role in spectral analysis. They then constructed the Jost solution for the discrete Dirac system and established its representation and convergence properties. They showed that the Jost solution had an analytic continuation and satisfied particular

asymptotic conditions.

The researchers also examined the continuous and discrete spectrum of the Dirac operator and proved that the continuous spectrum and the discrete spectrum consisted of a finite number of simple real eigenvalues. They used the properties of the Jost solution to demonstrate that the determinant of a related function had finite number of simple zeros that correspond to the eigenvalues of the system. In summary, the document provided a detailed analysis of the spectral properties of the matrix-valued discrete Dirac system, contributing to the understanding of such systems in the context of spectral theory. The results have potential applications in quantum scattering theory as well as areas where discrete systems are studied.

Eastham and Schmidt (M. S. Eastham & Schmidt, 2008) studied the asymptotics of the spectral density of one-dimensional Dirac systems on the half-line with an angular momentum term and a potential tending to infinity at infinity. The problem in this case had two singular end points. However, since the spectrum is simple, the derivative of the spectral matrix has only one non-zero eigenvalue which was taken to be the spectral density. Their main result showed that, assuming sufficient regularity of the potential, there are no points of spectral concentration for large values of the spectral parameter outside a neighbourhood of a discrete set of exceptional points.

Spectral analysis of Dirac operators on bounded domains has been studied by Benhellal (Benhellal, 2022a). Benhellal considered the coupling of the Dirac operator having delta-shell interactions of electrostatic combined with Lorentz scalar and magnetic type supported on regular compact surfaces or locally deformed hyperplanes. An approach based on regularisation techniques that allow description of the self-adjoint realization of the perturbed Dirac operator for a combination of the coupling constants was developed. They then investigated the qualitative spectral properties of various models by use of Birman-Schrodinger principle and a Krein-type formula to relate the perturbed operator to that of free Dirac operator. A lot of interest was on the case of critical combinations of coupling constants and those which result to the phenomenon of confinement.

Benhellal (Benhellal, 2022b) further studied the coupling of the Dirac operator with

non critical combinations of delta interactions supported on non-regular compact surfaces. The results obtained were generalized in the context of regular surfaces to the case of surfaces locally coincident with the graph of Lipschitz functions having a bounded gradient and mean oscillations that vanish. This was done by use of techniques from harmonic analysis, potential theory and Fredholm's theory. Besides, for the case of Hölder surfaces, they showed how smoothness of the surface supporting the delta interactions affect Sobolev regularity of the domain of the operator in consideration. Moreover, the Dirac operator coupled with the electrostatic and Lorentz scalar delta-shell interactions which are supported on uniformly rectifiable surfaces was studied. Under given certain conditions on the coupling constants, the self-adjointness of the perturbed operator was proved and various spectral properties for the Lipschitz case were established. The essential spectrum of the perturbed operator was determined and that at most a finite number of eigenvalues may appear in the gap. Besides, they linked the results to other delta-shell interactions and developed a number of models of Dirac operators which result to the confinement phenomenon.

Bairamov and Celebi (Bairamov & Okay Çelebi, 1999) worked on spectrum and spectral expansion for non self-adjoint discrete Dirac operators. They proved that under certain condition, the Dirac operator has finite number of eigenvalues and finite number of spectral singularities and that each one of them is of finite multiplicity. They obtained the integral representation for the Weyl function of the Dirac operator. Using that representation, they established the spectral expansion of the Dirac operator in terms of the principal vectors.

Arai (Arai, 1987) did asymptotic analysis of fundamental solutions of Dirac operators on even dimensional Euclidean spaces. The short distance asymptotic behaviour of quantities formed out of fundamental solutions of Dirac operators coupled minimally to a finite dimensional matrix-valued gauge potentials on even dimensional Euclidean space was analysed. The analysis gave mathematically rigorous proof as well as a mathematically self-consistent description to the 'chiral anomaly' in quantum gauge field theories.

Gu and Ma (Gu & Ma, 2020) investigated properties of solutions to Dirac operators

having gradient potentials, monogenic functions with respect to potential functions. They did establish a Borel-Pompeiu formula and obtained Cauchy integral formula as well as mean value formula about monogenic functions. The integral formulas were applied in the proof of geometric properties of monogenic functions with respect to potential functions and construction of integral transforms. Equally, the boundedness of integral transforms in Hölder space was shown. Moreover, Plemelj-Sokhotski formula and Painlevé theorem were proven. They showed that some kind of Riemann-Hilbert problem for monogenic functions with respect to the potential function is solvable. The solvability conditions of general Riemann-Hilbert problem for monogenic functions regarding the potential functions were established and some examples towards the Riemann-Hilbert problem given.

Fang et. al (Fang, Levitin, & Vassiliev, 2016) did spectral analysis of the Dirac operator on a 3-sphere and analysed the behaviour of eigenvalues for a case when the metric is perturbed in an arbitrary smooth fashion from the standard one. An explicit perturbation formula for the two eigenvalues which are closest to zero was derived. Fang showed that after first approximation, the two eigenvalues maintain symmetry about zero and are fully determined by the increment of Riemann volume. However, spectral symmetry was observed after second approximation of the perturbation process. As a reinforcement of these results, Fang illustrated an example considering a family of metrics, generalized Berger spheres whereby the eigenvalues can be explicitly evaluated.

Spectral properties of Dirac operators with singular potentials which are constructed by cut-off procedure were investigated by (Klaus & Wüst, 1979) and the focus was on the essential spectrum, norm resolvent convergence and spectral gaps. The authors extended earlier results by proving that the essential spectrum of the Dirac operator with a singular potential was the same as that of the free Dirac operator as long as the potential satisfied some given conditions. They also proved that the Dirac operator with a singular potential could be obtained as a norm resolvent limit of operators with cut-off potentials hence simplifying the analysis of such operators.

Detailed spectral gap formulas that improved on earlier results by considering the sign of the potential were demonstrated. Particularly, they showed that for poten-

tials with a negative part, the spectrum of the Dirac operator excludes the interval $(-1, (1 - \mu^2)^{\frac{1}{2}})$, while for potentials with a positive part, the spectrum excludes $(-(1 - \mu^2)^{\frac{1}{2}}, 1)$. These results were derived using monotonicity properties and perturbation theory, which are specifically interesting given that the Dirac operator is not semi-bounded.

Besides, the authors also discussed the case where the potential has both positive and negative parts, showing that the spectral gap could be determined by analysing the positive and negative components separately. The study concluded with a theorem summarizing the spectral properties of the Dirac operator with singular potentials hence providing a clear characterisation of the spectrum in terms of the potential's properties. This gave significant intuitions into the spectral theory of Dirac operators with singular potentials and extended earlier results as well as providing new tools for analysing such operators. The results have implications for the study of quantum systems with strongly singular potentials in relativistic quantum mechanics.

Puyda (Puyda, 2012) investigated the direct and inverse spectral problems for self-adjoint Dirac operators with matrix-valued potentials on the interval $(0, 1)$. The main interest was on operators with potentials with entries belonging to $L_p(0, 1)L$ for $p \in [1, \infty)$. The concept of spectral data, consisting of eigenvalues and norming matrices was introduced. This provided a complete description of these spectral data for the operators under consideration. The necessary and sufficient conditions for a sequence to be the spectral data of such an operator were established. The study extended previous results from the case of square-integrable potentials to a more generalised case of summable potentials, addressing several technical challenges that arise in the broader context. Puyda introduced an auxiliary object, a restriction of the Fourier transform of the spectral measure which played an important role in characterising the spectral data. The study also developed a method for reconstructing the potential from the spectral data, based on solving a certain integral equation referred to as the Krein equation.

The key results included a detailed description of the spectral data, a proof of the uniqueness of the potential corresponding to given spectral data and an efficient algorithm for reconstruction of the potential. The study was theoretical but has

practical implications, especially in fields such as quantum graphs, where inverse problems for operators with matrix-valued potentials are relevant. Therefore, the study provided a comprehensive solution to the inverse spectral problem for Dirac operators with summable matrix-valued potentials, extending earlier results and providing new insights and techniques for handling the challenges posed by more general potentials.

Clark and Gesztesy (S. Clark & Gesztesy, 2002) did a comprehensive study of the asymptotic behaviour of Weyl-Titchmarsh m -functions associated with Dirac operators both on half-lines and on the whole real line. They derived high energy asymptotics for these m -functions and proved that they exhibit leading order behaviour of the form in certain sectors of the complex plane. This result was extended to higher order asymptotic expansions which were derived under optimal smoothness conditions on the potential matrix $B(x)$. Besides, they established new local uniqueness results which showed that exponentially small differences in the Weyl-Titchmarsh matrices implied the equality of the potential matrices $B(x)$ on intervals. These results were then applied to prove trace formulas and Borg-type theorems for Dirac operators, which are the analogs of classical uniqueness results in spectral theory. The study was concluded with applications to periodic potential thus demonstrating that if a Dirac operator is reflectionless and has spectrum equal to the whole real line, then the potential $B(x)$ must vanish almost everywhere. The methods employed in the study included the use of Riccati equations and Volterra integral equations and therefore provided a good framework for analysing the spectral properties of Dirac operators and extending earlier results from scalar Schrödinger operators to the matrix-valued case. Generally, the study made huge contributions to the inverse spectral theory of Dirac operators, giving new insights into their asymptotic behaviour and uniqueness properties.

Sakhnovich (Sakhnovich, 2002) solved direct spectral problem for matrix Dirac-type systems with locally summable potentials as well as inverse spectral problem for matrix Dirac systems with locally bounded potentials on the interval and on the half-line. A direct procedure for recovering the potential by use of Weyl-Titchmarsh function was given. Some results on the high-energy asymptotics of the

Weyl-Titchmarsh functions and the local uniqueness for the corresponding inverse problem were given. Moreover, new results for the general type canonical systems were obtained.

Mamedov et al (Mamedov & Akçay, 2014) looked at a problem for the first order canonical Dirac differential equations system having piecewise continuous coefficient and a spectral parameter dependent on the boundary condition. They investigated the asymptotic behaviour of eigenvalues, eigenfunctions and normalising numbers for the system. Spectral expansion formula regarding eigenvector functions was obtained. Also, the Weyl solution and Weyl function for the problem were constructed and uniqueness theorem for the inverse problem by the Weyl function and by the spectral data were proved.

In the year 2017, Ackay and Mamedov (Akçay & Mamedov, 2017) investigated the inverse spectral problem for Dirac operators with piecewise continuous coefficients and a spectral parameter contained in the boundary condition. They established the necessary and sufficient conditions for the solvability of the inverse problem and provided an algorithm for reconstructing the potential function from a given spectral data. Detailed proofs and lemmas that led to the main theorem were presented. The main result stated the necessary and sufficient conditions for the sequences to be the spectral data of a specific boundary value problem. An algorithm was provided for construction of the potential function from the spectral data. This involved the construction of functions hence solving the main equation and finally calculation of the potentials.

A summary of the findings, with emphasis of the importance of the derived conditions and the provided algorithm in solving inverse problems for Dirac operators was provided. The results contributed greatly to the field by offering a clear and systematic way to reconstruct potentials from spectral data, which has implications in both theoretical and applied physics. The rigorous mathematical treatment and the detailed proofs ensured the reliability and applicability of the proposed methods.

Martin (Martin, 2003) showed that the absolutely continuous spectrum of Dirac operators on half-line with square integrable potentials fills the whole real axis.

They also established an estimate on the number of eigenvalues for Coulomb-like potentials.

Baskakov, Krishtal and Uskova (Baskakov, Krishtal, & Uskova, 2021), studied the asymptotics of the spectrum of the Dirac operator on the real line with potential in L_2 . They showed that the spectrum of an operator of this type lies in the domain of the complex plane symmetric about the real axis and is bounded by the graph of some continuous real-valued square integrable function. They proved this by use of L_1 functional calculus for self-adjoint operators and some suitable similarity transformation.

The asymptotic behaviour of the fundamental solutions of Dirac system (1.4) has been investigated by the authors, see [(Gomilko & Rzepnicki, 2020),(Rzepnicki, 2021)]. As such, a variety of asymptotic formulae for the solutions to the system (1.4) in different settings and under a wide range of assumptions on the potentials have been generated. Rzepnicki (Rzepnicki, 2021) considered the Dirac system on the interval $[0, 1]$ with spectral parameter $\mu \in \mathbb{C}$ and a complex valued potential with entries from the space $L_p[0, 1]$ for $1 \leq p$. The study of the asymptotic behaviour of its solutions was done in the strip $|Im\mu| \leq d$ for $|\mu| \rightarrow \infty$. From the results, sharp asymptotic formulas for eigenvalues and eigengfunctions of Sturm-Lioville operators associated with the continuous Dirac system with integrable potentials were obtained.

In this study, we have considered the discretised version of the Dirac system (1.4). The asymptotic behaviour of solutions of the discretised Dirac system with bounded and unbounded potentials has been analysed by use of asymptotic summation based on the discretised Levinson's theorem.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Introduction

In this chapter, we have outlined the methodological framework and theoretical tools employed to achieve the objectives of this study. The research focuses on the stability analysis of positive self-adjoint operator extensions of symmetric difference operators under unbounded perturbations and the asymptotic behavior of solutions of discretised Dirac operators. The methodology is rooted in functional analysis, operator theory, and discrete spectral theory, with a particular emphasis on the application of asymptotic summation techniques and the Levinson-Benzaid-Lutz theorem. We have stated the procedure that we have used and core theorems that were required to evaluate deficiency indices of the difference operator, prove the dichotomy conditions and investigate the spectral properties of order four difference operators. In addition, we have explained the transformation required to convert any higher order equation to appropriate first order and obtained preliminary formulation needed to prove main results. We did set some growth conditions on the coefficients of L_1 in equation (1.1) and L_2 in equation (1.2) which equally affect those of L in equation (1.3). Suppose that zL_2 , the perturbing term, is L_1 bounded then there exists $a > 0$ such that $\|zL_2\| < a\|L_1\|$ for $0 < a < 1$. It has been shown that discreet spectrum and deficiency indices are stable under bounded perturbations (Behncke & Focke, 1977). Thus, if the coefficients of L_1 dominate those of L_2 , then it would be interesting to check if positivity of L will remain the

same as positivity of L_1 and whether deficiency indices of L will be equal to deficiency indices of L_1 for a finite $z \in \mathbb{R}$. Moreover, we were interested in finding out if the point spectrum of self-adjoint operator extension of L will be equal to the point spectrum of self-adjoint operator extension of L_1 . In order to solve symmetric difference equations generated by L and L_1 , we have used asymptotic summation based on Levinson-Benzaid-Lutz's theorem. The original theorem was first formulated by Benzaid and Lutz in 1987 (Benzaid & Lutz, 1987). However, the theorem has undergone significant transformation through either strengthening or weakening dichotomy or decay conditions.

This Chapter is organized as follows: Section 3.2 introduces the concept of asymptotic summation and its application in the study of difference operators. Section 3.3 discusses the system formulation process, which involves converting higher-order difference equations into first-order systems. Section 3.4 introduces the Cayley transform, a crucial tool for analyzing symmetric operators and their extensions. Section 3.5 details the construction of positive symmetric difference operators while Section 3.6 explores the spectral properties of self-adjoint operator extensions. Section 3.7 presents von Neumann's extension theory, which is fundamental to understanding the existence of self-adjoint extensions. Section 3.8 discusses the existence of positive self-adjoint extensions and the Krein-von Neumann extensions. Finally, Section 3.9 focuses on the methodology for analyzing Dirac operators, particularly in the context of unbounded potentials.

3.2 Asymptotic Summation

Asymptotic summation is a powerful tool used to analyse the behaviour of solutions to difference equations, especially when the coefficients are unbounded. The technique is based on the Levinson-Benzaid-Lutz theorem, which provides conditions under which the solutions of a perturbed system look like those of an unperturbed system.

Levinson's Theorem is one of the most useful result in the asymptotic integration of the linear differential equations. Asymptotic integration is usually aimed at deter-

mining the asymptotics of the eigenfunctions of differential operators. This theorem states that the solutions of a system

$$\acute{u}(x) = \{\Lambda(x) + R(x)\}u(x), \quad \Lambda(x) = \text{diag}(\lambda_i(x))$$

look like those solutions of the unperturbed system

$$\acute{u} = \Lambda u$$

whenever $R(x)$ is sufficiently small, that is, absolutely integrable and $\Lambda(x) = \text{diag}(\lambda_i(x))$ satisfies a dichotomy condition. In this particular case, the dichotomy condition amounts to: $\text{Re}\lambda_i(x)$ and $\text{Re}[\lambda_k(x) - \lambda_j(x)]$ for $k \neq j$, have nearly constant sign modulo $\mathcal{L}^1$ for a large x . This theorem has since been extended in various directions by modifying the dichotomy conditions as well as by allowing perturbations which are referred to as Levinson's terms. In the field of spectral theory, the matrix elements and $\lambda_i(x)$ generally depend on some spectral parameter w and therefore one would write $\lambda_i = \lambda_i(x, w)$.

Theorem 3.2.1 (Levinson's Theorem ((Behncke & Remling, 1997),(Behncke & Remling, 2001) and (Remling, 1999))). *Let $\Lambda(x, w) = \text{diag}(\lambda_1(x, w), \dots, \lambda_{2n}(x, w))$ and $R(x, w)$ be $2n \times 2n$ matrices which for all x , are analytic functions of $w \in \Omega \subset \mathbb{C}$. For any unequal pair of indices i and j , $i, j \in [1, \dots, 2n]$, assume that $\Lambda(x, w) = \text{diag}(\lambda_1(x, w), \dots, \lambda_{2n}(x, w))$ satisfies dichotomy condition uniformly in w , that is, for every unequal pair i, j , $a \leq t \leq x < \infty$, we have that $\text{Re}\{\lambda_i(x, w) - \lambda_j(x, w)\}$ has a constant sign modulo $\mathcal{L}^1([a, \infty))$ for all $w \in \Omega$. Moreover, assume that $\|R(x, w)\| \leq \rho(x)$ with $\rho \in \mathcal{L}^1([a, \infty))$. Then*

$$\acute{Y}(x, w) = [\Lambda(x, w) + R(x, w)]Y(x, w)$$

has solutions $y_k(x, w)$, $1 \leq k \leq 2n$, with asymptotic form

$$y_k(x, w) = [e_k + r_k(x, w)] \cdot \exp\left[\int_a^x \lambda_k(t, w) dt\right]$$

where e_k denotes the k^{th} unit vector and $r_k(x, w)$ depends analytically on $w \in \Omega$ and tends to 0 w -uniformly as x tends to infinity.

The proof of this Theorem follows closely from that of Theorem 1.3.1 in (M. S. P. Eastham, 1989). Similar proofs can also be found in [(Behncke & Remling, 1997),(Behncke

& Remling, 2001) and (Remling, 1999)]. Levinson's theorem therefore gives the form of solutions of the perturbed system whenever the eigenvalues satisfy uniform dichotomy condition and the perturbing terms are sufficiently small.

Levinson's theorem gives the conditions under which the solutions of a perturbed system resemble those of an unperturbed system, particularly when the perturbing term is "small" in the operator sense. The dichotomy condition is a key assumption in the theorem. It requires that the real parts of the differences between the eigenvalues $\lambda_i(x, w)$ and $\lambda_j(x, w)$ have a constant sign (either positive or negative) modulo an integrable function. This ensures that the eigenvalues do not interfere with each other in a way that would destabilise the system. The uniformity in w implies that the condition holds for all values of the spectral parameter w in the domain Ω . The perturbation $R(x, w)$ represents a small deviation from the unperturbed system $Y(x, w) = \Lambda(x, w)Y(x, w)$. The requirement here is that the norm of $R(x, w)$ is bounded by an integrable function $\rho(x)$ and as such, the perturbation decays sufficiently fast as x tends to infinity.

The solutions $y_k(x, w)$ of the perturbed system are expressed in terms of the solutions of the unperturbed system with an additional correction term $r_k(x, w)$. The correction term tends to zero as x tends to infinity implying that the solutions of the perturbed system asymptotically look like those of the unperturbed system. The solutions $y_k(x, w)$ depend analytically on the spectral parameter w . This is crucial for applications in spectral theory where the behaviour of solutions is studied as functions of w .

From the above result, one can therefore expect that a number of results for linear systems extend from differential calculus set up to difference calculus set up. This is true because Benzaid-Lutz results are the difference equations counterparts of Levinson's Theorem. The results are normally concerned with systems of linear difference equations of the form

$$Y(t + 1, w) = A(t)Y(t, w) \text{ for } t \geq t_0$$

and $A(t)$ is an invertible square matrix. Here, we have stated the spectral parameter

version of the theorem as given in (Benzaid & Lutz, 1987). This version has been applied widely in obtaining asymptotic behaviour of solutions and spectral results of higher difference equations and associated operators.

Theorem 3.2.2. (*Levinson's-Benzaid-Lutz Theorem (Benzaid & Lutz, 1987)*) Let

$$\Lambda(t, w) = \text{diag}\{\lambda_1(t, w), \dots, \lambda_{2n}(t, w)\} \quad \text{for} \quad t \geq t_0. \quad (3.1)$$

Assume

- (i) $\lambda_i(t, w) \neq 0$ for all $1 \leq i \leq 2n$ and $t \geq t_0$
- (ii) $R(t, w)$ is a $2n \times 2n$ matrix defined for all $t \geq t_0$, satisfying $\sum_{t=t_0}^{\infty} \left| \frac{1}{\lambda_i(t, w)} \right| \|R(t, w)\| < \infty$, for all $i = 1, 2, \dots, 2n$
- (iii) $\Lambda(t, w)$ satisfies the following uniform dichotomy condition. For any pair of indices i and j , such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t, w)| \geq \delta$ for all $t \geq t_0$. Then, either $\left| \frac{\lambda_i(t)}{\lambda_j(t)} \right| \geq 1$ or $\left| \frac{\lambda_i(t)}{\lambda_j(t)} \right| \leq 1$ for a large t .

Then the linear system

$$Y(t+1, w) = [\Lambda(t, w) + R(t, w)]Y(t, w)$$

has a fundamental matrix satisfying, as $t \rightarrow \infty$

$$Y(t, w) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l, w).$$

The proof of this theorem closely follows from Lemma 2.1 and Theorem 3.3 in (Benzaid & Lutz, 1987).

The dichotomy condition is an essential assumption in the theorem. It requires that the ratios of the eigenvalues $\lambda_i(t, w)$ and $\lambda_j(t, w)$ are either bounded above by $1 - \delta$ or below by $1 + \delta$ for large t . This ensures that the eigenvalues do not interfere with each other in a way that the system would be unstable. The condition $|\lambda_i(t, w)| \geq \delta$ ensures that the eigenvalues do not become too small, which could lead to instability.

Besides, the perturbation $R(t, w)$ must be small such that its norm, weighted by the inverse of the eigenvalues, is summable. This ensures that the perturbation decays sufficiently fast as t tends to infinity hence enabling the solutions of the perturbed system to asymptotically look like those of the unperturbed system. The fundamental matrix $Y(t, w)$ is a matrix whose columns are linearly independent solutions of the system. The theorem says that the fundamental matrix of the perturbed system asymptotically look like that of the unperturbed system up to a correction term $\circ(1)$ that tends to zero as t tends to infinity.

The theorem is a crucial tool in the technique of asymptotic summation which is used to determine the deficiency indices and spectral properties of higher-order symmetric difference operators. By applying the Levinson-Benzaid-Lutz Theorem, we have analysed the asymptotic behaviour of solutions to these operators especially when the coefficients are unbounded.

The theorem says that the solutions of a system

$$Y(t+1, w) = [\Lambda(t) + R(t)]Y(t, w), \quad (3.2)$$

where $\Lambda(t)$ is diagonal and invertible, look like the solutions of the unperturbed system

$$Y(t+1, w) = \Lambda(t)Y(t, w), \quad (3.3)$$

if $R(t)$ is sufficiently small, that is, absolutely summable and $\Lambda(t) = \text{diag}(\lambda_i(t))$ satisfies a dichotomy condition. The dichotomy condition here states that: for any pair of indices i and j , such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ so that $|\lambda_i(t)| \geq \delta$ for every $t \geq t_0$, then either $|\frac{\lambda_i(t)}{\lambda_j(t)}| \geq 1 + \delta$ or $|\frac{\lambda_i(t)}{\lambda_j(t)}| \leq 1 - \delta$ for a large t . In the spectral theory of difference operators, the matrix elements and $\lambda_i(t)$ generally depend on the spectral parameter w . Hence we write $\lambda_i(t) = \lambda_i(t, w)$. In this case, the form of solutions shall be given by

$$y_k(t, w) = [e_k(t, w) + r_{kk}(t, w)] \prod_{l=t_0}^{t-1} \lambda_k(l, w)$$

which is continuous in w , the spectral parameter, with $r_{kk}(t, w) \rightarrow 0$ as $t \rightarrow \infty$ uniformly for w . The proof of dichotomy condition has been greatly simplified as

given in the results of Behncke (Behncke, 2010a). The theorem below, which is the core result of Behncke's paper (Behncke, 2010a) had impacted on the proof of dichotomy condition.

Theorem 3.2.3. *Let*

$$Y(t+1, w, z) = [\Lambda(t, w, z) + R(t, w, z)]Y(t, w, z) \quad t \geq t_o, \quad (3.4)$$

$$\Lambda(t, w, z) = \text{diag}(\lambda_1(t, w, z), \dots, \lambda_{2n}(t, w, z))$$

be asymptotically constant difference equation such that

$$\sum_{t=t_o}^{t-1} \|R(t, w, z)\| |\lambda_i^{-1}(t, w, z)| < \infty.$$

Assume the eigenvalues $\lambda_i(t, w, z)$ for $i = 1, \dots, 2n$ satisfy $\lambda_i(t, w, z) = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2}$, with $\lambda_{i,0}$ constant, $\lambda_{i,1}(t, w, z) \rightarrow 0$ as $t \rightarrow \infty$, $\lambda_{i,2}$ is conditionally summable and $\lambda_{i,0}$ distinct. Let $h(t) > 0$ be a nonsummable, monotonic function on $\mathbb{N}$ and assume the eigenvalues can be sorted into classes $C_1, \dots, C_n$ so that if $\lambda_i(t, w, z), \lambda_j(t, w, z) \in C_k$, then $(\frac{|\lambda_i(t, w, z)|}{|\lambda_j(t, w, z)|} - 1) = o(h(t))$; if $\lambda_i(t, w, z) \in C_k, \lambda_j(t, w, z) \in C_l, k \neq l$ then $(\frac{|\lambda_i(t, w, z)|}{|\lambda_j(t, w, z)|} - 1) \leq 1 - h(t)$ or $(\frac{|\lambda_i(t, w, z)|}{|\lambda_j(t, w, z)|} - 1) \geq 1 + h(t)$. For each $\lambda(t, w, z)$, now write $|\lambda(t, w, z)| = 1 + \nu(t, w, z)$ with $\nu_+ = \max(0, \nu)$ and $\nu_- = \min(0, \nu)$ and for each class k , define,

$$a_k(t, w, z) = \max \nu_+(t, w, z) \text{ and } b_k(t, w, z) = \max \nu_-(t, w, z).$$

The above conditions are assumed to hold uniformly for $w \in K$. Then associated to each C_k there are $|C_k|$ ($|C_k|$ is the number of elements in the k^{th} class) solutions satisfying

$$K_1 \prod_{s=t_o}^{t-1} (1 - b_k(s, w, z)) \leq \|y(t, w, z)\| \leq K_2 \prod_{s=t_o}^{t-1} (1 + a_k(s, w, z))$$

for all $t \geq t_o$.

Theorem (3.2.3) implies that those eigenvalues $\lambda_k(t, w, z)$ such that $|\lambda_k(t, w, z)| > 1$, for $t \geq t_o$ lead to eigensolutions which are uniformly non-square summable whereas the eigenvalues $|\lambda_k(t, w, z)| < 1$ lead to solutions which are uniformly square

summable. These sets of eigenvalues need no establishment of uniform dichotomy conditions. However, the eigenvalues that lie on the unit circle on the complex plane requires the establishment of uniform dichotomy condition. This results has since been extended in various directions by either strengthening the dichotomy conditions and weakening the decay conditions of $R(t)$ and vice-versa. Just like in the continuous case, in the spectral theory of difference operators, the matrix elements and $\lambda_i(t)$ generally depend on the spectral parameter w and hence $\lambda_i = \lambda_i(t, w)$.

Asymptotic summation is a way of finding the forms of solutions of a higher order symmetric difference system. It involves, first system formulation from higher order to first order by use of quasi-differences defined in (Y. Shi, 2006). This is followed by computation of the characteristic polynomial and hence the characteristic equation. Under some given growth conditions, the eigenvalues of the characteristic equation are computed and uniform dichotomy conditions established. The dichotomy conditions are fundamental for proving stability and convergence results. The spectral dichotomy condition guarantees that the solutions to the equations associated with the operators in question exhibit stable behavior in terms of their spectral components. Then by spectral analysis, deficiency indices are obtained. The eigenvectors associated with each eigenvalue are computed and normalized and hence the diagonalising or transforming matrix obtained from those normalised eigenvectors. Repeated diagonalisations are performed to obtain the remainder matrix with the correction terms, to the eigenvalues which are absolutely summable as the independent parameter tends to infinity. However, it is possible that there may be correction terms not yet summable after second diagonalisation and hence the need for $[I + Q]$ -transformation which is applied so as to reduce such off-diagonal terms into summable terms.

3.3 System Formulation

Suppose that one is given an even higher order difference equation of the form

$$Ly(t) = \sum_{k=0}^n (-1)^k \Delta^k [p_k(t) \Delta^k y(t - k)] \quad (3.5)$$

on $\ell^2(\mathbb{N})$. In order to solve the equation $Ly(t) = 0$, one needs to convert equation (3.5) into its first order system. This is done by use of quasi-differences of the form

$$x_k(t) = \Delta^{k-1}y(t-k) \quad (3.6)$$

$$u_k(t) = \sum_{l=k}^n (-1)^{l-k} \Delta^{l-k} [p_l(t) \Delta^l y(t-l)]$$

for $k = 1, 2, \dots, n$. Then for the vector valued functions

$$x(t) = [x_1(t), x_2(t), \dots, x_n(t)]^{tr} \text{ and}$$

$$u(t) = [u_1(t), u_2(t), \dots, u_n(t)]^{tr}$$

where tr denotes transpose, the forward difference results into

$$\Delta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix} \quad (3.7)$$

with the non-zero matrix elements of $n \times n$ matrices A, B and C given by

$$A_{k,k+1} = 1, B_{n,n} = p_n^{-1}, C = \text{diag}(p_0, p_1, \dots, p_{n-1})$$

and all the remaining elements are zeros. A direct formulation to convert equation (3.7) into its first order form results into

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = S(t) \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} E & EB \\ CE & I_n - A^* + CEB \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad (3.8)$$

where $S(t) = \begin{bmatrix} E & EB \\ CE & I_n - A^* + CEB \end{bmatrix}$ and $E = [I_n - A]^{-1}$. One can therefore, compute explicitly the matrix entries of $S(t)$. Computation of the associated characteristic polynomial gives

$$\det(S - \lambda \cdot I_{2n}) = \mathcal{P}(t, \lambda).$$

Since the interest is on the zeros of the polynomial $\mathcal{P}(t, \lambda)$, one may equate it to zero and then do the right scaling by multiplying by $(-1)^n \frac{p_n}{\lambda^n}$ to obtain a Fourier polynomial

$$\mathcal{F}(t, \lambda) = \sum_{k=0}^n p_k (1 - \lambda)^k (1 - \lambda^{-1})^k$$

which is symmetric and hence if λ is a root, then λ^{-1} is also a root. This implies that if λ is replaced with λ^{-1} in the polynomial then we obtain the same polynomial.

3.4 The Cayley Transform of an Operator

Let λ denote a fixed complex number such that $\operatorname{Im}\lambda > 0$. Also let $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. The Möbius transform

$$t \rightarrow (t - \lambda)(t - \bar{\lambda})^{-1}$$

usually maps the real axis onto $\mathbb{T} \setminus \{1\}$ and the upper (respectively lower) onto the set inside (respectively outside) of $\mathbb{T}$. The Cayley transform is the operator analog of the Möbius transformation. Usually, it does map densely defined symmetric operator onto isometric operator V with which $\mathcal{R}(I - V)$ is dense. Therefore, the Cayley transform relates symmetric operators and isometric operators.

An isometric operator is a linear operator V on a Hilbert space $\mathcal{H}$ so that $\|Vx\| = \|x\|$ for all $x \in D(V)$. Using polarisation formula, we have that $\langle Vx, Vy \rangle = \langle x, y \rangle$ for all $x, y \in D(V)$. The domain $D(V)$ may be an arbitrary linear subspace of the Hilbert space $\mathcal{H}$. If V is isometric, it is invertible and its inverse V^{-1} is also isometric. Therefore, V is closed if and only if $D(V)$ is a closed linear subspace of the Hilbert space $\mathcal{H}$.

Suppose that V is an isometric operator on the Hilbert space $\mathcal{H}$. From the inequality

$$\|(V - \lambda I)x\| \geq \| \|Vx\| - |\lambda|\|x\| \| = \|1 - |\lambda|\| \|x\|$$

for $x \in D(V)$ and $\lambda \in \mathbb{C}$, we have that $\mathbb{C} \setminus \mathbb{T} \subseteq \Pi(T)$. Hence by the polarisation identity [(Schmüdgen, 2012), Proposition 2.4], the defect numbers of V are constant on the interior of $\mathbb{T}$ and on the exterior of $\mathbb{T}$. The cardinal numbers

$$d^i(V) := d_\lambda(V) = \mathcal{R}(V - \lambda I)^\perp$$

for $|\lambda| < 1$ and

$$d^e(V) := d_\lambda(V) = \mathcal{R}(V - \lambda I)^\perp$$

for $|\lambda| > 1$ are referred to as deficiency indices of the isometric operator V . The best description of these numbers is given in the following results:

Lemma 3.4.1. ((Schmüdgen, 2012), Lemma 13.1) $d^i(V) = \dim \mathcal{R}(V)^\perp$ and $d^e(V) = \dim \mathcal{D}(V)^\perp$.

The deficiency indices are important when it comes to understanding the self-adjoint extensions of symmetric operators. From this result, we establish that if V is an isometric operator, then its deficiency indices determine whether V can be extended to a unitary operator, a particular case of a self-adjoint operator. This lemma is applied in the analysis of the Cayley transform of symmetric operators, which map symmetric operators to isometric operators. The deficiency indices of the isometric operator V are required to determine the existence and properties of self-adjoint extensions of the original symmetric operator. It provides an approach to compute the deficiency indices of an isometric operator V by examining the orthogonal complements of its range and domain. The indices are key for understanding the spectral properties and self-adjoint extensions of the operator. The lemma is used in the analysis of symmetric operators and their extensions especially in the study of the Cayley transform.

Lemma 3.4.2. *((Schmüdgen, 2012), Lemma 13.2) If V is an isometric operator on a Hilbert space $\mathcal{H}$ and $\mathcal{R}(I - V)$ is dense in $\mathcal{H}$, then $\mathcal{N}(I - V) = \{0\}$.*

The lemma says that if V is an isometric operator and the range of $(I - V)$ is dense in $\mathcal{H}$, then the only solution to $Vx = x$ is the zero vector. This is to say that the only vector x that satisfies $Vx = x$ is $x = 0$. It is very key in the context of operator theory especially when dealing with symmetric operators and their extensions. It aids in understanding the structure of the null space of some operators that is crucial when it comes to the determination of the deficiency indices and the existence of self-adjoint extensions. Particularly, it ensures that the operator $(I - V)$ does not have any non-trivial solutions in its null space under some given conditions. This usually simplifies the analysis of the spectral properties of the operator. The lemma is foundational in the study of symmetric operators and their extensions, particularly in the context of von Neumann's extension theory.

Let T be a densely defined symmetric linear operator on the Hilbert space $\mathcal{H}$. Given that $\text{Im}\lambda > 0$, by the results in [(Schmüdgen, 2012), Proposition 3.2i] we have that $\bar{\lambda} \in \Pi(T)$. Therefore, $T - \bar{\lambda}I$ is invertible. The Cayley transform of T is the operator

$$V_T = (T - \lambda I)(T - \bar{\lambda}I)^{-1}$$

with domain

$$D(V_T) = \mathcal{R}(T - \bar{\lambda}I).$$

Hence, V_T is defined by

$$V_T(T - \bar{\lambda}I)x = (T - \lambda I)x$$

for $x \in D(T)$.

3.5 Construction of Positive Symmetric Difference Operator

Positive symmetric difference operators are essential in the study of stability and spectral properties. A difference operator L is said to be positive if: $\langle Ly(t), y(t) \rangle \geq 0$ for all $y(t) \in \ell^2(N)$. To construct such operators, we consider the Hilbert space $\mathcal{H}$, the space of square summable sequences $x = x_n$ for $n \in \mathbb{Z}$ where

$$\|x\|^2 = \sum_{n \in \mathbb{Z}} |x_n|^2 < \infty.$$

Define a second order finite difference operator as:

$$(Lx)_n = x_{n+1} - 2x_n + x_{n-1}$$

for all $n \in \mathbb{Z}$. The domain of L is described as follows:

$$D(L) = \{x \in \mathcal{H} \mid Lx \in \mathcal{H}\}.$$

We need to show that L is symmetric. This requires that we compute the inner product $\langle Lx, y \rangle$ and show that it is equal to $\langle x, Ly \rangle$ for all $x, y \in D(L)$. By use of summation, we have that

$$\langle Lx, y \rangle = \sum_{n \in \mathbb{Z}} \bar{y}_n (x_{n+1} - 2x_n + x_{n-1}). \quad (3.9)$$

When the right hand side of equation (3.9) is split into three sums, we obtain that

$$\langle Lx, y \rangle = \sum_{n \in \mathbb{Z}} \bar{y}_n x_{n+1} - 2 \sum_{n \in \mathbb{Z}} \bar{y}_n x_n + \sum_{n \in \mathbb{Z}} \bar{y}_n x_{n-1}. \quad (3.10)$$

Using change of indices in the first and last sums in equation (3.10), we obtain that

$$\langle Lx, y \rangle = \sum_{n \in \mathbb{Z}} \bar{x}_n (y_{n+1} - 2y_n + y_{n-1}) = \langle x, Ly \rangle.$$

Hence the operator L is symmetric. We then check on its positivity by computing explicitly $\langle Lx, x \rangle$ for all $x \in D(L)$. Then

$$\langle Lx, x \rangle = \sum_{n \in \mathbb{Z}} \bar{x}_n (x_{n+1} - 2x_n + x_{n-1}).$$

Expanding and rearranging, we have that

$$\langle Lx, x \rangle = \sum_{n \in \mathbb{Z}} |x_{n+1} - x_n|^2 > 0.$$

Since $|x_{n+1} - x_n|^2 > 0$, the operator L is positive.

Positive symmetric operator can also be constructed by use of the Cayley transform. This would require a unitary operator U whose spectrum lies in the right half of the unit circle in the complex plane. This condition ensures that the corresponding symmetric operator will be positive. The inverse Cayley transform is applied to obtain the symmetric operator

$$L = i(I + U)(I - U)^{-1}.$$

The operator L will be positive symmetric, meaning that $\langle Lx, x \rangle \geq 0$ for all $x \in D(L)$. In this particular case, the domain of L is given by

$$D(L) = \{x \in \mathcal{H} \mid (I - U)x \in \mathcal{H}\}.$$

As such, the operator L is densely defined and symmetric.

3.6 Spectral Properties of Self-adjoint Operator Extension

Deficiency Indices

The understanding of deficiency indices is very important when one requires to determine the spectral properties of the operator and identifying possible self-adjoint extensions, which are key in a number of applications, including but not limited to quantum mechanics and numerical analysis.

The deficiency indices of a difference operator are computed by analyzing the solutions to the adjoint operator's equation in the complex plane. The process generally involves definition of the difference operator, finding the adjoint of the operator, solving the deficiency equation, checking for square summability of the solutions and lastly, determining self-adjointness.

Consider a symmetric difference operator L acting on the Hilbert space $\mathcal{H}$, of the form:

$$(Ly)_n = a_n y_{n+1} + b_n y_n + c_n y_{n-1}$$

where a_n, b_n and c_n are sequences defining the operator. The adjoint L^* of L is determined by analyzing the inner product:

$$\langle Ly, z \rangle = \langle y, L^* z \rangle \quad (3.11)$$

for all $y \in D(L)$ and $z \in D(L^*)$. Identification of L^* usually involves extending the domain of L to include those functions for which the relationship (3.11) holds. This always requires that boundary conditions are set.

One then solves the equation

$$L^* y = \lambda y \quad (3.12)$$

for $\lambda = i$ and $\lambda = -i$ where i is the imaginary unit. The number of square summable solutions to the equation (3.12) are referred to as the deficiency indices. The next step is usually the identification of solutions y_n to the deficiency equations that stay square summable, that is, they belong to the Hilbert space $\mathcal{H}$. This requires that

$$\sum_n |y_n|^2 < \infty.$$

The number of linearly independent, square summable solutions for $\lambda = i$ and for $\lambda = -i$ are denoted by d_+ and d_- respectively. These numbers are the deficiency indices of the operator L . The deficiency indices (d_+, d_-) provides an insight about the self-adjointness of L : whether L is essentially self-adjoint, has self-adjoint extensions or does not have self-adjoint extensions.

Let L be a closed linear operator on a Hilbert space $\mathcal{H}$ and λ be in the regularity domain of L such that there exists a number $c_\lambda > 0$ and $\|(L - \lambda I)x\| > c_\lambda \|x\|$ for

all $x \in D(L)$. The deficiency subspace of L at λ is the linear subspace $R(L - \lambda I)^\perp$ of H and its dimension

$$d_\lambda(L) = \dim R(L - \lambda I)^\perp \quad (3.13)$$

is the deficiency index of L at λ . Let L^* be the Hilbert adjoint of L . Then

$$\begin{aligned} R(L - \lambda I)^\perp &= N((L - \lambda I)^*) \\ &= N(L^* - \bar{\lambda} I) \end{aligned}$$

Hence the deficiency index of L at λ can also be defined using the nullspace of L^* as

$$\begin{aligned} d_\lambda(L) &= \dim R(L - \lambda I)^\perp \\ &= \dim N(L^* - \bar{\lambda} I) \end{aligned}$$

3.7 Von Neumann's Extension Theory of Symmetric Operators

The symmetric extension of an operator T is an operator S acting on the same Hilbert space $\mathcal{H}$ so that $T \subseteq S$. The self-adjoint operator S on the Hilbert space $\mathcal{H}$ refers to an extension of a densely defined symmetric operator T if and only if S is some restriction of the adjoint operator T^* . Given $T \subseteq S$, we have that $S = S^* \subseteq T^*$. On the other hand, if $S \subseteq T^*$, then we have that $S = S^* \supseteq T^{**} = \bar{T} \supseteq T$. This means that any self-adjoint extension S of T is fully described by its domain $D(S)$ given that $S = T^* \upharpoonright D(S)$.

Theorem 3.7.1. (*(Schmüdgen, 2012), Theorem 13.9*) Let T be a densely defined symmetric operator on $\mathcal{H}$. Suppose that $\mathcal{G}_+ \subseteq \mathcal{N}(T^* - \lambda I)$ and $\mathcal{G}_- \subseteq \mathcal{N}(T^* - \bar{\lambda} I)$ are closed linear subspaces on the Hilbert space $\mathcal{H}$ such that $\dim \mathcal{G}_+ = \dim \mathcal{G}_-$ and U is an isometric linear mapping of $\mathcal{G}_+$ onto $\mathcal{G}_-$. Define

$$D(T_U) = D(\bar{T}) \dot{+} (I - U)\mathcal{G}_+ \text{ and } T_U = T^* \upharpoonright D(T_U),$$

that is

$$T_U(x + (I - U)y) = \bar{T}x + \lambda y - \bar{\lambda}Uy$$

for all $x \in D(\bar{T})$ and $y \in \mathcal{G}_+$. Then T_U is a closed symmetric operator such that $T \subseteq T_U$. Any closed symmetric extension of T on $\mathcal{H}$ is of this form. Moreover, $d_{\pm}(T) = d_{\pm}(T_U) + \dim \mathcal{G}_{\pm}$.

This theorem provides an approach for constructing all the possible closed symmetric extensions of a densely defined symmetric operator. This is important because it enables one to fully understand the possible ways in which a symmetric operator can be extended while preserving its symmetry. It relates the deficiency indices of the symmetric operator T to its extension T_U . Particularly, it states that $d_{\pm}(T) = d_{\pm}(T_U) + \dim \mathcal{G}_{\pm}$. This relationship is crucial for understanding how the deficiency indices change under extensions. While the theorem deals with closed symmetric extensions, it forms a basis towards understanding self-adjoint extensions. To be specific, if the symmetric operator T has equal deficiency indices $d_+ = d_-$, then one is able to construct self-adjoint extensions of T using similar methods.

In the study of quantum mechanics, a number of physical observables are represented by symmetric operators. The ability to extend these operators to self-adjoint operators is always fundamental for ensuring that the observables have real eigenvalues and well defined spectral decompositions. This theorem is also relevant in the context of stability analysis, especially when doing a study of the stability of solutions to differential or difference equations under perturbations. By understanding the possible extensions of symmetric operators, one is able to analyse how the spectral properties of these operators change under perturbations.

Generally, Theorem 3.7.1 is a key result in the theory of symmetric operators and their extensions. It provides an approach for constructing all closed symmetric extensions of a densely defined symmetric operator and gives the relationship between the deficiency indices of the operator and those of its extensions. This theorem is crucial for understanding the spectral properties of symmetric operators and their extensions in the context of quantum mechanics and stability analysis. Besides, the theorem opens up avenues for further research into the spectral properties of symmetric operators in the context of unbounded operators and their extensions. It

also provides a framework for studying the stability of solutions to differential and difference equations under perturbations.

The result below is J. von Neumann's theorem on self-adjoint extensions.

Theorem 3.7.2. *((Schmüdgen, 2012), Theorem 13.10) A densely defined symmetric operator T on $\mathcal{H}$ possesses a self-adjoint extension on $\mathcal{H}$ if and only if $d_+(T) = d_-(T)$. If $d_+(T) = d_-(T)$, then all self-adjoint extensions of T are given by the operator T_U , where U is an isometric linear mapping of $\mathcal{N}(T^* - \lambda I)$ onto $\mathcal{N}(T^* - \bar{\lambda}I)$ and*

$$D(T_U) = D(\bar{T}) \dot{+} (I - U)\mathcal{N}(T^* - \lambda I),$$

$$T_U(x + (I - U)z) = \bar{T}x + \lambda z - \bar{\lambda}Uz$$

for all $x \in D(T)$ and $z \in \mathcal{N}(T^* - \lambda I)$.

This theorem states that a closed symmetric operator possesses self-adjoint operator extension if and only if the deficiency indices in the upper half-plane is equal to the deficiency indices in the lower half-plane. Hence, we have that for a closed densely defined symmetric operator T :

- (i). If $d_+(T) = d_-(T) = 0$, then $\bar{T}$ is the unique self-adjoint extension of T on the Hilbert space $\mathcal{H}$ and therefore T is essentially self-adjoint.
- (ii). If $d_+(T) = d_-(T) \neq 0$, then there are an infinite number of isometric mappings of $\mathcal{N}(T^* - \lambda I)$ onto $\mathcal{N}(T^* - \bar{\lambda}I)$ and hence an infinite number of self-adjoint extensions of T on the Hilbert space $\mathcal{H}$.
- (iii). If $d_+(T) \neq d_-(T)$, then there is no self-adjoint extension of T on the Hilbert space $\mathcal{H}$.

The theorem is usually used to study the stability of self-adjoint extensions of symmetric difference operators under perturbations. By ensuring that the deficiency indices remain equal, the theorem provided a way for us to analyse the stability of

these extensions. The theorem is usually essential for the understanding of the spectral properties of self-adjoint extensions of symmetric difference operators. Given that self-adjoint operators have real spectra, the theorem ensured that the operators studied in this research had very well defined spectral decompositions.

3.8 Existence of Positive Self-adjoint Extensions and Krein-von Neumann Extensions

Positive self-adjoint operator extensions are a key concept, especially in the study of unbounded operators in Hilbert spaces. These extensions enable the definition of certain operators with wider domains while preserving specific mathematical properties like positivity and self-adjointness. Operators are usually defined on a dense subset of the Hilbert space and the challenge is usually to extend them to a broader domain while retaining their properties.

Let T be a positive, closed, densely defined symmetric operator on a Hilbert space $\mathcal{H}$. Since T is densely defined, it admits a positive self-adjoint extension on $\mathcal{H}$ and so there exists a largest such extension called Friedrich's extension T_F . On the other hand, if T is not densely defined, then it is possible that it does not have any positive self-adjoint extension on the Hilbert space $\mathcal{H}$.

For an operator T , a positive extension $\tilde{T}$ is an operator such that $T \subseteq \tilde{T}$. This means that T agrees with $\tilde{T}$ on $D(T)$ and $\tilde{T}$ is positive. The extension of symmetric operators to self-adjoint operators has been a standard problem. As such, the Friedrich's extension and the Krein-von Neumann extension are two canonical approaches providing minimal and maximal extensions respectively. The Friedrich's extension is particularly important for positive symmetric operators. It is constructed to be the smallest self-adjoint extension in the sense of quadratic form hence ensuring that

$$\langle \tilde{T}x, x \rangle \geq \langle Tx, x \rangle$$

for all $x \in \mathcal{H}$.

Suppose that T is a densely defined symmetric operator on a Hilbert space $\mathcal{H}$ and

has a positive lower bound $m_T > 0$ such that $\langle Tx, x \rangle \geq m_T \|x\|^2$, for all $x \in D(T)$, then T is a positive operator. Given $0 \leq m_T = m_{T_F}$, we have that $0 \in \rho(T_F)$. The self-adjoint operator extensions of T on $\mathcal{H}$ are the operators T_B as defined in Theorem 14.12 in (Schmüdgen, 2012) with $B \in \mathcal{S}(\mathcal{N}(T^*))$. We have that

$$D(T_B) = \{x + (T_F)^{-1}(Bu + v) + u : x \in D(\bar{T}), u \in D(B), v \in \mathcal{N}(T^*) \cap D(B)^\perp\}$$

$$T_B(x + (T_F)^{-1}(Bu + v) + u) = \bar{T}x + Bu + v.$$

Let $\mathcal{S}(\mathcal{N}(T^*))_+$ be the set of positive operators in $\mathcal{S}(\mathcal{N}(T^*))$. By Theorem 14.24 in (Schmüdgen, 2012), the set $\mathcal{S}(\mathcal{N}(T^*))_+$ contains a largest operator and smallest operator. The largest operator is when $D(B) = \{0\}$ meaning $T_B \subseteq T_F$ and so $T_B = T_F$. T_B is the Friedrich's extentsion T_F of T . Therefore we have that $D(T_F) = D(\bar{T}) + T_F^{-1}\mathcal{N}(T^*)$. The smallest operator is when $B = 0$ and $D(B) = \mathcal{N}(T^*)$. Given it is the smallest element of $\mathcal{S}(\mathcal{N}(T^*))_+$, the corresponding operator T_B is the smallest of all positive self-adjoint extensions. The operator T_B therefore coincides with Krein-von Neumann extension T_N of T defined by

$$D(T_N) = D(\bar{T}) \dot{+} \mathcal{N}(T^*),$$

$$T_N(x + u) = Tx \text{ for } x \in D(\bar{T}), u \in \mathcal{N}(T^*),$$

$$D(T_N) = D(T_F) \dot{+} \mathcal{N}(T^*),$$

$$T_N(y + u, y' + u') = T_F(y, y') \text{ for } y, y' \in D(T_F) \text{ and } u, u' \in \mathcal{N}(T^*).$$

Assume $y \in \mathcal{H}$ and define

$$\nu_T := \sup_{x \in D(T)} \frac{|\langle Tx, y \rangle|^2}{\langle Tx, x \rangle}.$$

Denote by $\varepsilon(T)$ the set of vectors $y \in \mathcal{H}$ such that $\nu_T(y) < \infty$, or equivalently, there exists a number $c_y \geq 0$ such that

$$|\langle Tx, y \rangle|^2 \leq c_y \langle Tx, x \rangle \tag{3.14}$$

for all $x \in D(T)$. This set is crucial since it characterises the domain of the Krein-on-Neumann extension T_N . This implies that $\nu_T(y)$ is the smallest number c_y satisfying the equation (3.14) for $y \in \varepsilon(T)$.

The following theorem provides the necessary and sufficient condition for the existence of positive self-adjoint extension on the Hilbert space $\mathcal{H}$.

Theorem 3.8.1. *(Ando-Nishio Theorem, see [(Schmüdgen, 2012), Theorem 13.12])*
A positive symmetric operator L on a Hilbert space $\mathcal{H}$ admits a positive self-adjoint extension on H if and only if $\varepsilon(T)$ is dense in $\mathcal{H}$. If this is true, there exists a unique smallest among all positive self-adjoint extensions of T on $\mathcal{H}$, called Krein-von Neumann extension of T and is denoted by T_N . Then

$$\begin{aligned} D(T_N) &= D(T_N^{\frac{1}{2}}) = \varepsilon(T), \\ T_N(y) &= \|T_N^{\frac{1}{2}}y\|^2 = \nu_T(y), y \in \varepsilon(T). \end{aligned}$$

It states that if the operator T is densely defined, then there is always a smallest positive self-adjoint extension.

This theorem has significant implications for spectral theory. Given that T_N is a positive self-adjoint operator, it has a well-defined spectral decomposition. This makes it possible the analysis of the spectrum of T_N , which is vital in applications such as quantum mechanics and stability analysis. The Krein-von Neumann extension also ensures that the spectrum of T_N is real and positive, which is a desirable property in many physical and mathematical contexts. It is particularly important in the context of perturbation theory, where one studies how the properties of an operator (such as its spectrum) vary under some small perturbations. The existence of a positive self-adjoint extension ensures that the operator remains well-behaved even under perturbations.

In the context of this study, Theorem 3.8.1 was applied to analyse the stability of positive self-adjoint extensions of symmetric difference operators under unbounded perturbations. Particularly, it provides the theoretical basis for the determination of when a symmetric difference operator (such as L_1 or L) could be extended to a positive self-adjoint operator. This was crucial for understanding the stability of

these operators under perturbations.

This study focused on difference operators with unbounded coefficients, which are more challenging to analyse than bounded operators. The theorem ensured that under some conditions, these operators could still be extended to positive self-adjoint operators hence preserving their spectral properties. The Krein-von Neumann extension played a crucial role in the stability analysis of the operators in this study. By ensuring the existence of a minimal positive self-adjoint extension, the theorem allowed for a careful analysis of the spectral properties and stability of the operators.

Therefore, Theorem 3.8.1 is a key result in the theory of unbounded operators as it provides the necessary and sufficient conditions for the existence of positive self-adjoint extensions. It was essential for analysing the stability of symmetric difference operators under unbounded perturbations and for understanding their spectral properties. It was therefore a cornerstone of this study's theoretical framework.

3.9 The M-Matrix

The Weyl-Titchmarsh m -function $M(w) = M$, where w is the spectral parameter, is the Borel transform of the spectral measure. Therefore, the density of the absolutely continuous spectrum of the self-adjoint operator extension is given by

$$\left(\frac{1}{\Pi}\right) \lim_{\eta \rightarrow 0^+} (M(\mu + i\eta)) = \left(\frac{1}{\Pi}\right) M(\mu_+) = \varrho(\mu),$$

where we have that $w = \mu + i\eta$, $\mu, \eta \in \mathbb{R}$ and $\eta > 0$ is very small. Furthermore, the spectrum is absolutely continuous if M has finite limits $M(\mu_+)$. As such the eigenvalues of the self-adjoint operator extensions correspond to the poles of M . Therefore, for a system that satisfies the Levinson-Benzaid-Lutz form and has order $2n$, the solutions are of the form

$$y_k(t, w) = (e_k(t, w) + r_{kk}(t, w)) \prod_{l=t_0}^{t-1} \lambda_k(l, w)$$

that is continuous in w (the spectral parameter) with $r_{kk}(t, w) \rightarrow 0$ as $t \rightarrow \infty$ uniformly for w . Thus, in order to determine the m -function, we have to apply the

M-matrix method. Let

$$V = \begin{bmatrix} v_1(t, w) \\ v_2(t, w) \end{bmatrix}$$

be a set of square summable eigenfunctions of a system and satisfy some boundary conditions. By definition, the square summable functions are given by

$$Y_{t_0}(t, w) \begin{bmatrix} I_n \\ M_{t_0}(t, w) \end{bmatrix}$$

for an initial value $t_0 \in \mathbb{N}$. On the other hand, by asymptotic analysis, we have an $n \times n$ matrix that relates the systems

$$Y_{t_0}(t, w) \begin{bmatrix} I_n \\ M_{t_0}(t, w) \end{bmatrix} = \begin{bmatrix} v_1(t, w) \\ v_2(t, w) \end{bmatrix} C.$$

$Y_{t_0}(t_0)$ satisfies the appropriate boundary conditions and in this case shall be identity matrix at $t = t_0$. Thus

$$M_{t_0}(w) = v_2(t_0, w)v_{1^{-1}}(t_0, w).$$

In this case $v_{1^{-1}}(t_0, w)$ is essentially the Vandermonde matrix for the eigenvalues of the square summable solutions. Thus v_1^{-1} exists and $M_{t_0}(w)$ is continuous in w . This in turn shows that the spectrum is not singular continuous there and other than the isolated poles, the density of the spectral measure of the self-adjoint operator extension is continuous. The m -function can therefore be extended to $t_0 = 0$ by the results of Remling [(Remling, 1999)].

This means that for the first order system of a difference equation of order $2n$ whose characteristic polynomial is

$$\mathcal{P}(\lambda, t, w) = \det(S(t, w) - \lambda \cdot I_{2n})$$

that can be converted into its Fourier polynomial by taking

$$\frac{(-1)^n p_n}{\lambda^n} \mathcal{P}(\lambda, t, w)$$

to obtain

$$\mathcal{F}(\lambda, t, w) = \sum_{k=0}^n p_k (1 - \lambda)^k (1 - \lambda^{-1})^k$$

shows that if λ is a root, then λ^{-1} is also a root. This implies that for $w = \mu$, μ belongs to the associated absolutely continuous spectrum if and only if there is at least one eigenvalue $\lambda(t, \mu)$ for which $t \rightarrow \infty$ satisfies

$$|\lambda(\infty, \mu) + \lambda^{-1}(\infty, \mu)| < 2.$$

The multiplicity of the absolutely continuous spectrum is given by the number of such pairs. Thus, the M-matrix generalises the Weyl-Titchmarsh m -function and so it relates the asymptotics of the eigenfunction of higher order difference operators to the spectrum of their self-adjoint extensions. We fix two $n \times n$ matrices α_1 and α_2 , that is, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^{n \times n}$, with

$$\alpha_1 \alpha_1^* + \alpha_2 \alpha_2^* = I_n, \quad (3.15)$$

$$\alpha_1 \alpha_2^* - \alpha_2 \alpha_1^* = 0_n$$

and set the boundary condition for the left regular end point a as

$$(\alpha_1, \alpha_2)Y(a) = 0. \quad (3.16)$$

On the other hand, the M-matrix can be obtained from the inner products of square summable solutions explained here below (Remling, 1998): Let $Y_\alpha(., w) = (U_\alpha(., w), V_\alpha(., w))$ be the fundamental matrix with initial values

$$Y_\alpha(a, w) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}$$

with α_1 and α_2 satisfying the conditions (3.15) and (3.16). U_α and V_α are n by $2n$ matrices whose every column solves $Ly(t) = wy(t)$ and $V_\alpha(., w)$ satisfies the self-adjoint boundary conditions at a . In the limit point case with $Imw > 0$, we have an invertible matrix $M(w) \in \mathbb{C}^{n \times n}$ such that

$$Y_\alpha(t, w) \begin{bmatrix} I_n \\ M(w) \end{bmatrix} = U_\alpha(t, w) + V_\alpha(t, w)M(w)$$

The matrix $M(w)$ is determined from those solutions which stay absolutely square summable as $Imw \searrow 0$. It has been shown before in (Y. Shi, 2006) that if L is

at limit point at $t = \infty$, then it is possible to construct the M-matrix $M(w)$ for the Hamiltonian system restricted to $[a, \infty)$ having Dirichlet boundary conditions. Here, we let

$$\begin{bmatrix} X_1(a, w) \\ X_2(a, w) \end{bmatrix}$$

to be a system of n square summable solutions for $\text{Im}w > 0$. By the theory of Hinton and Shaw (Hinton & Shaw, 1981) that was extended to the discrete setting by (Y. Shi, 2006), these solutions also arise from

$$Y(t, w) \begin{bmatrix} I_n \\ M(w) \end{bmatrix}$$

where $Y_a(t, w)$ is the fundamental solution of the system satisfying the appropriate boundary conditions at a . In this case, these are $Y_a(t, w) = I_{2n}$. A comparison of the two sets of solutions shows that there is an invertible n by n matrix C so that

$$\begin{bmatrix} X_1(a, w) \\ X_2(a, w) \end{bmatrix} C = Y(t, w) \begin{bmatrix} I_n \\ M(w) \end{bmatrix}.$$

This means that $M(w) = X_2(a, w)X_1^{-1}(a, w)$. The existence of the absolutely continuous spectrum can be determined as follows: Let $F_\alpha(., w)$ be a $2n$ by n system of square summable solutions of the Hamiltonian system satisfying boundary conditions at a and at infinity and $w, w' \in \mathbb{R}$, then

$$F_\alpha(., w) - F_\alpha(., w') \in D(H)$$

and by results of Remling [(Remling, 1999), Lemma 3.1] and Shi [(Y. Shi, 2006), Theorem 6.3], we have that

$$\langle F_\alpha(., w), F_\alpha(., w) \rangle = (\text{Im}w)^{-1} \text{Im}M(w)$$

if $w' = w$. Therefore, if $w = w_0 + i\eta$ for $\eta > 0$, then we have for $w_{0+} = \lim_{\eta \rightarrow 0+} w_0 + i\eta$,

$$\begin{aligned} \text{Im}M(w_{0+}) &= \lim_{\eta \rightarrow 0+} \text{Im}M(w_0 + i\eta) \\ &= \lim_{\eta \rightarrow 0+} \eta \langle F_\alpha(., w_{0+} + i\eta), F_\alpha(., w_{0+} + i\eta) \rangle. \end{aligned}$$

The absolutely continuous spectrum of the self-adjoint extension of the minimal difference operator exists if the M-matrix is finite and bounded. Moreover, its spectral multiplicity is equal to the rank of the M-matrix.

3.10 Dirac Operators

In this Section, we begin by formulating the discretised version of the system (1.4) and explain the methodology that have been employed to obtain the main results. The Dirac operator considered is of the form:

$$\Delta D(t) + J(t)RD(t) = A_\mu RD(t), \quad D(t_0) = I, \quad t_0 \in \mathbb{N}, \quad (3.17)$$

t_0 is the left regular end point where the necessary boundary conditions are imposed. $D(t)$ is a 2×2 matrix defined by

$$D(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

Δ refers to a forward difference operator defined by

$$\Delta f(t) = f(t+1) - f(t),$$

$f(t)$ a complex valued function, $t \in \mathbb{N}$. Here $f(t) = p_{kj}(t)$, $k, j = 1, 2$. R in this case is a partial forward shift operator whose action in $D(t)$ is defined by

$$R(D(t)) = \begin{bmatrix} p_{11}(t+1) & p_{12}(t+1) \\ p_{21}(t) & p_{22}(t) \end{bmatrix}.$$

Just like in the case of continuous version where J_0 was taken as the value of $J(x)$ at the initial left regular end point $x = 0$, here $J(t_0)$ will be its equivalent with t_0 being the left regular end point. Hence we have that

$$J_{t_0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J(t) = \begin{bmatrix} 0 & \delta_1(t) \\ \delta_2(t) & 0 \end{bmatrix}, \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$A_\mu = i\mu J_{t_0}$ and $\mu \in \mathbb{C}$ is a spectral parameter. In the sequel, we shall define the inner products and the space of operation as follows:

$$\ell^2(\mathbb{N}) = \{D(t) : \{D(t)\}_{t=t_0}^\infty \subset \mathbb{C}^{2 \times 2} : \sum_{t=t_0}^\infty (RD(t))^*(RD(t)) < \infty\}$$

and

$$\langle D_1(t), D_2(t) \rangle = \sum_{t=t_0}^\infty D_1(t) \overline{D_2(t)}, \quad D_1(t), D_2(t) \in \ell^2(\mathbb{C}^{2 \times 2}).$$

In the case of summable potentials, there exists $t_1 \in \mathbb{N}$, $t_1 > t_0$, such that the integral domain $\mathcal{I} = [t_0, t_1]$ has the property

$$\sum_{t=t_0}^{t_1-1} (RD(t))^* (RD(t)) < \infty.$$

This integral domain leads to the Hilbert space $\ell^2(\mathbb{C}_I^{2 \times 2})$. In the first part of the analysis, we will assume that the potentials $\delta_j(t)$, $j = 1, 2$ are summable and spectral parameter $\mu \in \mathbb{C}$ is in the set P_d defined by

$$P_d = \{\mu = z_0 + iz_1 : z_0, z_1 \in \mathbb{R}, |z_1| < d\}$$

though we will allow $|\mu| \rightarrow \infty$ in the case $|z_0| \rightarrow \infty$. Here, $d > 0$. This will be adjusted appropriately in the case of unbounded potentials. The methodology largely applied to obtain the solutions of the system (3.17) in the next Sections under different growth and decay conditions of the potentials and spectral parameter is asymptotic summation based on the discretised Levinson's theorem [(Benzaid & Lutz, 1987), (F. O. Nyamwala, 2015), (F. O. Nyamwala, 2017)]. Since we are interested in the form of solutions for (3.17) that we are going to solve using asymptotic summation, it is advisable that we convert it to a form which we can easily apply the discretised Levinson's theorem (3.2.2). Thus reformulation of (3.17) leads to the following relations:

$$p_{11}(t+1) = (1 - i\mu)^{-1} p_{11}(t) - (1 - i\mu)^{-1} \delta_1 p_{21}(t),$$

$$p_{12}(t+1) = (1 - i\mu)^{-1} p_{12}(t) - (1 - i\mu)^{-1} \delta_1 p_{22}(t),$$

$$p_{21}(t+1) = -(1 - i\mu)^{-1} \delta_2 p_{11}(t) + [(1 - i\mu)^{-1} \delta_1 \delta_2 + (1 - i\mu)] p_{21}(t), \quad (3.18)$$

$$p_{22}(t+1) = -(1 - i\mu)^{-1} \delta_2 p_{12}(t) + [(1 - i\mu)^{-1} \delta_1 \delta_2 + (1 - i\mu)] p_{22}(t).$$

Here, $\delta_j = \delta_j(t)$, $j = 1, 2$. Application of the equations (3.18) in (3.17) and re-writing (3.17) in its first order transfer form [(Behncke & Nyamwala, 2011b), (Y. Shi, 2006)]

$$D(t+1, \mu) = C(t, \mu) D(t, \mu) \quad (3.19)$$

where,

$$C(t, \mu) = \frac{1}{1 - i\mu} \begin{bmatrix} 1 & -\delta_1 \\ -\delta_2 & 1 + \delta_1\delta_2 - \mu^2 - 2i\mu \end{bmatrix}. \quad (3.20)$$

Equation (3.19) is now solved based on discretised version of Levinson's theorem. In its simplest form, the theorem can be restated in its fundamental matrix form, see [(Behncke & Nyamwala, 2011b), (Benzaid & Lutz, 1987), (Y. Shi, 2006)] for details.

The following is the modified version of Theorem (3.2.2) in line with the results in (Benzaid & Lutz, 1987).

Theorem 3.10.1. *Let*

$$D(t+1, \mu) = [\Lambda(t, \mu) + \mathfrak{R}(t, \mu)]D(t, \mu), \quad (3.21)$$

$D(t_0) = I$, be the first order system of (3.17) and assume that $\|\mathfrak{R}_{kj}(t, \mu)\| \in \ell^1$ with $\Lambda(t, \mu) = \text{diag}\{\lambda_1(t, \mu), \lambda_2(t, \mu)\}$ satisfying the μ -uniform dichotomy condition, that is

$$| \frac{\lambda_k(t, \mu)}{\lambda_j(t, \mu)} | < 1 - \gamma \text{ or } | \frac{\lambda_k(t, \mu)}{\lambda_j(t, \mu)} | > 1 + \gamma, \quad k \neq j, \quad \gamma > 0, \quad \mu \in P_d,$$

then the solutions of this system and those of (3.17) are of the form

$$D(t, \mu) = E(t, \mu) \prod_{l=t_0}^{t_1-1} (\Lambda(l, \mu)).$$

Here, $E(t, \mu)$ is a normalized 2×2 matrix with its first normalised components given by

$$E_{1k}(t, \mu) = e_k(t),$$

$k = 1, 2$, $e_k(t)$ is a normalized eigenvector. The proof of this theorem is done using the variation of constant approach and for details, see [(Behncke, 2013), (Behncke & Nyamwala, 2011b), (Benzaid & Lutz, 1987)].

Theorem (3.10.1) is the discrete analogue of Levinson's Theorem tailored for analysing the asymptotic behaviour of matrix solutions to discretised Dirac systems. It provided conditions under which the matrix solutions of a perturbed discrete Dirac

system resembled those of an unperturbed discrete Dirac system, specifically when the perturbation was small enough and the eigenvalues satisfied a uniform dichotomy condition. It was therefore essential for analysing the asymptotic behaviour and stability of solutions to the discretised Dirac system when in the presence of unbounded potentials. Its application in spectral theory and stability analysis made it a crucial result in the theoretical framework of this particular study.

Behncke and Nyamwala [(Behncke & Nyamwala, 2011b)], greatly simplified the analysis of uniform dichotomy condition to the extent that it was established for eigenvalues with unit magnitude since it would be satisfied for the other set of eigenvalues (3.4).

Our interest in this study was the solution form and asymptotic behaviour of the solutions of (3.17) as well as (3.19) when either the spectral parameter, μ , is unbounded in some strip within the complex plane and the potentials are bounded within some integral interval or when the potentials are unbounded and the spectral parameter is within a bounded strip. The method used to transform (3.19) into its Levinson's form, see [(Behncke & Nyamwala, 2011b), (Benzaid & Lutz, 1987), (Y. Shi, 2006)] for more details, involve a chain of transformations and application of discretised Levinson's theorem. Here, because of the decay conditions imposed on the entries of $D(t)$ and the potentials $\delta_j(t)$, we have applied two diagonalisations and $[I + Q(t, \mu)]$ -transformations, see [(Benzaid & Lutz, 1987)] to change (3.19) into the required form (3.21).

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Difference Operators

4.1.1 Introduction

We have considered the fourth and second order symmetric difference equations defined respectively in (1.1) and (1.2). In this case, the scalar product for any two vectors $y_1(t), y_2(t) \in \ell_\omega^2(\mathbb{N})$ is defined by

$$\langle y_1, y_2 \rangle_\omega = \sum_{t=t_0}^{\infty} y_1(t) \omega(t) \overline{y_2(t)} < \infty, \quad y_1 = y_1(t) \text{ and } y_2 = y_2(t)$$

for some fixed left regular end point $t_0 \in \mathbb{N}$. In order to simplify the computations, we will assume that $\omega(t) = 1$, for all $t \in \mathbb{N}$. This results into the Hilbert space $\ell^2(\mathbb{N})$ which is isomorphic to $\ell_\omega^2(\mathbb{N})$. We have considered difference operators generated by the difference maps $\mathcal{L}_1$ and $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$, where $\mathcal{L}$ is obtained by perturbing $\mathcal{L}_1$ by a scalar multiple of $\mathcal{L}_2$, $z \in \mathbb{R}$. We were interested in the stability of the existence of positive self-adjoint operator extensions of the minimal operators generated by the difference maps (1.1) and (1.3) on $\ell^2(\mathbb{N})$ under various growth and decay conditions imposed on the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$. The method applied in this case was asymptotic summation based on the discretised version of Levinson's theorem to establish the equality of deficiency indices and to determine the existence of positive self-adjoint operator extension of the minimal difference operators. First, we have

determined the growth conditions on the coefficients that guarantees that $\mathcal{L}_1$ and $\mathcal{L}$ are positive.

In order to apply the discretised version of Levinson's theorem to our systems, we begin by systems formulation so that the equations

$$\mathcal{L}_1 y(t) = wy(t) \text{ and } \mathcal{L} y(t) = wy(t)$$

can be converted into their first order systems. Thus for $\mathcal{L}_1 y(t) = wy(t)$ we require the following quasidifferences (Y. Shi, 2006): Let

$$\begin{aligned} x_1(t) &= y(t-1) \\ x_2(t) &= \Delta y(t-2) \\ u_1(t) &= p_1 \Delta y(t-1) - \Delta(p_2 \Delta^2 y(t-2)) \\ u_2(t) &= p_2 \Delta^2 y(t-2). \end{aligned}$$

We define the vector valued functions $x(t)$, $u(t)$ and $Y_1(t)$ by

$$\begin{aligned} x(t) &= (x_1(t), x_2(t)) \\ u(t) &= (u_1(t), u_2(t)) \\ Y_1(t) &= (x(t), u(t))^{tr} \end{aligned}$$

and the partial shift operator $R(Y_1)(t)$ by

$$(RY_1)(t) = (x(t+1), u(t))^{tr},$$

the superscript tr denotes the transpose. The maximal difference operator $L_1^* y(t)$ generated by $\mathcal{L}_1$ is defined via its domain by

$$\begin{aligned} D(L_1^*) &= \{y(t) \in \ell^2(\mathbb{N}) : \forall f(t) \in \ell^2(\mathbb{N}) \text{ such that} \\ &\quad \mathcal{J} \Delta Y_1(t) - P_1(t)(RY_1)(t) = F(t), t \in \mathbb{N}\} \end{aligned}$$

and $L_1^* y(t) = f(t)$, if

$$\mathcal{J} \Delta Y_1(t) - P_1(t)(RY_1)(t) = F(t)$$

where $\mathcal{J}$ is a symplectic matrix given by

$$\mathcal{J} = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix},$$

$$F(t) = \begin{bmatrix} f(t), & 0, & 0, & 0 \end{bmatrix}^{tr} \text{ and } P_1(t) = \begin{bmatrix} -C_1 & A_1^* \\ A_1 & B_1 \end{bmatrix}$$

for which the non-zero elements of the 2×2 matrices A_1, B_1, C_1 are given by

$$(A_1)_{j,j+1} = 1, (B_1)_{2,2} = p_2^{-1} \text{ and } (C_1)_{j,j} = p_{j-1}, \text{ for } j = 1, 2.$$

The pre-minimal operator L_1' is defined by

$$D(L_1') = \{y(t) \in D(L_1^*) : \text{there exists } n \in \mathbb{N} \text{ such that } y(0) = y(t) = 0, \text{ for all } t \geq n+1\},$$

$L_1^*y = L_1'y$. L_1' is symmetric and densely defined but not necessarily closed. The closure of the pre-minimal operator, L_1' , is the minimal difference operator, L_1 . Thus L_1 and L_1^* are symmetric, $L_1 \subset L_1^*$ and $L_1 = L_1^{**}$ as required. We thus obtain a first order forward difference of the form

$$\Delta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} -C_1 & A_1^* \\ A_1 & B_1 \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix}, \quad (4.1)$$

which can be reformulated into a simpler version,

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = \begin{bmatrix} E_1 & E_1 B_1 \\ C_1 E_1 & I - A_1^* + C_1 E_1 B_1 \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \quad (4.2)$$

where $E_1 = (I - A_1)^{-1}$ and can be written as

$$Y_1(t+1, w) = S_1(t, w)Y_1(t, w) \quad (4.3)$$

for which

$$Y_1(t) = [x(t), u(t)]^{tr}, x(t) = [x_1(t), x_2(t)]^{tr} \text{ and } u(t) = [u_1(t), u_2(t)]^{tr}.$$

Thus

$$S_1(t, w) = \begin{bmatrix} 1 & 1 & 0 & p_2^{-1} \\ 0 & 1 & 0 & p_2^{-1} \\ (p_0 - w) & (p_0 - w) & 1 & (p_0 - w)p_2^{-1} \\ 0 & p_1 & -1 & 1 + p_1 p_2^{-1} \end{bmatrix}. \quad (4.4)$$

In order to proceed with transformation of equation (4.3), we require the eigenvalues and eigenvectors of $S_1(t, w)$. The associated characteristic polynomial is

$$\begin{aligned}\mathcal{P}_1(t, \lambda, w) &= \det(S_1(t, w) - \lambda \cdot I_4) \\ &= (1 - \lambda)^4 + p_1 p_2^{-1} (1 - \lambda)^3 + (p_0 - w) p_2^{-1} (1 - \lambda)^2\end{aligned}$$

which we multiply by $\frac{p_2}{\lambda^2}$ to obtain the Fourier characteristic equation

$$\mathcal{F}_1(t, \lambda, w) = p_2(1 - \lambda)^2(1 - \lambda^{-1})^2 + p_1(1 - \lambda)(1 - \lambda^{-1}) + (p_0 - w). \quad (4.5)$$

For purposes of simplification, we let $\zeta = (1 - \lambda)(1 - \lambda^{-1})$, implying that if λ is a root then λ^{-1} is equally a root. This leads us to the quadratic equation

$$p_2 \zeta^2 + p_1 \zeta + (p_0 - w) = 0. \quad (4.6)$$

On the other hand, for $\mathcal{L}y(t) = wy(t)$, we require the following quasi-differences (Y. Shi, 2006): Let

$$\begin{aligned}x_1(t) &= y(t - 1) \\ x_2(t) &= \Delta y(t - 2) \\ \tilde{u}_1(t) &= (p_1 - zq_1)\Delta y(t - 1) - \Delta(p_2 \Delta^2 y(t - 2)) \\ \tilde{u}_2(t) &= p_2 \Delta^2 y(t - 2).\end{aligned}$$

Then define the vector valued functions $x(t)$, $\tilde{u}(t)$ and $Y(t)$ by

$$\begin{aligned}x(t) &= (x_1(t), x_2(t)) \\ \tilde{u}(t) &= (\tilde{u}_1(t), \tilde{u}_2(t)) \\ Y(t) &= (x(t), \tilde{u}(t))^{tr}\end{aligned}$$

and the partial shift operator $R(Y)(t)$ by

$$(RY)(t) = (x(t + 1), \tilde{u}(t))^{tr},$$

the superscript tr denoting the transpose. Let $\ell^2(\mathbb{N})$ be as defined earlier. One therefore defines, the maximal and minimal difference operators L^* and L respectively generated by $\mathcal{L}$. The maximal difference operator L^* generated by $\mathcal{L}$ on $\ell^2(\mathbb{N})$ where

$$\mathcal{L}y(t) = (\mathcal{L}_1 - z\mathcal{L}_2)y(t) = wy(t) \quad (4.7)$$

is defined via its domain

$$D(L^*) = \{y(t) \in \ell^2(\mathbb{N}) : \forall f(t) \in \ell^2(\mathbb{N}) \text{ such that} \\ \mathcal{J}\Delta Y(t) - P(t)(RY)(t) = F(t), t \in \mathbb{N}\}$$

and $L^*y(t) = f(t)$, if

$$\mathcal{J}\Delta Y(t) - P(t)(RY)(t) = F(t) \text{ where } F(t) = \begin{bmatrix} f(t), & 0, & 0, & 0 \end{bmatrix}^{tr}.$$

One defines the pre-minimal operator generated by (4.7) by

$$D(L') = \{y(t) \in D(L^*) : \text{there exists } n \in \mathbb{N} \text{ such that } y(0) = y(t) = 0, \text{ for all} \\ t \geq n + 1\}, L^*y = L'y.$$

L' is symmetric and densely defined but not necessarily closed. The closure of the pre-minimal operator, L' , is the minimal difference operator generated by (4.7), meaning that the minimal difference operator is a restriction of the maximal difference operator by further Dirichlet boundary conditions at 0. In this case, we shall denote this minimal difference operator by L . It therefore follows that L and L^* are symmetric, $L \subset L^*$ in the operator sense and $L = L^{**}$ as required. Equation (4.7) can therefore be written in its linear discrete form

$$\mathcal{J}\Delta Y(t) = [wI_4 + P(t)]R(Y)(t) \quad (4.8)$$

where $t \in \mathbb{N}$, I_4 and $P(t)$ are square symmetric matrices and $\mathcal{J}$ is a canonical symplectic matrix, that is,

$$\mathcal{J} = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \text{ and } P(t) = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix}.$$

The non-zero elements of the 2×2 matrices A, B, C are given by

$$A_{j,j+1} = 1, B_{2,2} = p_2^{-1} \text{ and } c_{j,j} = p_{j-1} - zq_{j-1}, j = 1, 2$$

and $p_0 - zq_0$ interpreted as $p_0 - zq_0 - w$. Thus the fourth order difference equation (4.7) has a first order form which can be derived from (4.8) given by

$$\Delta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix} \quad (4.9)$$

The first order form in (4.9) therefore results into a simple first order form given by

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

where $E = (I - A)^{-1}$.

Since $Y(t) = (x(t), u(t))^{tr}$, we have that

$$Y(t+1, z, w) = S(t, z, w)Y(t, z, w) \quad (4.10)$$

with an explicit calculation of $S(t, z, w)$ giving

$$S(t, z, w) = \begin{bmatrix} 1 & 1 & 0 & p_2^{-1} \\ 0 & 1 & 0 & p_2^{-1} \\ (p_0 - zq_0 - w) & (p_0 - zq_0 - w) & 1 & (p_0 - zq_0 - w)p_2^{-1} \\ 0 & (p_1 - zq_1) & -1 & 1 + (p_1 - zq_1)p_2^{-1} \end{bmatrix}.$$

Computation of the associated characteristic polynomial gives

$$\begin{aligned} \mathcal{P}(t, \lambda, w) &= \det(S(t, z, w) - \lambda \cdot I_4) \\ &= (1 - \lambda)^4 + (p_1 - zq_1)p_2^{-1}(1 - \lambda)^3 + (p_0 - zq_0 - w)p_2^{-1}(1 - \lambda)^2 \end{aligned}$$

which when multiplied by $\frac{p_2}{\lambda^2}$ one obtains

$$\mathcal{F}(t, \lambda, w) = p_2(1 - \lambda)^2(1 - \lambda^{-1})^2 + (p_1 - zq_1)(1 - \lambda)(1 - \lambda^{-1}) + (p_0 - zq_0 - w)$$

In order to solve for the values of λ in the polynomial and to make simplification easier, we let $(1 - \lambda)(1 - \lambda^{-1})$ to be ζ so that the polynomial becomes

$$\mathcal{F}(t, \lambda, w) = p_2\zeta^2 + (p_1 - zq_1)\zeta + (p_0 - zq_0 - w) \quad (4.11)$$

In this case after computing the eigenvalues of $S(t, z, w)$, the roots of (4.11), the eigenvectors $v_k, k = 1, 2, 3, 4$ are computed as follows: Let λ_k be the root of (4.11), then in the quasidifferences, we substitute $y(t+r)$ by λ^r and Δ^r by $(\lambda_k - 1)^r$ so that

$$v_k = [\lambda_k^{-1}, (\lambda_k - 1)\lambda_k^{-2}, (p_1 - zq_1)(\lambda_k - 1)\lambda_k^{-1} - (\lambda_k - 1)p_2(\lambda_k - 1)^2\lambda_k^{-2}, p_2(\lambda_k - 1)^2\lambda_k^{-2}]^{tr}.$$

Explicit calculations of these eigenvectors based on various growth conditions of the coefficients have been done in Section 4.1.3. The matrix

$$T(t, z, w) = [v_1, v_2, v_3, v_4]$$

is now used to diagonalise equation (4.10). A similar approach is used to obtain the transforming matrix $T_1(t, z, w)$ for equation (4.1.1.5). We make the transformation

$$\chi(t, z, w) = T(t, z, w)Y(t, z, w)$$

which results into

$$\begin{aligned}\chi(t+1, z, w) &= T^{-1}(t+1, z, w)S(t, z, w)T(t, z, w)\chi(t, z, w) \\ &= [\Lambda(t, z, w) + \mathfrak{R}(t, z, w)]\chi(t, z, w)\end{aligned}$$

where,

$$\mathfrak{R}(t, z, w) = -T^{-1}(t+1, z, w)\Delta T(t, z, w)\Lambda(t, z, w) \quad (4.12)$$

and

$$\Lambda(t, z, w) = \text{diag}(\lambda_k(t, z, w)), \quad k = 1, \dots, 4.$$

Here, $\mathfrak{R}(t, z, w)$ consists of terms in $\ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N})$. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t, z, w)$, $k = 1, 2$. In (4.12),

$$\Delta T(t, z, w) = T(t+1, z, w) - T(t, z, w).$$

The second diagonalisation is done by use of the eigenvectors of the matrix

$$[\Lambda(t, w) + \mathfrak{R}(t, z, w)].$$

Using the results of Behncke (Behncke & Hinton, 2011), a matrix of the form

$$[I + B(t, z, w)]$$

with

$$B_{kk}(t, z, w) = 0, \quad B_{kj}(t, z, w) = (\lambda_j - \lambda_k)^{-1}\mathfrak{R}_{kj}, \quad (4.13)$$

$$k \neq j, \quad k, j = 1, \dots, 4, \quad t \geq t_0.$$

will be required for the second diagonalisation. The second diagonalisation results into correction terms added to the diagonals given by

$$(\Lambda_2)_{kk} = \text{diag}((\Re B)_{kk}).$$

The second diagonalisation is thus done using the transformation

$$\mathcal{W}(t, z, w) = [I + B(t, z, w)]\chi(t, z, w) \quad (4.14)$$

and which results into a system of the form

$$\mathcal{W}(t+1, w) = \{[\Lambda(t, z, w) + \Lambda_2(t, w)] + [I + B(t+1, z, w)]^{-1}\Re(t, w)[I + \quad (4.15)$$

$$B(t, z, w)]\}\mathcal{W}(t, z, w)$$

where

$$(\Lambda_2) = \text{diag}(\Re B)_{kk}.$$

A similar transformation can be carried out for the case of (4.15). One, therefore, obtains LBL-form of

$$\mathcal{W}(t+1, z, w) = [\Lambda(t, z, w) + R(t, z, w)]\mathcal{W}(t, z, w)$$

to which we apply Theorem (3.2.2) to obtain the solutions of the form

$$Y(t, w) = T(t, w)[I + B(t, w)][E(t, w) + o(1)] \prod_{t_0}^{t-1} \Lambda(t, w)$$

if we apply backward transformations. Note that the matrices $T(t, w)[I + B(t, w)]$ shall be bounded and the asymptotics of the eigensolutions shall depend on the matrix $\prod_{t_0}^{t-1} \Lambda(t, w)$. Explicitly, we have

$$y_k(t, z, w) = \rho_k(t, z, w)[e_k(t, z, w) + r_{kk}(t, z, w)] \prod_{t_0}^{t-1} \lambda_k(t, z, w)$$

where $\rho_k(t, z, w)$ is a bounded function from $T(t, z, w)[I + B(t, z, w)]$ and $e_k(t, z, w)$ is a normalised vector while $r_{kk}(t, z, w) \in \ell^1(\mathbb{N})$ and tends to zero as $t \rightarrow \infty$.

4.1.2 Construction of Positive Operators with Unbounded Coefficients

In this Section, we have established conditions for positivity of the maps $\mathcal{L}_1$ and $\mathcal{L}$. We have assumed that

$$p_2(t) > 0, \quad p_0 > zq_0, \quad p_2(t) \nearrow \infty, \quad p_0, p_1, q_1, q_0 = o(p_2), \quad (4.16)$$

for every $t \in \mathbb{N}$.

Theorem 4.1.1. *Suppose that (4.16) is satisfied in addition to*

$$(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t)) > z(q_1(t) + q_0(t)). \quad (4.17)$$

Then the minimal difference operators L and L_1 are positive.

Proof. Let $Ly(t)$ be the difference expression generated by (1.3) acting on $y(t)$, where $y(t)$ is a summable sequence of real valued functions on $\ell^2(\mathbb{N})$. Then L is symmetric since

$$D(L) \subseteq D(L^*) \text{ and}$$

$$\begin{aligned} \langle Ly(t), y(t) \rangle &= \sum_{t=1}^{\infty} Ly(t) \cdot \overline{y(t)} \\ &= \sum_{t=1}^{\infty} y(t) \cdot \overline{(Ly(t))} \\ &= \langle y(t), Ly(t) \rangle. \end{aligned}$$

By construction, we have equation (1.3) and for any $y(t) \in \ell^2(\mathbb{N})$, it suffices to show that $\langle Ly(t), y(t) \rangle > 0$ for all $t \in \mathbb{N}$ and $y(t) \in D(L)$. However, the expansion of the inner product leads to

$$\begin{aligned} \langle Ly(t), y(t) \rangle &= \langle (L_1 - zL_2)y(t), y(t) \rangle \\ &= \langle L_1y(t), y(t) \rangle - \langle zL_2y(t), y(t) \rangle \end{aligned}$$

and this inner product is positive if

$$\langle L_1y(t), y(t) \rangle > \langle zL_2y(t), y(t) \rangle.$$

Hence this condition implies that

$$\{\Delta^2[p_2(t)\Delta^2y(t-2)] - \Delta[p_1(t)\Delta y(t-1)] + p_0(t)y(t)\}y(t) > \\ \{-z\Delta[q_1(t)\Delta y(t-1) + q_0(t)y(t)]\}y(t).$$

An explicit expansion of both sides of this inequality leads to

$$2p_2(t) - p_2(t+2) - p_0(t) > z(q_1(t) + q_0(t))$$

which we can rewrite as

$$(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t)) > z(q_1(t) + q_0(t)).$$

As $t \rightarrow \infty$, the summand $(p_2(t) - p_2(t+2))$ goes to zero. With the assumptions in (4.16) where $p_2(t)$ dominates $p_1(t)$, $p_0(t)$, $q_1(t)$ and $q_0(t)$, the inequality (4.17) is true for any $t \in \mathbb{N}$. Therefore, the summation $\sum_{t=t_0}^{\infty} Ly(t) \cdot \overline{y(t)} > 0$ implies that $\langle Ly(t), y(t) \rangle > 0$ as required.

In a similar way one can show that L_1 is positive with these assumptions. To see this we have for any $y(t) \in \ell^2(\mathbb{N})$, $L_1y(t)$ is positive if

$$\langle L_1y(t), y(t) \rangle > 0. \tag{4.18}$$

An explicit expansion shows that the inequality (4.18) holds if

$$p_2(t) > p_0(t). \tag{4.19}$$

Given that $p_2(t)$ dominates $p_1(t)$ and $p_0(t)$, the inequality (4.19) is true for any $t \in \mathbb{N}$. Therefore, the summation $\sum_{t=t_0}^{\infty} (L_1y(t) \cdot \overline{y(t)}) > 0$ and hence $\langle L_1y(t), y(t) \rangle > 0$. $\square$

4.1.3 Bounded Perturbations of Unbounded Operators

We assume both the growth conditions (4.16) and the decay conditions,

$$\Delta g(t), \quad p_2^{-1}\Delta(g(t)) \in \ell^2(\mathbb{N}), \quad g = p_0, p_1, q_1, q_0, \tag{4.20}$$

$$\Delta^2(g(t)), \quad (p_2^{-1}\Delta(g(t)))^2, \quad p_2^{-1}\Delta^2(g(t)), \quad (\Delta g(t))^2 \in \ell^1(\mathbb{N}).$$

Theorem 4.1.2. *Let L_1 be the minimal operator generated by $\mathcal{L}_1$ (1.1) on $\ell^2(\mathbb{N})$. Assume conditions (4.16) and (4.20) are satisfied. If $|\frac{g}{p_2}|^{\frac{1}{4}}$, $g = p_0, p_1$ is absolutely*

summable, then L_1 is in limit circle case, that is, $\text{def} L_1 = (4, 4)$ and has positive self-adjoint operator extension and spectrum of its self-adjoint operator extension is discrete. However, if $|\frac{q}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $\text{def} L_1 = (2, 2)$ and $\sigma_{ac}(H) \subseteq [\bar{s}, \infty)$ where $\bar{s} = \limsup \frac{p_0(t)}{p_2(t)}$ of spectral multiplicity 2.

Proof. We solve $L_1 y(t) = w y(t)$ so as to determine the deficiency indices and spectral multiplicity. In this case w is our spectral parameter. We begin by determining the zeros of equation (4.6) up to the term $O(p_2^{-3})$ to obtain

$$\zeta_{\pm} \approx \pm i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} - \frac{p_1}{p_2} \mp \frac{1}{8} i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} \frac{(p_1)^2}{p_2(p_0)} \mp \frac{1}{128} i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} \frac{(p_1)^4}{p_2^2(p_0)^2} + O(p_2^{-3})$$

with p_0 interpreted as $p_0 - w$. This leads to the following roots:

$$\begin{aligned} \lambda_1 &\approx 1 + \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{4}} - i \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_2 &\approx 1 - \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{4}} - i \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_3 &\approx 1 + \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{4}} + i \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_4 &\approx 1 - \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{4}} + i \left| \frac{p_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}). \end{aligned}$$

The absolute value of λ_k tend to 1 as $t \rightarrow \infty$, in the limiting sense. The dichotomy conditions are established off the real axis by application of Theorem (3.2.2). Let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$ and $\eta > 0$ and small. Then $\text{Re} \lambda_1 = O(1 + (\frac{\eta}{p_2})^{\frac{1}{4}})$ while $\text{Re} \lambda_2 = O(1 - (\frac{\eta}{p_2})^{\frac{1}{4}})$ and hence $|\lambda_1| > 1$ while $|\lambda_2| < 1$ as $t \rightarrow \infty$. Similarly, $|\lambda_3| > 1$ and $|\lambda_4| < 1$. This is the dichotomy condition required by Theorem (3.2.2). We replace in the quasi-differences, $y(t+k)$ by λ^k and Δ by $\lambda - 1$ and normalize the first component through multiplication by λ to obtain the eigenvectors given as

$$v_k = \{1, (\lambda_k - 1)\lambda_k^{-1}, (p_1 - zq_1)(\lambda_k - 1) - p_2(\lambda_k - 1)^3\lambda_k^{-1}, p_2(\lambda_k - 1)^2\lambda_k^{-1}\}^{tr}.$$

Using only the leading terms, the approximated eigenvectors are of the form:

$$\begin{aligned}
v_1 &= [1, (\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, -ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}}]^{tr} \\
v_2 &= [1, -(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, -ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}}]^{tr} \\
v_3 &= [1, (\frac{1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, p_1(\frac{1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}}]^{tr} \\
v_4 &= [1, -(\frac{1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}}, ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}}]^{tr}.
\end{aligned}$$

Hence the matrix $T_1(t, \lambda, w)$ is given by

$$\begin{bmatrix}
1 & 1 & 1 & 1 \\
(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & (\frac{-1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & (\frac{1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & (\frac{-1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} \\
p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & p_1(\frac{1+i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} & -p_1(\frac{1-i}{\sqrt{2}})|\frac{p_0-w}{p_2}|^{\frac{1}{4}} \\
-ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}} & -ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}} & ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}} & ip_2|\frac{p_0-w}{p_2}|^{\frac{1}{2}}
\end{bmatrix}.$$

Two diagonalisations as outlined in Section 4.1.1 are enough to convert (4.3) into LBL-form because of (4.20). The solutions of $L_1 y(t) = w y(t)$ by application of Theorem (3.2.2) are of the form

$$y_k(t, w) = \rho_k(t, w)[e_k(t, w) + o(1)] \prod_{l=t_0}^{t-1} \lambda_k(l, w)$$

where $\rho_k(t, w)$ is bounded and obtained from $T_1(t, w)[I + B_1(t, w)]$.

Given the conditions (4.16), the transformation matrix is approximately

$$T_1(t, \lambda, w) = \begin{bmatrix}
1 & 1 & 1 & 1 \\
O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) \\
O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) & O(p_2^{-\frac{1}{4}}) \\
O(p_2^{\frac{1}{2}}) & O(p_2^{\frac{1}{2}}) & O(p_2^{\frac{1}{2}}) & O(p_2^{\frac{1}{2}})
\end{bmatrix}$$

and the determinant, $Det T_1(t, \lambda, w) \approx O(1)$. The inverse of this transforming matrix is therefore approximately equal to

$$T_1^{-1}(t, \lambda, w) \approx \begin{bmatrix}
O(1) & O(1) & O(1) & O(1) \\
O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) \\
O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) & O(p_2^{\frac{1}{4}}) \\
O(p_2^{-\frac{1}{2}}) & O(p_2^{-\frac{1}{2}}) & O(p_2^{-\frac{1}{2}}) & O(p_2^{-\frac{1}{2}})
\end{bmatrix}$$

having entries with alternating signs. By equation (4.12), the remainder matrix after first diagonalisation is given by

$$\mathfrak{R} \approx \begin{bmatrix} O(p_2^{-\frac{1}{2}} \Delta p_2) & O(p_2^{-\frac{1}{2}} \Delta p_2) & O(p_2^{-\frac{1}{2}} \Delta p_2) & O(p_2^{-\frac{1}{2}} \Delta p_2) \\ O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) \\ O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) & O(p_2^{-\frac{1}{4}} \Delta p_2) \\ O(p_2^{-1} \Delta p_2) & O(p_2^{-1} \Delta p_2) & O(p_2^{-1} \Delta p_2) & O(p_2^{-1} \Delta p_2) \end{bmatrix}.$$

Hence the correction terms to the eigenvalues after first diagonalisation are in ℓ^2 and are given by:

$$O(p_2^{-\frac{1}{2}} \Delta p_2), O(p_2^{-\frac{1}{4}} \Delta p_2) \text{ and } O(p_2^{-1} \Delta p_2).$$

As such, a second diagonalisation is performed as described in Section 4.1.1. This leads to the remainder matrix

$$\mathfrak{R}_B \approx \begin{bmatrix} O(p_2^{-\frac{3}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{3}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{3}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{3}{4}} (\Delta p_2)^2) \\ O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) \\ O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) & O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) \\ O(p_2^{-\frac{5}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{5}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{5}{4}} (\Delta p_2)^2) & O(p_2^{-\frac{5}{4}} (\Delta p_2)^2) \end{bmatrix}.$$

Thus the eigenvalues after two diagonalisation are approximately equal to

$$\begin{aligned} \tilde{\lambda}_1 &\approx \lambda_1 + O(p_2^{-\frac{1}{2}} \Delta p_2) + O(p_2^{-\frac{3}{4}} (\Delta p_2)^2) \\ \tilde{\lambda}_2 &\approx \lambda_2 + O(p_2^{-\frac{1}{4}} \Delta p_2) + O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) \\ \tilde{\lambda}_3 &\approx \lambda_3 + O(p_2^{-\frac{1}{4}} \Delta p_2) + O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) \\ \tilde{\lambda}_4 &\approx \lambda_4 + O(p_2^{-1} \Delta p_2) + O(p_2^{-\frac{5}{4}} (\Delta p_2)^2) \end{aligned}$$

and the correction terms

$$O(p_2^{-\frac{3}{4}} (\Delta p_2)^2), O(p_2^{-\frac{1}{2}} (\Delta p_2)^2), O(p_2^{-\frac{1}{2}} (\Delta p_2)^2) \text{ and } O(p_2^{-\frac{5}{4}} (\Delta p_2)^2)$$

are in ℓ^1 .

The asymptotics and summability of $y_k(t, w)$ thus depends on $\prod_{l=t_0}^{t-1} \lambda_k(l, w)$. We compute

$$\lim_{\eta \rightarrow 0} \langle y_k(t, w), y_k(t, w) \rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l, w)|^2$$

so that when $|\frac{g}{p_2}|^{\frac{1}{4}}$ is absolutely summable, then all the four solutions are square summable and $\text{def} L_1 = (4, 4)$, limit circle case and hence the spectrum is discrete. However, if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability while $y_2(t)$ and $y_4(t)$ stay square summable as $\eta \rightarrow 0^+$. The M-matrix can be shown to be finite and bounded using Euler's formula. To see this, we have

$$\begin{aligned}
\text{Im} M(t, w) &= \lim_{\eta \rightarrow 0^+} \langle y_j(t, w), y_j(t, w) \rangle, j = 2, 4 \\
&= \ln \prod_{l=t_0}^{t-1} |\lambda_j(l)|^2 \\
&\approx -2\eta \sum_{t_0}^{t-1} |p_0(l)|^{\frac{1}{4}} |p_2(l)|^{-\frac{1}{4}} \\
&\approx -2\eta \int_{t_0}^{t-1} |p_0(l)|^{\frac{1}{4}} |p_2(l)|^{-\frac{1}{4}} dl < \infty.
\end{aligned}$$

Hence the spectral measure is continuous and the deficiency indices of L_1 , $\text{def} L_1 = (2, 2)$. This implies that L_1 possesses positive self-adjoint operator extension, H_1 , by application of Theorem (3.7.2), defined using boundary conditions (3.15) and (3.16) at $a = 0$. The leading term associated with spectral parameter w is $|\frac{p_0-w}{p_2}|^{\frac{1}{4}}$ such that as $p_2(t) \nearrow \infty$ and $p_2(t) > 0$, $\bar{s} < w < \infty$ where $\bar{s} = \limsup(\frac{p_0}{p_2})$. The absolutely continuous spectrum is a subset of $[\bar{s}, \infty)$ and of spectral multiplicity 2. $\square$

Theorem 4.1.3. *Assume the conditions (4.16) and (4.20) are satisfied with $z \in \mathbb{R}$ finite. If $|\frac{g}{p_2}|^{\frac{1}{4}}$, $g = p_0, q_0, p_1, q_1$, is absolutely summable, then L is in limit circle case, that is, $\text{def} L = (4, 4)$ and has positive self-adjoint operator extension and spectrum of its self-adjoint operator extension is discrete. However, if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $\text{def} L = (2, 2)$ and $\sigma_{ac}(H) \subseteq [\bar{r}, \infty)$ where $\bar{r} = \limsup(\frac{p_0-zq_0}{p_2})(t)$ of spectral multiplicity 2.*

Proof. In order to establish the deficiency indices and spectral properties, we solve the equation $Ly(t) = wy(t)$ where w is the spectral parameter. This now requires the construction of the minimal and maximal difference operators generated by $\mathcal{L}$, eigenvalues and performance of asymptotic summation based on Levinson-Benzaid-Lutz theorem [(Behncke & Nyamwala, 2011b), (Benzaid & Lutz, 1987), (Y. Shi, 2006)]. Application of quadratic formula and binomial expansion on (4.11) up to the term

$O(p_2^{-3})$ lead to values of ζ that can be approximated by

$$\zeta_{\pm} \approx \pm i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} - \frac{p_1}{p_2} \mp \frac{1}{8} i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} \frac{(p_1)^2}{p_2(p_0)} \mp \frac{1}{128} i \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} \frac{(p_1)^4}{p_2^2(p_0)^2} + O(p_2^{-3})$$

where p_0 and p_1 are interpreted as $p_0 - zq_0 - w$ and $p_1 - zq_1$ respectively. A backward substitution for the values of ζ in terms of λ , that is,

$$\lambda^2 + (\zeta - 2)\lambda + 1 = 0 \quad (4.21)$$

leads to λ -values that can be written in terms of ζ given by

$$\lambda_{\pm} \approx 1 \pm (-\zeta)^{\frac{1}{2}} - \frac{\zeta}{2} \mp \frac{1}{8} (-\zeta)^{\frac{1}{2}} \zeta \mp \frac{1}{128} (-\zeta)^{\frac{1}{2}} \zeta^2 + O(-\zeta^{-3}). \quad (4.22)$$

Explicitly, one has

$$\begin{aligned} \lambda_1(t, z, w) &\approx 1 + \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{4}} - i \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_2(t, z, w) &\approx 1 - \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{4}} - i \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_3(t, z, w) &\approx 1 + \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{4}} + i \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}) \\ \lambda_4(t, z, w) &\approx 1 - \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{4}} + i \left| \frac{p_0 - zq_0 - w}{p_2} \right|^{\frac{1}{2}} + O(p_2^{-1}). \end{aligned}$$

By replacing in quasi-differences $y(t+k)$ by λ^k and Δ by $\lambda - 1$ and if we normalize the first component through multiplication by λ , we have the eigenvectors given as

$$v_k = \{1, (\lambda_k - 1)\lambda_k^{-1}, (p_1 - zq_1)(\lambda_k - 1) - p_2(\lambda_k - 1)^3\lambda_k^{-1}, p_2(\lambda_k - 1)^2\lambda_k^{-1}\}^{tr}.$$

Thus approximating eigenvectors using only the leading terms, we obtain eigenvectors of the form:

$$\begin{aligned} v_1 &= [1, \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, p_1 \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, -ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}}]^{tr} \\ v_2 &= [1, -\left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, -p_1 \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, -ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}}]^{tr} \\ v_3 &= [1, \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, p_1 \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}}]^{tr} \\ v_4 &= [1, -\left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, -p_1 \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}}, ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}}]^{tr}. \end{aligned}$$

Hence the matrix $T(t, \lambda, w)$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & \left(\frac{-1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & \left(\frac{-1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} \\ p_1 \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & -p_1 \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & p_1 \left(\frac{1+i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} & -p_1 \left(\frac{1-i}{\sqrt{2}} \right) \left| \frac{p_0}{p_2} \right|^{\frac{1}{4}} \\ -ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} & -ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} & ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} & ip_2 \left| \frac{p_0}{p_2} \right|^{\frac{1}{2}} \end{bmatrix}.$$

In this case, p_1 and p_0 have been interpreted as $p_1 - zq_1$ and $p_0 - zq_0 - w$ respectively. As $t \rightarrow \infty$, that is, in the limiting sense, the absolute value of λ_k are tending to 1. By application of Levinson's-Benzaid-Lutz theorem, the dichotomy conditions can only be established off the real axis. Now let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$, $\eta > 0$ and small. Then

$$Re\lambda_1 = O(1 + (\frac{\eta}{p_2})^{\frac{1}{4}}) \text{ while } Re\lambda_2 = O(1 - (\frac{\eta}{p_2})^{\frac{1}{4}})$$

and hence

$$|\lambda_1| > 1 \text{ while } |\lambda_2| < 1 \text{ as } t \rightarrow \infty.$$

A similar analysis is true for λ_3 and λ_4 . This is the required dichotomy condition by [(Behncke & Nyamwala, 2011b), Theorem 1.1].

The diagonalisation process follows from Theorem (4.1.2) and as outlined in Section 4.1.1. The two diagolisations are enough to convert (4.10) into LBL-form because of (4.20) and the correction terms are similar to those in Theorem (4.1.2). An explicit computation shows that

$$\lim_{\eta \rightarrow 0} \langle y_k(t, z, w), y_k(t, z, w) \rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l, w)|^2$$

so that when $|\frac{g}{p_2}|^{\frac{1}{4}}$ is absolutely summable, all the four solutions are square summable and $defL = (4, 4)$, hence limit circle case and the spectrum is discrete. On the other hand if $|\frac{g}{p_2}|^{\frac{1}{4}}$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability as $\eta \rightarrow 0^+$.

We need to show that the M-matrix is finite and bounded. We compute the M-matrix using the eigenvectors associated with $\lambda_2(t, z, w)$ and $\lambda_4(t, z, w)$ whose eigensolutions are of the form :

$$y_k(t, z, w) = \rho_k(t, z, w)[e_k(t, z, w) + o(1)] \prod_{l=t_0}^{t-1} \lambda_k(l) \quad (4.23)$$

where $\rho_k(t, z, w)$ are the diagonal terms of $T(t, z, w)[I+B(t, z, w)]$ which are bounded. Let $w = w_0 + i\eta$, $\eta > 0$, small and $w_0 \in \mathbb{R}$, then from the relation

$$ImM(t, z, w) = \lim_{\eta \rightarrow 0^+} \langle F(t, z, w), F(t, z, w) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|p_2(l)|^{\frac{1}{4}}} \approx -2\eta \int_{t_0}^{t-1} |p_2(l)|^{-\frac{1}{4}} dl.$$

The one for $y_4(t)$ follows in a similar way. Therefore, we have that $def L = (2, 2)$ and hence by application of Theorem (3.7.2), L possesses self-adjoint operator extensions defined using boundary conditions (3.15) and (3.16). With $a = 0$, the self-adjoint extension, H of L is defined by

$$D(H) = \{y(t) \in D(L^*) : (\alpha_1, \alpha_2)y(0) = 0\}, L^*y(t) = Hy(t)$$

and $L \subset H = H^* \subset L^*$.

The absolutely continuous spectrum is contributed by those eigenfunctions that lose their square summability. Note that the leading term associated with spectral parameter w is given by $|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{4}}$ so that as $p_2(t) \nearrow \infty$ and $p_2(t) > 0$, $\bar{r} < w < \infty$ where $\bar{r} = \limsup(\frac{p_0 - zq_0}{p_2})$. The absolutely continuous spectrum is a subset of $[\bar{r}, \infty)$ and of spectral multiplicity 2. $\square$

Corollary 4.1.4. *Assume the conditions (4.16), (4.17) and (4.20) are satisfied, then the minimal operators generated by $\mathcal{L}y(t)$ and $\mathcal{L}_1y(t)$ are positive, have positive self-adjoint operator extensions with the same location of absolutely continuous spectrum and spectral multiplicity of respective self-adjoint operator extensions equal.*

Proof. The proof follows immediately from Theorems (4.1.1), (4.1.2) and (4.1.3). $\square$

The example below has been used to verify the results obtained in Theorems (4.1.2) and (4.1.3).

Example 4.1.5. We have considered the minimal and maximal difference operators generated by (1.1) and (1.2) given by

$$p_2(t) = t, p_1(t) = t^{-1}, p_0(t) = t^{-2}, q_1(t) = e^{-t}, q_0(t) = t^{-4}, z < \infty.$$

This implies that $p_0, p_1, q_0, q_1 = o(p_2)$, $p_2(t) > 0$ and $|p_2(t)| \nearrow \infty$ as required by (4.16). We compute the range of z as required by (4.17)

$$\begin{aligned} \{(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t))\} &> z(q_1(t) + q_0(t)) \\ z &< \frac{\{(p_2(t) - p_2(t+2)) + (p_2(t) - p_0(t))\}}{(q_1(t) + q_0(t))} \\ &= \frac{(t - (t+2)) + (t - t^{-2})}{e^{-t} + t^{-4}} \end{aligned}$$

As $t \rightarrow \infty$, $e^{-t} \rightarrow 0$ faster than t^{-4} . Thus we have $z < \frac{t^4(t-2-t^{-2})}{(1+e^{-t}t^4)}$ so that as $t \rightarrow \infty$, then $z < \infty$. This implies that L and L_1 are positive operators. We now solve the roots of the associated $\mathcal{F}(t, z, w)$ polynomial. In the case of L_1 we have $t\zeta^2 + t^{-1}\zeta + t^{-2} - w = 0$. Hence by use of quadratic formula we obtain that $\zeta_{\pm} \approx \pm it^{-\frac{3}{2}} + O(t^{-2})$. The corresponding λ -roots are obtained from (4.6) and hence

$$\begin{aligned} \lambda_1(t, z, w) &\approx 1 + \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\ \lambda_2(t, z, w) &\approx 1 - \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\ \lambda_3(t, z, w) &\approx 1 + \left(\frac{1+i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}) \\ \lambda_4(t, z, w) &\approx 1 - \left(\frac{1+i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) + O(t^{-\frac{3}{2}}). \end{aligned}$$

The diagonalising matrix, $T_1(t, \lambda, w)$ is obtained from normalised eigenvectors of the associated eigenvalues and is given by

$$T_1(t, \lambda, w) \approx \begin{bmatrix} 1 & 1 & 1 & 1 \\ \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) & \left(\frac{-1+i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) & \left(\frac{1+i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) & \left(\frac{-1-i}{\sqrt{2}}\right)(t^{-\frac{3}{4}} - w) \\ \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{7}{4}} - w) & \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{7}{4}} - w) & \left(\frac{1+i}{\sqrt{2}}\right)(t^{-\frac{7}{4}} - w) & \left(\frac{1-i}{\sqrt{2}}\right)(t^{-\frac{7}{4}} - w) \\ -i(t^{-\frac{1}{2}} - w) & -i(t^{-\frac{1}{2}} - w) & i(t^{-\frac{1}{2}} - w) & i(t^{-\frac{1}{2}} - w) \end{bmatrix}.$$

Here, $(t^{-\frac{3}{4}} - w)$ are bounded and hence summable. However, the value of w can shift the magnitude of λ_k to either less or greater than 1. A careful analysis shows that if $|\lambda_1| < 1$ then $|\lambda_2| > 1$ and vice versa. Similarly, if $|\lambda_3| < 1$ then $|\lambda_4| > 1$ and vice versa. Thus $\text{def } L_1 = (2, 2)$, L_1 has positive self-adjoint operator extension with absolutely continuous spectrum contained in $(0, \infty)$ of spectral multiplicity 2.

A similar analysis also reveals that the eigenvalues of L are similar to those of L_1 approximated to $O(t^{-\frac{3}{2}})$ and hence similar results in deficiency indices, existence of positive self-adjoint operator extension and spectral analysis are obtained.

4.1.4 Unbounded Perturbations

In our subsequent analysis in this Section, we show that unlike in the case of deficiency indices and spectral results, positive self-adjoint operator extension is stable under unbounded perturbations. For that case we assume the following growth conditions on the coefficients:

$$z < 0, \quad p_2(t) > 0, \quad q_1(t) < 0, \quad |q_1(t)| \nearrow \infty \quad \text{as} \quad t \rightarrow \infty, \quad (4.24)$$

$$z \neq 0 \quad \text{and} \quad z \in \mathbb{R}.$$

$$p_0, p_1, p_2, q_0 = o(q_1), \quad p_2(t) > p_0(t), \forall t \in \mathbb{N} \quad \text{and} \quad \Delta p_2(t) < 0, \forall t \in \mathbb{N}.$$

Note that by assumption $p_2(t)$ is a decreasing function. Since we are solving the equation $L = L_1 - zL_2$, the above growth conditions in (4.24) ensures that L_1 and L are positive symmetric operators with L_1 perturbed by unbounded operator $-zL_2$ because $|q_1(t)| \nearrow \infty$ as $t \rightarrow \infty$. For the decay conditions, we shall assume that

$$\Delta g(t), \quad q_1^{-1} \Delta(g(t)) \in \ell^2(\mathbb{N}), \quad g = p_0, p_1, p_2, q_0, \quad (4.25)$$

$$\Delta^2(g(t)), \quad (q_1^{-1} \Delta(g(t)))^2, \quad q_1^{-1} \Delta^2(g(t)), \quad (\Delta g(t))^2 \in \ell^1(\mathbb{N}).$$

With these conditions we obtain the following results:

Theorem 4.1.6. *Suppose that L_1 is the minimal difference operator generated by $\mathcal{L}_1$ on $\ell^2(\mathbb{N})$. Assume that the coefficients $p_k(t)$, $k = 0, 1, 2$ are bounded and almost constant with the condition (4.17) then at limiting point L_1 has positive self-adjoint operator H with the absolutely continuous spectrum coinciding with that of constant coefficient operator with boundary condition α at 0 and spectral multiplicity equal to half the eigenvalues with unit magnitude.*

Proof. Positivity of the operator L_1 follows from Theorem (4.1.1) while the other claims are immediate from the results of Behncke and Nyamwala (Behncke & Nyamwala, 2011a). $\square$

Theorem 4.1.7. *Assume (4.17), (4.24) and (4.25) are satisfied with $z \in \mathbb{R}$ finite. If $|\frac{g}{q_1}|^{\frac{1}{2}}$, $g = p_0, q_0, p_1, p_2$ is absolutely summable, then $\text{def } L = (3, 3)$ and has positive self-adjoint operator extension and spectrum of its self-adjoint operator extension is discrete. On the other hand if $|\frac{g}{q_1}|^{\frac{1}{2}}$ is not absolutely summable, then $\text{def } L = (2, 2)$ and $\sigma_{ac}(\mathcal{H}) \subseteq [\bar{u}, \infty)$ where $\bar{u} = \limsup(\frac{p_0 - zq_0}{p_1 - zq_1})(t)$ of spectral multiplicity 1.*

Proof. In order to compute the deficiency indices and hence establish spectral multiplicity, we solve the equation $Ly(t) = wy(t)$ where w is the spectral parameter. As before, we introduce quasidifferences as stated by Shi (Y. Shi, 2006) and perform similar computations as before to obtain the polynomial

$$\mathcal{P}(t, \lambda, w) = (1 - \lambda)^4 + (p_1 - zq_1)p_2^{-1}(1 - \lambda)^3 + (p_0 - zq_0 - w)p_2^{-1}(1 - \lambda)^2$$

and hence the characteristic equation

$$p_2(1 - \lambda)^2(1 - \lambda^{-1})^2 + (p_1 - zq_1)(1 - \lambda)(1 - \lambda^{-1}) + (p_0 - zq_0 - w) = 0.$$

Assume $\zeta = (1 - \lambda)(1 - \lambda^{-1})$ and we obtain a biquadratic equation which we solve and make backward substitution to $O(p_2^{-2})$ for simplicity with $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$ and $\eta > 0$. This results to the following roots:

$$\begin{aligned} \lambda_{1/2}(t, z, w) &\approx 1 \mp i(p_0 - zq_0 - w)^{\frac{1}{2}}z^{-\frac{1}{2}}q_1^{-\frac{1}{2}} + O(q_1^{-\frac{3}{2}}) \\ \lambda_3(t, z, w) &\approx -p_2z^{-1}q_1^{-1} + z^{-2}q_1^{-2}p_1p_2 + O(q_1^{-3}) \\ \lambda_4(t, z, w) &\approx -p_2^{-1}zq_1 + p_2^{-1}p_1 + O(1). \end{aligned}$$

We note that $|\lambda_3| < 1$ and $|\lambda_4| > 1$. If $|\frac{g}{q_1}|^{\frac{1}{2}}$ is absolutely summable, then $\text{def } L = (3, 3)$ and the spectrum is discrete. As $t \rightarrow \infty$, in the limiting sense, the absolute value of $\lambda_{1/2}$ are tending to 1. The dichotomy conditions for the case of $\lambda_{1/2}$ are established off the real axis by application of Levinson's-Benzaid-Lutz theorem. Let $w = w_0 + i\eta$, $w_0, \eta \in \mathbb{R}$, $\eta > 0$ and small. Then $\text{Re } \lambda_1 = O(1 - (\frac{\eta}{q_1})^{\frac{1}{2}})$ while $\text{Re } \lambda_2 = O(1 + (\frac{\eta}{q_1})^{\frac{1}{2}})$. Hence, $|\lambda_1| < 1$ while $|\lambda_2| > 1$ as $t \rightarrow \infty$. The solutions of $Ly(t) = wy(t)$ are of the form

$$y_k(t, z, w) = \rho_k(t, z, w)[e_k(t, z, w) + o(1)] \prod_{l=t_0}^{t-1} \lambda_k(l, z, w)$$

and as before,

$$\lim_{\eta \rightarrow 0} \langle y_k(t, z, w), y_k(t, z, w) \rangle \approx \prod_{l=t_0}^{t-1} |\lambda_k(l, z, w)|^2.$$

If $|\frac{g}{q_1}|^{\frac{1}{2}}$ is not absolutely summable, then $y_2(t, z, w)$ loses its square summability as $\eta \rightarrow 0^+$ and $\text{def } L = (2, 2)$. This implies that L possesses self-adjoint operator extensions, which can be defined as before. By Euler's formula, the M-matrix is finite and bounded. The leading term associated with spectral parameter w is given by $|\frac{p_0 - zq_0 - w}{p_2}|^{\frac{1}{2}}$ so that as $q_1(t) \nearrow \infty$, $q_1(t) > 0$, then $\bar{u} < w < \infty$ where $\bar{u} = \limsup(\frac{p_0 - zq_0}{p_1 - zq_1})$. Therefore the absolutely continuous spectrum is a subset of $[\bar{u}, \infty)$ of spectral multiplicity 1. $\square$

In this case, unbounded perturbation leads to stability in the existence of positive self-adjoint operator extension which is similar to the case of bounded perturbation. However, the deficiency indices, absolutely continuous spectrum and spectral multiplicities are not stable. The following example reiterates the results of Theorem (4.1.6) and Theorem (4.1.7).

Example 4.1.8. Assume the following coefficients

$$z < 0, p_2(t) = t^{-1}, p_1(t) = e^{-t}, p_0(t) = t^{-2}, q_1(t) = \ln t, \text{ and } q_0(t) = c$$

for some constant c , $c > 0$, $t \in \mathbb{N}$ and $t \geq 2$. Then it follows that $p_k(t) > 0$ for all $t \in \mathbb{N}$ and hence L_1 is a positive map. Similarly, L is positive since by application of condition (4.17) we need that

$$z \leq (t^2 + 3t - 2)\{(\ln t + c)(t^3 + 2t^2)\}^{-1}$$

which tends to zero as $t \rightarrow \infty$. Because $z \neq 0$, since otherwise there will be no perturbation, the positivity of L implies that $z < 0$. This is the assumption made at the beginning. Thus (4.17) now implies that both L and L_1 are positive. We now proceed to perform the deficiency indices analysis to determine the stability

of the existence of the positive self-adjoint operator extension. The analysis of the eigenvalues now lead to

$$\begin{aligned}
\lambda_1(t, w, z) &\approx 1 - i|c|^{\frac{1}{2}}(z \ln t)^{-\frac{1}{2}} + O((\ln t)^{-1}) \\
\lambda_2(t, w, z) &\approx 1 + i|c|^{\frac{1}{2}}(z \ln t)^{-\frac{1}{2}} + O((\ln t)^{-1}) \\
\lambda_3(t, w, z) &\approx -t^{-1}(z \ln t)^{-1} + O((\ln t)^{-2}) \\
\lambda_4(t, w, z) &\approx -tz \ln t + 2 + t^{-1}(\ln t)^{-1} + O((\ln t)^{-2}).
\end{aligned}$$

Suppose that $\eta|c|^{\frac{1}{2}}(z \ln t)^{-\frac{1}{2}}$ is summable, then $\lambda_1(t, w, z)$, $\lambda_2(t, w, z)$ and $\lambda_3(t, w, z)$ lead to eigensolutions which are square summable and hence $\text{def} L = (3, 3)$ with discrete spectrum at most. On the other hand, if $\eta|c|^{\frac{1}{2}}(z \ln t)^{-\frac{1}{2}}$ is non-summable, then off the real axis, i.e taking $w = w_0 + i\eta$, $\eta > 0$ and small and $w, \eta \in \mathbb{R}$, then only two eigensolutions associated with $\lambda_1(t, w, z)$ and $\lambda_3(t, w, z)$ will be square summable leading to $\text{def} L = (2, 2)$. The operator L has self-adjoint operator extension with absolutely continuous spectrum of multiplicity 1 contained in $[0, \infty)$. A similar analysis of the deficiency indices of L_1 leads to the following eigenvalues:

$$\begin{aligned}
\lambda_1(t, w) &\approx 1 + \left(\frac{i-1}{\sqrt{2}}\right)t^{-\frac{1}{4}} - it^{-\frac{1}{2}} \\
\lambda_2(t, w) &\approx 1 - \left(\frac{i-1}{\sqrt{2}}\right)t^{-\frac{1}{4}} - it^{-\frac{1}{2}} \\
\lambda_3(t, w) &\approx 1 + \left(\frac{i-1}{\sqrt{2}}\right)t^{-\frac{1}{4}} + it^{-\frac{1}{2}} \\
\lambda_4(t, w) &\approx 1 - \left(\frac{i-1}{\sqrt{2}}\right)t^{-\frac{1}{4}} + it^{-\frac{1}{2}}.
\end{aligned}$$

This shows that if $O(\eta t^{-\frac{1}{4}})$ is summable, then $\text{def} L_1 = (4, 4)$. However, if $O(\eta t^{-\frac{1}{4}})$ is non-summable, then $\text{def} L_1 = (2, 2)$ and has absolutely continuous spectrum of multiplicity 2 contained in $[0, \infty)$. Hence L_1 has positive self-adjoint operator extension.

4.1.5 Power Coefficients

In this section, we have discussed two examples of maps $\mathcal{L}$ and $\mathcal{L}_1$ with power coefficients as reinforcement of results in Sections 4.1.3 and 4.1.4. Let $f(t) = ct^\varepsilon$, $t \in \mathbb{N}$ and c a non-zero constant. Then $f(t)$ is known as a power coefficient. $f(t)$ is bounded if $\varepsilon \leq 0$ otherwise it is unbounded.

Example 4.1.9. Let $p_2(t) = t^{\alpha_2}$, $p_1(t) = t^{\alpha_1}$, $p_0(t) = t^{\alpha_0}$, $q_1(t) = t^{\beta_1}$ and $q_0(t) = t^{\beta_0}$ such that the following conditions are satisfied:

$$\alpha_1, \alpha_0, \beta_1, \beta_0 < 0 \quad \text{and} \quad \alpha_2 > 0 \quad \text{so} \quad \text{that} \quad t^{\alpha_2} \rightarrow \infty \quad (4.26)$$

$$\text{as} \quad t \rightarrow \infty.$$

We obtain the equation

$$Ly(t) = \Delta^2[t^{\alpha_2}\Delta^2y(t-2)] - \Delta[(t^{\alpha_1} - zt^{\beta_1})\Delta y(t-1)] + [t^{\alpha_0} - zt^{\beta_0}]y(t) \quad (4.27)$$

Solving the equation $\mathcal{L}y(t) = wy(t)$ leads to the characteristic equation

$$t^{\alpha_2}(1-\lambda)^2(1-\lambda^{-1})^2 + (t^{\alpha_1} - zt^{\beta_1})(1-\lambda)(1-\lambda^{-1}) + (t^{\alpha_0} - zt^{\beta_0} - w) = 0 \quad (4.28)$$

Thus approximating the roots of (4.28) to $O(t^{-\alpha_2})$ leads to the following roots:

$$\begin{aligned} \lambda_1 &\approx 1 + \left(\frac{1-i}{\sqrt{2}}\right)t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} - it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\ \lambda_2 &\approx 1 - \left(\frac{1-i}{\sqrt{2}}\right)t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} - it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\ \lambda_3 &\approx 1 + \left(\frac{1+i}{\sqrt{2}}\right)t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} + it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{2}} + O(t^{-\alpha_2}) \\ \lambda_4 &\approx 1 - \left(\frac{1+i}{\sqrt{2}}\right)t^{-\frac{1}{4}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - w|^{\frac{1}{4}} + it^{-\frac{1}{2}\alpha_2}|t^{\alpha_0} - zt^{\beta_0} - wt|^{\frac{1}{2}} + O(t^{-\alpha_2}). \end{aligned}$$

The magnitudes of $\lambda_1(t, z, w)$ and $\lambda_3(t, z, w)$ are greater than 1 while the magnitudes of $\lambda_2(t, z, w)$ and $\lambda_4(t, z, w)$ are less than 1 as $t \rightarrow \infty$. Therefore $\text{def } L = (2, 2)$ and the spectrum is absolutely continuous since two eigensolutions stay w -uniformly square summable. Therefore the minimal difference operator generated by L has a positive self-adjoint operator extension under bounded perturbation.

Example 4.1.10. We consider Example (4.1.9) with the growth conditions

$$\alpha_2, \alpha_1, \alpha_0, \beta_0 < 0, \quad \beta_1 > 0, \quad , \quad q_1(t) = -t^{\beta_1}, \quad z \neq 0, \quad \text{and} \quad (4.29)$$

$$z \in \mathbb{R}.$$

These conditions imply that $p_2, p_1, p_0, q_0 = o(q_1)$ as $t \rightarrow \infty$. Further, $p_2(t) > 0$ while $\Delta p_2(t) < 0$, for all $t \in \mathbb{N}$. Thus by the results of Theorem (4.1.7), $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$

is a positive map and hence generates a positive operator. It remains to check the nature of the roots associated with the difference operator $\mathcal{L} = \mathcal{L}_1 - z\mathcal{L}_2$ under these new conditions. In line with equation (4.28) we obtain a modified polynomial of the form

$$t^{\alpha_2}(1 - \lambda)^2(1 - \lambda^{-1})^2 + (t^{\alpha_1} + zt^{\beta_1})(1 - \lambda)(1 - \lambda^{-1}) + (t^{\alpha_0} - zt^{\beta_0} - w) = 0. \quad (4.30)$$

This yields the roots

$$\begin{aligned} \lambda_{1/2}(t, z, w) &\approx 1 \mp (t^{\alpha_0} - zt^{\beta_0} - w)^{\frac{1}{2}} z^{-\frac{1}{2}} t^{-\frac{1}{2}\beta_1} + O(t^{-\frac{3}{2}\beta_1}) \\ \lambda_3(t, z) &\approx z^{-1} t^{\alpha_2 - \beta_1} - z^{-2} t^{\alpha_2 + \alpha_1 - s\beta_1} + O(t^{-3\beta_1}) \\ \lambda_4(t, z) &\approx zt^{\beta_1 - \alpha_2} + t^{\alpha_1 - \alpha_2} + O(1). \end{aligned}$$

The absolute value of λ_3 is less than one while that of λ_4 is greater than one. The analysis for λ_1 and λ_2 is done off the real axis using spectral parameter $w = w_o + i\eta$, $w_o, \eta \in \mathbb{R}$, $w_o > 0$ and $\eta > 0$ is small. Thus

$$\begin{aligned} \lambda_1 &\approx 1 + \frac{1}{2}\eta|w_o|^{-\frac{1}{2}}t^{-\frac{1}{2}\beta_1} - i|w_o|^{\frac{1}{2}}t^{-\frac{1}{2}\beta_1} \\ \lambda_2 &\approx 1 - \frac{1}{2}\eta|w_o|^{-\frac{1}{2}}t^{-\frac{1}{2}\beta_1} + i|w_o|^{\frac{1}{2}}t^{-\frac{1}{2}\beta_1} \end{aligned}$$

Computation of the absolute values of λ_1 and λ_2 now shows that $|\lambda_1| > 1$ while $|\lambda_2| < 1$ as $t \rightarrow \infty$, meaning that the solution $y_2(t, z, w) \approx c_2 \prod_{t=t_o}^{t-1} \lambda_2(t, z, w)$, where $c_2 = 1 \pm \eta|w_o|^{-\frac{1}{2}}t^{-\frac{1}{2}\beta_1}$, is absolutely summable. This implies that L_1 and L have similar deficiency indices, that is $(2, 2)$ and therefore there exists positive self-adjoint operator extensions under these unbounded perturbation. The absolutely continuous spectrum is $(0, \infty)$ of spectral multiplicity one.

4.2 Dirac Operators

4.2.1 System Formulation for Dirac Operators

In the Section, the expansion of the characteristic polynomial and eigenvalues is done with the assumption that μ is unbounded and $\delta_j(t)$, $j = 1, 2$ are summable. The case of unbounded potentials are discussed in Section 4.3. Though, the analysis largely

remain the same with some slight difference being in the eigenvalue expansion either in μ or in $\delta_1(t)$ which are the unbounded elements in Sections 4.2.-4.3 respectively.

Asymptotic summation based on Theorem (3.10.1) requires that we find the eigenvalues of the matrix (3.20) which are determined from the polynomial

$$\mathcal{P}_\lambda(t, \mu) = \det(C(t, \mu) - \lambda I_2). \quad (4.31)$$

Thus, we obtain the characteristic polynomial

$$\mathcal{P}_\lambda(t, \mu) = (1 - i\mu)^2 \lambda^2 - (1 - i\mu)(2 + \delta_1 \delta_2 - \mu^2 - 2i\mu) \lambda + (1 - \mu^2 - 2i\mu) \quad (4.32)$$

whose zeros are the eigenvalues of (3.20) and which when approximated to the term $O(\mu^{-5})$ results into the following roots:

$$\lambda_1(t, \mu) \approx -i\mu + i\mu^{-1} \delta_1(t) \delta_2(t) - i\mu^{-3} \delta_1(t) \delta_2(t) + O(\mu^{-5}), \quad (4.33)$$

$$\lambda_2(t, \mu) \approx i\mu^{-1} + i\mu^{-3} \delta_1(t) \delta_2(t) + O(\mu^{-5}).$$

The polynomial (4.31) has two distinct roots. The corresponding eigenvectors are then approximated from the relations

$$\nu_1(t, \mu) \approx \begin{bmatrix} 1 & 1 - \lambda_2(t, \mu) \end{bmatrix}^{tr} \quad (4.34)$$

$$\nu_2(t, \mu) \approx \begin{bmatrix} 1 & 1 - \lambda_1(t, \mu) \end{bmatrix}^{tr}.$$

Here ' tr ' means transpose. Thus a matrix of the form

$$T(t, \mu) \approx \begin{bmatrix} \nu_1(t, \mu) & \nu_2(t, \mu) \end{bmatrix}$$

diagonalises the matrix $C(t, \mu)$ (3.20). The matrix for second diagonalisation and the transformation, $[I + Q(t)]$, that accelerates the decay of the off-diagonal terms that are not summable after second diagonalisation, are discussed later on in this Section. The analysis as carried out here can be extended to any discretised Dirac system of order n in a straight forward way. The only task shall be computation of eigenvalues and establishment of uniform dichotomy conditions.

We now give preliminary results required in transforming the discretised Dirac system (3.17) to achieve the desired LBL-form. The assumptions made here are that

$$\delta_j(t) \in \ell^1([t_0, t_1]), \quad |\mu| \rightarrow \infty, \quad \mu \in P_d. \quad (4.35)$$

$\delta_j(t)$ have second forward difference with

$$\Delta(p_{kj}(t)), \Delta(\delta_j(t)) \in \ell^2[t_0, t_1]$$

$$\Delta^2(\delta_j(t)), (\Delta(\delta_j(t)))^2, \Delta^2(p_{kj}(t)), (\Delta(p_{kj}(t)))^2 \in \ell^1[t_0, t_1]. \quad (4.36)$$

In the Lemma below, we have established that the roots of the polynomial (4.31) as given in (4.33) satisfy the required μ -uniform dichotomy condition.

Lemma 4.2.1. *Suppose (4.35) and (4.36) are satisfied, then the eigenvalues of the matrix (3.20), $C(t, \mu)$ which are the zeros of the polynomial (4.31) are given in (4.33) satisfy the μ -uniform dichotomy condition.*

Proof. We need to show that the value

$$\left| \frac{\lambda_1(t, \mu)}{\lambda_2(t, \mu)} \right| < 1 - \gamma \text{ or } \left| \frac{\lambda_1(t, \mu)}{\lambda_2(t, \mu)} \right| > 1 + \gamma$$

as $|\mu| \rightarrow \infty$ for some $\gamma > 0$, however small. This should be true for all $\mu \in P_d$. Here, $\lambda_1(t, \mu)$ and $\lambda_2(t, \mu)$ are as given in (4.33). Therefore, computing $\left| \frac{\lambda_1(t, \mu)}{\lambda_2(t, \mu)} \right|$ with the assumptions in (4.35) we obtain

$$\left| \frac{\lambda_1(t, \mu)}{\lambda_2(t, \mu)} \right| \approx |\mu|^2 + c$$

where $c = O(|\delta_1 \delta_2 - 2\mu^{-2} \delta_1^2 \delta_2^2|)$. As $|\mu| \rightarrow \infty$ and invoking the results of [(Behncke & Nyamwala, 2011b)], the required dichotomy condition is satisfied since

$$|\lambda_1(t, \mu)| > 1 \text{ while } |\lambda_2(t, \mu)| < 1$$

uniformly in μ . □

For us to reduce (3.19) into (3.21) we will require two diagonalisations because of (4.36). Therefore, the required diagonalising matrix which is now used to transform (3.19) is given by

$$T(t, \mu) \approx \begin{bmatrix} 1 & 1 \\ 1 - \lambda_2(t, \mu) & 1 - \lambda_1(t, \mu) \end{bmatrix}. \quad (4.37)$$

Hence, we make a transformation of the form

$$D(t, \mu) = T(t, \mu)W(t, \mu) \quad (4.38)$$

that takes (3.19) into

$$\begin{aligned} W(t+1, \mu) &= T^{-1}(t+1, \mu)C(t, \mu)T(t, \mu)W(t, \mu) \\ &= [\Lambda(t, \mu) + \mathfrak{R}(t, \mu)]W(t, \mu) \end{aligned}$$

where,

$$\mathfrak{R}(t, \mu) = -T^{-1}(t+1, \mu)\Delta T(t, \mu)\Lambda(t, \mu). \quad (4.39)$$

$$\Lambda(t, \mu) = \text{diag}(\lambda_k(t, \mu)), \quad k = 1, 2.$$

Here, $\mathfrak{R}(t, \mu)$ consists of ℓ^2 and ℓ^1 terms. This suggests that the eigenvalues of $C(t, \mu)$ (3.20) are responsible for the asymptotic behaviour of the solutions of (3.19) although there will be some correction terms as a result of the first transformation which are given by $\mathfrak{R}_{kk}(t, \mu)$, $k = 1, 2$. In (4.39),

$$\Delta T(t, \mu) = T(t+1, \mu) - T(t, \mu).$$

Explicitly, the correction terms are of the form

$$O(i\mu^{-3}\Delta(\delta_1\delta_2)).$$

Because of the decay assumptions in (4.36) on the functions $\delta_j(t)$, $j = 1, 2$, one requires second diagonalisation. This is done by use of the eigenvectors of the matrix

$$[\Lambda(t, \mu) + \mathfrak{R}(t, \mu)].$$

Using the results of Behncke [(Behncke, 2013)], a matrix of the form $[I + B(t, \mu)]$ with

$$\begin{aligned} B_{kk}(t, \mu) &= 0, & B_{kj}(t, \mu) &= (\lambda_j - \lambda_k)^{-1} \mathfrak{R}_{kj}, \\ k &\neq j, & k, j &= 1, 2, & t &\geq t_0. \end{aligned} \quad (4.40)$$

will be required for the second diagonalisation. The second diagonalisation will result into correction terms added to the diagonals given by

$$(\Lambda_2)_{kk} = \text{diag}((\mathfrak{R}B)_{kk})$$

which explicitly are given by

$$\text{diag}(O(\mu^{-7}(\Delta(\delta_1\delta_2))^2)).$$

The second diagonalisation is thus done using the transformation

$$W(t, \mu) = [I + B(t, \mu)]X(t, \mu) \quad (4.41)$$

and which results into a system of the form

$$X(t+1, \mu) = \{[\Lambda(t, \mu) + \Lambda_2(t, \mu)] + [I + B(t+1, \mu)]^{-1} \mathfrak{R}(t, \mu)[I + B(t, \mu)]\}X(t, \mu), \quad (4.42)$$

where $(\Lambda_2) = \text{diag}(\mathfrak{R}B)_{kk}$. One, therefore obtains the following result on the corrected eigenvalues with their expansion on μ .

Lemma 4.2.2. *Define Dirac system in (3.19) with (4.35)- (4.36) satisfied. The eigenvalues of the form $\tilde{\lambda}_k(t, \mu) \approx \lambda_k(t, \mu) + O(i\mu^{-3}\Delta(\delta_1\delta_2)) + O(\mu^{-7}(\Delta(\delta_1\delta_2))^3)$ determines the asymptotic behaviour of the solutions when two diagonalisations are performed.*

Proof. In this case we assume that after two diagonalisations, the Dirac system will be converted into the form in (3.21). One therefore, computes the eigenvalues of the matrix, $C(t, \mu)$ (3.20) which are the zeros of the polynomial (4.31) and are of the form:

$$\begin{aligned} \lambda_1(t, \mu) &\approx -i\mu + i\mu^{-1}\delta_1(t)\delta_2(t) - i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}), \\ \lambda_2(t, \mu) &\approx i\mu^{-1} + i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}). \end{aligned}$$

Thus the eigenvectors can be computed and approximated to the term $O(\mu^{-5})$ and then a transformation using the matrix of the eigenvectors applied. In our case, this is the matrix $T(t, \mu)$ defined by

$$T(t, \mu) = \begin{bmatrix} 1 & 1 \\ 1 - \lambda_2(t, \mu) & 1 - \lambda_1(t, \mu) \end{bmatrix}$$

and explicitly is approximately given by

$$T(t, \mu) \approx \begin{bmatrix} 1 & 1 \\ 1 - i\mu^{-1} - i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}) & 1 + i\mu - i\mu^{-1}\delta_1(t)\delta_2(t) + i\mu^{-3}\delta_1(t)\delta_2(t) + O(\mu^{-5}) \end{bmatrix}.$$

$T(t, \mu)$ has inverse since its determinant is $O(\mu)$. Using the transformation as given in (4.38), the corrections to the diagonals are given by the diagonals of the matrix $\mathfrak{R}(t, \mu)$ and hence

$$(\mathfrak{R}(t, \mu))_{kk} \approx i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-5}\delta_1\delta_2\Delta(\delta_1\delta_2)).$$

In order to perform a second diagonalisation as explained earlier, we absorb $(\mathfrak{R}(t, \mu))_{kk}$ into $\Lambda(t, \mu) = \text{diag}(\lambda_k(t, \mu))$, $k = 1, 2$ to obtain

$$W(t+1, \mu) = [\tilde{\Lambda}(t, \mu) + \tilde{\mathfrak{R}}(t, \mu)]W(t, \mu) \quad (4.43)$$

where

$$\tilde{\Lambda}(t, \mu) = \text{diag}\{\lambda_k(t, \mu) + O(i\mu^{-3}\Delta(\delta_1\delta_2))\}$$

and

$$\tilde{\mathfrak{R}}(t, \mu) = \mathfrak{R}(t, \mu) - \text{diag}((\mathfrak{R}(t, \mu))_{kk}, k = 1, 2).$$

A second diagonalisation is then performed using the matrix $[I + B(t, \mu)]$ with

$$(B(t, \mu))_{kk} = 0, \quad (B(t, \mu))_{kj} = (\lambda_j - \lambda_k)^{-1}(\tilde{\mathfrak{R}}(t, \mu))_{kj}, \quad k, j = 1, 2, \quad k \neq j.$$

The matrices $\tilde{\mathfrak{R}}(t, \mu)$ and $B(t, \mu)$ are approximately given by

$$\begin{bmatrix} 0 & -i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-4}) \\ -i\mu^{-3}\Delta(\delta_1\delta_2) + O(\mu^{-4}) & 0 \end{bmatrix} \quad (4.44)$$

and

$$\begin{bmatrix} 0 & -\mu^{-4}(\Delta(\delta_1\delta_2))^2 + O(\mu^{-5}) \\ \mu^{-4}(\Delta(\delta_1\delta_2))^2 + O(\mu^{-5}) & 0 \end{bmatrix} \quad (4.45)$$

respectively. Thus by making a transformation of the form (4.41), a second diagonalisation as outlined above will now result into correction terms of the form $(\tilde{\mathfrak{R}}B)_{kk}$, $k = 1, 2$ that explicitly are of the form

$$\begin{aligned}(\tilde{\mathfrak{R}}B)_{11} &\approx -i\mu^{-7}(\Delta(\delta_1\delta_2))^3, \\(\tilde{\mathfrak{R}}B)_{22} &\approx i\mu^{-7}(\Delta(\delta_1\delta_2))^3.\end{aligned}$$

Hence, after two diagonalisations, the corrected eigenvalues are $\tilde{\lambda}_k(t, \mu)$ given by

$$\tilde{\lambda}_k(t, \mu) \approx \lambda_k(t, \mu) + O[\mu^{-3}\Delta(\delta_1\delta_2)] + O[\mu^{-7}(\Delta(\delta_1\delta_2))^3]. \quad (4.46)$$

Each summand after $\lambda_k(t, \mu)$ contributed by each diagonalisation. By assumptions in (4.35) and (4.36), the system (4.42) will be in the form of (3.21) since the off diagonal terms are absolutely summable and by application of the discretised Levinson's theorem the asymptotics of solutions of (3.19) are determined by $\tilde{\lambda}_k(t, \mu)$. $\square$

Finally, we discuss transformations that accelerates the decay of the off-diagonal terms, that is, $[I + Q(t, \mu)]$ -transformations as discussed by Behncke [(Behncke, 2013), (Benzaid & Lutz, 1987)]. It should be noted that after two diagonalisations, not all the off-diagonal terms may be absolutely summable, especially when $\delta_j(t)$ are unbounded. This transformation is required to convert any ℓ^2 -terms that might remain after two diagonalisations into ℓ^1 -terms and accelerate the decay of ℓ^1 -terms. Now re-write the system (4.42) in the form

$$X(t+1, \mu) = [\tilde{\Lambda}(t, \mu) + V(t, \mu)]X(t, \mu) \quad (4.47)$$

where,

$$\begin{aligned}\tilde{\Lambda}(t, \mu) &= \Lambda(t, \mu) + \Lambda_2(t, \mu) \\ \Lambda(t, \mu) &= \text{diag}(\lambda_k + O(\mu^{-3}\Delta(\delta_1\delta_2))) \\ \Lambda_2(t, \mu) &= \text{diag}(O(\mu^{-7}(\Delta(\delta_1\delta_2))^2)) \\ V_{kk}(t, \mu) &= 0 \\ V_{kj}(t, \mu) &= (\mathfrak{R}B)_{kj}(t, \mu)\end{aligned}$$

for $k, j = 1, 2$. Applying the transformation

$$X(t, \mu) = [I + Q(t, \mu)]Z(t, \mu)$$

on the system (4.47), we obtain

$$Z(t+1, \mu) = [I + Q(t+1, \mu)]^{-1} [\tilde{\Lambda}(t, \mu) + V(t, \mu)] [I + Q(t, \mu)] Z(t, \mu). \quad (4.48)$$

Here,

$$Q_{12}(t, \mu) \approx - \sum_{t_0}^{t_1-1} \frac{V_{12}(s, \mu)}{\tilde{\lambda}_2(t, \mu)} \prod_{l=s}^t \frac{\tilde{\lambda}_2(t, \mu)}{\tilde{\lambda}_1(t, \mu)}$$

since $|\frac{\lambda_1(t, \mu)}{\lambda_2(t, \mu)}| > 1 + \gamma$, $\gamma > 0$ and

$$Q_{21}(t, \mu) \approx \sum_{t_0}^{t_1-1} \frac{V_{21}(s, \mu)}{\tilde{\lambda}_2(s, \mu)} \prod_{l=s}^t \frac{\tilde{\lambda}_2(t, \mu)}{\tilde{\lambda}_1(t, \mu)}$$

because $|\frac{\lambda_2(t, \mu)}{\lambda_1(t, \mu)}| < 1 - \gamma$, $\gamma > 0$ small, as $|\mu| \rightarrow \infty$. Explicit computation now gives

$$\begin{aligned} V_{kk}(t, \mu) &= 0 \\ V_{kj}(t, \mu) &= (\Re B)_{kj}(t, \mu) \\ &\approx O(\mu^{-7}(\Delta(\delta_1 \delta_2))^2). \end{aligned}$$

Therefore, applying the relations in [(Benzaid & Lutz, 1987), (3.16a-3.16b), (3.19-3.20)] together with the eigenvalues in (4.33) and the workings in the proof of [(Benzaid & Lutz, 1987), Theorem 2.1] we have $Q_{kj}(t, \mu)$ as follows:

$$\begin{aligned} Q_{12}(t, \mu) &= -Q_{21}(t, \mu) \\ &= - \sum_{t_0}^{t_1-1} \frac{V_{21}(s, \mu)}{\tilde{\lambda}_2(s, \mu)} \prod_{l=s}^t \frac{\tilde{\lambda}_2(t, \mu)}{\tilde{\lambda}_1(t, \mu)} \\ &\approx - \sum_{s=t_0}^{t_1-1} O(\mu^{-6}(\Delta \delta_1 \delta_2)^2)(s) \prod_{l=s}^t [O(\mu^{-2}) + O(\mu^{-4} \delta_1 \delta_2)](l). \end{aligned}$$

Any subsequent $[I + Q_j(t, \mu)]$ -transformations are applied with assumptions that $Q_j(t, \mu) = o(Q(t, \mu))$ for $j = 1, 2, \dots$. Hence, mathematically, only the first $[I + Q(t, \mu)]$ -transformation has significance in the asymptotic behaviour of solutions of (3.19). In line with the arguments in Benzaid-Lutz paper [(Benzaid & Lutz, 1987)] and highlighted by Behncke [(Behncke, 2013)], $Q_{kj}(t, \mu)$ are bounded since $|Q_{kj}(t, \mu)|$ are majorised by convergent geometric series that can be shown using Hölder's inequality. Thus in the subsequent workings, we will assume the following growth conditions on $Q_k(t, \mu)$.

$$Q_k(t, \mu) = o(Q(t, \mu)), \quad k = 1, 2, 3, \dots \quad t \in [t_0, t_1], \quad \mu \in P_d. \quad (4.49)$$

4.2.2 Dirac Operators with Summable Potentials

Here, we have proved main results based on different growth and decay conditions imposed on the spectral parameters and potentials.

Theorem 4.2.3. *Suppose all the conditions in Lemma (4.2.1) are satisfied in addition to (4.49) then the solutions of the Dirac system (3.17) after two diagonalisations and a chain of $[I + Q(t, \mu)]$ -transformations are given by*

$$D(t, \mu) = T(t, \mu)[I + B(t, \mu)][I + Q(t, \mu)][E(t, \mu) + o(I)] \prod_{s=t_0}^{t-1} [\tilde{\Lambda}(s, \mu)]$$

where $E(t, \mu)$ is a 2×2 matrix of the normalized eigenvectors, $\tilde{\Lambda}(s, \mu) = \text{diag}[\tilde{\lambda}_1(s, \mu), \tilde{\lambda}_2(s, \mu)]$ and

$$\tilde{\lambda}_k(s, \mu) = \lambda_k(s, \mu) + O(\mu^{-3}\Delta(\delta_1\delta_2)^2) + O(\mu^{-7}\Delta(\delta_1\delta_2)^2).$$

The asymptotic behaviour of these solutions are determined by the leading power of μ .

Proof. The proof of this theorem follow closely from that of Lemma (4.2.1). The Dirac system in (3.17) is converted into its first order system (3.19) and the eigenvalues of the 2×2 matrix $C(t, \mu)$ are computed. The uniform dichotomy condition now follows from Theorem (3.10.1). The corresponding eigenvectors are computed and diagonalising matrix $T(t, \mu)$ is obtained with the columns of $T(t, \mu)$ as the eigenvectors of $C(t, \mu)$. A transformation of the form (4.38) is then applied on the system (3.19) to obtain a system with remainder terms $\mathfrak{R}(t, \mu)$ given by (4.39) and $\Lambda(t, \mu)$ basically the diagonal matrix $\text{diag}(\lambda_k(t, \mu))$, $k = 1, 2$. Because of the decay assumptions in (4.36), the remainder matrix can be split as a sum of two matrices $\mathfrak{R}_1(t, \mu)$ and $\mathfrak{R}_2(t, \mu)$ with elements of the former in ℓ^1 and those of the latter in ℓ^2 . A second diagonalisation can then be carried out using the transformation

$$W(t, \mu) = [I + B(t, \mu)]X(t, \mu)$$

with the matrix $B(t, \mu)$ computed as explained in Section 4.2.1. This will result into a system of the form (4.42). Any elements in the remainder matrix

that is not small enough, that is, not in ℓ^1 can be made smaller using a chain of $[I + Q(t, \mu)]$ -transformation based on the assumption (4.49). In this case, it is assumed that the first $[I + Q(t, \mu)]$ -transformation will dominate the other $[I + Q_k(t, \mu)]$ -transformation, $k = 1, 2, \dots$. Hence the transformation

$$X(t, \mu) = [I + Q(t, \mu)]Z(t, \mu)$$

has an effect on the asymptotics of the solutions of the system (3.19). The asymptotics of the solutions of the Dirac system (3.19) as well as (3.17) now depend on the two diagonalisations and the first $[I + Q(t, \mu)]$ -transformation. Therefore, by performing backward transformation after obtaining the transformed system (4.48) which is assumed to be in LBL-form, the solutions of the system (3.17) as well as (3.19) are given by the product of the matrices

$$T(t, \mu), [I + B(t, \mu)] \text{ and } [I + Q(t, \mu)]$$

applied on the product of the matrix of normalised eigenvectors with remainder elements that tend to zero as $|\mu| \rightarrow \infty$ and the direct product of corrected diagonal system. Hence, the solution of (3.17) is the matrix $D(t, \mu)$ given by

$$D(t, \mu) = T(t, \mu)[I + B(t, \mu)][I + Q(t, \mu)][E(t, \mu) + o(I)] \prod_{s=t_0}^{t_1-1} \tilde{\Lambda}(s, \mu)$$

with $D(t_0, \mu) = I$. The matrix $T(t, \mu)[I + B(t, \mu)][I + Q(t, \mu)][E(t, \mu) + o(I)]$ does not affect the asymptotic behaviour of the solution but rather the matrix $\prod_{s=t_0}^{t_1-1} \tilde{\Lambda}(s, \mu)$.

The matrix $\prod_{s=t_0}^{t_1-1} \tilde{\Lambda}(s, \mu)$ has dominant unbounded diagonal and thus of the corresponding $D(t, \mu)$. The associated discretised Dirac operator has compact resolvent and the values of μ in the strip P_d are isolated points hence the spectrum is discrete at most. □

4.3 Dirac Operators with Unbounded Potentials

We extend the results of Theorem (4.2.3) to the case of bounded spectral parameter and unbounded potential $\delta_1(t)$. Hence, we assume

$$\mu = o(\delta_k), |\delta_1(t)| \nearrow \infty, \delta_2(t) = o(\delta_1(t)) \quad \text{and} \quad 0 \leq |\mu| < 1 \quad (4.50)$$

for $k = 1, 2$ and as $t \rightarrow \infty$ and the Hilbert space which is $\ell^2(\mathbb{N})$. Note that unlike in Section 4.2.2 where $t \in [t_0, t_1]$, here we will have $t \in \mathbb{N}$. We also assume that

$$\Delta(\delta_2(t)) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \quad (4.51)$$

The assumption in (4.51) is to ensure that the correction terms to the eigenvalues after first diagonalisation are controllable. The analysis carried out in this section can also be done in a similar way without alteration of results when $|\delta_2(t)| \nearrow \infty$ as $t \rightarrow \infty$ while $\delta_1(t) = o(\delta_2(t))$ and $\Delta(\delta_1(t)) \rightarrow 0$ as $t \rightarrow \infty$. The characteristic polynomial (4.31) yields the characteristic roots approximated to $O(\delta_1^{-3})$ given by:

$$\lambda_1(t, \mu, \delta_k) \approx (1 - i\mu)^{-1}(1 + \delta_2\delta_1) - (1 - i\mu)\delta_2^{-1}\delta_1^{-1} - (1 - i\mu)^3\delta_2^{-3}\delta_1^{-3} + O(\delta_1^{-3}) \quad (4.52)$$

$$\lambda_2(t, \mu, \delta_k) \approx (1 - i\mu)\delta_2^{-1}\delta_1^{-1} + (1 - i\mu)(1 + (1 - i\mu)^2)\delta_2^{-2}\delta_1^{-2} + O(\delta_1^{-3}).$$

After computing and normalizing the first component of the respective eigenvectors, we obtain the transformation matrix of the form:

$$T(t, \mu, \delta_k) \approx \begin{bmatrix} 1 & 1 \\ \delta_2(t) + O(\delta_1^{-1}(t)) & \delta_1^{-1}(t) + O(\delta_1^{-2}(t)) \end{bmatrix} \quad (4.53)$$

with the determinant $O(\delta_2(t))$ and whose inverse is approximately

$$T^{-1}(t, \mu, \delta_k) \approx -[\delta_2(t) + O(\delta_1^{-1}(t))]^{-1} \begin{bmatrix} \delta_1^{-1}(t) + O(\delta_1^{-2}(t)) & -1 \\ -\delta_2(t) + O(\delta_1^{-1}(t)) & 1 \end{bmatrix}. \quad (4.54)$$

Just like in Section 4.2.2, after the first diagonalisation we obtain

$$\begin{aligned} D(t+1, \mu, \delta_k) &= T^{-1}(t+1, \mu, \delta_k)C(t, \mu, \delta_k)T(t, \mu, \delta_k)Y(t, \mu, \delta_k) \\ &= (\Lambda + R)(t, \mu, \delta_k)D(t, \mu, \delta_k) \end{aligned}$$

where

$$R(t, \mu, \delta_k) = -T^{-1}(t+1, \mu, \delta_k)\Delta T(t, \mu, \delta_k)\Lambda(t, \mu, \delta_k). \quad (4.55)$$

The elements of $R(t, \mu, \delta_k)$ are approximately given by

$$R(t, \mu, \delta_k) \approx \begin{bmatrix} -\delta_2^{-1}(1 + \delta_2\delta_1)\Delta\delta_2 & -(1 - i\mu)\delta_2^{-2}\delta_1^{-1}\Delta\delta_1^{-1} \\ \delta_2^{-1}(1 + \delta_2\delta_1)\Delta\delta_2 & (1 - i\mu)\delta_2^{-2}\delta_1^{-1}\Delta\delta_1^{-1} \end{bmatrix} \quad (4.56)$$

whereas $\Lambda(t, \mu, \delta_k)$ is approximately given by

$$\begin{bmatrix} (1 + \delta_2\delta_1) - (1 - i\mu)\delta_2^{-1}\delta_1^{-1} + O(\delta_1^{-2}) & 0 \\ 0 & (1 - i\mu)\delta_2^{-1}\delta_1^{-1} + O(\delta_1^{-2}) \end{bmatrix}. \quad (4.57)$$

Explicitly, working on the off-diagonal terms of the matrix $R(t, \mu, \delta_k)$ in (4.55) we have

$$\delta_2^{-1}(t)\Delta(\delta_2(t)), \quad \delta_1(t)\Delta(\delta_2(t)), \quad \delta_2^{-1}(t)\delta_1^{-1}(t)\Delta(\delta_1^{-1}(t)) \in \ell^2. \quad (4.58)$$

The correction terms to the diagonals are given by

$$R_{11}(t, \mu, \delta_k) \approx -\delta_2^{-1}(1 + \delta_2\delta_1)\Delta(\delta_2(t)) \text{ and } R_{22}(t, \mu, \delta_k) \approx (1 - i\mu)\delta_2^{-1}\delta_1^{-1}\Delta(\delta_1^{-1}(t)).$$

It is the term $R_{11}(t, \mu, \delta_k)$ that warranted the assumption in (4.51). This results into corrected eigenvalues of the form:

$$\begin{aligned} \tilde{\lambda}_1(t, \mu, \delta_k) &\approx \lambda_1(t, \mu, \delta_k) + R_{11}(t, \mu, \delta_k) \\ \tilde{\lambda}_2(t, \mu, \delta_k) &\approx \lambda_2(t, \mu, \delta_k) + R_{22}(t, \mu, \delta_k). \end{aligned}$$

Making transformation similar to (4.38) in Section 4.2.2, we obtain the following after first diagonalisation

$$W(t+1, \mu, \delta_k) \approx [\tilde{\Lambda}(t, \mu, \delta_k) + \tilde{R}(t, \mu, \delta_k)]W(t, \mu, \delta_k) \quad (4.59)$$

with

$$\tilde{\Lambda}(t, \mu, \delta_k) \approx \text{diag}(\tilde{\Lambda}(t, \mu, \delta_k))$$

and

$$\tilde{R}(t, \mu, \delta_k) \approx \begin{bmatrix} 0 & -(1 - i\mu)\delta_2^{-2}\delta_1^{-1}\Delta(\delta_1^{-1}(t)) \\ \delta_2^{-1}(1 + \delta_2\delta_1)\Delta\delta_2(t) & 0 \end{bmatrix}$$

where $\tilde{R}_{kj}(t, \mu, \delta_k) \in \ell^2$, $k \neq j$. Now as in Section 4.2.2, we proceed with second diagonalisation by computing the matrix $B(t, \mu, \delta_k)$. Thus, we use $B_{kk}(t, \mu, \delta_k) = 0$ and for $k \neq j$, $B_{kj}(t, \mu, \delta_k) = (\lambda_j - \lambda_k)R_{kj}$, which implies that

$$\begin{aligned} B_{12}(t, \mu, \delta_k) &\approx (1 - i\mu)^2\delta_2^{-3}\delta_1^{-2}\Delta(\delta_1^{-1}) \\ B_{21}(t, \mu, \delta_k) &\approx (1 - i\mu)\delta_2^{-1}\Delta(\delta_2). \end{aligned}$$

The correction terms to the diagonals are

$$\begin{aligned} (\tilde{R}B)_{11} &\approx O((1 - i\mu)\delta_2^{-3}\delta_1^{-1}\Delta(\delta_1^{-1})\Delta(\delta_2)) \\ (\tilde{R}B)_{22} &\approx O((1 - i\mu)\delta_2^{-3}\delta_1^{-1}\Delta(\delta_1^{-1})\Delta(\delta_2)). \end{aligned}$$

The second diagonalisation is then obtained through transformation similar to (4.41) so that a form similar to (4.42) can be obtained. We perform the $(I + Q(t, \mu, \delta_k))$ transformation as discussed earlier in Section 4.2.2. Hence we obtain that

$$\begin{aligned} V_{kk}(t, \mu, \delta_k) &= 0 \\ V_{kj}(t, \mu, \delta_k) &= (RB)_{kj}(t, \mu, \delta_k) \\ &= O(\delta_2^{-3}\delta_1^{-1}\Delta(\delta_2)\Delta(\delta_1^{-1})) \end{aligned}$$

while

$$\begin{aligned} Q_{12}(t, \mu, \delta_k) &= -Q_{21}(t, \mu, \delta_k) \\ &\approx -\sum_{s=t_0}^{t_1-1} O(\delta_2^{-2}\Delta(\delta_2)\Delta(\delta_1^{-1})) \prod_{l=s}^t [O(\delta_2^{-2}\delta_1^{-2})]. \end{aligned}$$

In the next result, we have extended the results of Lemma (4.2.1) to the case of unbounded potentials and bounded spectral parameter is thus formulated.

Theorem 4.3.1. *Let the discretised Dirac systems be given by (3.19) and assume that (4.50) and (4.51) are satisfied then the corrected eigenvalues of (3.20) are given by*

$$\begin{aligned} \tilde{\lambda}_1(t, \mu, \delta_k) &\approx \lambda_1(t, \mu, \delta_k) - \delta_2^{-1}(1 + \delta_2\delta_1)\Delta(\delta_2(t)) - (1 - i\mu)\delta_2^{-3}\delta_1^{-1} \cdot \\ &\quad \Delta(\delta_1^{-1})\Delta(\delta_2) \\ \tilde{\lambda}_2(t, \mu, \delta_k) &\approx \lambda_{t,\mu,\delta_k}(t) + (1 - i\mu)\delta_2^{-1}\delta_1^{-1}\Delta(\delta_1^{-1}(t)) + (1 - i\mu)\delta_2^{-3}\delta_1^{-1} \cdot \\ &\quad \Delta(\delta_1^{-1})\Delta(\delta_2). \end{aligned}$$

Here, $\lambda_k(t, \mu, \delta_k)$ are as given in (4.52).

Proof. The proof follows closely that of Lemma (4.2.1). The only difference is the expansion of the eigenvalues based on the assumptions in (4.50). Thus the magnitude of the eigenvalues are determined by the leading value of $\delta_1(t)$. The rest follow from Lemma (4.2.1). $\square$

The next result is for the case of unbounded potential and bounded spectral parameter.

Theorem 4.3.2. *The asymptotic behaviour of the Dirac system (3.17) under the conditions (4.50)-(4.51) are determined by the magnitude of $\delta_1(t)$ in the corrected eigenvalues and the solutions are of the form*

$$D(t, \mu, \delta_k) = T(t, \mu, \delta_k)(I + B(t, \mu, \delta_k))(I + Q(t, \mu, \delta_k)) \cdot (E(t, \mu, \delta_k) + o(I_2)) \prod_{s=t_0}^{t-1} \tilde{\Lambda}(s, \mu, \delta_k), \quad (4.60)$$

where $\tilde{\Lambda}(t, \mu, \delta_k) = \text{diag}(\tilde{\lambda}(t, \mu, \delta_k))$ with

$$\begin{aligned} \tilde{\lambda}_1(t, \mu, \delta_k) &= \lambda_1(t, \mu, \delta_k) + O(\delta_1 \Delta(\delta_2)) + O(\delta_2^{-3} \delta_1^{-1} \Delta(\delta_2) \Delta(\delta_1^{-1})) \\ \tilde{\lambda}_2(t, \mu, \delta_k) &= \lambda_2(t, \mu, \delta_k) + O(\delta_2^{-2} \delta_1^{-1} \Delta(\delta_1^{-1})) + O(\delta_2^{-3} \delta_1^{-1} \Delta(\delta_2) \Delta(\delta_1^{-1})). \end{aligned}$$

Proof. The proof closely follows from that of Theorem (4.2.3). Thus diagonalisations based on the assumption (4.51) will make off-diagonal terms to be ℓ^1 and those that are still ℓ^2 - terms their decay can be accelerated using $[I + Q(t, \mu)]$ -transformations. The rest now follow from the proof of Theorem (3.10.1). Here, the asymptotic behaviour of the solutions are determined by the matrix $\prod_{s=t_0}^{t-1} \tilde{\Lambda}(s, \mu, \delta_k)$ whose diagonals are the eigenvalues of $C(t, \mu, \delta_k)$ and hence the leading term of $\delta_1(t)$ plays a great role. Since the diagonal of $\tilde{\Lambda}(s, \mu, \delta_k)$ has unbounded term in $\tilde{\lambda}_1(t, \mu, \delta_k)$, the resolvent of the corresponding discretised Dirac operator is compact and the spectrum is discrete as expected. $\square$

The example below illustrates stability of national income of a capitalist state.

Example 4.3.3. Let the national income, $Y(t)$ be given by

$$Y(t) = C(t) + I(t) + G(t)$$

where $C(t)$ is consumer expenditure, $I(t)$ is induced private investment for buying capital equipment and $G(t)$ is government expenditure. $C(t)$ is proportional to national income of previous year $Y(t-1)$ via

$$C(t) = \alpha Y(t-1),$$

$\alpha > 0$ is marginal propensity to consume. $I(t)$ is proportional to the increase in consumption,

$$I(t) = \beta [C(t) - C(t-1)],$$

$\beta > 0$ is the relation. $G(t)$ is constant over the years with negligible variations and we may chose $G(t) = 1$. Reducing the equations we obtain

$$Y(t+2) - \alpha[1 + \beta]Y(t+1) + \alpha\beta Y(t) = 1, \quad t \in \mathbb{N}.$$

We form the vector valued equations (quasidifferences):

$$Y_1(t) = Y(t), \quad \text{and} \quad Y_2(t) = Y(t+1).$$

This implies that

$$\begin{aligned} Y_1(t+1) &= Y(t+1) = Y_2(t) \\ Y_2(t+1) &= Y(t+2) = \alpha[1 + \beta]Y(t+1) - \alpha\beta Y(t). \end{aligned}$$

Therefore,

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \quad \text{and} \quad Y(t+1) = \begin{bmatrix} Y_1(t+1) \\ Y_2(t+1) \end{bmatrix}$$

and we obtain

$$Y(t+1) = A(t)Y(t)$$

where $A(t)$ is a 2×2 matrix. Explicitly we have that

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\alpha\beta & \alpha[1 + \beta] \end{bmatrix}$$

Then we get

$$\det(A - \lambda I_2) = 0.$$

This yields the characteristic equation

$$\lambda^2 - (\alpha + \alpha\beta)\lambda + \alpha\beta = 0$$

whose roots are calculated as follows:

$$\lambda_{1/2} = \frac{\alpha + \alpha\beta}{2} \pm \left\{ \frac{[-(\alpha + \alpha\beta)]^2}{4} - \alpha\beta \right\}^{\frac{1}{2}}$$

Hence we obtain that

$$\begin{aligned} \lambda_1 &\approx \alpha[1 + \beta] - \beta[1 + \beta]^{-1} - \alpha^{-1}\beta^2[1 + \beta]^{-3} \\ \lambda_2 &\approx \beta[1 + \beta]^{-1} + \alpha^{-1}\beta^2[1 + \beta]^{-3} \end{aligned}$$

The equilibrium state of the national income

$$Y^*(t) = \frac{1}{1 - \alpha}$$

is asymptotically stable (stable in theory of economics) if and only if

$$\alpha < 1, \quad 1 + \alpha + 2\alpha\beta > 0, \quad \alpha\beta < 1.$$

Here, the spectrum is pure discrete. $Y(t)$ fluctuates (oscillatory) around the equilibrium state if

$$\alpha < \frac{4\beta}{(1 + \beta)^2}.$$

If $\alpha = 0.5$ and $\beta = 1$, $Y^*(t) = 2G(t)$ irrespective of initial boundary conditions. Other components of the spectrum are present.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

This research was concerned with the asymptotic behaviour of solutions of the discretised unbounded positive symmetric and Dirac operators on Hilbert spaces. The focus was particularly on the stability of existence of positive self-adjoint operator extensions for the case of unbounded perturbations. It has specifically added onto existing knowledge in functional analysis, operator theory and discrete spectral theory. We constructed and analysed the perturbed difference operators with unbounded coefficients and provided insights into the necessary and sufficient conditions for stability of of existence positive self-adjoint operator extensions, their spectral properties as well as the asymptotic behaviour of solutions of the discretised Dirac system with unbounded potentials.

We managed to construct positive perturbed difference operators with unbounded potentials by use of inner products as defined in Hilbert spaces. The coefficient growth conditions necessary for the construction of positive symmetric difference operators were successfully derived. It was shown that the leading coefficient of the fourth order symmetric difference operator L_1 dominated the coefficients of the second order symmetric difference operator L_2 ensuring that both the perturbed and unperturbed operators L and L_1 respectively remained positive and symmetric. This construction was central in establishing the stability of those operators under unbounded perturbations.

The necessary and sufficient conditions for the stability of positive self-adjoint operator extensions were also considered under unbounded perturbations. We established that the deficiency indices of the perturbed and unperturbed difference operators were (n, n) implying that each operator had positive self-adjoint extensions even if the coefficients were unbounded. The study showed that the absolutely continuous spectra of the self-adjoint extensions of the perturbed and unperturbed difference operators were subsets of $(0, \infty)$ having spectral multiplicity of two whenever the leading coefficient of the unperturbed difference operator was unbounded. This finding emphasises the stability of the operators' spectral properties under unbounded perturbations.

We also investigated on the spectral properties of self-adjoint operator extensions of the perturbed difference operators with unbounded coefficients. This was done by application of the M-matrix of square summable eigensolutions so as to determine the spectral measures of positive self-adjoint operator extensions. We established that the absolutely continuous spectrum of the self-adjoint extensions remained invariant under unbounded perturbations and that their spectral multiplicity was determined by the rank of the M-matrix. These findings give a comprehensive understanding of the spectral stability of the operators considered in this study.

Besides, we examined the asymptotic behaviour of solutions of the discretised Dirac system having unbounded potential. This was done by use of matrix asymptotic summation based on the Levinson-Benzaid-Lutz's theorem. We established that the solutions of the discretised Dirac system were as a result of the product of their transforming matrices and the direct product of the eigenvalues matrix. The asymptotic behaviour of these solutions were determined by the unbounded potentials. This thesis shows that the solutions exhibited asymptotic behaviour under various growth and decay conditions of the potentials and the spectral parameter. This analysis was important for the understanding of the stability of solutions for systems with unbounded potentials.

Implications of the Research

This research has advanced the understanding of stability of existence of positive self-adjoint operator extensions under unbounded perturbations. It has contributed

to the broader theory of unbounded operators in Hilbert spaces by establishing the conditions under which the self-adjoint operator extensions remained stable.

The results of this research have practical applications in analysis of stability of national income and other economic models. The stability of national income implies a number of aspects of quality of life, like economic security, employment opportunities and access to essential services. These results can be used in analysis of stability of national income overtime hence providing valuable insights to economists and those charged with the responsibility of making policies. Besides, the results on asymptotic behaviour of solutions of the discretised Dirac systems are applicable in fields like quantum mechanics where Dirac operators play an essential role.

In addition, this study has introduced and applied mathematical techniques like asymptotic summation with regards to the discretised Levinson's theorem and the M-matrix to analyse the stability and spectral properties of unbounded operators. These approaches could be extended to various areas of mathematical physics and applied mathematics thus giving a basis for analysing similar problems.

Limitations and Recommendations

In this study, we considered fourth and second order symmetric difference operators. However, analysis of higher order, sixth order and above, presents additional challenges as a result of the complexity of computations involved in determining the roots of the polynomials of higher degree. We recommend that future research could consider the use of numerical analysis techniques to investigate the stability and spectral properties of higher order operators.

We particularly considered symmetric operators in this study. We therefore recommend that this analysis could be extended to non-symmetric operators so as to provide a more comprehensive understanding of the stability and spectral properties of a broader class of operators. Such extension would require that new theoretical frameworks and methodologies are developed.

The results of this study have potential applications in various physical systems, including quantum field theory and general relativity. We recommend that future research could explore the application of these results in analysis of the stability of

fermionic fields in the presence of external fields like electromagnetic or gravitational fields.

Finally, we recommend that numerical simulations and experimental validation could offer more insights into the practical implications of these findings. As such, future research could involve the development of numerical algorithms to simulate how the operators behave under different conditions and hence validate the theoretical predictions.

Conclusion

In conclusion, this study has made huge contributions to the understanding of the asymptotic behaviour of solutions of the discretised unbounded positive symmetric and Dirac operators on Hilbert spaces. Through construction of positive perturbed difference operators with unbounded coefficients, analysis of the stability of positive self-adjoint operator extensions and investigation of the spectral properties of these extensions, this study has given valuable insights into the stability and the behaviour of unbounded operators. The results have significant implications for both theoretical and applied mathematics with potential applications in analysis of economic stability as well as in quantum mechanics. While there are limitations and areas of future research, the results of this study provide strong foundation for further exploration and development in the field of operator theory and spectral analysis.

This study has illustrated that the stability of existence of positive self-adjoint operator extensions is robust under unbounded perturbations and that the spectral properties of these extensions are very well defined even under unbounded coefficients. The asymptotic behaviour of the solutions of the discretised Dirac systems have been analysed and provide a comprehensive understanding of the stability of solutions under various growth and decay conditions. These results advance the theoretical understanding of unbounded operators as well as the practical applications in different fields. Hence this study was a major contribution to mathematical sciences.

In summary, this research has addressed the challenges connected with stability and spectral analysis of unbounded operators. It has provided a good framework

for future research. The results have the potential to influence a broad range of applications, from economic stability analysis to quantum mechanics, outlining the need for continued research in the field of operator theory and spectral analysis.

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