

**THE MINIMAL NILPOTENT COVER OF THE FINITE
SYMMETRIC, ALTERNATING AND DIHEDRAL GROUPS**

BY

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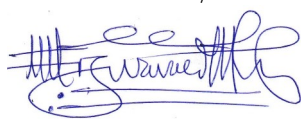
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DECLARATION

Declaration by the candidate

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DEDICATION

I dedicate this work to my family, parents and supervisors.

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ABSTRACT

Given a group G , a minimal nilpotent cover is a family of subgroups of G of minimal size subject to the property that all of the subgroups are nilpotent and their union is equal to the group G . A maximal non-nilpotent subset is a subset of G of largest size such that for any two distinct elements, the subgroup they generate is not nilpotent. There has been effort to investigate covers of groups with particular structure, specifically normal and abelian covers; much has not been done on nilpotent covers. The main objective of this study was to conduct an analysis of minimal nilpotent cover for each of the three families of groups; finite symmetric group S_n , alternating group A_n and dihedral group D_{2k} , which are ubiquitous throughout the discipline of mathematics. The specific objectives were to; investigate nilpotent covers, analyse their relationship with non-nilpotent subsets, investigate the size of minimal nilpotent cover and maximal non-nilpotent subset and apply group theoretic properties in determining a general formula for the size of the minimal nilpotent cover for each family. The analysis was carried out by means of mathematical proofs based on logical approach together with results from computer algebra package **GAP** and properties of permutations as well as group action on a k -gon. It has been established that except for A_9 , the size of minimal nilpotent cover for each of S_n , A_n for $n = 3, 4, \dots, 10$, and D_{2k} , coincide with size of maximal non-nilpotent subset. For D_{2k} , a general formula is produced for $k = 3, 4, \dots$. The results of this study benefit scientists in identifying redundancy in the analysis of possible organic molecular structures, solve molecular conformation problems as well as mathematicians in expanding knowledge in group theoretic concepts. For S_n , only for $n = 3, 4, \dots, 10$ were analyzed since as n grows large, it becomes complex and even computer algebra package **GAP** cannot generate any results. We therefore recommend an investigation on a general formula for minimal nilpotent cover of the finite symmetric group S_n as well as the finite alternating group, A_n .

NOTATIONS AND TERMINOLOGIES

Our terminology and notation in this thesis is entirely standard. Below we list key pieces of notation that have been used throughout.

A_n	Alternating group on n letters.
S_n	Symmetric group on n letters.
D_{2k}	The dihedral group of order $2k$ (The symmetry group of the k -gon).
$\sigma_c(G)$	The maximum size of a non- c -nilpotent subset of G .
$\Sigma_c(G)$	The minimum size of a c -nilpotent cover of G .
$\sigma_\infty(G)$	The maximum size of a non-nilpotent subset of G .
$\Sigma_\infty(G)$	The minimum size of a nilpotent cover of G .
$\sigma(G)$	The covering number of a finite group G .
$\rho(G)$	The maximum size of a subset of G for which any pair of elements generates G .
$\Phi(G)$	Fratini subgroup of G .
$N_G(H)$	The normalizer of subgroup H in G .
$\mu(G)$	The least positive integer k for which G has more than one maximal subgroup of index k .
C_n	Cyclic group of order n .
C_p	Cyclic group of prime order p .
$\xi(G)$	The normal covering number of G .
$\delta(G)$	The size of maximal non-commuting subset of G .
$\Delta(G)$	The size of minimal abelian cover G .
$P_{p,k}$	Sylow p -subgroup of S_k .

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CHAPTER ONE

INTRODUCTION

1.1 Background

Throughout this chapter G is a group. A *cover* for a group G is a finite set of subgroups of G whose union equals G . A *minimal cover* of G is a cover in which all the redundancies have been removed. The *covering number* for a finite group G , is the size of the smallest number of proper subgroups whose union equals G (Cohn, 1994). Certain group theoretic properties have been imposed on these minimal covers in order to establish more meaningful and useful information of the group G .

One of the earliest types of covers with restrictive properties imposed on it was the normal cover which contains conjugacy classes of subgroups of G . Bubboloni (Bubboloni & Praeger, 2011) studied normal coverings of symmetric and alternating groups. Indeed the *normal covering number*, which we denote by $\xi(G)$ is the smallest number of conjugacy classes of subgroups in a normal cover for G . The normal covering number is defined for all finite non-cyclic subgroups. Furthermore there always exist minimal normal covers consisting of maximal subgroups.

A study of abelian covers with a bias on alternating groups was conducted recently by Barrantes, *et al* (Barrantes, Gill, & Ramírez, 2016).

In 2015 Nick Gill, *et al* (Azad, Britnell, & Gill, 2015) worked on nilpotent covers and non-nilpotent subsets of finite groups of Lie type. In the study we have just cited the connection between nilpotent cover and non-nilpotent set is established.

Little or no attention has been given to covers of dihedral groups let alone minimal cover or even minimal nilpotent cover which is our area of study in this

research.

The minimal nilpotent covers of the finite symmetric group S_n , which is the group of permutations of n elements, the alternating group A_n , the group of all even permutations of n elements and the dihedral group D_{2k} , the group of symmetries of a regular k -gon have not been investigated and hence our study. Note that A_n is a normal subgroup of the corresponding symmetric group S_n . Indeed S_n contains both even and odd permutations. While the order of S_n is $n!$, that of A_n is $\frac{n!}{2}$.

The relationship between nilpotent subgroup structure and its maximal non-nilpotent set is key in this study as found in (Azad et al., 2015).

1.2 Basic definitions

1.2.1 Covers

Definition 1.2.1. A *cover* of a group G is a set $\mathcal{F} = \{G_1, \dots, G_k\}$, which is a family of subgroups of G such that $\bigcup_{i=1}^n G_i = G$.

Example 1.2.1. Let $G = S_3$, the symmetric group on 3 symbols, 1, 2, 3. That is,

$$S_3 = \{(1), (123), (132), (12), (13), (23)\}.$$

(Using disjoint cycle notation). The following are some subgroups of G :

(i) $G_1 = \{(1), (123), (132)\},$

(ii) $G_2 = \{(1), (12)\},$

(iii) $G_3 = \{(1), (13)\},$

(iv) $G_4 = \{(1), (23)\},$

The following are some *covers* of G : $\mathcal{F}_1 = \{G_1, G_2, G_3, G_4\}$, $\mathcal{F}_2 = \{S_3, G_2, G_3, G_4\}$, $\mathcal{F}_3 = \{G_1, G_2, G_3, S_3\}$, while the following are not *covers* of G : $\mathcal{F}_4 = \{G_1, G_2\}$,

$\mathcal{F}_5 = \{G_3, G_4\}$, $\mathcal{F}_6 = \{G_1, G_2, G_3\}$. For example, $\mathcal{F}_4$ is not a cover of G because the permutation (13) does not belong to G_1 or G_2 .

1.2.2 Abelian groups

In this subsection we give the definition of an abelian group and exhibit some examples. Abelian subgroups of a group plays an important role in the construction of nilpotent covers of a group. The concepts of abelian series, upper and lower central series, derived series of a group (all to come later) require a clear understanding of an abelian group.

Definition 1.2.2. Let G be a group. G is said to be *abelian* or *commutative* if for every $x, y \in G$, $xy = yx$.

Example 1.2.2. The set $\mathbb{Z}$ of integers with the usual composition operation of addition is an infinite abelian group with identity element 0.

Example 1.2.3. The alternating group $A_3 = \{(1), (123), (132)\}$, is a finite abelian group. It is clear that $(123)(132) = (132)(123) = (1)$.

Definition 1.2.3. Let $a \in G$. If there exists a least positive integer n such that $a^n = 1$, where 1 is the identity in G , then n is called the *order* of a . We say that the element a is of finite order. If no such integer exists then the elements $a^1, a^2, \dots$ are all different. We say the element a is of an infinite order.

Definition 1.2.4. An element $a \in G$ is called an *involution* if it is of order 2.

Definition 1.2.5. A group G is said to be *cyclic* if it is generated by one of its elements, that is, for some element $a \in G$,

$$G = \langle a \rangle = \{a^n : n \in \mathbb{Z}^+\}$$

In this case a is called a *generator* for G . If there exists a smallest positive integer m such that $a^m = 1$, then $|\langle a \rangle| = |G| = m$, the order of G . Otherwise G is of infinite order.

Example 1.2.4. The alternating group $A_3 = \{(1), (123), (132)\}$ is cyclic. Each of the elements $(123), (132)$ is a generator, so one can write $A_3 = \langle (123) \rangle = \langle (132) \rangle$.

Example 1.2.5. The group $\mathbb{Z}$ of integers under addition is cyclic of infinite order where, $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$. The only generators are 1 and -1 .

Example 1.2.6. The group $\mathbb{Z}_n$ of integers under addition modulo n is cyclic of order n . For instance, $\mathbb{Z}_4 = \{0, 1, 2, 3\} = \langle 1 \rangle = \langle 3 \rangle$.

Example 1.2.7. $W_n = \langle \omega \rangle$, the multiplicative group of the n^{th} roots of unity in $\mathbb{C}$ (the field of complex numbers) is cyclic of order n , where $\omega = \cos(\frac{2\pi}{n}) + i \sin(\frac{2\pi}{n})$ is the primitive n^{th} root of unity. For instance when $n = 4$, $W_4 = \{1, i, -1, -i\} = \langle i \rangle = \langle -i \rangle$.

Theorem 1.2.1. (Fraleigh, 2002, Theorem 6.1) *Let G be a cyclic group, then G is abelian.*

Proof. Let $G = \langle a \rangle = \{a^n : n \in \mathbb{Z}^+\}$. If $g_1, g_2 \in G$, then there exists integers m, n such that $g_1 = a^m, g_2 = a^n$. Clearly $g_1 g_2 = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m = g_2 g_1$. Thus G is abelian. $\square$

The division algorithm plays an important role in the study of cyclic groups and we state and prove it here formally.

Theorem 1.2.2. (Fraleigh, 2002, Theorem 6.3) *Let $m \in \mathbb{Z}^+$, that is m is a positive integer and $n \in \mathbb{Z}$, that is n is any integer, then there exist unique integers p and r such that $n = mp + r$, where $0 \leq r < m$. This is called the division algorithm.*

Proof. Intuitively we may think of the real x -axis and mark off the multiples of m and the position of n , then either n coincides with a multiple pm of m so in this case $r = 0$ or n lies between any two multiples of m . Suppose in the latter case, we let pm be the first multiple of m to the left hand side of n . Then r is such that $0 \leq r < m$. It is now clear from this argument that p and r are unique because if n is not a multiple of m so we can take $r = 0$ and so there is a unique

multiple pm of m to the left of n and at a distance less than m from n . This completes the proof. $\square$

Note that in the expression $n = mp + r$, p is called the quotient while r is the remainder which must be non negative when we divide n by m . This expression is important when we want to examine whether a given subgroup H of a cyclic group G is also cyclic.

Example 1.2.8. Find the quotient p and remainder r when -20 is divided by 6 according to the division algorithm. Solution: The negative multiples of 6 are $-6, -12, -18, -24$, choosing the multiple to leave a non negative remainder less than 6 , we write $-20 = -24 + 4 = 6(-4) + 4$, hence the quotient is $p = -4$ and the remainder is $r = 4$.

Theorem 1.2.3. (Rose, 1994) Let G be a cyclic group. If H is any subgroup of G then H is cyclic.

Proof. Suppose $G = \langle x \rangle$, that is the cyclic group G is generated by an element x . Let H be a subgroup of G . If $H = \{e\}$, here e is the identity in G then H is cyclic. On the other hand if $H \neq \{e\}$, then $x^n \in H$ for some $n \in \mathbb{Z}^+$. Let k be the smallest integer in $\mathbb{Z}^+$ such that $x^k \in H$ our claim is that $t = x^k$ generates H ; that is, $H = \langle x^k \rangle = \langle t \rangle$. We need show that every $y \in H$ is a power of t . Since $y \in H$ and H is a subgroup of G , we have $y = x^n$ for some n . We seek for p and r such that $n = kp + r$ for $0 \leq r < k$ applying the division algorithm. Then $x^n = x^{kp+r} = (x^k)^p x^r$, so $x^r = (x^k)^{-p} x^n$. Further more, $x^n \in H$, $x^k \in H$ and H is a group, both $x^{(k)-p}$ and x^n are in H . Therefore $(x^k)^p x^n \in H$; that is, $x^r \in H$. Note that we chose k to be the smallest positive integer such that $x^k \in H$ and $0 \leq r < k$, so we must have $r = 0$. Hence $n = pk$ and $y = x^n = (x^k)^p = t^p$, so y is a power of t . $\square$

Theorem 1.2.4. (Herstein, 1975) Let $G_1, G_2, \dots, G_n$ be groups. For $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ in $\prod_{i=1}^n G_i$, define $(x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n)$ to be the element

$(x_1y_1, x_2y_2, \dots, x_ny_n)$. Then $\prod_{i=1}^n G_i$ is a group, the direct product of the groups $G_i, i = 1, 2, \dots, n$, under this binary operation.

Proof. Let $x_i \in G_i, y_i \in G_i$, and that each G_i is a group, with $x_iy_i \in G_i$. The definition of the binary operation on $\prod_{i=1}^n G_i$ which is given in the assertion of the theorem makes sense, because $\prod_{i=1}^n G_i$ is closed under the binary operation. The associative law in $\prod_{i=1}^n G_i$ holds and is similar to the associative law in each component as the following shows:

$$\begin{aligned}
 \prod_{i=1}^n G_i &= (x_1, x_2, \dots, x_n)[(y_1, y_2, \dots, y_n)(z_1, z_2, \dots, z_n)] \\
 &= (x_1, x_2, \dots, x_n)(y_1z_1, y_2z_2, \dots, y_nz_n) \\
 &= (x_1(y_1z_1), x_2(y_2z_2), \dots, x_n(y_nz_n)) \\
 &= ((x_1y_1)z_1, (y_2z_2)z_2, \dots, (y_nz_n)z_n) \\
 &= (x_1y_1, y_2z_2, \dots, y_nz_n)(z_1, z_2, \dots, z_n) \\
 &= (x_1, x_2, \dots, x_n)(y_1, y_2, \dots, y_n)(z_1, z_2, \dots, z_n)
 \end{aligned}$$

If e_i is the identity element in G_i , then clearly, with multiplication by components, $(e_1, e_2, \dots, e_n)$ is an identity in $\prod_{i=1}^n G_i$. Finally, an inverse of $(x_1, x_2, \dots, x_n)$ is $(x_1^{-1}, x_2^{-1}, \dots, x_n^{-1})$; compute the product by components. Hence $\prod_{i=1}^n G_i$ is a group. $\square$

It is important to note that the external direct product of two groups is really a construction of a large group out of two smaller groups. Another of product of groups related to the direct product of groups is the internal direct product of two proper subgroups of the group.

Definition 1.2.6. Let G be a group with subgroups H and K such that $G = HK = \{hk : h \in H, k \in K\}$; $H \cap K = \{e\}$; $hk = kh$ for all $k \in K$ and $h \in H$. Then G is called the internal direct product of H and K .

Example 1.2.9. The dihedral group D_{12} is an internal direct product of its two

subgroups $H = \{1, x^3\} \cong Z_2$ and $K = \{1, x^2, x^4, y, x^2y, x^4y\} \cong S_3$, therefore $HK \cong Z_2 \times S_3$.

We note that, not every group can be written as the internal direct product of two of its proper subgroups. For example if the group S_3 were an internal direct product of its proper subgroups H and K , then one of the subgroups, say H , would be of order 3. It is easy to see that H is the subgroup $\{(1), (123), (132)\}$. The subgroup K must have order 2. Now, whichever subgroup K we choose, the condition that $hk = kh$ will never be satisfied for $h \in H$ and $k \in K$.

Theorem 1.2.5. (*Rose, 1994*) *Let G be the internal direct product of subgroups H and K . Then G is isomorphic to $H \times K$.*

Proof. Suppose that G is an internal direct product, then any element $g \in G$ can be written as $g = hk$ for some $h \in H$ and some $k \in K$. Let us define a map $\Phi : G \rightarrow H \times K$ by $\Phi(g) = (h, k)$ for some $h \in H$ and some $k \in K$. We need to show that Φ is a well-defined map; that is, we must show that h and k are uniquely determined by g . Let $g = hk = h'k'$. Then $h^{-1}h' = k(k')^{-1}$ is in both H and K , so it must be the identity. Therefore, $h = h'$ and $k = k'$, which proves that Φ is, in fact a well-defined map. The map Φ preserves the group operation, since if we let $g_1 = h_1k_1$ and $g_2 = h_2k_2$ we observe that $\Phi(g_1g_2) = \Phi(h_1k_1h_2k_2) = \Phi(h_1h_2, k_1k_2) = \Phi(h_2h_1k_2k_1) = (h_1, k_1)(h_2, k_2) = \Phi(g_1)\Phi(g_2)$. $\square$

Naturally we can extend the above definition to a collection of subgroups $\mathcal{C} = \{H_1, H_2, \dots, H_n\}$ of G where $G = H_1H_2\dots H_n = \{h_1h_2\dots h_n / h_i \in H_i\}$, $H_i \cap (\cup_{i \neq j} H_j) = \{e\}$ and $h_ih_j = h_jh_i$ for all $h_i \in H_i$ and $h_j \in H_j$.

Theorem 1.2.6. (*Herstein, 1975, Theorem 2.14.1*) *Every finite abelian group is a direct product of cyclic groups of prime power order.*

This theorem is also referred to as the fundamental theorem of finitely generated abelian groups. To illustrate the theorem, suppose that $G = \{e, x, y, xy\}$ is an abelian group of order 4, where we have the generator relations: $x^2 = y^2 =$

$e, xy = yx$. We have the following cyclic subgroups of G : $A = \langle x \rangle$, $B = \langle y \rangle$, $C = \langle xy \rangle$. We can decompose G as a direct product of these cyclic subgroups. For example:

$$G = A \times B = C \times B = A \times C.$$

Notice that while the decomposition of G into cyclic subgroups is not unique the orders of the cyclic subgroups are.

Example 1.2.10. If G is a group such that $|G| = n = p_1 p_2 \cdots p_r$ is a product of distinct primes $p_i, i = 1, 2, \dots, r$, then there is only one abelian group of order n up to isomorphism namely, C_n . For the decomposition $G = C_{p_1} \times C_{p_2} \times \cdots \times C_{p_r}$ is isomorphic to C_n . In the symmetric group S_5 the abelian subgroup

$$H = \{(1), (12), (345), (354), (12)(345), (12)(354)\}$$

of order 6 is a direct product of two cyclic subgroups, that is $H = \langle (12) \rangle \times \langle (345) \rangle$.

Theorem 1.2.7. (Aschbacher, 1986) *Let p be a prime number. Then every group G of order p^2 is abelian.*

Proof. If G is not cyclic, then every element except e must be of order p . Let x be such an element. Then the cyclic subgroup $\langle x \rangle$ of order p does not exhaust G . Also let $y \in G$ with $y \notin \langle x \rangle$. Then $\langle x \rangle \cap \langle y \rangle = \{e\}$, since an element z in $\langle x \rangle \cap \langle y \rangle$ with $z \neq e$ would generate both $\langle x \rangle$ and $\langle y \rangle$, giving $\langle x \rangle = \langle y \rangle$, contrary to construction. Now $\langle x \rangle$ is normal in some subgroup of order p^2 of G , that is, normal in all of G . Likewise $\langle y \rangle$ is normal in G . Now $\langle x \rangle \vee \langle y \rangle$ is a subgroup of G properly containing x and of order dividing p^2 . Hence $\langle x \rangle \vee \langle y \rangle$ must be all of G . Thus G is isomorphic to $\langle x \rangle \times \langle y \rangle$ and therefore abelian. $\square$

Theorem 1.2.8. (Fraleigh, 2002) *Let G be a finite abelian p -group and suppose that $g \in G$ has maximal order. Then G is isomorphic to $\langle g \rangle \times H$ for some subgroup H of G*

Proof. Let the group G be of order p^n . Using induction on n we have that for $n = 1$, the group G is cyclic of order p and is therefore generated by the element g . Next assume the statement holds for all integers k with $1 \leq k < n$ and suppose g is of maximal order in G , say $|g| = p^m$. Then $x^{p^m} = e$ for all $x \in G$. Now choose y in G such that y is not in $\langle g \rangle$, where y has the smallest possible order. Certainly such a y exists; otherwise, $G = \langle g \rangle$ and we are done. Let $H = \langle y \rangle$. We claim that $\langle g \rangle \cap H = \{e\}$. We show that $|H| = p$. Since $|y^p| = |y|/p$ the order of y^p is smaller than the order of h and must be in $\langle g \rangle$ by the minimality of y ; that is, $y^p = g^r$ for some number r . Hence, $(g^r)^{p^{m-1}} = (y^p)^{p^{m-1}} = y^p = e$, and the order of g^r must be less than or equal to p^{m-1} . Therefore, g^r cannot generate $\langle g \rangle$. Notice that p must occur as a factor of r , say $r = ps$, and $y^p = g^r = g^{ps}$. Define x to be $g^s h$. Then x cannot be in $\langle g \rangle$; otherwise, y would also have to be in $\langle g \rangle$. Also, $x^p = g^{sp} h^p = g^{-r} h^p = h^p h^p = e$. Certainly we have formed an element x with order p such that x is not in $\langle g \rangle$. Moreover by choice of y as the element with the smallest order of all of the elements that are not in $\langle g \rangle$, $|H| = p$. Now we will show that the order of gH in the factor group G/H must be the same as the order of g in G . If $|gH| < \langle g \rangle = p^m$, then $H = (gH)^{p^{m-1}} = g^{p^{m-1}} H$; hence, $g^{p^{m-1}}$ must be in $\langle g \rangle \cap H = \{e\}$, which contradicts the fact that the order of g is p^m . Therefore, gH must have maximal order in G/H . Using the well known correspondence Theorem and our induction hypothesis, $G/H \cong gH \times K/H$, for some subgroup K of G containing H . We claim that $\langle g \rangle \cap K = \{e\}$. If $b \in \langle g \rangle \cap K$, then $bH \in \langle gH \rangle \cap K/H = \{H\}$ and $b \in \langle g \rangle \cap H = \{e\}$. It follows that $G = \langle g \rangle K$ implies that $G \cong \langle g \rangle \times K$ □

We close this subsection by giving the definition of a Lie group.

Definition 1.2.7. A Lie group G is a group that is also a smooth manifold such that the group operation $\mu : G \times G \rightarrow G$ defined by $\mu(g, h) = gh$ and the inversion map $i : G \rightarrow G$ defined by $i(g) = g^{-1}$ are smooth maps.

Lie groups describe continuous symmetries in mathematics and physics. Some examples include $\mathbb{R}^n$, the n dimensional Euclidean space under addition, $GL(n, \mathbb{R})$

the group of invertible $n \times n$ real matrices.

1.2.3 Nilpotent groups

In this subsection we give definitions and some examples of p -groups, a review of Sylow theorems and Sylow p -subgroups. In a study of this scope it is natural to make use of normal subgroups. The role played by normal subgroups in the construction of quotient groups is hard to ignore in a study involving groups. We find the “abelianess” of a group; that is, how close a group is to being abelian. To do this we determine how close the center of the group is to the whole group by looking at its upper central series. This can equivalently be done by finding how the derived group or the commutator subgroup $[G : G]$ is close to the trivial group. We achieve this by looking at the derived series or the lower central series of the group.

The definition of nilpotent groups and subgroups with some examples follow, but first we give a review of Lagrange’s Theorem, a normal subgroup, normalizer and centralizer of a group G . Through out this subsection p is a prime.

Definition 1.2.8. Let G be a group and H a subgroup of G . The subset $xH = \{xh : h \in H\}$ of G is the left co set of H containing x , while the subset $Hx = \{hx : h \in H\}$ is the right co set of H containing x .

Theorem 1.2.9. (*Rose, 1994*) Let G be a finite group and H a subgroup of G . Then the order of H is a divisor of the order of G .

Proof. Let $|G| = t$ be the order of G , and let $|H| = k$ be the order of H . Every left or right co set of a subgroup H of a group G has the same number of elements as H . Suppose m is the number of cells in the partition of G into left co sets of H . Then $t = mk$, so k is indeed a divisor of t . $\square$

Let H be a subgroup of the group G . H is called a normal subgroup of G if $g^{-1}hg \in H$ for each $g \in G$ and $h \in H$. We write $H \trianglelefteq G$ to mean that H is a normal subgroup of G . Suppose $\mu : G \rightarrow X$ is a group homomorphism, we define

the Kernel of μ as $\text{Ker}(\mu) = \{g \in G : \mu(g) = e\}$, where e is the identity element of G . It is well known that the Kernel is a normal subgroup of G . We may also conclude that $\mu(G) = \{\mu(g) : g \in G\}$ is the image of G in X . It is clear that $\mu(G)$ is a subgroup of X .

If we let $H \trianglelefteq G$, and define the map $\pi : T \rightarrow T/H$, then π is a bijection between the set of subgroups of G containing H and the set of all subgroups of G/H . Under this bijective map, normal subgroups correspond to normal subgroups. Suppose we let $a, b \in G$, and we write $a^b = b^{-1}ab$. Now for $X \subseteq G$, we set $X^b = \{a^b : a \in X\}$. Then we know that X^b is the conjugate of X under y . We define X^G for the set $\{X^g : g \in G\}$, as the set of conjugates of X under G . If we define $N_G(X) = \{g \in G : X^g = X\}$, then $N_G(X)$ is the normalizer in G of X . It turns out that if X is a subgroup of G then $N_G(X)$ is the largest subgroup of G in which X is normal. The normalizer $N_G(X)$ of a subgroup X of a group G is the set of all inner automorphisms that carry X onto itself. Finally we define $C_G(X) = \{g \in G : xg = gx \forall x \in X\}$. Then $C_G(X)$ is the centralizer in G of X which is also a subgroup of G .

Theorem 1.2.10. (*Aschbacher, 2000*) *Let G be a group containing normal subgroups H and K such that $H \cap K = \{e\}$ and $H \vee K = G$. Then G is isomorphic to $H \times K$.*

Proof. Let $h \in H$ and $k \in K$, we show that $hk = kh$. First note that the commutator $hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} = h(kh^{-1}k^{-1})$. Now H and K are normal subgroups of G so (hkh^{-1}) and $(kh^{-1}k^{-1})$ imply that $hkh^{-1}k^{-1}$ is an element of K and also an element H . But then $H \cap K = \{e\}$, and therefore $hkh^{-1}k^{-1} = e$, so $hk = kh$. Let $\pi : H \times K \rightarrow G$ be defined by $\pi(h, k) = hk$. Then $\pi((h, k)(h', k')) = \pi(hh', kk') = hh'kk' = hkh'k' = \pi(h, k)\pi(h', k')$, so π is a homomorphism. If $\pi(h, k) = e$, then $hk = e$, so $h = k^{-1}$, and both h and k are in $H \cap K$. Therefore $h = k = e$, so $\text{Ker}(\pi) = \{(e, e)\}$ and π is one to one. Note that $HK = H \vee K$, and $H \vee K = G$, and so π is onto G , and $H \times K \simeq G$. $\square$

Definition 1.2.9. A finite group G is called a *p-group* if it has order a power of

a prime p .

A subgroup of a group G is a p -subgroup of G if the subgroup is itself a p -group. A finite group G has a subgroup of every prime power order dividing $|G|$.

Lemma 1.2.1. (Aschbacher, 1984) Let G be a non trivial p -group, then the centre of G that is, $Z(G)$ is not trivial.

Proof. Let us suppose that G acts on itself by conjugation. If $t \in G$ we have that $G_t = \{g \in G : g^{-1}tg = t\} = C_G(t)$. Since we then have that $G/C_G(t) \cong G/G_t$, $t \in Z(G)$ if and only if $G_t = \{t\}$. Then the centre $Z(G)$ is the union of the orbits of G containing only one element. We know that the orbits of G partition G , and if $\{G(t_i)\}_{i \in \mathbb{N}}$ is a disjoint collection of all orbits of G containing at least two elements, then $|G| = |Z(G)| + \sum_{i \in \mathbb{N}} |G(t_i)|$. If $G = Z(G)$ and from our assumption that G is non trivial, $Z(G)$ is non trivial, so we may assume that $Z(G) < G$. Then $\mathbb{N}$ is not empty. $[G : C_G(t_i)] = |G(t_i)| > 1$ and G is a p -group, so $p |G(t_i)|$ for all $i \in \mathbb{N}$. Thus $p |Z(G)|$ so $Z(G)$ is not trivial. $\square$

The equation $|G| = |Z(G)| + \sum_{i \in \mathbb{N}} [G : C_G(t_i)]$, where $\{G(t_i)\}_{i \in \mathbb{N}}$ is a disjoint collection of all orbits of G containing at least two elements, is called the class equation of G .

Lemma 1.2.2. (Artin, 1957) If G has order p^2 where p is a prime, then G is abelian.

Proof. From the previous Lemma 1.2.1, $Z(G)$ is not trivial, so $Z(G)$ has order p or p^2 . Suppose for a contradiction that $Z(G)$ has the order p . Then $G/Z(G)$ also has order p , so it is cyclic. If we let $Z(G)k$ to be a generator, and further let $g, h \in G$. Then $g = xk^m$ and $h = yk^n$ for some positive integers m and n and for some $x, y \in Z(G)$. Then $gh = xk^m yk^n = yk^n xk^m = hg$ since $x, y \in Z(G)$, so G is commutative, that is, abelian, hence a contradiction. Thus $Z(G)$ has order p^2 , so $Z(G) = G$ and G is abelian. $\square$

Theorem 1.2.11. (*Aschbacher, 1986*) Let G be a finite group and p be a prime where p divides the order of G , then G must contain an element of order p .

Proof. We begin the proof by assuming that the group G is abelian. let $|G| = pn$, where $n \geq 1$. We proceed by mathematical induction on n . If $|G| = p$, then G is cyclic and therefore must contain an element of order p . Let us assume that the result is true for some $n > 1$. Let $g \in G$ with order $m > 1$. If p divides m , then $g^{m/p}$ has order p , and the result follows. So we may assume that p does not divide m . Note that $G/\langle g \rangle$ is an abelian group of order $\frac{pn}{m}$. Since p does not divide m and $\frac{pn}{m}$ is an integer, $\frac{n}{m}$ must be an integer. But $\frac{n}{m} < n$, as $m > 1$, so by induction, $G/\langle g \rangle$ contains an element of order p , say $\langle g \rangle h$. Then if h has order k , $(\langle g \rangle h)^k = \langle g \rangle$ so p divides k . Again, G contains an element of order p , and the result follows. Suppose now that G is any finite group with p dividing the order of G . The proof is by induction on $|G|$. If p divides $|Z(G)|$, then since $Z(G)$ is abelian, $Z(G)$ contains an element of order p , and immediately the result follows. Further more if we assume that p does not divide $|Z(G)|$, then by the class equation, there exists a $g \in G$ for which $[G : C_G(g)] > 1$ and p does not divide $[G : C_G(g)]$. Since p divides $|G|$ and p divides $|C_G(g)|$, but $C_G(g) < G$, then by induction, $C_G(g)$ contains an element of order p , this completes the proof. $\square$

Example 1.2.11. The group V (the Klein 4-group), the group Q_8 (Quaternion group) are examples of p -groups.

The first of the Sylow theorems, talks of the existence of prime-power subgroups of G , for any prime power dividing $|G|$. The following Lemma will be used.

Lemma 1.2.3. Let H be a p -subgroup of a finite group G . Then $(N_G(H) : H) \equiv (G : H) \pmod{p}$, where $(N_G(H))$ is the normalizer of H in G

Theorem 1.2.12. (*Herstein, 1975*) (**First Sylow Theorem**). Let G be a finite group and let $|G| = p^r m$ where $r \geq 1$ and where p does not divide m . Then

- (a) G contains a subgroup of order p^i for each i where $1 \leq i \leq r$,
- (b) Every subgroup H of G of order p^i is a normal subgroup of a subgroup of order p^{i+1} for $1 \leq i \leq r$.

Proof.

- (a) By the well known Cauchy's theorem (Meo, 2004), G contains a subgroup of order p . We use an induction on p^i for $i < n$ to show that the existence of a subgroup of order p^i implies the existence of a subgroup of order p^{i+1} . Let H be a subgroup of order p^i . Since $i < n$, we see p divides $(G : H)$. By Lemma 1.2.3 we know that p divides $(N_G(H) : H)$, Since H is a normal subgroup of $(N_G(H))$, we can form $(N_G(H)/H)$, and we see that p divides $(N_G(H)/H)$. By Cauchy's theorem, the factor group $(N_G(H)/H)$ has a subgroup K which is of order p . If $\varphi : (N_G(H)(N_G(H)/H))$ is the canonical homomorphism, then $\varphi^{-1}[K] = xN_G(H)/\varphi(x) \in K$ is a subgroup of $N_G(H)$ hence of G . This subgroup contains H and is of order p^{i+1} .
- (b) Using part(a) above we see that $H < \varphi^{-1}[K] \leq N[H]$, where $|\varphi^{-1}[K]| = p^{i+1}$. Since H is normal in $N_G(H)$, it is clear that it is normal. $\square$

Definition 1.2.10. A Sylow p -subgroup P of a group G is a maximal p -subgroup of G , that is, a p -subgroup contained in no larger p -subgroup.

Let G be a finite group, where $|G| = p^r m$ as in Theorem 1.2.12 which shows that the Sylow p -subgroups of G are precisely those subgroups of order p^r . If P is a Sylow p -subgroup, every conjugate gPg^{-1} of P is also a Sylow p -subgroup. The second Sylow theorem states that every Sylow p -subgroup can be obtained from P by conjugation; that is, any two Sylow p -subgroups are conjugate.

Theorem 1.2.13. (Herstein, 1975)(**Second Sylow Theorem**). Let P_1 and P_2 be Sylow p -subgroups of a finite group G . Then P_1 and P_2 are conjugate subgroups of G .

Proof. Suppose we let one of the subgroups act on left co sets of the other and further let $\mathcal{C}$ be the collection of left co sets of P_1 , and let P_2 act on $\mathcal{C}$ by $y(xP_1) = (yx)P_1$ for $y \in P_2$. Then $\mathcal{C}$ is a P_2 -set. Then $|\mathcal{C}P_2| \equiv |\mathcal{C}| \pmod{p}$, and $|\mathcal{C}| = (G : P_1)$ is not divisible by p , so $|\mathcal{C}P_2| = 0$. Let $xP_1 \in \mathcal{C}P_2$. Then $yxP_1 = xP_1$ for all $y \in P_2$, so $x^{-1}yxP_1 = P_1$ for all $y \in P_2$. Thus $x^{-1}yx \in P_1$ for all $y \in P_2$, so $x^{-1}P_2x \leq P_1$. Since $|P_1| = |P_2|$, we must have $P_1 = x^{-1}P_2x$, so P_1 and P_2 are indeed conjugate sub-groups. $\square$

The final Sylow theorem gives information on the number of Sylow p -subgroups. A few illustrations are given after the theorem, and many more are given in the next section.

Theorem 1.2.14. (*Herstein, 1975*)(**Third Sylow Theorem**). *If G is a finite group and p divides $|G|$, then the number of Sylow p -subgroups is congruent to 1 modulo p and divides $|G|$.*

Proof. Let P be one Sylow p -subgroup of G . Let $\mathcal{L}$ be the set of all Sylow p -subgroups and let P act on $\mathcal{L}$ by conjugation, so that $x \in P$ carries $T \in \mathcal{L}$ into xTx^{-1} . But then $|\mathcal{L}| \equiv |\mathcal{L}_P| \pmod{p}$. If $T \in \mathcal{L}_P$, then $xTx^{-1} = T$ for all $x \in P$. Thus $P \leq N[T]$, the normalizer of T . It is straightforward that $T \leq N[T]$ and since P and T are both Sylow p -subgroups of G , they are also Sylow p -subgroups of $N[T]$. Note also that they are conjugate in $N[T]$. Since T is a normal subgroup of $N[T]$, it is its only conjugate in $N[T]$. Thus $T = P$. Then $\mathcal{L}_P = \{P\}$. Since $|\mathcal{L}| \equiv |\mathcal{L}_P| \equiv 1 \pmod{p}$, we see the number of Sylow p -subgroups is congruent to 1 modulo p . Now let G act on $\mathcal{L}$ by conjugation. Since all Sylow p -subgroups are conjugate, there is only one orbit in $\mathcal{L}$ under G . If $P \in \mathcal{L}$, then $|\mathcal{L}| = |\text{Orb}(P)| = (G : G_P)$, where G_P is the normalizer of P . But $(G : G_P)$ is a divisor of $|G|$, so the number of Sylow p -subgroups divides $|G|$. $\square$

Theorem 1.2.15. (*Neumann, 1976*) *If p and q are distinct primes with $p < q$, then every group G of order pq has a single subgroup of order q and this subgroup is normal in G . Hence, G cannot be simple. Furthermore, if $q \equiv 1 \pmod{p}$, then*

G is cyclic.

Proof. We know that G contains a subgroup H of order q . The number of conjugates of H divides pq and is equal to $1 + kq$ for $k = 0, 1, \dots$. However, $1 + q$ is already too large to divide the order of the group; hence, H can only be conjugate to itself. That is, H must be normal in G . The group G also has a Sylow p -subgroup, say K . The number of conjugates of K must divide q and be equal to $1 + kp$ for $k = 0, 1, \dots$. Since q is prime, either $1 + kp = q$ or $1 + kp = 1$. If $1 + kp = 1$, then K is normal in G . It can be verified easily that G is the internal direct product of H and K . Furthermore, H is isomorphic to Z_q and K is isomorphic to Z_p , $G \cong Z_p \times Z_q \cong Z_{pq}$ $\square$

Definition 1.2.11. Let G be a group. Write $|G| = p^r m$, the order of G , where p does not divide m . A *Sylow p -subgroup* of G is a p -subgroup P of G having the maximal order p^r . Equivalently, a p -subgroup P of G is a *Sylow p -subgroup* exactly when p does not divide the index $[G : P]$.

Example 1.2.12. In the symmetric group S_3 , we have $|S_3| = 6 = 2 \times 3$. Thus the Sylow 2-subgroups are $\{(1), (12)\}$, $\{(1), (13)\}$ and $\{(1), (23)\}$. The only Sylow 3-subgroup is $\{(1), (123), (132)\}$. In the alternating group A_4 , we have $|A_4| = 12 = 2^2 \times 3$. Thus the Sylow 2-subgroup is $V = \{(1), (12)(34), (13)(24), (14)(23)\}$. The Sylow 3-subgroup are the cyclic subgroups $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$, $\langle(234)\rangle$.

Example 1.2.13. By applying the Sylow Theorems, we can determine that A_5 has subgroups of orders 2, 3, 4, and 5. Any Sylow p -subgroups of A_5 must have order 3 or 4 or 5. The third Sylow Theorem gives the number of Sylow p -subgroups contained in A_5 . We know that the number of Sylow 5-subgroups must divide 60 and are also congruent to 1 (mod 5). Therefore we expect either one or six Sylow 5-subgroups in A_5 , but then all Sylow 5-subgroups are conjugate. Clearly if there were only a single Sylow 5-subgroup then it would be conjugate to itself; that is, it would be a normal subgroup of A_5 . We also know that A_5 has no normal subgroups, so this cannot happen. We therefore conclude that there are exactly six distinct Sylow 5-subgroups of A_5 .

Example 1.2.14. Let $G = D_{2k}$, $k \geq 3$, be the dihedral group of order $2k$. Write $|D_{2k}| = m2^r$, where m is odd. The Sylow m -subgroup C_m is the cyclic subgroup generated by a rotation of order m . The Sylow 2-subgroups are isomorphic to D_{2^r} , where the reflections in D_{2^r} act by inversion on C_m .

Definition 1.2.12. If G is a group and H a subgroup of G , H is said to be a *normal* subgroup of G if H is self conjugate in G ; that is $g^{-1}Hg = H$, for every $g \in G$. We write $H \trianglelefteq G$ to mean that H is a normal subgroup of G .

Definition 1.2.13. A group G is said to be *simple* if it has no proper normal subgroups. That is, the only normal subgroups are the identity subgroup and the whole group G .

Definition 1.2.14. A map ϕ of a group G into a group G' is a homomorphism if the homomorphism property $\phi(xy) = \phi(x)\phi(y)$ holds for all $x, y \in G$.

Example 1.2.15. Let Γ be the natural map of $\mathbb{Z}$ into $\mathbb{Z}_n$ given by $\Gamma(p) = r$, where r is the remainder given by the division algorithm when p is divided by n , then Γ is a homomorphism. For $s, t \in \mathbb{Z}$, we show that $\Gamma(s+t) = \Gamma(s) + \Gamma(t)$. Using the division algorithm, we let $s = q_1n + r_1$ and $t = q_2n + r_2$, where $0 \leq r_i < n$ for $i = 1, 2$. Now suppose that $r_1 + r_2 = q_3n + r_3$ and for $0 \leq r_3 < n$. It is clear that $s + t = (q_1 + q_2 + q_3)n + r_3$. Therefore, $\Gamma(s + t) = r_3$. Consequently $\Gamma(s) = r_1$ and $\Gamma(t) = r_2$. The sum $(r_1 + r_2) \in \mathbb{Z}_n$ is equal to r_3 . Finally we conclude that $\Gamma(s + t) = \Gamma(s) + \Gamma(t)$, which is clearly a homomorphism.

It can be shown that a composition of group homomorphisms is also a group homomorphism. That is, if $\pi : G \rightarrow G$ and $\theta : G \rightarrow G$ are both group homomorphisms then their composition $(\pi \circ \theta) : G \rightarrow G$, where $(\pi \circ \theta)(x) = \pi(\theta(x))$ for $x \in G$, is also a homomorphism.

Theorem 1.2.16. (Aschbacher, 2000) Let $\phi : G \rightarrow G'$ be a group homomorphism with kernel H . Then the cosets of H form a factor group, G/H , where $(xH)(yH) = (xy)H$. Also, the map $\mu : G/H \rightarrow \phi(G)$ defined by $\mu(xH) = \phi(x)$ is

an isomorphism. Both co set multiplication and μ are well defined, independent of the choices x and y from the co sets.

A factor group G/H is often called the factor group of $G \pmod{H}$. Elements in the same co set of H are often said to be *congruent* $\pmod{H}$.

Theorem 1.2.17. (*Aschbacher, 1984*) *Let H be a normal subgroup of G . Then the co sets of H form a group G/H under the binary operation $(xH)(yH) = (xy)H$.*

Proof. Observe that $(xH)[(yH)(zH)] = (xH)[(yz)H] = [x(yz)]H$, and similarly, we have $[(xH)(yH)](zH) = [(xy)z]H$, hence associativity in G/H follows from associativity in G . It is also easy to see that $(xH)(eH) = (xe)H = xH = (ex)H = (eH)(xH)$, therefore $eH = H$ is the identity element in G/H . Finally, existence of inverse can be seen by noting that $(x^{-1}H)(xH) = (x^{-1}x)H = eH = (xx^{-1}H) = (xH)(x^{-1}H)$ and this shows that $x^{-1}H = (xH)^{-1}$. $\square$

There are indeed easier ways to check normality other than determining both the left and the right co set decompositions.

If N is a subgroup of G such that $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$. Then $gNg^{-1} = \{gng^{-1} : n \in N\} \subseteq N$ for all $g \in G$. Our claim is that $gNg^{-1} = N$. We must show that $N \subseteq gNg^{-1}$ for all $g \in G$. Let $n \in N$. Now if we replace $g \in G$ by g^{-1} in the relation $gng^{-1} \in N$, we obtain $g^{-1}n(g^{-1})^{-1} = g^{-1}ng = n_1$, where $n_1 \in N$. Consequently, $n = gn_1g^{-1} \in gNg^{-1}$, and the result follows. Suppose that $gN = Ng$ for all $g \in G$. Then $gn = n_1g$, so $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$. By the previous assertion, we see that $gNg^{-1} = N$ for all $g \in G$. Conversely, if $gNg^{-1} = N$ for all $g \in G$, then $gng^{-1} = n_1$ so $gn = n_1g \in Ng$ and $gN \subseteq Ng$. But also, $g^{-1}Ng = N$ giving $g^{-1}ng = n_2$, so that $ng = gn_2$ and $Ng \subseteq gN$.

Theorem 1.2.18. (*Herstein, 1975*) *The following are three equivalent conditions for a subgroup N of a group G to be a normal subgroup of G .*

(i) $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$.

(ii) $gNg^{-1} = N$ for all $g \in G$.

(iii) $gN = Ng$ for all $g \in G$.

It is important to note that a subgroup N of an abelian group G is normal. We need only note that $gn = ng$ for all $n \in N$ and all $g \in G$, clearly $gng^{-1} \in N$ for all $g \in G$ and all $n \in N$.

Definition 1.2.15. An isomorphism $\theta : G \rightarrow G$ of a group G with itself is an automorphism of G . The automorphism $i_g : G \rightarrow G$, where $i_g(x) = gxg^{-1}$ for all $x \in G$, is the inner automorphism of G by g . Performing i_g on x is called conjugation of x by g .

Note that from Theorem 1.2.18, $gN = Ng$ for all $g \in G$ if and only if $i_g[N] = N$ for all $g \in G$, that is, if and only if N is invariant under all inner automorphisms of G . One observes immediately that $i_g[N] = N$ is an equation in sets; we need not have $i_g(n) = n$ for all $n \in N$. That is i_g may perform a non trivial permutation of the set N . We see that the normal subgroups of a group G are precisely those that are invariant under all inner automorphisms. A subgroup T of G is a conjugate subgroup of N if $T = i_g[N]$ for some $g \in G$.

Theorem 1.2.19. (Rose, 1994) *A factor group of a cyclic group is cyclic.*

Proof. Let G be cyclic group with generator x , and let H be a normal subgroup of G . We claim that the co set H generates G/H . We must compute all powers of xH . But this amounts to computing, in G , all powers of the representative x and all these powers give all elements in G . Hence the powers of xH certainly give all co sets of H and G/H is cyclic. $\square$

Theorem 1.2.20. (Abdollahi & Mohammadi Hassanabadi, 2004) *Every group of prime order is cyclic.*

Proof. Suppose the group G is of prime order p , and that x is an element of G different from the identity. Then the cyclic subgroup $\langle x \rangle$ of G generated by x has at least two elements, x and e . Clearly by Theorem 1.2.9 the order of $\langle x \rangle$ must

be more than or equal to 2 and must divide the prime p . It therefore follows that the order of $\langle x \rangle$ must be p and $\langle x \rangle = G$, so G is cyclic $\square$

Theorem 1.2.21. (*Aschbacher & Scott, 1985*) *Let p and q be distinct primes with $p < q$, then every group G of order pq has a single subgroup of order q and this subgroup is normal in G . Hence G is not simple. If $q \not\equiv 1 \pmod{p}$, then G is abelian and cyclic.*

Proof. By Theorem 1.2.12 and Theorem 1.2.14 we conclude that G has a Sylow q -subgroup and that the number of such subgroups is congruent to 1 (mod q) and divides pq , and therefore must divide p . Since $p < q$, the only possibility is the number 1. Therefore there is only one Sylow q -subgroup P of G . This group P must be normal in G , for under an inner automorphism it would be carried into a group of the same order, hence itself. Thus G is not simple. Using a similar argument, there is a Sylow p -subgroup T of G , and the number of these divides pq and is congruent to 1 (mod p). This number must be either 1 or q . If q is not congruent to 1 (mod p), then the number must be 1 and T is normal in G . Suppose we assume that $q \equiv 1 \pmod{p}$. Since every element in P other than e is of order q and every element in T other than e is of order p , we have $P \cap T = \{e\}$. Also $P \vee T$ must be a subgroup of G properly containing P and of order dividing pq . Hence $P \vee T = G$ and is isomorphic to $P \times T$. $\square$

Definition 1.2.16. Let $H \leq G$. Suppose that there is a finite sequence $(H_i)_{0 \leq i \leq n}$ of subgroups of G , such that: $H = H_0 \leq H_1 \leq \dots \leq H_{n-1} \leq H_n = G$. Then this is called a series of length n from H to G (ascending) or from G to H (descending). The subgroups $H_0, H_1, \dots, H_n$ are called the terms of the series and the quotient groups $H_i/H_{i-1}, i = 1, 2, \dots, n$ the factors of the series.

The series in the above definition is called proper if $H_{i-1} \triangleleft H_i$ for every $i = 1, 2, \dots, n$. Another series $H = L_0 \trianglelefteq L_1 \trianglelefteq \dots \trianglelefteq L_m = G$ from H to G is called a refinement of $H = H_0 \leq H_1 \leq \dots \leq H_{n-1} \leq H_n = G$ if $n \leq m$ and there are no negative integers $k_0 < k_1 < \dots < k_n \leq m$ such that $H_i = L_{k_i}$ for $i = 1, 2, \dots, n$.

, that is the former can be obtained from the latter by deleting terms of the latter. In this case $H = L_0 \trianglelefteq L_1 \trianglelefteq \dots \trianglelefteq L_m = G$ is called a proper refinement of $H = H_0 \leq H_1 \leq \dots \leq H_{n-1} \leq H_n = G$ if there is a $k \in \{i = 1, 2, \dots, m\}$ such that $H_i \neq L_k$ for $i = 0, 1, \dots, n$.

A proper series of G which does not have a proper refinement is called a composition series of G . The factors of a composition series of the group G are called composition factors of G . It is well known that every finite group has a composition series.

Definition 1.2.17. A subnormal (or sub invariant) series of a group G is a finite sequence $G_0, G_1, \dots, G_n$ of subgroups of G such that $G_i < G_{i+1}$ and G_i is a normal subgroup of G_{i+1} with $G_0 = \{e\}$ and $G_n = G$. A normal (or invariant) series of G is a finite sequence $G_0, G_1, \dots, G_n$ of normal subgroups of G such that $G_i < G_{i+1}$, $G_0 = \{e\}$ and $G_n = G$.

Notice that for abelian groups the concept of subnormal and normal series coincide, since every subgroup is normal. A normal series is always subnormal, but the converse is not necessarily true.

Definition 1.2.18. A subnormal (normal) series $\{H_j\}$ is a refinement of a subnormal (normal) series G_i of a group G if $\{G_i\} \subseteq \{H_j\}$, that is, if each G_i is one of the H_j .

Definition 1.2.19. A subnormal series $\{G_i\}$ of a group G is a composition series if all the factor groups $G_i < G_{i+1}$ are simple. A normal series $\{G_i\}$ of G is a principal or chief series if all the factor groups $G_i < G_{i+1}$ are simple.

Again for abelian groups the two notions of composition and principal series coincide. It is also true that since every normal series is subnormal, then every principal series turns out to be a composition series for any group, whether it is abelian or non-abelian.

Example 1.2.16. For $S_n, n \geq 5$, the series $\{e\} < A_n < S_n$ is both a composition

series and a principal series of S_n . This is because $A_n/\{e\}$ is isomorphic to A_n , which is simple for $n \geq 5$, and S_n/A_n is isomorphic to $\mathbb{Z}_2$, which is simple.

Lemma 1.2.4. (Rose, 1994) A series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$ of G is called a *composition series* of G if and only if the factors $G_i/G_{i-1}, i = 1, 2 \dots n$ of the series are all simple groups.

Proof. If the series is a composition series of the group G then it is by definition proper, and so the factors G_i/G_{i-1} are all non trivial for $i = 1, 2 \dots n$. Indeed if some factor G_i/G_{i-1} were not simple, then it would have a non trivial normal subgroup for instance H/G_{i-1} . this implies that we would have $G_{i-1} \triangleleft H \triangleleft G_i$ and therefore obtain a proper refinement of the original series by inserting H as an extra term between G_{i-1} and G_i . This is clearly a contradiction to the definition of composition series. Therefore the factors of the composition series are simple. On the other hand if the given series is not a composition series then either the series is not proper in which case one of its factors is trivial and therefore simple or the series is not proper and has a refinement, for example $1 = H_0 \leq H_1 \leq \dots \leq H_{m-1} \leq H_m = G$. In this case let k be the largest positive integer for which H_k is not equal to any term of the original series. Then $0 < k < m$ and $H_{k+1} = l$ for some integer l with $0 < l \leq m$. Furthermore the series $(H_j)_{0 \leq j \leq m}$ is a refinement of $(G_i)_{0 \leq i \leq n}$ and by choice of k $G_{l-1} < H_k \triangleleft H_{k+1} = G_l$. Clearly H_k/G_{l-1} is a proper non trivial normal subgroup of G_l/G_{l-1} and so the factor group G_l/G_{l-1} is not simple. $\square$

Example 1.2.17. The series $1 < A_n < S_n$ for $n \geq 5$ is a composition series since A_n is a normal subgroup of S_n and $A_n/1 \cong A_n$, $S_n/A_n \cong \mathbb{Z}_2$ are simple.

Every finite group G has a composition series. It is also interesting to observe that $G_i < G_{i+1}$ is simple if and only if G_i is a maximal normal subgroup of G_{i+1} . Thus for a composition series, each G_i must be a maximal normal subgroup of G_{i+1} . To form a composition series of a group G , we must find a maximal normal subgroup G_{n-1} of G , then for a maximal normal subgroup G_{n-2} of G_{n-1} , and so

on. If this process terminates in a finite number of steps, we have a composition series. A composition series cannot have any further refinement. To form a principal series, we have to look for a maximal normal subgroup G_{n-1} of G , then for a maximal normal subgroup G_{n-2} of G_{n-1} that is also normal in G , and so on.

One more remark : For a finite group, one may think of a composition series as some kind of factorization of the given group into simple factor groups, which resembles the well known basic prime factorization of a positive integer. In the two scenarios the factorization is unique, up to order of the factors.

Definition 1.2.20. Let G be a group. The *centre* of G is the subgroup $Z(G) = \{z \in G : zg = gz, \forall g \in G\}$.

The centre $Z(G)$ is a normal subgroup of G .

Theorem 1.2.22. (Conway, Curtis, Norton, Parker, & Wilson, 1985) Let G be a group of order p^k , where k is a positive integer, then the centre of G cannot be trivial.

Proof. Suppose $x \in G$, then we know that the conjugacy class of x contains just one element if and only if the centralizer of x in G is the whole of G . In other words if and only if $x \in Z(G)$. Therefore if $Z(G) = \{e\}$, then the class equation gives $p^n = 1 + m_1 + m_2 + \dots + m_t$, where each of the positive divisors $m_1, m_2, \dots, m_t$ is a divisor of p^n and must be greater than one. Now $m_i = |G : C_G(x_i)|$. It follows that because p is prime, each of $m_1, m_2, \dots, m_t$ is a positive power of p and the sum $m_1 + m_2 + \dots + m_t$ is divisible by p . This then implies from $p^n = 1 + m_1 + m_2 + \dots + m_t$ has p as a factor or is divisible by P , which is clearly a contradiction. We therefore conclude that the center of G cannot be trivial. $\square$

The above property is important in the investigation of groups of prime power orders. As a consequence of this theorem we state that if G is a finite non-abelian simple group, then the order of G is divisible by at least two distinct primes. It

can be shown that the order of a finite non-abelian simple group is divisible by at least three distinct primes.

Definition 1.2.21. Let G be a group. Then the *ascending or upper central series* of normal subgroups of G is defined by $1 = Z_0 \trianglelefteq Z_1 \trianglelefteq Z_2 \trianglelefteq \cdots$, where $Z_1 = Z(G)$, $Z_{n+1}/Z_n = Z(G/Z_n)$.

Example 1.2.18. Let G be the group D_8 , the group of symmetries of a square. Then $Z_0 = 1$, $Z_1 = Z(D_8) = \{1, (13)(24)\}$, $Z_2/Z_1 = Z(D_8/Z_1) = D_8$, so $Z_2 = D_8$. Thus the ascending central series for D_8 is $1 \trianglelefteq \{1, (13)(24)\} \trianglelefteq D_8 \trianglelefteq D_8 \trianglelefteq \cdots$. The series terminates at $D_8 = G$.

Example 1.2.19. For the group S_3 , it is clear that $Z_0 = 1$, $Z_1 = Z(S_3) = 1$, $Z_2/Z_1 = Z(S_3/Z_1) = 1$. Thus the ascending central series for S_3 is: $1 \trianglelefteq 1 \trianglelefteq 1 \trianglelefteq 1 \cdots$. This is the case since S_3 is a non-abelian group with a trivial center.

Definition 1.2.22. Let H, K be subgroups of a group G . The *commutator subgroup* $[H, K]$ is the subgroup of G generated by all commutators $[h, k] = hkh^{-1}k^{-1}$, where $h \in H, k \in K$. Thus, the elements of $[H, K]$ are finite products of such commutators and their inverses. Since $[h, k]^{-1} = [k, h]$, we have $[H, K] = [K, H]$. The particular subgroup $[G, G]$ generated by all the commutators in G is denoted by G' and is called the *derived subgroup* or *commutator subgroup* of G .

Notice that if H and K are normal in G then $[H, K]$ is normal in G . This is clear from the fact that $g[h, k]g^{-1} = [ghg^{-1}, gkg^{-1}]$. We have $[H, K] = 1$ if and only if $hk = kh$ for all $h \in H, k \in K$. The derived group G' is the unique smallest normal subgroup K of G such that G/K is abelian. Thus G/G' is the abelian residual of G , also called G made abelian.

Theorem 1.2.23. (Conway et al., 1985) Let G be a group, then the set of all commutators $xyx^{-1}y^{-1}$ for $x, y \in G$ generates a subgroup C (the commutator subgroup) of G . This subgroup C is a normal subgroup of G . Furthermore, if N is a normal subgroup of G , then G/N is abelian if and only if $C \leq N$.

Proof. The commutators certainly generate a subgroup C . We must show that it is normal in G . Note that the inverse $(xyx^{-1}y^{-1})^{-1}$ of a commutator is again a commutator, namely, $xyx^{-1}y^{-1}$. Clearly $e = eee^{-1}e^{-1}$ is a commutator. For $z \in C$, we must show that $g^{-1}zg \in C$ for all $g \in G$, or that if z is a product of commutators, so is $g^{-1}zg$ for all $g \in G$. By inserting $e = gg^{-1}$ between each product of commutators occurring in z , we see that it is sufficient to show for each commutator that $g^{-1}aba^{-1}b^{-1}g$ is in C . But $g^{-1}(aba^{-1}b^{-1})g = (g^{-1}aba^{-1})(e)(b^{-1})g = (g^{-1}aba^{-1})(gb^{-1}bg^{-1})(b^{-1}g) = [(g^{-1}a)b(g^{-1}a)^{-1}b^{-1}][bg^{-1}b^{-1}g]$ which is in C . Thus C is normal in G . $\square$

Definition 1.2.23. An ascending series $1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \dots \trianglelefteq G$ is called an *abelian series* if all the factors G_{i+1}/G_i in the series are abelian.

If $G_i \triangleleft G$ for all i and $G_{i+1}/G_i \subset Z(G/G_i)$ the abelian series is called a *central series*.

Definition 1.2.24. Consider a collection of commutator subgroups $G^{(i)}$ for $i = 0, 1, 2, \dots$ defined recursively by $G^{(0)} = G, G^{(1)} = G' = [G : G], G^{(2)} = G'' = [G^{(1)} : G^{(1)}]$, in general $G^{(i+1)} = G^{(i)'} = [G^{(i)} : G^{(i)}]$ (This is a commutator subgroup of $G^{(i)}$, where $G^{(i+1)} \triangleleft G^{(i)}$ and $G^{(i+1)}/G^{(i)}$ is abelian). Then the series $G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \dots \supseteq G^{(n)}$, where $G^{(i+1)}/G^{(i)}$ is abelian for all i is called the *derived series* of G .

If G is an abelian group, the derived series $G' = [G : G] = 1$, terminates at the identity subgroup. The converse is not true in general as the following examples show.

Example 1.2.20. Let $G = S_n$. We have $G^{(0)} = [S_n : S_n] = A_n$. When $n = 3$, the derived series is $S_3 \triangleright A_3 \triangleright 1$ even though S_3 is not abelian. When $n = 4$, the derived series for S_4 is $S_4 \triangleright A_4 \triangleright V \triangleright 1$. When $n \geq 5$, the derived series for S_n terminates at A_n , because that of A_n terminates at A_n for each $n \geq 5$.

Definition 1.2.25. Consider the descending abelian series $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq 1$ where all the factors G_i/G_{i+1} in the series are abelian. If $G_i \triangleleft G$ for all

i and $G_i/G_{i+1} \subset Z(G/G_{i+1})$ the abelian series turns out to be a central series. Define recursively the subgroups F_i such that for $i > 0$, $F_{i+1} = [G : F_i]$ with $F_0 = G$, $F_1 = [G : G]$ and $F_2 = [G : F_1] = [G : [G : G]]$ so that $F_0 = G$, $F_1 = G'$ but F_i in general is not equal to $G^{(i)}$ for $i > 1$. The series $G = F_0 \supseteq F_1 \supseteq F_2 \dots \supseteq F_n = 1$, where $F_i \triangleleft G$ is called the *lower central series* of G .

Example 1.2.21. Let $G = S_n$. We have that $F_0 = G = S_n$, $F_1 = [S_n, S_n] = S_n' = A_n$. When $n = 3$, the lower central series is $S_3 \supset A_3 \supset A_3 \supset \dots$, since $F_2 = [S_3 : A_3] = A_3$. The lower central series for A_3 is $A_3 \supset 1$ since A_3 is abelian. When $n = 4$, the lower central series for S_4 is $S_4 \supset A_4 \supset A_4 \supset \dots$ since $F_2 = [S_4 : A_4] = A_4$. The lower central series for A_4 is $A_4 \supset V \supset V \supset \dots$ It terminates at V since $F_2 = [A_4 : V] = V$. When $n \geq 5$ the lower central series for S_n terminates at $F_1 = A_n$ for each n since $F_2 = [S_n : A_n] = A_n$ so $F_i = F_1 = A_n$ for $i \geq 1$.

Theorem 1.2.24. (Bereczky, 2000) *If G has a composition (principal) series, and if H is a proper normal subgroup of G , then there exists a composition (principal) series containing H .*

Proof. The series $\{e\} < H < G$ is both a subnormal and a normal series. Since G has a composition series $\{K_i\}$, then there exists a refinement of $\{e\} < H < G$ to a subnormal series isomorphic to a refinement of $\{K_i\}$. But as a composition series, $\{K_i\}$ can have no further refinement. Thus $\{e\} < H < G$ can be refined to a subnormal series all of whose factor groups are simple, that is, to a composition series. A similar argument holds if we start with a principal series $\{L_i\}$ of G . $\square$

Definition 1.2.26. Let G be a group. G is called *solvable* if one of the following equivalent conditions holds:

- (i) G has an abelian series.
- (ii) $G^{(n)}$ is trivial for some n (i.e the derived series terminates at 1).

Theorem 1.2.25. (Cameron, 2022) *Every finite p -group (group of prime-power order) is solvable.*

Proof. By Theorem 1.2.12, if G has order p^r , then G has a subgroup H_i of order p^i normal in a subgroup H_{i+1} of order p^{i+1} for $1 \leq i < r$. Then $\{e\} = H_0 < H_1 < H_2 < \dots < H_r = G$, is a composition series, where the factor groups are of order p and hence abelian and actually cyclic. Thus, G is solvable. $\square$

Definition 1.2.27. Let G be a group. G is said to be a *nilpotent group* if it satisfies any one of the the following equivalent conditions:

- (i) G has a normal central series.
- (ii) $F_n = 1$ for some $n \in \mathbb{N}$, where F_n is the lower central series for G .
- (iii) $Z_n = G$ for some $n \in \mathbb{N}$, where $(Z_n$ is the upper central series for G .

Definition 1.2.28. The least n such that $F_n = 1$ or equivalently $Z_n = G$, is called the *nilpotency class* of G . This number corresponds to the number of factors in each of the normal series of G , and it is the same in all the three cases.

Example 1.2.22. The trivial group is the only group of nilpotency class 0. Non-trivial abelian groups are of nilpotency class 1; an example is A_3 . Non-abelian groups in which G/Z is non-trivial and abelian, where Z is the center of G are of nilpotency class 2. For example D_8 .

Theorem 1.2.26. (Rose, 1994, Theorem 7.44) Let G be a finite p -group. Then G is nilpotent.

Proof. We argue by induction on G . The assertion is trivial when $|G| = 1$. Suppose $G \neq \{e\}$ and by induction that all finite p -groups which have smaller orders than G are nilpotent, then we know that $Z(G) \neq \{e\}$. Therefore $|G/Z(G)| < |G|$. Since $G/Z(G)$ is a finite p -group, it immediately follows by the inductive assumption that $G/Z(G)$ is nilpotent and hence G is nilpotent. $\square$

Theorem 1.2.27. (Rose, 1994, Theorem 7.46)

- (i) If G is nilpotent then all the subgroups and all quotient groups of G are nilpotent.

(ii) If G is solvable then all the subgroups and all quotient groups of G are solvable.

Proof. Assume the following series of G : $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$. Assume that $H \leq G$ and $K \trianglelefteq G$, then $1 = (G_0 \cap H) \trianglelefteq (G_1 \cap H) \trianglelefteq \dots (G_n \cap H) = H$ also $K/K = G_0K/K \trianglelefteq G_1K/K \trianglelefteq \dots \trianglelefteq G_nK/K = G/K$. Now suppose that G is nilpotent and that the first series is a central series then by definition of central series and for each $i = 1, \dots, n$, $[G_i, G] \leq G_{i-1}$, therefore $[G_i \cap H, H] \leq H \cap [G_i, g] \leq H \cap G_{i-1}$ and so $[G_iK/K, G/K] = [G_i, G]K/K \leq G_{i-1}K/K$. Now the second and the third series are central series and so H and G/K are nilpotent. Next suppose that G is soluble and that the first series is an abelian series then for each $i = 1, 2, \dots, n$, $(G_i \cap H)/(G_{i-1} \cap H) \cong G_{i-1}(G_i \cap H)/G_{i-1} \leq G_i/G_{i-1}$, so that $(G_i \cap H)/(G_{i-1} \cap H)$ is abelian. Again $G_iK/K/G_{i-1}K/K \cong G_iK/G_{i-1}K \cong G_i/G_{i-1}(G_i \cap K) \cong$ a quotient group of G_i/G_{i-1} , so that $G_iK/K/G_{i-1}K/K$ is abelian group. Therefore the first and the second series are abelian series so that H and G/K are soluble. $\square$

It is important to note that the family of soluble groups has the property that every extension of a soluble group by a soluble group is also soluble, a property which is not a common characteristic to nilpotent groups.

Theorem 1.2.28. (Rose, 1994, Theorem 7.49) Let G be a finite group such that $G = H \times K$.

(i) If H and K are both nilpotent then G is nilpotent.

(ii) If H and K are both solvable then G is solvable.

Proof. For part one, if H and K are both nilpotent then there are central series $1 = H_0 \trianglelefteq H_1 \trianglelefteq \dots \trianglelefteq H_m = H$ and $1 = K_0 \trianglelefteq K_1 \trianglelefteq \dots \trianglelefteq K_n = K$ and by inserting repetitions of terms if necessary, we may assume without loss of generality that $m = n$. Now $1 = (H_0 \times K_0) \trianglelefteq (H_1 \times K_1) \trianglelefteq \dots \trianglelefteq (H_n \times K_n) = G$ and for each $i = 1, 2, \dots, n$, $[H_1 \times K_1] = ([H_1, H] \times [K_1, K]) \leq (H_{i-1} \times K_{i-j})$. and therefore G is

nilpotent. For part two, suppose that both H and K are both soluble, by a similar argument $H \cong (H \times 1) \trianglelefteq G$ and $K \cong (1 \times K) \trianglelefteq G$ and so $G = (H \times 1)(1 \times K)$ $\square$

Theorem 1.2.29. (Rose, 1994, Theorem 7.50) *Let both Q and R be normal subgroups of P .*

(i) *If P/Q and P/R are both nilpotent then $P/(Q \cap R)$ is nilpotent.*

(ii) *If P/Q and P/R are both solvable then $P/(Q \cap R)$ is solvable.*

(iii) *Every finite group has a nilpotent residual and soluble residual*

Proof. $P/(Q \cap R)$ can be embedded in $P/Q \times P/R$. Therefore if P/Q and P/R are both nilpotent then it follows that $P/(Q \cap R)$ is nilpotent. By the same reasoning, we see that $Q/Q \cap R \cong QR/R \leq P/R$. Thus if P/R is soluble, it follows from above that $Q/Q \cap R$ is soluble. If P/Q is also soluble it can be shown that $P/(Q \cap R)$ is soluble. Now for part (iii) if P is a finite group and let R be a normal subgroup of P , such that it is of the smallest possible order such that P/R is nilpotent. Then if $Q \trianglelefteq P$ and P/Q is nilpotent, $Q \cap R \leq R$ and so by part (i) $P/(Q \cap R)$ is nilpotent, therefore by choice R , $Q \cap R = R$ so $R \leq Q$. So P/R is nilpotent residual of G . Using part (iii), a similar argument shows that G has a soluble residual. Of course among all the abelian series of soluble group there is a series that descends most rapidly and like wise among all the central series of a nilpotent group there is one which descends most rapidly and one which ascends most rapidly. $\square$

The following theorem includes some terms that will be defined later.

Theorem 1.2.30. (Rose, 1994, Theorem 11.3) *Let G be a finite group. Then the following seven statements are equivalent.*

(i) *G is nilpotent.*

(ii) *Every subgroup of G is subnormal in G .*

(iii) *Whenever $H < G$, $H < N_G(H)$.*

(iv) Every maximal subgroup of G is normal in G

(v) $G' \leq \Phi(G)$, where G' and $\Phi(G)$ are respectively the Derived and Frattini subgroups of G

(vi) Every Sylow subgroup of G is normal in G .

(vii) G is a direct product of groups of prime power orders.

Proof. We first suppose that the group G is non trivial. Now all the statements are trivially true and obviously (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) for if we let $H < G$, then by (ii), there is a series from H to G so there is a proper series from H to G , $H = H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_n = G$. Since $H < G$, $n > 0$. Then $H < H_1 \leq N_G(H)$. Also (iii) $\Rightarrow$ (iv). If we let M to be a maximal subgroup of G , by (iii), $M < N_G(M) \leq G$. Now the maximality of M implies that $N_G(M) = G$, thus $M \triangleleft G$ again (iv) $\Rightarrow$ (v), let M to be a maximal subgroup of G , by (iv), $M \triangleleft G$, then G/M is cyclic, of prime order, hence $G' \leq M$. This fact is true for every maximal subgroup M of G and so by definition the Frattini subgroup $\Phi(G)$, $G' \leq \Phi(G)$. Next (v) $\Rightarrow$ (vi), let M to be a maximal subgroup of G , then (v) implies that $G' \leq M$. Hence M/G' is a subgroup of the abelian group G/G' and so $M/G' \triangleleft G/G'$ and so $M \triangleleft G$. Notice that (iv) $\Rightarrow$ (vi), for if we let P be a Sylow p -subgroup of G , and suppose that $N_G(P) < G$, then there is a maximal subgroup M of G which contains $N_G(P)$, but then $N_G(M) = M$, that is $P \not\leq G$. We see that (vi) $\Rightarrow$ (vii), by letting the distinct prime divisors of $|G|$ be $p_1, \dots, p_s$, where s is a positive integer and by part (vi), G has a normal Sylow p_i subgroup P_i for each $i = 1, 2, \dots, s$. Therefore $G = D_r \prod P_i, i = 1, 2, \dots, s$. Clearly (vii) $\Rightarrow$ (i). This completes the proof. $\square$

We close this section by noting the containments between different classes of groups, namely: Cyclic groups $\subset$ Abelian groups $\subset$ Nilpotent groups $\subset$ Solvable groups $\subset$ all groups. We remark from this observation that the results that are valid for our object of study (nilpotent covers of groups) will by containment be valid for abelian groups and cyclic groups as well.

1.2.4 Nilpotent and Abelian covers

In this subsection we give the definitions of a maximal nilpotent subgroup of a group. We define nilpotent covers, minimal nilpotent and abelian covers and offer some examples. The subsection concludes by a definition of a c -nilpotent cover and a description of non- c -nilpotent subsets of a group.

Definition 1.2.29. Let G be a non-trivial group. A proper subgroup M of G is said to be a *maximal subgroup* of G if there is no proper subgroup L of G such that $M < L < G$.

Definition 1.2.30. Let G be a non-trivial group. A proper nilpotent subgroup M of G is said to be a *maximal nilpotent subgroup* of G if there is no proper nilpotent subgroup L of G such that $M < L < G$.

Definition 1.2.31. Let $\mathcal{F} = \{G_1, \dots, G_k\}$ be a family of subgroups of G whose union equals G . Then $\mathcal{F}$ is said to be a *nilpotent cover* for G if $G_1, \dots, G_k$ are nilpotent.

Example 1.2.23. In Example 1.2.1, a *nilpotent cover* for $G = S_3$ would be the family $\mathcal{F} = \{G_1, G_2, G_3, G_4\}$, since each of the subgroups is a p -group.

Definition 1.2.32. Let $\mathcal{F} = \{G_1, \dots, G_k\}$ be a family of nilpotent subgroups of G whose union equals G , then $\mathcal{F}$ is called a *minimal nilpotent cover* of G if it is of smallest size.

The set of all maximal nilpotent subgroups of G forms a nilpotent cover for G which in general is not a minimal cover.

Example 1.2.24. In the family $\mathcal{F} = \{G_1, G_2, G_3, G_4\}$ of Example 1.2.23, all the subgroups are maximal nilpotent.

Definition 1.2.33. Let $\mathcal{F} = \{G_1, \dots, G_k\}$ be a family of subgroups of G whose union equals G . Then $\mathcal{F}$ is said to be an *abelian cover* for G if $G_1, \dots, G_k$ are abelian.

Definition 1.2.34. Let $\mathcal{F} = \{G_1, \dots, G_k\}$ be a family of abelian subgroups of G whose union equals G . Then $\mathcal{F}$ is called a *minimal abelian cover* of G if it is of smallest size.

Definition 1.2.35. Let $c \in \mathbb{Z}^+ \cup \{\infty\}$. A *c-nilpotent cover* of G is a family $\mathcal{F} = \{G_1, \dots, G_k\}$ of nilpotent subgroups of G whose union is G and such that each G_i is of nilpotency class at most c . A *minimal c-nilpotent cover* is a c -nilpotent cover of least size. We denote its size by $\Sigma_c(G)$.

One observes immediately that a 1-nilpotent cover is an *abelian cover* of G and that an ∞ -nilpotent cover is a *nilpotent cover* of G .

Note that Example 1.2.23 implies that $\Sigma_\infty(S_3) \leq 4$.

1.2.5 Non- c -nilpotent subsets

Definition 1.2.36. A *non- c -nilpotent subset* is a subset $\{g_1 \dots g_k\}$ of G such that for any two distinct elements g_i, g_j the subgroup $\langle g_i, g_j \rangle$ they generate is not nilpotent of class at most c . A *maximal non- c -nilpotent subset* of G is a non- c -nilpotent subset of G of largest size. We denote this size by $\sigma_c(G)$.

A non-1-nilpotent subset is a *non-commuting subset* of G while a non- ∞ -nilpotent subset is a *non-nilpotent subset* of G .

Example 1.2.25. In S_3 , define $H = \{(12), (13), (23), (123)\}$. Any pair of elements in H generates all of S_3 (which is not a nilpotent group) and so we conclude that H is a non-nilpotent subset.

Lemma 1.2.5. Let G be a group and let $c \in \mathbb{Z}^+ \cup \infty$. Then $\sigma_c(G) \leq \Sigma_c(G)$.

Proof. Suppose $k = \Sigma_c(G)$. Then there exist c -nilpotent subgroups $G_1 \dots G_k$ of G such that their union is G . Let H be a maximal non- c -nilpotent subset of G . If h_1, h_2 are distinct elements of H it follows that they cannot belong to the same c -nilpotent subgroup G_i . Thus the largest possible size of H occurs possibly when each element $h_i \in H$ is an element of a distinct $G_i \in G$, for $i = 1, \dots, k$. Thus $|H| = \sigma_c(G) \leq k = \Sigma_c(G)$. $\square$

A natural question that arises from this lemma is: For which groups G and which $c \in \mathbb{Z}^+ \cup \{\infty\}$ do we have $\sigma_c(G) = \Sigma_c(G)$?

Example 1.2.25 implies that $\sigma_\infty(S_3) \geq 4$. But we have already seen that $\Sigma_\infty(S_3) \leq 4$. Then Lemma 1.2.5 implies that

$$4 \leq \sigma_\infty(S_3) \leq \Sigma_\infty(S_3) \leq 4.$$

We conclude immediately that $\sigma_\infty(S_3) = \Sigma_\infty(S_3) = 4$.

1.2.6 Conjugacy

Definition 1.2.37. Let G be a group. Two elements g and h are said to be *conjugate* in G when $h = xgx^{-1}$ for some $x \in G$.

When $xgx^{-1} = h$ we say x conjugates g to h . It is a matter of notation whether one writes $x^{-1}gx = h$ instead of $h = xgx^{-1}$ to mean ‘ x conjugates g to h ’. It is easy to see that this conjugacy relation is symmetric for we can have $g = aha^{-1}$ with $a = x^{-1}$.

Definition 1.2.38. Let G be a group and g an element of G . The *conjugacy class* $Cl(g)$ of g is the set of elements in G conjugate to g :

$$Cl(g) = \{xgx^{-1} : x \in G\}$$

Definition 1.2.39. The *cycle type* of a permutation in S_n is a partition of n corresponding to the cycle lengths when the permutation is written as a disjoint product of cycles. Cycles of length one represent fixed points.

Example 1.2.26. Given the permutation $\Phi = (3)(12)(456) \in S_6$, we say that it is of type $(1, 2, 3)$. Notice that one can also write this as $\Phi = (12)(456)$. Since this permutation fixes the point 3, we can also say that it is of type $(2, 3)$. The permutation $\Phi = (12)(34)(56) \in S_6$ is of type $(2, 2, 2)$.

Corollary 1.2.1. (*Rose, 1994, Corollary 5.27*) Let n be a positive integer and let $g_1, g_2 \in S_n$. Then g_1 and g_2 are conjugate in S_n if and only if g_1 and g_2 have the same cycle type; that is, if and only if the expressions of g_1 and g_2 as products of disjoint cycles contain the same number of cycles of length m , for each integer m such that $1 \leq m \leq n$.

Example 1.2.27. In S_3 , the conjugates of (23) constitutes the set:

$$\begin{aligned} Cl(23) &= \{h(23)h^{-1}, \forall h \in S_3\} \\ &= \left\{ \begin{array}{l} (1)(23)(1)^{-1}, (12)(23)(12)^{-1}, (13)(23)(13)^{-1}, \\ (23)(23)(23)^{-1}, (123)(23)(123)^{-1}, (132)(23)(132)^{-1} \end{array} \right\} \\ &= \{(23), (13), (12), (23), (12), (13)\} \\ &= \{(23), (13), (12)\}. \end{aligned}$$

Clearly all the transpositions i.e cycles of length 2 in S_3 belong to the same set.

Lemma 1.2.6. (Herstein, 1975, Lemma 2.11.3) Let $p(n)$ denote the number of partitions of the positive integer n . Then the number of conjugacy classes in S_n is $p(n)$, and each conjugacy class is determined by its cycle type.

Example 1.2.28. There are 7 partitions of the positive integer 5. These are $(5; 1 + 4; 2 + 3; 1 + 1 + 3; 1 + 2 + 2; 1 + 1 + 1 + 2; 1 + 1 + 1 + 1 + 1)$. This says that S_5 has $p(5) = 7$ conjugacy classes. It can be verified that $p(1) = 1$, $p(2) = 2$, $p(3) = 3$, $p(4) = 5$, $p(6) = 11$.

Each time a given permutation in S_n is written as a product of disjoint cycles, we always obtain a partition of n . We notice that if the cycles that arise are respectively of lengths $l_1, l_2, \dots, l_r$, where $l_1 \leq l_2 \leq \dots \leq l_r$, then $n = l_1 + l_2 + \dots + l_r$. We often say that a permutation $\rho \in S_n$ has the cycle decomposition $\{l_1, l_2, \dots, l_r\}$ if it is written as a product of disjoint cycles of lengths $l_1, l_2, \dots, l_r$, $l_1 \leq l_2 \leq \dots \leq l_r$. This is illustrated in Example 1.2.26.

Example 1.2.29. In S_3 there are 3 *conjugacy classes*. These are:

$$Cl(1) = \{1\}, Cl(12) = \{(12), (13), (23)\}, Cl(123) = \{(123), (132)\}.$$

Example 1.2.30. In $A_3 = \{(1), (123), (132)\}$, each element is its own conjugacy class thus:

$$Cl(123) = \{(123)\}, Cl(132) = \{(132)\}, Cl(1) = \{(1)\}.$$

These two examples say that two elements in the same conjugacy class in a symmetric group may not be in the same conjugacy class in the corresponding alternating subgroup.

1.2.7 Conjugacy of subgroups

The conjugation relation can also be defined for subgroups.

Definition 1.2.40. Let G be a group and H a subgroup of G . Then for all g in G , the set $Cl(H) = gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a *conjugate subgroup* of H in G .

We can therefore partition the subgroups of G into classes of conjugate subgroups.

Example 1.2.31. In the symmetric group S_3 take the non-trivial subgroups:

$$H_1 = \{(1), (12)\}, H_2 = \{(1), (13)\}, H_3 = \{(1), (23)\}, H_4 = \{(1), (123), (132)\}.$$

All the subgroups fall into four conjugacy classes of subgroups namely:

- (i) $Cl(1) = \{(1)\}$, the trivial subgroup .
- (ii) $Cl(H_1) = \{H_1, H_2, H_3\}$.
- (iii) $Cl(H_4) = \{H_4\}$.
- (iv) $Cl(S_3) = \{S_3\}$, the group itself .

Notice that H_4 is self conjugate since it is a normal subgroup of S_3 .

1.3 Statement of the problem

The problem addressed in this research is on nilpotent covers of finite groups, specifically symmetric group S_n , alternating group A_n and dihedral group D_{2k} . In particular, nilpotent covers with more restrictive properties imposed on them have been studied. Bryce and Serena (Bryce, Fedri, & Serena, 1999), for example, worked on groups with nilpotent minimal covers. Bubboloni (Bubboloni & Praeger, 2011) studied normal coverings of symmetric and alternating groups. Garonzi (Garonzi, 2013) computed the covering number of nilpotent groups. Barrantes, (Barrantes et al., 2016), studied abelian covers of the alternating groups. The connection between nilpotent cover and non-nilpotent set is established for a group G by the results in (Azad et al., 2015). Nick Gill, *et al* (Azad et al., 2015), for example, worked on nilpotent covers and non-nilpotent subsets of finite groups of Lie type.

Consequently the only groups for which nilpotent covers and non-nilpotent subsets have had their size and relationship established are finite groups of lie type.

In this research we investigated the minimal nilpotent cover for each of the three families of groups; finite symmetric, alternating and dihedral groups, which are ubiquitous throughout the discipline of mathematics.

When n in S_n , A_n and k in D_{2k} become large, the study of these minimal nilpotent covers becomes complex and the analysis gets harder. This study makes use of the investigations on the relationship between nilpotent subgroup structure and its maximal non-nilpotent set and also group theoretic properties as found in (Azad et al., 2015). The **GAP** program was used where applicable (GAP, 2013).

1.4 Justification of the study

The concept of a minimal nilpotent cover of a group G as the family of subgroups of G of minimal size subject to the property that all of the subgroups are nilpotent and their union is equal to the group G is not a straight forward adventure. There are many unanswered questions on minimal nilpotent cover of a group G that need to be investigated. The questions surrounding this group theoretic property are either out of mathematical curiosity and expansion of knowledge in group theory or real life application. In fact very little about minimal nilpotent cover of each of the three families of groups; finite symmetric group S_n , alternating group A_n and dihedral group D_{2k} , which are ubiquitous throughout the discipline of mathematics is known.

Bryce and Serena researched on groups with nilpotent minimal covers in (Bryce et al., 1999), Bubboloni studied normal coverings of symmetric and alternating groups in (Bubboloni & Praeger, 2011), Garonzi (Garonzi, 2013) computed the covering number of nilpotent groups, Barrantes, *et al* (Barrantes et al., 2016), studied abelian covers of the alternating groups and most recently Nick Gill, *et al* (Azad et al., 2015), worked on nilpotent covers and non-nilpotent subsets of finite groups of Lie type. Minimal nilpotent cover of each of the three families: S_n , A_n , D_{2k} has not been done and therefore we research on minimal nilpotent cover of the finite symmetric, alternating and dihedral groups.

1.5 Objectives of the Study

In this section we have given both the general and specific objectives of this study.

1.5.1 General Objective

The general objective of this study was to conduct an investigation and characterize the minimal nilpotent covers of each of the three families of groups; finite

symmetric, alternating and dihedral groups.

1.5.2 Specific Objectives

The specific objectives of this study were for each family to:-

- (i) Investigate nilpotent covers and analyse their relationship with non-nilpotent subsets.
- (ii) Determine the minimal number of nilpotent subgroups required to cover each group and the size of maximal non-nilpotent subset.
- (iii) Apply group theoretic properties in determining a general formula for the size of the minimal nilpotent cover.

1.6 Significance of the study

The study generalizes existing results on minimal nilpotent covers of symmetric group S_n , alternating group A_n and dihedral group D_{2k} and looks at the size of their covers. This contributes significantly to better understanding of minimal nilpotent covers of groups. The results expand the scope of study in the field of group theory and positively contribute to the discipline of mathematics today.

Our results have relevance wherever the notion of “symmetry” enters. Rose, (Rose, 1994, page 28, 2.25), notes that in modern mathematics the investigation of the symmetries of a given mathematical structure has always yielded the most powerful results.

This study benefits scientists in identifying redundancy in the analysis of possible organic molecular structures and solve molecular conformation problems. In physics the Lorenz group, Poincare group and the like all enter in the study of cosmology, while string theory leans heavily on the use groups. The study of biological functions involving protein interactions as well as gene regulation inherent in DNA proceeds formally via group symmetry. In mathematics a better

understanding of group theoretic concepts as well as easy computation of the size of minimal nilpotent cover of each of these three families is achieved. Indeed there are many more similar applications across the whole realm of fundamental science.

CHAPTER TWO

LITERATURE REVIEW

In this chapter we review earlier work on covers of groups. We have three sections: In §2.1, we review earlier work on minimal covers and covering numbers which are related to our study; in §2.2 and §2.3 we review work on covers with additional properties imposed on them. Specifically, in §2.2 we review work on normal covers and the size of minimal normal covers; in section §2.3 we set out a review on abelian covers, nilpotent covers and the size of minimal abelian and nilpotent covers. The chapter closes with identification of gaps in the literature which our research addresses.

2.1 Minimal covers and Covering numbers

Recall that a *cover* of a group G is a set of proper subgroups of G whose union is equal to G . In 1926, Scorza(Scorza, 1926) is credited for answering a basic question on finite covers of groups by proper subgroups. He succeeded in answering the question: Which groups are the union of three proper subgroups? His result is summarized in the following theorem.

Theorem 2.1.1. *[Scorza] Let G be a finite group. The minimal cover for G contains exactly 3 proper subgroups if and only if there exists a normal subgroup N of G such that the factor group $G/N \cong C_2 \times C_2$, the direct product of cyclic subgroups of order 2.*

Observe here that $G/N \cong C_2 \times C_2$ is simply the Kleins 4-group.

It is now known that a finite group G is the union of any three irredundant proper subgroups if and only if it is isomorphic to $C_2 \times C_2$ or Q_8 . Indeed there is a natural condition which is obtained by replacing the irredundant proper subgroups with distinct proper subgroups. Further, if a finite group G is the

union of any three distinct proper subgroups, then it is also the union of any three irredundant proper subgroups. $G = C_2 \times C_2$ or $G = Q_8$. Since Q_8 does not verify our condition (the collection consisting of any two distinct maximal subgroups and their intersection is not a covering of Q_8 , we infer that it is verified only by $C_2 \times C_2$.

The following result clarifies why Scorza was interested in covers consisting of three proper subgroups – because this is the smallest possible size of a cover.

Lemma 2.1.1. (Scorza, 1926) If a group G is a union of t proper subgroups, then $t \geq 3$.

Proof. We must show that $t \neq 2$. Suppose, for a contradiction, that $t = 2$, i.e. that $G = G_1 \cup G_2$, where G_1, G_2 are distinct proper subgroups of G . We use the well known lagrange's theorem and observe that $\frac{|G|}{|G_i|} \geq 2$, i.e. $|G_i| \leq \frac{1}{2}|G|$ for $i = 1, 2$. But we know that $|G_1 \cap G_2| \geq 1$, thus we obtain that:

$$|G_1 \cup G_2| = |G_1| + |G_2| - |G_1 \cap G_2| \leq \frac{1}{2}|G| + \frac{1}{2}|G| - 1 = |G| - 1 < |G|,$$

a contradiction. □

This theorem simply says that given any group G and two of its distinct proper subgroups M and N , G can never be expressed as the union, $M \cup N$. To see this, we suppose $M \not\subseteq N$ and $N \not\subseteq M$. Now since $M \neq N$, there exists an arbitrary element $m \in M \setminus N$. In a similar manner pick another element $n \in N \setminus M$. We see that the element mn is not in M because $m^{-1}(mn) = n$ which is not in M . Again $(mn)n^{-1} = m$ is not in N meaning that the element mn is not in N . It therefore follows that the elements m and n are not in $M \cup N$ and so $G \neq M \cup N$.

Much of the earliest work on covers was carried out by Cohn (Cohn, 1994) and Tomkinson (Tomkinson, 1992). In 1994 Cohn (Cohn, 1994) defined the covering number for a finite group G , to be denoted $\sigma(G)$, as the smallest number of proper subgroups whose union equals G . The statistic $\sigma(G)$ has received a great deal of attention from mathematicians in recent years. This great attention is due to

the fact that there is a connection to sets of pairwise generators of a group since a finite non cyclic group can be generated by two elements. We may define $\rho(G)$ to be the largest integer t such that there exists a set T of size t consisting of elements of G such that every pair of distinct elements of T generates G . The covering number $\sigma(G)$ provides a natural and often reliable upper bound for $\rho(G)$. It is sufficient to focus our attention to finite groups when determining covering numbers, since, by a well known result a group is the union of finitely many proper subgroups if and only if it has a finite non cyclic homomorphic image. Furthermore, if for a group G a finite cover exists, then we may find a finite homomorphic image of G . This is obtained from the quotient over the normal core of the intersection of the subgroups belonging to the finite cover. If a group is cyclic we will say that its covering number is infinite. A finite group is primitive if it admits a maximal subgroup H with trivial normal core, and the index of H in G is called the primitivity degree of G with respect to H . A finite group that has a unique minimal normal subgroup is called monolithic.

Cohn (Cohn, 1994) characterized the groups with covering number $\sigma(G) \in \{4, 5, 6\}$. He conjectured that no finite group G can have $\sigma(G) = 7$ and that if G is a finite non cyclic solvable group then $\sigma(G) = p^n + 1$ where p is a prime and n is a positive integer. The case of seven subgroups was studied by Tomkinson (Tomkinson, 1992)

A paper written by Garonzi, et.al (Garonzi & Lucchini, 2015) made a major contribution in this direction where strong evidence suggests that there are infinitely many integers that are not covering numbers. In this paper they give an infinite sequence of numbers that are covering numbers for finite groups. They proved for an integer $m \neq 3$ and p a prime power, an integer of the form $\frac{p^m - 1}{p - 1}$ is a covering number.

Example 2.1.1. The minimal cover for S_3 is

$$\mathcal{C} = \{\{1, (12)\}, \{1, (13)\}, \{1, (23)\}, \{1, (123), (132)\}\}.$$

It is therefore clear that the covering number for the symmetric group S_3 is $\sigma(S_3) = 4$.

According to Scorza, $\sigma(C_2 \times C_2) = 3$ and $\sigma(G/N) = 3$ if and only if $N \trianglelefteq G$ and $G/N \cong C_2 \times C_2$.

Example 2.1.2. Suppose G_1 and G_2 are isomorphic to the cyclic group C_2 . Let $G_1 = \langle x \rangle$, $G_2 = \langle y \rangle$, where x, y are the generators of G_1 and G_2 respectively. It is clear that $G_1 \times G_2 \cong C_2 \times C_2$. Let the identities in G_1 and G_2 be e and f respectively. $G_1 \times G_2 = \{e, x\} \times \{f, y\} = \{(e, f), (e, y), (x, f), (x, y)\}$. The minimal cover for $G_1 \times G_2$ is $\mathcal{C} = \{\{(e, f), (e, y)\}, \{(e, f), (x, f)\}, \{(e, f), (x, y)\}\}$. Thus the covering number for $G_1 \times G_2$ is $\sigma(G_1 \times G_2) = \sigma(C_2 \times C_2) = 3$.

The covering number, $\sigma(G)$, is defined for any non-cyclic finite group. To see this, simply observe that the set of all cyclic subgroups of G must form a cover for G . For cyclic groups, the covering number is undefined. Lemma 2.1.1 implies that $\sigma(G) \geq 3$ for all groups, and Theorem 2.1.1 characterizes finite groups G for which $\sigma(G) = 3$.

The previous paragraph deals with lower bounds for $\sigma(G)$. To obtain an easy upper bound, note first that if $\mathcal{F}$ is a cover for G then $\sigma(G) \leq |\mathcal{F}|$. Now if G is any non-cyclic group then, by considering again the cover given by the set of all cyclic subgroups of G , we obtain that $\sigma(G) < |G|$. A *minimal cover* for G is a collection of $\sigma(G)$ subgroups of G whose union equals G .

Theorem 2.1.2. *Let G be a finite non-cyclic group. Then there exists a minimal cover of G consisting of maximal subgroups of G .*

Proof. Let $\mathcal{F} = \{G_1 \dots G_k\}$ be a cover of a finite group G , where $G_i \in G$ is a proper subgroup of G . Let G_i^* be a maximal subgroup of G such that $G_i < G_i^* \subset G$ for each i . On replacing each G_i by G_i^* in $\mathcal{F}$ and removing redundancies, we obtain $\mathcal{H} = \{G_1^* \dots G_j^*, j < k\}$. This is a cover of G consisting of maximal subgroups. $\square$

The theorem above implies that when calculating the covering number for a finite group G , it is economical to look at covers formed by maximal subgroups.

Recall that given a group G a proper subgroup M of G is called a maximal subgroup if it is not contained in any other proper subgroup of G . Maximal subgroup or Maximal subgroups of a group G exist. One immediately observes that if G is the trivial group, then it has no maximal subgroups. In finite, non-trivial groups, the lattice of subgroups comprises a poset in which a partial ordering is by set inclusion. It is well known that every finite poset has a maximal element.

Theorem 2.1.3. *(Cooperstein, 1981) Let G be a non-trivial finite group. Then there exists some maximal subgroup of G*

Proof. Suppose G_0 is a proper subgroup of G . Since G is a non-trivial group then it is true that proper subgroups of G exist. If G_0 is maximal then we are done. If not, then there exists some proper subgroup G_1 such that $G_0 \leq G_1$. If G_1 is maximal, we are done. If not, there must exist some proper subgroup G_2 such that $G_1 \leq G_2$. We thus obtain a chain of subgroups $G_0 \leq G_1 \leq G_2 \dots$, which we assert that it is finite because by Lagrange's Theorem, we have the following chain of factors. $|G_0|, |G_1|, |G_2|, \dots, |G|$, but then $|G|$ is finite and so has a finite number of factors, and hence the chain is finite. We therefore conclude that every finite non-trivial group has a maximal subgroup. $\square$

Recall that the Frattini subgroup of G , denoted by $\Phi(G)$, is the intersection of all maximal subgroups of G .

Recently studies on covers of finite groups with proper subgroups have caught the attention of modern mathematicians. A number of areas that have been studied include some of the following types of covers, for instance minimal covers, Hall covers, Sylow covers, subgroup covers of symmetric groups, alternating groups and linear groups. Covers of groups with co-sets or conjugates of proper subgroups have also been studied.

2.1.1 Upper and lower bounds

Let us consider the literature on lower and upper bounds for the covering number, $\sigma(G)$, of a finite group G . Cohn(Cohn, 1994) proved the following lower bound for $\sigma(G)$. Let $\mu(G)$ be the least positive integer k such that G has more than one maximal subgroup of index k .

Proposition 2.1.1. *(Lucchini & Garonzi, 2016, Proposition 1) If G is a finite group, then $\sigma(G) \geq \mu(G) + 1$.*

For our next result we need a definition.

Definition 2.1.1. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a normal series for a finite group G . If there are no normal subgroups in between G_i and G_{i+1} of G , then such a series is called a *chief series* of G . The quotient groups G_{i+1}/G_i are called chief factors of G .

Tomkinson (Tomkinson, 1992) gave the following theorem on the covering number $\sigma(G)$ of solvable groups:

Theorem 2.1.4. *(Tomkinson, 1992, Theorem 2) Let G be a finite non-cyclic solvable group with chief series $1 = G_0 < G_1 < \cdots < G_n = G$. Then the covering number $\sigma(G) = |G_{i+1}/G_i| + 1$, where $|G_{i+1}/G_i|$ is the order of the smallest chief factor G_{i+1}/G_i of G with more than one complement in G/G_i .*

The number $|G_{i+1}/G_i| = t$ is a prime power. i.e. $t = p^k, t, p, k \in \mathbb{N}$ with p being prime. Note that not every $\sigma(G)$ is of the form $t + 1$. As an example, $\sigma(S_6) = 13 \neq p^k + 1$ for any prime p .

Many group theorists have studied the covering number of finite groups. In particular, and for our focus in this study we enumerate a few. In 2005, Maróti(Maróti, 2005), was interested in studying minimal covers of symmetric,

alternating groups and some classes of linear groups. He proved that:

$$\sigma(S_n) \leq \begin{cases} 2^{n-1}, & \text{if } n \text{ is odd and } n \neq 9. \\ 2^{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

He also showed that $\sigma(A_n) \geq 2^{n-2}$ with equality if and only if $n \equiv 2 \pmod{4}$.

For the alternating groups, a number of other results exist: Cohn (Cohn, 1994) showed that $\sigma(A_5) = 10$. Bryce *et al* (Bryce et al., 1999) calculated $\sigma(A_6) = 16$. Others included Kappe *et al* (Kappe & Redden, 2010) who came up with results showing $\sigma(A_7) = 31$, $\sigma(A_8) = 71$ and managed to prove that $\sigma(A_9)$ is a natural number lying between 127 and 157.

All of these studies concentrated on minimal covers with no restrictive properties. There has been interest in coverings where certain requirements are imposed on the proper subgroups of the cover for the group G . For instance covers by cyclic subgroups up to conjugacy have been studied. One observes that every cyclic subgroup is contained in a maximal cyclic subgroup. This is to say that any covering by cyclic subgroups must contain the maximal cyclic subgroups because the only cyclic subgroups containing the generators of the maximal cyclic subgroups are the maximal cyclic subgroups. Indeed the collection of the maximal cyclic subgroups of G form the only irredundant covering of G by cyclic subgroups. It is therefore immediate that the study of the set of maximal cyclic subgroups of G is worth the effort. It is important to note that the collection of conjugacy classes of maximal cyclic subgroups form an irredundant normal covering where all subgroups are cyclic.

2.1.2 Covers of quotients

A conjecture by Lucchini and Detomi (Detomi & Lucchini, 2008) stated that, if G is a non-abelian group and N a non-trivial normal subgroup of G such that $\sigma(G) \leq \sigma(G/N)$, where G/N is a factor group, then G is *monolithic* (admits a

unique minimal normal subgroup). This leads to the following definition.

Definition 2.1.2. A finite group G is called σ -elementary if $\sigma(G) \leq \sigma(G/N)$ for every non-trivial normal subgroup N of G .

Every group G has a σ -elementary quotient G/N , where N is a non-trivial normal subgroup of G with the same covering number. Thus if there exists $N \trianglelefteq G$ with $\sigma(G) = \sigma(G/N)$ we may as well consider G/N instead of G . Based on the above conjecture Martino Garonzi (Garonzi, 2013), worked on covers of monolithic groups. More studies were done on the covering number of the factor group G/N with lots of success as demonstrated in the theorem below.

Theorem 2.1.5. (Garonzi, 2013) Let G be a non-cyclic finite group. Let $N \trianglelefteq G$ and $\mathcal{H}$ be a cover for G/N . Define $\eta : G \rightarrow G/N$ to be the quotient map. Then the family $\{\eta^{-1}(H) : H \in \mathcal{H}\}$ is a cover for G of size $|\mathcal{H}|$. It follows that $\sigma(G) \leq \sigma(G/N)$.

This theorem says that the covering number of a group G can be obtained from the covering number of the factor group G/N where N is a normal subgroup of G . That is $\sigma(G) = \sigma(G/N)$.

Definition 2.1.3. (Rose, 1994, Definition 11.1) Let G be a finite group. We define $\Phi(G)$ to be the intersection of all maximal subgroups of G . $\Phi(G)$ is called the *Frattini subgroup* of G .

Lemma 2.1.2. (Garonzi, 2013) Let G be a finite group. If $\Phi(G)$ is the Frattini subgroup of G then $\sigma(G) = \sigma(G/\Phi(G))$.

2.2 Normal covers

Definition 2.2.1. Let G be a finite group and let $\mathcal{F} = \{G_1 \dots G_k\}$ be a family of proper subgroups of G whose union equals G . A *normal cover* for G is a cover $\mathcal{F}$ of G such that $xG_ix^{-1} \in \mathcal{F}$ for every $x \in G, G_i \in \mathcal{F}$.

Normal covers are covers which are invariant under conjugation by G . The least number of conjugacy classes of proper subgroups in a normal cover of G is the normal covering number of the group G .

The above definition simply says that a normal cover of a finite group G is the minimal collection of proper normal subgroups of G whose union is G . The size of such collection is called the normal covering number of G and denoted by $\xi(G)$. If a group G has no such cover, that is, G cannot be written as a union of proper normal subgroups, then we set $\xi(G) = \infty$.

Lemma 2.2.1. (Bubboloni, Praeger, & Spiga, 2014) Let G be a finite group G for which $\sigma(G) = n$, then $\xi(G) \geq n$

Proof. Let G be a finite group and $\sigma(G) = n$. Then, G can be written as a union of proper subgroups, and the minimal size of such collection is n . If each subgroup in this collection is normal, then this collection is a normal covering of G . So, $\xi(G) = n$. There are two cases. One, there is a collection of subgroups such that replacing this collection by normal subgroups, implies $\xi(G) \geq n$ and two there is no such collection for G and hence $\xi(G)$ is unbounded. $\square$

We observe that every proper subgroup of a group $G = \langle a \rangle$ does not contain the element a . We may conclude that every cyclic group has no normal cover. Thus any cyclic group has no ordinary covering. It is also straight forward to see that every subgroup of an abelian or commutative group is normal.

Definition 2.2.2. The *normal covering number*, which we denote by $\xi(G)$ is the smallest number of conjugacy classes of subgroups in a normal cover for G .

A normal cover for G therefore contains conjugacy classes of subgroups of G . Again it is clear that the normal covering number is defined for all finite non-cyclic subgroups. Furthermore there always exist minimal normal covers consisting of maximal subgroups.

Theorem 2.2.1. (Britnell & Maróti, 2013) Let G be a non-cyclic finite group. The normal covering number $\xi(G) \geq 2$.

Proof. Let H be a proper subgroup of G . Let $|H| = k$. Choose $g_1, g_2, \dots, g_n \in G$ such that $G = g_1H \cup g_2H \cup \dots \cup g_nH$, where $g_1H, g_2H, \dots, g_nH$ are distinct cosets. Then $\bigcup_{g \in G} gHg^{-1} = \bigcup_{i=1}^n g_iHg_i^{-1}$. So, $|\bigcup_{g \in G} gHg^{-1}| = n(k-1) + 1$. But then $|G| = nk$ and we see that $nk - n + 1 < nk$, provided $n > 1$. So $G \neq \bigcup_{g \in G} gHg^{-1}$. Therefore $\xi(G) > 1$ $\square$

Example 2.2.1. The non-trivial subgroups of S_3 are

$$H_1 = \{(1), (12)\}, H_2 = \{(1), (13)\}, H_3 = \{(1), (23)\}, H_4 = \{(1), (123), (132)\}.$$

A normal cover for S_3 contains conjugacy class of H_1 and conjugacy class of H_4 which is self-conjugate because it is of index 2 in S_3 and hence normal. That is $Cl(H_1) = \{H_1, H_2, H_3\}$, $Cl(H_4) = \{H_4\}$. Thus S_3 has only two non-trivial conjugacy classes of subgroups. The normal covering number for the symmetric group S_3 is $\xi(S_3) = 2$.

Lemma 2.2.2. (Bubboloni, Praeger, & Spiga, 2020) If G is a finite abelian non cyclic group, then $\sigma(G) = \xi(G)$.

The converse to the above lemma is not always true, namely the condition $\sigma(G) = \xi(G)$ for a finite group does not imply that G is abelian. As an example, the dihedral group $G = D_{12}$ is non abelian group even though $\sigma(G) = \xi(G) = 3$

Example 2.2.2. The dihedral group D_{12} is non abelian group even though $\sigma(G) = \xi(G) = 3$. If we remember that the dihedral group D_{12} can be written as $G = \{x, y/x^n = y^2 = (x^k y)^2\}$ for $k = 1, 2, \dots, n$. There are n rotations $x^k, k = 1, 2, \dots, n$ and n reflections $x^k y, k = 1, 2, \dots, n$.

Theorem 2.2.2. (Bubboloni, Praeger, & Spiga, 2013) Let $G = D_{2k}$ be the dihedral group of order $2k, k \geq 3$. Then each of the following holds:

- (i) $\sigma(G) = \xi(G) = 3$, for n is even.
- (ii) $\sigma(G) = k + 1$, and $\xi(G) = \infty$ for $n \geq 3$ is prime.

(iii) $\sigma(G) = k + 1$, and $\xi(G) = \infty$ for $n = km$ is prime, and m is an odd integer

Proof. Let $G = D_{2k}$, $k \geq 3$. Then, $C = \langle x \rangle$ is a subgroup of index 2. Let $R = \{y, xy, x^2y, \dots, x^{(k-1)}y\}$ be the set of all reflections. The covering number of G can be computed as follows:

- (i) If k is even, then R contains an even number of elements. Set $H_1 = \langle x^2, xy \rangle$ and $H_2 = \langle x^2, x^2y \rangle$. Each of these subgroups is normal of order k and $H_1 \cap H_2 = \{e, x^2, x^4\}$ which is not H_1 or H_2 . Moreover, $C \cup H_1 \cup H_2 = G$. This implies that $\sigma(G) = \xi(G) = 3$.
- (ii) If k is an odd prime, then the only proper subgroup of G that contains $x \in R$ is $H_i = \{e, x\}$. Therefore, there are k such subgroups, for which $\cap H_i = \{e\}$ and $\cup H_i = R$. This implies that $\sigma(G) = k + 1$. Furthermore, the only proper non trivial normal subgroup of G is C which is not a covering for G . Therefore $\xi(G) = \infty$.
- (iii) If $k = tm$ for a prime $t > 2$ and an odd integer m where $t \leq m$, then the subgroups $H_i = \{x^t, x^iy\}$, $i = 1, 2, \dots, t$ are covering R and each of these subgroups is of order $2m$. In addition, $\cap H_i = \langle x^k \rangle$. Therefore, C and $\{H_i/i = 1, \dots, k\}$ are covers of G . Hence, $\xi(G) = k + 1$. Furthermore, the proper non trivial normal subgroups of G are subsets of C . Clearly, these subgroups are not the cover of G . Therefore, $\xi(G) = \infty$.

□

Corollary 2.2.1. *Let $G = D_{2k}$ be the dihedral group of order $2k$, where k is even and $\mathcal{N}$ be a normal cover of G . Then $G/H \cong Z(G)$ for all $H \in \mathcal{N}$.*

Proof. Let $G = D_{2k}$ be the dihedral group of order $2k$, where k is even. Then G has a normal cover $\mathcal{N}$ of minimal size 3. In fact, $Z(G) = \{e, x^{n/2}\} \cong \mathbb{Z}_2$. Moreover, $[G, H] = 2 = |G/H|$ for all $H \in \mathcal{N}$. □

Theorem 2.2.3. *(Bubboloni et al., 2020) Every non cyclic p -group G , has a normal cover and $\xi(G) = p + 1$.*

Proof. Let G be a non cyclic p -group of order p^n . Then G has at least two generators. Therefore, the number of the normal subgroups of G of order p^{n-1} is at least $N(G) = p^{2-1} + 1 = p + 1$. Suppose $\mathcal{N}$ is be the collection of all such subgroups. Then none of these subgroups is a subgroup of any other, otherwise they would be identically equal to $H = p^{n-1}$. Furthermore $|\cap_{H \in \mathcal{N}} H| = p^{n-2}$. Therefore $|\cap_{H \in \mathcal{N}} H| = (p + 1)p^{n-1} - (p + 1)p^{n-2} + p^{n-2} = p$. Thus, $\xi(G) = |\mathcal{N}| \leq p + 1$. We observe that any other system of normal subgroups of order less than p^{n-1} that needs to cover G should contain at least p^2 subgroups. Hence, $\xi(G) = p + 1$. $\square$

Lemma 2.2.3. Let G be a p -group and $\mathcal{N}$ be the normal cover of G . Then $G/H \cong \mathbb{Z}_p$ for all $H \in \mathcal{N}$.

Proof. Let G be a p -group of order p^n , then $\mathcal{N}$ is a collection of $p + 1$ normal subgroups each of order p^{n-1} . Clearly if $H \in \mathcal{N}$, then $[G : H] = \frac{p^n}{p^{n-1}}$. Thus, $G/H \cong \mathbb{Z}_p$ for all $H \in \mathcal{N}$. We are done. $\square$

Note that, not every non cyclic group which has a normal cover is a p -group. For example, let $G = D_{12}$ be the dihedral group of order 12, which can be written as $G = \{x, y/x^6 = e = (x^i y)^2 \forall i = 1, 2, \dots, n\}$. Clearly, G is a non cyclic group. Furthermore, the subgroups $H_1 = \{e, x^2, x^4, y, x^2 y, x^4 y\}$, $H_2 = \{e, x^2, x^4, xy, x^3 y, x^5 y\}$ and $H_3 = \langle x \rangle$, are proper normal subgroups each of index 2 and $H_1 \cup H_2 \cup H_3 = G$, then $\xi(G) = 3$. We easily conclude that G is not a p -group.

The alternating group A_n has no normal cover for any $n \geq 3$. As A_n is a simple group for all $n > 4$. For the case $n = 4$, the group A_n has only one non-trivial proper normal subgroup of index 3. The Symmetric group S_n has no normal cover for any $n \geq 3$. This is because S_n has one maximal normal subgroup H of index 2, and the other normal subgroups if they exist are subgroups of H .

Cohn (Cohn, 1994) hypothesized that the covering number of any solvable group has the form $p^k + 1$, where p is a prime and k a positive integer. This hypothesis does not always hold.

Based on Tomkinson's theorem, (Bubboloni & Praeger, 2011), Bubboloni studied normal coverings of symmetric and alternating groups. The calculation of minimal normal covers varies significantly from the calculation of minimal covers. For example there is no similar argument for $\xi(G)$ as Theorem 2.1.4. In fact the lower bound for the covering number of the minimal normal cover for non-cyclic finite group G differs from that of minimal cover for G . It turns out that the lower bound for minimal covering number for G is the upper bound for minimal normal covering number for G . This is summarized in the following proposition by (Lucchini & Garonzi, 2016). Recall that $\mu(G)$ was the least positive integer k such that G has more than one maximal subgroup of index k .

Proposition 2.2.1. *(Lucchini & Garonzi, 2016, Proposition 2) If G is a finite group, then $\xi(G) \leq \mu(G) + 1$. Moreover $\xi(G) = \mu(G) + 1$ if and only if $\mu(G)$ is a prime, G contains at least two normal subgroups of index $\mu(G)$ and $\xi(G) = \xi(G/G')$.*

The following corollary by (Lucchini & Garonzi, 2016) singles out conditions under which $\sigma(G) = \xi(G)$.

Corollary 2.2.2. *(Lucchini & Garonzi, 2016, corollary 3) Suppose that G is a non-cyclic finite group. If $\sigma(G) = \xi(G)$, then $p = \sigma(G) - 1$ is a prime and G has a minimal cover consisting of normal subgroups of index p . In particular $\xi(G) = \xi(G/G') = \sigma(G/G')$.*

It is clear from the above corollary that, if all maximal subgroups of G are normal, i.e. if G is nilpotent, then $\sigma(G) = \xi(G)$. We note that if $G' = [G, G]$, is a commutator subgroup of a solvable group G and G/G' is cyclic then in general $\xi(G) = 2$.

The following theorem gives the upper bound of the covering number of the minimal normal cover of a non-cyclic permutation group.

Theorem 2.2.4. *(Garonzi, 2013, Theorem 4) If G is a non-cyclic permutation group of degree n , then $\xi(G) \leq (n + 2)/2$.*

Example 2.2.3. (Garonzi, 2013) Suppose $n = 2p$, where p is any prime and G is a subgroup of S_{2p} generated by two disjoint p -cycles. Then $G \cong C_p \times C_p$ and $\xi(G) = (n + 2)/2 = (2p + 2)/2 = p + 1$.

2.3 Nilpotency

In recent times other restrictive properties have been imposed on the covers of groups. We start this section by noting from Sylow's theorems that any finite nilpotent group is the direct product of its Sylow subgroups.

2.3.1 Covers of nilpotent groups

Martino Garonzi (Garonzi, 2013) computed the covering number of nilpotent groups.

Lemma 2.3.1. (Garonzi, 2013, page 2) Let A, B be two finite groups of co-prime order. Then $\sigma(A \times B) = \text{minimum}(\sigma(A), \sigma(B))$.

From this lemma, a cover of a nilpotent group $C_p \times C_p$, p -prime, (a direct product of cyclic groups) gives $\sigma(C_p \times C_p) = p + 1$. This is due to the fact that for prime p the group $C_p \times C_p$ admits exactly $p + 1$ proper cyclic subgroups, of order p and index p . A unique cover of $C_p \times C_p$, contains all of its non-trivial proper subgroups.

Proposition 2.3.1. (Garonzi, 2013, Proposition 1, page 2) Let G be a finite nilpotent group. Then $\sigma(G) = p + 1$ where p is the smallest prime divisor of $|G|$ such that the Sylow p -subgroup of G is not cyclic.

We have seen in Lemma 2.1.2 that, given the Frattini subgroup $\Phi(G)$ of a group of G , $\sigma(G) = \sigma(G/\Phi(G))$. If $\mathcal{H}$ is a minimal cover of G containing maximal subgroups then, every member contains $\Phi(G)$. If G is a non-cyclic p -group and $G = \langle X \rangle$, where X is the smallest subset that generates G with $|X| = k$, then it is known that $G/\Phi(G) \cong C_p^k$. Since $\sigma(G) = \sigma(G/\Phi(G))$, we have that $\sigma(G) =$

$\sigma(C_p^k)$. This covering number can be calculated using the lemma on *minimal index lower bound* by (Garonzi, 2013) which follows.

Lemma 2.3.2. (Garonzi, 2013, Lemma 1, page 3) Let $\mathcal{H}$ be a minimal cover of a finite group G . Then

$$\min\{|G : H| : H \in \mathcal{H}\} < \sigma(G)$$

.

2.3.2 Nilpotent minimal covers

In 2007, Bryce and Serena (Bryce et al., 1999), worked on groups with *nilpotent minimal covers*. By this we mean a cover of a group G of size $\sigma(G)$ such that every group in the cover is nilpotent. Such a cover does not necessarily exist, and so Bryce and Serena spelt out conditions necessary for a group to admit a nilpotent minimal cover. One of their results is summarized in the following theorem.

Theorem 2.3.1. *A group with a minimal cover of abelian subgroups is solvable of very restricted structure.*

They conjectured that if a group admits a minimal cover with all subgroups being nilpotent (a nilpotent minimal cover) then it is solvable. They showed that if there is an insolvable group admitting a nilpotent minimal cover there is a finite, almost simple one and that certain classes of almost simple groups violate these conditions.

2.3.3 Minimal nilpotent covers

Our major concern in this thesis was on minimal nilpotent covers of groups. A *minimal nilpotent cover* is a nilpotent cover of smallest size whose union is equal to G . In general a minimal nilpotent cover has size greater than $\sigma(G)$ (and so this is a different concept to that studied in the previous subsection).

Note that, by considering the set of all cyclic subgroups of a group G as before, we see that all finite non-cyclic groups G admit a minimal nilpotent cover. Indeed in this setting we do not require that our subgroups are *proper* – thus if a group G is itself nilpotent, then it has a minimal nilpotent cover of size 1 (containing only itself), and so we have that *all* finite groups admit minimal nilpotent covers.

In 2014, Nick Gill, *et al* (Azad et al., 2015), worked on nilpotent covers and non-nilpotent subsets of finite groups of Lie type. They gave the following definition of *c-nilpotent* group G and *non-c-nilpotent* subset X of a group G . In the introduction to their main results a *c-nilpotent* group G is described thus:

(Azad et al., 2015, page.1) ‘Let G be a group and c an element of $\mathbb{Z}^+ \cup \infty$. For $c \in \mathbb{Z}^+$, we define G to be *c-nilpotent* if G is nilpotent of nilpotency class at most c . We define G to be ∞ -nilpotent if G is nilpotent’.

Definition 2.3.1. (Azad et al., 2015, Definition 1.1) A subset X of G said to be *non-c-nilpotent* if for any two distinct elements x and y in X the subgroup $\langle x, y \rangle$ of G which they generate is not *c-nilpotent*. The subset X is said to be *non-nilpotent* if it is non- ∞ -nilpotent.

(Azad et al., 2015) also defined a quantity $\omega_c(G)$, which is the size of the largest non-*c-nilpotent* subset of finite group G of Lie type. The researchers also identify covers of G by *c-nilpotent* subgroups and use them to construct large non-*c-nilpotent* sets in G . The connection between non-*c-nilpotent* sets and *c-nilpotent* covers is established for a group H and c an element of $\mathbb{Z}^+ \cup (\infty)$. This is given by the following results in (Azad et al., 2015).

Lemma 2.3.3. (Azad et al., 2015, Proposition 1.1) A *c-nilpotent* cover has size at least $\omega_c(G)$.

Lemma 2.3.4. (Azad et al., 2015, Lemma 3.1) Let $Y_1 \dots Y_k$ be subsets of H , and suppose that $Y_1 \cup \dots \cup Y_k = H$. Then $\omega_c(H) \leq \omega_c(Y_1) + \dots + \omega_c(Y_k)$.

Lemma 2.3.5. (Azad et al., 2015, Lemma 3.2) Let $X_1 \dots X_k, Y_1 \dots Y_k$ be subsets of H . Suppose the following hold:

- (1) For $i = 1 \dots k$, X_i is a non- c -nilpotent subset of Y_i of maximal order,
- (2) If $1 \leq i < j \leq k$ and $x_i \in X_i, x_j \in X_j$, then $\langle x_i, x_j \rangle$ is not c -nilpotent.

Then $X_1 \cup \dots \cup X_k$ is a non- c -nilpotent subset of H and $\omega_c(H) \geq \omega_c(Y_1) + \dots + \omega_c(Y_k)$.

Note that a 1-nilpotent group is abelian. Thus a 1-nilpotent cover of a group G is precisely an *abelian cover*, and a non-1-nilpotent subset of G is precisely a non-commuting subset. Let us be explicit:

Definition 2.3.2. (Barrantes et al., 2016, Definition 1.1) Let G be a finite group. Let Ω be a subset of the group G . A *non-commuting subset* of Ω is a subset N of Ω such that for some $x, y \in N \Rightarrow xy \neq yx$.

In 2015, Barrantes, *et al* (Barrantes et al., 2016), studied abelian covers of the alternating groups. From this definition they give $\delta(\Omega)$ as the maximum possible cardinality of N . Further a *maximal non-commuting subset* of Ω is a non-commuting subset of cardinality $\delta(\Omega)$. An *abelian cover* of Ω is a set $\mathcal{C}$ of abelian subgroups of G whose union contains Ω . They define $\Delta(\Omega)$ to be the minimum possible cardinality of an abelian cover of Ω .

Lemma 2.3.6. (Barrantes et al., 2016, Lemma 1.1) For any finite group G and subset $\Omega \subseteq G$, $\delta(\Omega) \leq \Delta(\Omega)$.

A *minimal abelian cover* of Ω is an abelian cover that has cardinality $\Delta(\Omega)$. They based their result of minimal abelian covers of the alternating group using Brown's (Brown, 1988) result on minimal abelian covers of the symmetric group which is stated in the following theorem:

Theorem 2.3.2. (Barrantes et al., 2016, Theorem 1.3) Let S_n be the symmetric group on n letters with n a positive integer.

(1) If $n \leq 7$ or $n = 11$, then $\delta(S_n) = \Delta(S_n)$

(2) If $n = 8, n = 13$ or $n \geq 15$, then $\delta(S_n) \neq \Delta(S_n)$.

This lead to their result on minimal abelian covers of the alternating group which is also stated in the following theorem :

Theorem 2.3.3. (*Barrantes et al., 2016, Theorem 1.2*) Let A_n be the alternating group on n letters,

(1) If $n \leq 11$ or $n = 15$, then $\delta(A_n) = \Delta(A_n)$.

(2) If $n = 12, 16, 18$ or $n \geq 20$ then $\delta(A_n) \neq \Delta(A_n)$.

2.3.4 Gaps in the literature

The results on minimal nilpotent covers of finite groups in the previous section have some obvious gaps. We have remedied some of these, as outlined in §1.3. In addition we note the following open problems.

Problem 2.3.4. Ascertain whether or not $\delta(S_n) = \Delta(S_n)$ for $n = 9, 10, 12, 14$.

Problem 2.3.5. Ascertain whether or not $\delta(A_n) = \Delta(A_n)$ for $n = 13, 14, 17, 19$.

Problem 2.3.6. Calculate $\delta(A_n)$, $\delta(S_n)$, $\Delta(A_n)$ and $\Delta(S_n)$.

Problem 2.3.7. For $c = 1, 2, \dots$ and $c = \infty$, and $G = D_{2k}$, the dihedral group of order $2k$, calculate the size of the largest non- c -nilpotent subset of G ; calculate the size of the smallest c -nilpotent cover of G ; and ascertain when these two numbers coincide.

Problem 2.3.8. For $c = 1, 2, \dots$ and $c = \infty$, and $G = A_n$, calculate the size of the largest non- c -nilpotent subset of G ; calculate the size of the smallest c -nilpotent cover of G ; and ascertain when these two numbers coincide.

Problem 2.3.9. For $c = 1, 2, \dots$ and $c = \infty$, and $G = S_n$, calculate the size of the largest non- c -nilpotent subset of G ; calculate the size of the smallest c -nilpotent cover of G ; and ascertain when these two numbers coincide.

Note that the final two problems are (dramatic) generalizations of the first two. All of the listed problems are difficult open problems. It seems reasonable to aim for a complete solution to Problem 2.3.7; for the others, even partial results would be of significant interest.

CHAPTER THREE

RESEARCH METHODOLOGY

3.1 Introduction

This section outlines the procedure we used to determine the minimal size of a nilpotent cover and maximal non-nilpotent subset of the finite symmetric, alternating and dihedral groups. We give the logical steps we undertook to generalise the results into theorems and lemmas among others.

3.2 General methods

This research was in pure mathematics so our methodology was one of “pure thought”: This is summarized by Vladimir Arnold’s quote:

Vladimir Arnold(1937-2010) “Mathematics is part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap”

We sought to derive proofs using only logic and existing results in the literature. With that in mind, our methodology involved several basic components:

Literature on group theory with emphasis on covers was surveyed. Such literature survey appears above in Chapter 2.

Consultations from experts who have authority in the research area (supervisors of this research) was sought. Specifically consultations on the structure of nilpotent subgroups of S_n , A_n , D_{2k} nilpotent covers, abelian covers, their applications and extensions was done. This supervision team was chosen specifically for their expertise in this area.

Background reading was a vital component of our research methodology. It is important that there is a good grounding in foundational ideas from across

algebra, as well as with respect to the use of mathematical logic for better presentation of ideas and proofs of theorems and lemmas. A reading programme was done using, in particular, the classic textbook of Rose (Rose, 1994). This book contains all of the foundational group theoretic material that was needed for this programme of research. This foundational material was supplemented by study of modern research in specific cover-related questions.

Computer calculations were used to determine group and subgroup related properties like normality, “abelianness”, etc. as well as calculations involving orders of groups. The **GAP** program (GAP, 2013) was used where applicable by constructing **GAP** codes for S_n and A_n . By running the codes we were able to explore the structure, subgroups and properties which were fundamental in this study.

Tabulation was used to give a clear and concise presentation of data regarding our research. This helped in organization, comparison, analysis of complex relationships as well as visualization, interpretation and communication of our findings.

For clear presentation of this research, the **L^AT_EX** typesetting program was used.

3.3 Mathematical methods

For each of the families of groups we did the following:

1. Established subgroup structure. For A_n and S_n , this is given by the O’Nan-Scott Theorem (Liebeck, Praeger, & Saxl, 1987). For dihedral groups we worked it out using action on k -gon.
2. Established which subgroups were nilpotent (and their class). For A_n and S_n , properties of permutations were used. For dihedral groups this came down to number theory.
3. Established which elements of the group generate which subgroups. For

A_n and S_n , we studied permutations. For dihedral groups, we considered angles between reflections.

4. Used the information above to calculate $\sigma_c(G)$, $\Sigma_c(G)$ for $c = 1, 2, 3, \dots$ and $c = \infty$. For A_n and S_n , this was possible for small n since as n grows large, it becomes complex and even computer algebra package **GAP** cannot generate any results. For the dihedral group D_{2k} we produced a formula that works for all k .

CHAPTER FOUR

RESULTS AND DISCUSSIONS

In this chapter we collect our main results. These results are split into three parts according to the three families of groups we were interested in: S_n , A_n , D_{2k} . For each of these three families we calculated the following:

1. $\sigma_c(G)$, the size of a maximal non- c -nilpotent subset;
2. $\Sigma_c(G)$, the size of a minimal c -nilpotent cover;
3. A statement describing the conditions under which $\sigma_c(G) = \Sigma_c(G)$.

4.1 D_{2k} , the dihedral group of order $2k$

Throughout this section, $G = D_{2k}$.

4.1.1 Background material on D_{2k}

The set of symmetries of a regular k -sided plane figure, the so called k -gon, form a group, the dihedral group of order $2k$.

One observes immediately that D_{2k} consists of rotations and reflections: k of each (here, and henceforth, we regard the identity symmetry as a rotation). The first two lemmas are well-known:

Lemma 4.1.1. (Rose, 1994) Let $G = D_{2k}$ with $k > 1$, then D_{2k} has the presentation $D_{2k} = \{x, y : x^k = y^2 = 1, y^{-1}xy = x^{-1}\}$

Lemma 4.1.2. (Rose, 1994) If $G = D_{2k}$ with $k > 2$, then G has a unique cyclic subgroup C_k of order k – it contains all the rotations of G . Furthermore, every element of G that does not lie in C_k is a reflection and so has order 2.

This lemma, and a consideration of the two cosets of C_k in D_{2k} immediately allows us to observe that:

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It is worth noting to see that each element of D_{2k} with order greater than 2 is a power of a rotation, because the non-rotations are reflections and thus have order 2, it is false in general that the only elements of order 2 (involutions) are reflections. When k is even and $x^i, i = 1, 2, \dots, k$, are rotations, $x^{\frac{k}{2}}$ is a rotation of π radians, which is an involution. Clearly a rotation of π radians is the only rotation of order 2, and lies in D_{2k} only when n is even.

The following basic facts are worth noting: If x^i is a rotation then the inverse is x^{k-i} . A reflection y is its own inverse since $y^2 = 1 = yy^{-1} \Rightarrow y = y^{-1}$.

Let x be a rotation of $\frac{2\pi}{k}$ radians about the center of the k -gon. Then under x the invariant or fixed point is the center. Let y be a reflection, then y fixes a line. When k is odd the lines of reflection pass through a vertex and the midpoint of the opposite side. For even k the fixed lines either join opposite vertices or midpoints of opposite sides.

Lemma 4.1.3. Let $x, y \in G = D_{2k}$, where x is a rotation and y a reflection. Then $y^{-1}xy = x^{-1}$.

Proof. Let V_1 be a vertex such that the mirror line of y passes through V_1 . Write $x(V_1) = V_2$ and $x^{-1}(V_1) = V_2^*$. Note also that $y(V_1) = V_1$ and $y(V_2) = V_2^*$.

From these equations we see that $x^{-1}(y(V_1)) = y(x(V_1))$ that is $(x^{-1}y)(V_1) = (yx)(V_1)$. This implies in particular that $x^{-1}y^{-1}x^{-1}y$ fixes V_1 . But note that $x^{-1}y^{-1}x^{-1}y$ is a product of two rotations (x^{-1} and $y^{-1}x^{-1}y$) and so is a rotation.

Now, the only rotation that fixes a point is the identity. Hence $x^{-1}y^{-1}x^{-1}y = 1$ and we conclude that $x^{-1} = yxy^{-1}$ as required. $\square$

Lemma 4.1.4. Every subgroup of D_{2k} is either dihedral or cyclic.

Proof. Let $H \leq D_{2k}$ and let C_k be the group of rotations in D_{2k} . There are two possibilities for $H \cap C_k$. Suppose $H = H \cap C_k$, in this case $H \leq C_k$ and so H is cyclic. Suppose $|H : H \cap C_k| = 2$. Observe that $H \cap C_k$ is cyclic. Hence $H \cap C_k = \langle x \rangle$ for some rotation x .

Now let $y \in H \setminus C_k$, so y is a reflection. Lemma 4.1.3 implies that $y^{-1}xy = x^{-1}$. Therefore $H = \langle x, y \rangle$ is a group of order 2ℓ such that $x^\ell = y^2 = 1$ and $y^{-1}xy = x^{-1}$. If $\ell > 1$ then Lemma 4.1.1 implies that H is dihedral. If $\ell = 1$ then $|H| = 2$ and H is cyclic. $\square$

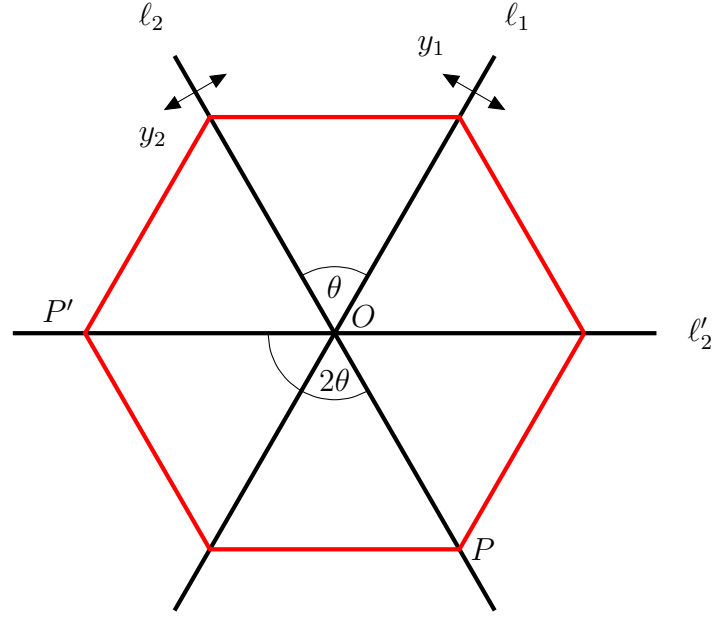
Lemma 4.1.5. Let $G = D_{2k}$ and let y_1, y_2 be reflections in G with lines of reflection ℓ_1 and ℓ_2 , respectively. Let θ be the angle between ℓ_1 and ℓ_2 and write $\theta = \frac{\pi a}{b}$ where a and b are co-prime integers. Then $y_1 y_2$ is a rotation by 2θ and $\langle y_1, y_2 \rangle \cong D_{2b}$.

Proof. First we know that $y_1 y_2$ is a rotation. Let P be a point on ℓ_2 on the edge of the k -gon. Let $P' = y_1 y_2(P) = y_1(P)$ and let ℓ'_2 be the line from center to P' , then ℓ_1 is the mediator of PP' . This implies that the angle between ℓ_1 and ℓ'_2 is equal to the angle between ℓ_1 and ℓ_2 which is θ . (See Figure 4.1).

Therefore $\angle POP' = 2\theta$ and hence $y_1 y_2$ is a rotation by $2\theta = \frac{2\pi a}{b}$. Clearly $y_1 y_2$ is a rotation of order b . Furthermore $\langle y_1, y_2 \rangle = \langle x, y_2 \rangle$. But now Lemma 4.1.3 implies that $y_2^{-1}xy_2 = x^{-1}$ and Lemma 4.1.1 implies that $\langle x, y_2 \rangle = D_{2b}$ as required. $\square$

A group D is of dihedral type if it is non abelian and has a set of two generators $\{x, y\}$ such that y is an involution and that $x^y = x^{-1}$, where x^y means $y^{-1}xy$. We have already seen that for $k \geq 3$, the dihedral group D_{2k} of order $2k$ is of dihedral type. It can be shown that for $k \geq 3$ any finite group of dihedral group is isomorphic to D_{2k} .

Theorem 4.1.1. Let G be a finite group of dihedral type, then G is isomorphic to D_{2k} for some integer $k \geq 3$.

Figure 4.1: Reflections y_1 and y_2 in D_{2k} .

Source: Research.

Proof. The group G is non abelian and there are elements $x, y \in G$ such that $G = \langle x, y \rangle$, where y is an element of order 2 and that $x^y = x^{-1}$. Suppose that the order of x is k , and let $T = \langle x \rangle$. Now since G is non abelian, T is a subgroup of G . Furthermore, $G = \langle x, y \rangle$ where $x^y = x^{-1}$ and this implies that $T \trianglelefteq G$. We know that $G/T = \langle xT, yT \rangle = \langle yT \rangle$ but since $x \in T$. Since y is not in T but $y^2 \in T$ we have that $|G/T| = 2$. We also see that $|G| = 2|T| = 2k$, where $G = \{1, x, x^2, \dots, x^{k-1}, y, xy, x^2y, \dots, x^{k-1}y\}$, because G is non abelian, with $k \geq 3$.

Finally if we let $G = D_{2k}$, the dihedral group of order $2k$ and if we let r be a rotation and s be a reflection, then $G = \{1, r, r^2, \dots, r^{k-1}, s, rs, r^2s, \dots, r^{k-1}s\}$ where $r^k = 1 = s^2$ and $r^s = r^{-1}$. It can be verified easily that the map $x^i y^j \mapsto r^i s^j$ for $i = 0, 1, 2, \dots, k-1$ and $j = 0, 1$ is a one to one and onto map (isomorphism) of G onto D_{2k} . $\square$

Every group G of dihedral type can be generated by two involutions. For suppose $G = \langle x, y \rangle$ where y is an involution and $x^y = x^{-1}$. Then again $G = \langle x, y \rangle$ and since $x = (xy)y$. It is clear $xy \neq 1$ because G is non abelian and $(xy)^2 = xx^y = 1$. Therefore xy is an involution. On the other hand if G is non abelian group which

can be generated by two involutions then G is of dihedral type. Observe that $G = \langle z, y \rangle$ where both z and y are of order 2. We note that $G = \langle zy, y \rangle$, because $z = (zy)y$, further $(zy)^y = yzy^2 = y^{-1}s^{-1} = (zy)^{-1}$ and we see immediately that G is of dihedral type. Notice that if H is an abelian group which can be generated by two involutions, for instance $H = \langle x, y \rangle$, where both x and y are elements of order two each, then we have two considerations. One, either $x = y$ and hence $H = \langle x \rangle \cong C_2$ or two, $x \neq y$ and so $H = \{1, x, y, xy\} \cong C_2 \times C_2$.

Theorem 4.1.2. *Let G be a group. Suppose x and y are involutions in G . Then either x and y are conjugate in G or there exists an involution $w \in \langle x, y \rangle$, such that w commutes with both x and y .*

Proof. Suppose $G = \langle x, y \rangle$. If G is abelian we are done. If G is non abelian it follows that G is of dihedral type. Now let $w = xy$. If G is finite, the element x has finite order say k but since $G = \langle x, y \rangle$ and $x^y = x^{-1}$ with $|G| = 2k$. Now for odd k both $\langle x \rangle$ and $\langle y \rangle$ are actually Sylow 2-subgroups of G . By Sylow theorems $\langle x \rangle$ and $\langle y \rangle$ are conjugate in G . In fact x and y are the only non trivial elements of $\langle x \rangle$ and $\langle y \rangle$ respectively. Obviously the conjugacy of the subgroups $\langle x \rangle$ and $\langle y \rangle$ implies the conjugacy of the elements x and y also. On the other hand if k is even and if we let $w = x^{\frac{k}{2}}$, then $w \in G$ and the element w is of order 2. This follows by observing that $w^x = yx = x^{-1}$ and $w^y = x^{-1}$, $(w^{\frac{k}{2}})^x = w^{-\frac{k}{2}} = (w^{\frac{k}{2}})^y$. This says that $w^x = w^{-1} = w = w^y$ and therefore w commutes with both x and y . $\square$

4.1.2 Conjugacy in Dihedral Groups

Recall by definition 1.2.38, that the *conjugacy class* $Cl(g)$ of g is the set of elements in G conjugate to g :

$$Cl(g) = \{xgx^{-1} : x \in G\}$$

In D_{2k} the geometric description of reflections depends on the parity of k .

When k is odd the mirror lines (lines of reflection) all look the same in that each mirror line joins a particular vertex to the midpoint on the opposite side, for example, Figure 4.3. When k is even the mirror lines fall into two types. Reflection lines through pairs of opposite vertices and reflection lines through midpoints of opposite sides, for example, Figure 4.6. This fact is important in determination of conjugacy classes of D_{2k} .

Theorem 4.1.3. *Let $x, y \in G = D_{2k}$, where x is a rotation and y a reflection. Then the conjugacy classes in $G = D_{2k}$ depend on whether k is odd or even.*

(i) *When k is odd, the conjugacy classes are:*

(a) *The conjugacy class of size 1 containing the identity element 1 i.e, the rotation of 2π radians.*

$$Cl(1) = \{1\}$$

(b) *The $\frac{k-1}{2}$ conjugacy classes of size 2, each containing a reflection and its inverse.*

$$Cl(y_1) = \{y_1, y_1^{-1}\}, Cl(y_2) = \{y_2, y_2^{-1}\} \dots Cl(y_{\frac{k-1}{2}}) = \{y_{\frac{k-1}{2}}, y_{\frac{k-1}{2}}^{-1}\}.$$

(c) *The conjugacy class of size k containing all the non trivial $(k-1)$ rotations.*

$$Cl(x_i) = \{x_0, x_1 \dots x_{k-1}\}$$

(ii) *When k is even, the conjugacy classes are:*

(a) *The 2 conjugacy classes of size 1, namely*

$$Cl(1) = \{1\}$$

and

$$Cl(x^{\frac{k}{2}}) = \{x^{\frac{k}{2}}\}$$

(b) The $\frac{k}{2} - 1$ conjugacy classes of size 2

$$Cl(x^i) = \{x^i, x^{-i}, i = 1, 2, \dots, \frac{k}{2} - 1\}$$

(c) The reflections fall into two conjugacy classes each of size $\frac{k}{2}$. These are:

$$Cl(x^{2i}y) = \{x^{2i}y, i = 0, 1, 2, \dots, \frac{k}{2} - 1\}$$

and

$$Cl(x^{2i+1}y) = \{x^{2i+1}y, i = 0, 1, 2, \dots, \frac{k}{2} - 1\}$$

Proof. An element of D_{2k} can either be a rotation x^i or a reflection $x^i y$ where i is an integer. Let $g \in D_{2k}$, then the conjugacy class of g is $x^i g x^{-i}$ and $(x^i y)g(x^i y)^{-1}$. If $g = x^m$, where m is an integer, then the only conjugates of x^m as i varies in D_{2k} are $x^i x^m (x^{-i}) = x^m$ and $(x^i y)x^m(x^i y)^{-1} = x^{-m}$. Recall that the basic relation in the definition of the dihedral group says that $yx^m y^{-1} = x^{-m}$, hence x^m and x^{-m} are conjugate. For the conjugacy class of any reflection y , and to be sure there is nothing else x^m is conjugate to we see that $x^i y (x^{-i}) = x^{2i} y$ and $(x^i y)y(x^i y)^{-1} = x^{2i} y$, where $x^{2i} y$ are reflections in which x occurs with a power a multiple of 2 as i varies. If k is odd then every $i \pmod{k}$ is a multiple of 2 (since 2 is invertible $\pmod{k}$ we can solve $t \equiv 2i \pmod{k}$ for i given t). Therefore when k is odd $\{x^{2i} y : i \in \mathbb{Z}\} = \{x^t y : t \in \mathbb{Z}\}$, therefore every reflection in D_{2k} is conjugate to y . In the case when k is even, we find that only half of the reflections are conjugates of y . The other half of the reflections are conjugate to xy . This is concluded from the fact that $x^i (xy) x^{-i} = x^{2i+1} y$ and $(x^i y)(xy)(x^i y)^{-1} = x^{2i-1} y$ which reduces to $\{xy, x^3 y, \dots, x^{k-1} y\}$ $\square$

Theorem 4.1.3 asserts that apart from the identity, and the rotation of π radians, namely $x^{\frac{k}{2}}$ when k is even, a given rotation is conjugate only to its inverse which is also a rotation. All the reflections are conjugate for odd k but when k is even they break up into two conjugacy classes as seen in Figure 4.4. They are

precisely of two types. Those in which the fixed mirror line joins opposite vertices ($x^{2i}y$) and the ones in which a fixed mirror line joins midpoints of opposite sides ($x^{2i+1}y$).

Conjugacy classes of size 1 contain those elements which belong in the center of a group. The above proof clarifies that D_{2k} has trivial center when k is odd, while when k is even the center is $\{1, x^{\frac{k}{2}}\}$. For odd k , we see again in Theorem 4.1.3 that the only conjugacy class of size 1 is $\{1\}$, while for even k the only conjugacy classes of size 1 are two, $\{1\}$ and $\{x^{\frac{k}{2}}\}$.

4.1.3 Nilpotency

In this subsection we investigate when the dihedral group, D_{2k} , is nilpotent and calculate its nilpotency class. To do this, we need some results about the center of D_{2k} .

Lemma 4.1.6. Let $G = D_{2k} = \{x, y : x^k = y^2 = 1, y^{-1}xy = x^{-1}\}$. Then

$$Z(D_{2k}) = \begin{cases} \{1\}, & \text{if } k \text{ odd;} \\ \{1, x^{\frac{k}{2}}\}, & \text{if } k \text{ is even and } k > 2; \\ D_{2k}, & \text{if } k = 2. \end{cases}$$

In the case where k is even and $k > 2$, $D_{2k}/Z(D_{2k}) \cong D_k$.

Proof. Lemma 4.1.3 says that if x_1 is a rotation and y is a reflection, then $x_1y = yx_1^{-1}$. These commute if $x_1 = x_1^{-1}$, i.e $x_1 = 1$ or $x_1 = x^{\frac{k}{2}}$. If k is odd then no non-trivial rotations and reflections commute. So, $Z(D_{2k}) = 1$. If $k > 2$ is even then no rotation of order more than 2 and no reflection commute. So, $Z(D_{2k})$ only contains rotations of orders 1 or 2. It is clear that these do lie in the centre and therefore $Z(D_{2k}) = \{1, x^{\frac{k}{2}}\}$. If $k = 2$, then $|D_{2k}| = 4$, so D_{2k} is abelian and hence $Z(D_{2k}) = D_{2k}$.

Let's prove the final statement, assuming that $k > 2$ is even. Take x and y as defined in the statement and write $Z = Z(D_{2k})$. Then $xZ \in D_{2k}/Z(D_{2k})$ and

$yZ \in D_{2k}/Z(D_{2k})$. Furthermore $(xZ)^{\frac{k}{2}} = x^{\frac{k}{2}}Z = Z = 1$ in $D_{2k}/Z(D_{2k})$. We also have that $(yZ)^2 = y^2Z = Z$. Finally we have that,

$$(yZ)^{-1}(xZ)(yZ) = (y^{-1}Z)(xZ)(yZ) = (y^{-1}xy)Z = x^{-1}Z = (xZ)^{-1}$$

and so, $D_{2k}/Z(D_{2k})$ is a group of order k satisfying the presentation in Lemma 4.1.1 (with k replaced by $\frac{k}{2}$). Thus $D_{2k}/Z(D_{2k}) \cong D_k$. $\square$

Lemma 4.1.7. The group $G = D_{2k}$ is nilpotent if and only if k is a power of 2.

Proof. If k is a power of 2, then $|G|$ is a power of 2, and hence G is nilpotent. If k is not a power of 2, then there exists a non-trivial rotation, x , of odd order. But now, if y is a reflection (and hence of even order), then Lemma 4.1.3 implies that x and y do not commute. Thus G is not nilpotent. $\square$

We investigate the situation where k is a power of 2 in more detail: We know that G is nilpotent, but what is its nilpotency class?

Lemma 4.1.8. Let $G = D_{2(2^c)}$ with $c \geq 1$. Then the nilpotency class of G is equal to c .

Proof by induction. If $c = 1$, then $G \cong K_4$, the Klein 4-group and so G is abelian and therefore has nilpotency class equal to 1. Now assume the result is true for c and prove that it is true for $c + 1$. So let $G = D_{2(2^{c+1})}$. The upper central series of G is, by definition,

$$1 \triangleleft Z(G) \triangleleft \cdots \triangleleft G.$$

In particular, the number of terms between $Z(G)$ and G in the upper central series for G is equal to the number of terms in the upper central series for $G/Z(G)$. But Lemma 4.1.6 implies that $Z(D_{2k}) \cong C_2$ and $G/Z(G) \cong D_{2(2^c)}$. By assumption $D_{2(2^c)}$ has nilpotency class equal to c . Therefore the nilpotency class of G is $c + 1$, as required. $\square$

4.2 Minimal c -nilpotent covers and Maximal non c -nilpotent subsets of D_{2k}

In this subsection we work out a formula for $\Sigma_c(D_{2k})$ and $\sigma_c(D_{2k})$, which are respectively the size of a minimal c -nilpotent cover and maximal non- c -nilpotent subset. (Here $c \in \mathbb{Z}^+ \cup \{\infty\}$). In order to do this, the material in §4.1.1 and §4.1.3 will be useful. For clarity of ideas, some diagrammatic illustrations and interpretations are presented.

4.2.1 The case when k is odd

Lemma 4.2.1. If $G = D_{2k}$ with k odd, then there exists a 1-nilpotent cover of G that contains exactly $k + 1$ subgroups.

Recall that a 1-nilpotent cover of G is precisely an abelian cover of G .

Proof. The Sylow 2-subgroups are precisely the subgroups of order 2 isomorphic to C_2 generated by distinct involutions (reflections). They are abelian. There are k Sylow 2-subgroups. Each of these subgroups contains an element which is not contained in any other Sylow 2-subgroup, namely the reflection itself.

Furthermore Lemma 4.1.2 says that every dihedral group has the unique cyclic subgroup generated by a rotation of $\frac{2\pi}{k}$ radians. It is of order k , contains all the rotations, and is abelian. We conclude that there exists 1-nilpotent cover of G of size $k + 1$. $\square$

Lemma 4.2.2. If $G = D_{2k}$ with k odd, then there exists a non-1-nilpotent subset of G containing exactly $k + 1$ elements.

Proof. Let τ_i, τ_j be any two distinct reflections in G . By Lemma 4.1.5 they generate a dihedral group. If they commute, $\langle \tau_i, \tau_j \rangle = D_4 \cong K_4$. This implies (by Lemma 4.1.6) that the lines of symmetry differ by $\frac{\pi}{2}$ radians. This cannot happen since k is odd. Therefore the set $\{\tau_1, \tau_2 \dots \tau_k\}$ is a non-commuting set.

Finally let x be a rotation of $\frac{2\pi}{k}$ radians. Suppose $\tau_i x = x \tau_i$ then $x = \tau_i^{-1} x \tau_i$. But by Lemma 4.1.3, $x^{-1} = \tau_i^{-1} x \tau_i$. So we conclude that $x = x^{-1}$. This is an impossibility since $k > 2$, hence $\{\tau_1, \tau_2, \dots, \tau_k, x\}$ is a non-commuting subset of size $k + 1$. $\square$

Lemma 4.2.3. If k is odd, $\sigma_1(G) = \Sigma_1(G) = k + 1$.

Proof. Lemma 4.2.1 implies that $\Sigma_1(G) \leq k + 1$ while Lemma 4.2.2 implies that $\sigma_1(G) \geq k + 1$. Lemma 1.2.5 asserts that $\sigma_1(G) \leq \Sigma_1(G)$. Thus $k + 1 \leq \sigma_1(G) \leq \Sigma_1(G) \leq k + 1$. Therefore $\sigma_1(G) = \Sigma_1(G) = k + 1$. $\square$

Proposition 4.2.1. If k is odd and $c \in \mathbb{Z}^+ \cup \infty$ then $\sigma_c(G) = \Sigma_c(G) = k + 1$.

Proof. Lemma 4.2.3 asserts that when k is odd, $\sigma_1(G) = \Sigma_1(G) = k + 1$. We also know by Lemma 4.1.4 that all subgroups of D_{2k} are either dihedral or cyclic. Furthermore cyclic groups are abelian. Lemma 4.1.7 says that D_{2k} is nilpotent if and only if its order is a power of 2. In our case k is odd and so any subgroup of G whose order is a power of 2 is C_2 .

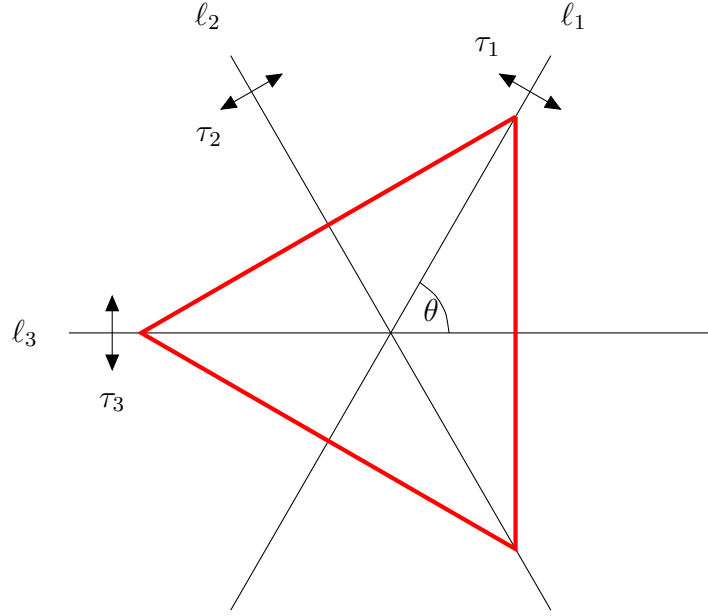
In particular there are no nilpotent dihedral subgroups of G , so all nilpotent subgroups of G are abelian. This implies that for all $c \in \mathbb{Z}^+ \cup \{\infty\}$, $\sigma_1(G) = \sigma_c(G)$. This also implies that $\Sigma_1(G) = \Sigma_c(G)$. Therefore $\sigma_c(G) = \Sigma_c(G) = k + 1$. $\square$

Example 4.2.1. (Refer to Figure 4.2 and Figure 4.3 when $k = 3$ and $k = 5$ respectively).

In the case when $k = 3$ we obtain D_6 , where τ_i , for $i = 1, 2, 3$, are the three reflections with mirror lines ℓ_i for $i = 1, 2, 3$, respectively.

Let x_j be the anticlockwise rotation of $\theta = \frac{2\pi j}{k}$ radians. This rotation generates the cyclic subgroup $\langle x \rangle \cong C_3$.

By Lemma 4.1.6, the center of D_6 is $Z(D_6) = \{1\}$. Since k is odd, no two mirror lines differ by an angle of $\frac{\pi}{2}$ radians, so no two reflections commute. Each reflection generates a subgroup isomorphic to C_2 . These are $\langle \tau_1 \rangle, \langle \tau_2 \rangle, \langle \tau_3 \rangle$. All

Figure 4.2: Reflections in D_6 .

Source: Research.

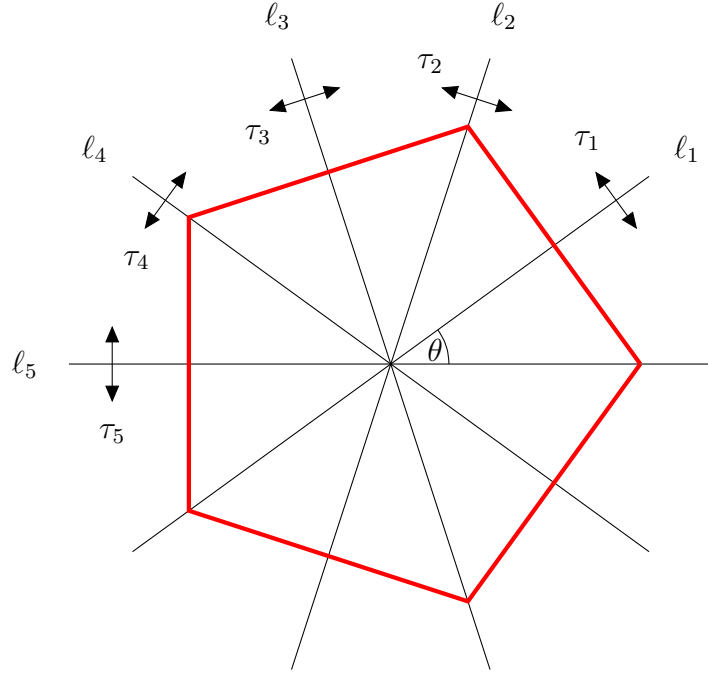
these subgroups namely, $\langle x \rangle, \langle \tau_1 \rangle, \langle \tau_2 \rangle, \langle \tau_3 \rangle$, are abelian and their union is G . This is to say that for all $c \in Z^+ \cup \{\infty\}$, $\Sigma_c(G) = \Sigma_1(G)$, $\sigma_c(G) = \sigma_1(G)$.

One observes that the union of the subgroups $\langle x \rangle, \langle \tau_1 \rangle, \langle \tau_2 \rangle, \langle \tau_3 \rangle$ is a 1-nilpotent cover of G . Each of the subgroups in this cover contains at least one element that is not contained in the others. We conclude that this is a minimal 1-nilpotent cover of G whose size is equal to 4, that is, $\Sigma_1(D_6) = 4$.

The set $\{x, \tau_1, \tau_2, \tau_3\}$, is a non-commuting subset of G . It contains the three reflections and a non-trivial rotation. It is a maximal non-1-nilpotent subset whose size is equal to 4, that is, $\sigma_1(D_6) = 4$. We conclude that $\sigma_1(D_6) = \Sigma_1(D_6) = 4$.

4.2.2 The case when $k \equiv 2 \pmod{4}$

First note that $k \equiv 2 \pmod{4}$ implies that $k = 4m + 2, m = 1, 2, \dots$. In this case one immediately observes that $|D_{2k}| = 2k = 8m + 4 = 4(2m + 1)$. Letting $t = 2m + 1$, it is clear that t is an odd positive integer. Observe that $|D_{2k}| = 4t = 2^2t$.

Figure 4.3: Reflections in D_{10} .

Source: Research.

Lemma 4.2.4. If $G = D_{2k}$ with $k \equiv 2 \pmod{4}$, there exists a 1-nilpotent cover of G which contains exactly $\frac{k}{2} + 1$ subgroups.

Proof. By Lemma 4.1.6, $Z(D_{2k}) = \{1, x^{\frac{k}{2}}\}$. Also, by Lemma 4.1.5, suppose τ_i, τ_j are reflections in G with lines of reflection ℓ_i and ℓ_j , respectively. Since k is even, we can consider the case when ℓ_i and ℓ_j differ by $\frac{\pi}{2}$ radians. This implies that τ_i, τ_j commute and $\tau_i\tau_j = x^{\frac{k}{2}}$ is a rotation of π radians. Again, $\langle \tau_i, \tau_j \rangle = \{1, \tau_i, \tau_j, x^{\frac{k}{2}}\} \cong D_4$. This is one of the Sylow 2-subgroups of order 4 in G . It is abelian. Furthermore, recall Lemma 4.1.2 and pick the cyclic subgroup of order k . We have a 1-nilpotent cover of size $\frac{k}{2} + 1$. $\square$

Lemma 4.2.5. If $G = D_{2k}$ with $k \equiv 2 \pmod{4}$, then there exists a non-1-nilpotent subset of G containing exactly $\frac{k}{2} + 1$ elements.

Proof. Let τ_i, τ_j be any two distinct reflections in G . By Lemma 4.1.5 they generate a dihedral group. Lemma 4.1.6 says that when k is even there exists lines of symmetry that differ by $\frac{\pi}{2}$ radians. If τ_i, τ_j differ by $\frac{\pi}{2}$ radians they commute and $\tau_i\tau_j = \tau_j\tau_i = x^{\frac{k}{2}}$ which is a rotation of π radians.

Furthermore, by the same lemma, the center of $Z(D_{2k}) = \{1, x^{\frac{k}{2}}\}$ and $\langle \tau_i, \tau_j \rangle = D_4 \cong K_4$. Clearly these are the Sylow 2-subgroups of order 4. Since the number of subgroups is $t = \frac{k}{2}$, the set $\{\tau_1, \tau_2, \dots, \tau_{\frac{k}{2}}\}$ in which no two reflections are from the same Sylow 2-subgroup is a non-commuting set.

Finally let x be a rotation of $\frac{2\pi}{k}$ radians. Suppose $\tau_i x = x \tau_i$ then $x = \tau_i^{-1} x \tau_i$. But by Lemma 4.1.3, $x^{-1} = \tau_i^{-1} x \tau_i$. So we conclude that $x = x^{-1}$. That is, x is either 1 or $x^{\frac{k}{2}}$. So we pick a non-trivial rotation $x \neq x^{\frac{k}{2}}$. Therefore the set $\{\tau_1, \tau_2, \dots, \tau_{\frac{k}{2}}, x\}$ is a non-commuting subset of size $\frac{k}{2} + 1$. $\square$

Proposition 4.2.2. *If $k \equiv 2 \pmod{4}$ and $c \in \mathbb{Z}^+ \cup \infty$ then $\sigma_c(G) = \Sigma_c(G) = \frac{k}{2} + 1$.*

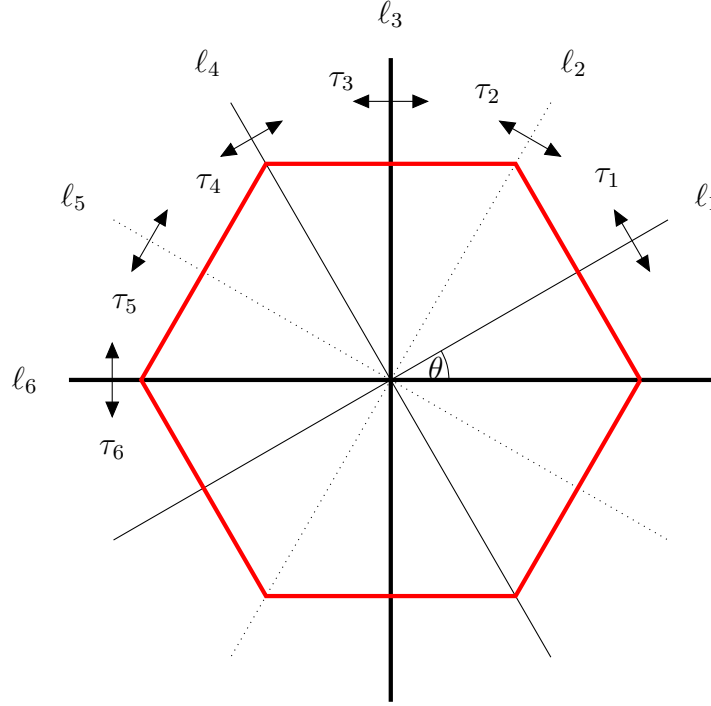
Proof. First note that Lemma 4.2.4 and Lemma 4.2.5 imply that when $k \equiv 2 \pmod{4}$, $\sigma_1(D_{2k}) = \Sigma_1(D_{2k}) = \frac{k}{2} + 1$. Lemma 4.1.4 says that all subgroups of D_{2k} are dihedral or cyclic. In addition Lemma 4.1.7 says that a dihedral group is nilpotent if and only if its order is a power of 2. But then $|D_{2k}| = 2^2 \times (\frac{k}{2})$, where $\frac{k}{2}$ is odd. Clearly a Sylow 2-subgroup of $G = D_{2k}$ is the Klein 4-group, $K_4 \cong D_4$. This group is abelian and so G does not contain nilpotent dihedral groups hence all nilpotent subgroups of G are abelian. Recalling the definitions of $\sigma_c(G)$ and $\Sigma_c(G)$ we have that for all c , $\sigma_1(G) = \sigma_c(G)$, $\Sigma_1(G) = \Sigma_c(G)$ and the result follows. $\square$

Example 4.2.2. (Refer to Figure 4.4 and Figure 4.5 when $k = 6$ and $k = 10$ respectively).

In the case when $k = 6$ we obtain D_{12} , where $\tau_i, i = 1, 2, \dots, 6$ are the six reflections with mirror lines $\ell_i, i = 1, 2, \dots, 6$ respectively.

By Lemma 4.1.5, let x_j be a rotation of $\theta = \frac{2\pi j}{k}$ radians, where j is relatively prime to k i.e. $j = 1, 5$. This rotation generates the cyclic subgroup $\langle x_j \rangle \cong C_6$.

Now the center $Z(D_{12}) = \{1, x^3\}$, where x^3 is a rotation of π radians. The following pairs of reflections differ by $\frac{\pi}{2}$ radians and therefore commute: (τ_1, τ_4) , (τ_2, τ_5) , (τ_3, τ_6) . The 3 Sylow 2-subgroups of order 4 are $\{1, x^3, \tau_1, \tau_4\}$, $\{1, x^3, \tau_2, \tau_5\}$,

Figure 4.4: Reflections in D_{12} .

Source: Research.

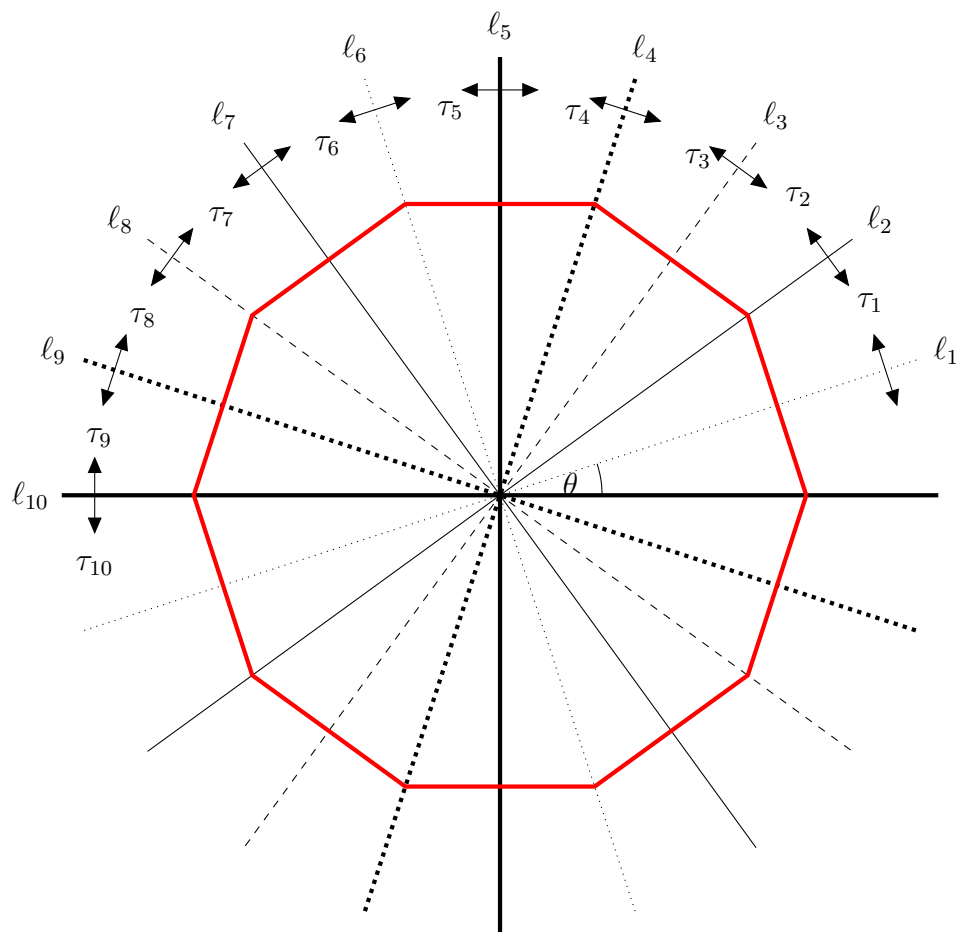
$\{1, x^3, \tau_3, \tau_6\}$. Each of these subgroups is isomorphic to the Klein 4-group $K_4 \cong D_4$ and so is abelian.

The cyclic subgroup together with the 3 Sylow 2-subgroups are all abelian and their union is G . No subgroup of G is nilpotent dihedral and so all the nilpotent subgroups of G are abelian. This union constitutes a 1-nilpotent cover of G because every element of G is contained in one of these subgroups. We conclude that this is a 1-nilpotent cover of G whose size is equal to 4.

Now the set $\{x_j, \tau_1, \tau_2, \tau_3\}$ is a non-commuting subset of G . It contains three reflections, one from each commuting pair and a non-trivial rotation. It is a non-1-nilpotent subset of size 4. We conclude that $\sigma_1(D_{12}) = \Sigma_1(D_{12}) = 4$.

4.2.3 The case when $k \equiv 4 \pmod{8}$

In this subsection we consider the case when $k \equiv 4 \pmod{8}$. This is the final individual case we need to examine before we prove the general theorem.

Figure 4.5: Reflections in D_{20} .

Source: Research.

Lemma 4.2.6. If $k \equiv 4 \pmod{8}$, then a nilpotent subgroup of D_{2k} has nilpotency class equal to 1 or 2.

Proof. Let H be a nilpotent subgroup of D_{2k} . Lemma 4.1.4 says that any subgroup of D_{2k} is either dihedral or cyclic; if H is cyclic, then it is nilpotent of nilpotency class 1 and the result holds. If H is dihedral, then Lemma 4.1.7 implies that $|H|$ is a power of 2, i.e. H is a 2-group. In particular H is a subgroup of a Sylow 2-subgroup of D_{2k} . Since $k \equiv 4 \pmod{8}$, a Sylow 2-subgroup of D_{2k} is of order 8. Thus H has order dividing 8. Then Lemma 4.1.8 implies that H is nilpotent of nilpotency class at most 2, as required. $\square$

Corollary 4.2.1. If $k \equiv 4 \pmod{8}$, then $\sigma_c(G) = \sigma_2(G)$ for all $c \geq 2$, and $\Sigma_c(G) = \Sigma_2(G)$ for all $c \geq 2$.

Lemma 4.2.7. If $k \equiv 4 \pmod{8}$, there exists a 1-nilpotent cover of $G = D_{2k}$ of size $\frac{k}{2} + 1$, and a 2-nilpotent cover of D_{2k} of size $\frac{k}{4} + 1$.

Proof. Let us number the rotations $\tau_1, \dots, \tau_k$ so that their lines of reflection move in sequence around the origin with $2\pi/k$ radians between successive lines.

We consider the 1-nilpotent cover first: Suppose that $k_1 \neq k_2$; we know that $T = \langle \tau_i, \tau_j \rangle$ is nilpotent. By Lemma 4.1.8, T is abelian if and only if it is of order 4. Also, by Lemma 4.1.5, T is abelian if and only if the lines of reflections τ_i and τ_j differ by $\pi/2$ radians. This, in turn, is if and only if $i \equiv j + \frac{k}{2} \pmod{k}$.

Take $T_1 = \langle \tau_1, \tau_{\frac{k}{2}+1} \rangle$, $T_2 = \langle \tau_2, \tau_{\frac{k}{2}+2} \rangle$, and so on. There are $\frac{k}{2}$ such groups; they are all abelian and they contain all reflections in G . By adding in the cyclic subgroup containing all rotations we obtain a 1-nilpotent cover of size $\frac{k}{2} + 1$.

Next, consider the 2-nilpotent cover: Since $k = 8m + 4 = 4(2m + 1)$, there exists in G pairs of reflections that differ by $\frac{\pi}{4}$ radians. Let τ_i, τ_j be any such pair. By Lemma 4.1.5, such a pair of reflections generates a dihedral subgroup of order 8, namely

$$U = \langle \tau_i, \tau_j \rangle = \{1, \tau_i, \tau_j, x^{\frac{k}{2}}, x^{\frac{k}{4}}, x^{\frac{3k}{4}}, (\tau_i)x^{\frac{k}{4}}, (\tau_i)x^{\frac{3k}{4}}\}.$$

By Lemma 4.1.7 and Lemma 4.1.8, U is nilpotent of nilpotency class equal to 2. Finally, observe that τ_i, τ_j are such a pair if and only if $i \equiv j + \frac{k}{4} \pmod{k}$ or $j \equiv i + \frac{k}{4} \pmod{k}$.

Take $U_1 = \langle \tau_1, \tau_{\frac{k}{4}+1} \rangle$, $U_2 = \langle \tau_2, \tau_{\frac{k}{4}+2} \rangle$, and so on. There are $\frac{k}{4}$ such groups; they are all 2-nilpotent and they contain all reflections in G . By adding in the cyclic subgroup containing all rotations we obtain a 2-nilpotent cover of size $\frac{k}{4} + 1$. $\square$

Lemma 4.2.8. If $k \equiv 4 \pmod{8}$, $k > 4$ there exists a non-1-nilpotent subset of D_{2k} of size $\frac{k}{2} + 1$, and a non-2-nilpotent subset of D_{2k} of size $\frac{k}{4} + 1$.

Proof. We know, by Lemma 4.1.5, that a distinct pair of reflections commute if and only if the angles between their lines of reflection differ by $\frac{\pi}{2}$ radians. Now G has k reflections and so contains exactly $\frac{k}{2}$ distinct commuting pairs of reflections. From each of these pairs, we pick one distinct reflection and obtain a non-commuting subset of reflections of size equal to $\frac{k}{2}$.

In addition, we add a rotation, x , of order k . Since $k > 2$, Lemma 4.1.3 implies that x does not commute with any reflection. We conclude that there exists in G a non-commuting subset (non-1-nilpotent subset) of D_{2k} of size $\frac{k}{2} + 1$.

Next, we know by Lemmas 4.1.5 and 4.1.8, that a distinct pair of reflections generate a 2-nilpotent group if and only if they differ by a multiple of $\frac{\pi}{4}$ radians. Further, D_{2k} contains $\frac{k}{4}$ sets of 4 rotations which differ from each other by a multiple of $\frac{\pi}{4}$ radians. We pick one reflection from each set and we obtain a non-2-nilpotent subset of reflections in $G = D_{2k}$ of size equal to $\frac{k}{4}$.

Once again, we add a rotation, x , of order k . Since $k > 2$, Lemma 4.1.3 implies that, if τ is any reflection, then $\langle x, \tau \rangle$ is dihedral of order $2k$ and so, by Lemma 4.1.7, is not nilpotent. We conclude that there exists in G a non-2-nilpotent subset of D_{2k} of size $\frac{k}{4} + 1$. $\square$

We summarize the above results by the proposition:

Proposition 4.2.3. *If $k \equiv 4 \pmod{8}$ and $k > 4$, then*

$$\Sigma_c(D_{2k}) = \sigma_c(D_{2k}) = \begin{cases} \frac{k}{2} + 1, & c = 1; \\ \frac{k}{4} + 1, & c \geq 2. \end{cases}$$

Example 4.2.3. (Refer to Figure 4.8 when $k = 12 \equiv 4 \pmod{8}$). In our case $k = 12$, so we obtain D_{24} , where $\tau_i, i = 1, 2, \dots, 12$ are the twelve reflections with mirror lines $\ell_i, i = 1, 2, \dots, 12$ respectively.

By Lemma 4.1.5, let x_j be a rotation of $\theta = \frac{2\pi j}{12}$ radians, where j is relatively prime to k that is, $j = 1, 5, 7, 11$. This rotation generates the cyclic subgroup $\langle x_j \rangle \cong C_{12}$.

The center $Z(D_{24}) = \{1, x^{12}\}$, where x^{12} is a rotation of π radians. The following pairs of reflections have mirror lines that differ by $\frac{\pi}{2}$ radians and therefore commute: $(\tau_1, \tau_7), (\tau_2, \tau_8), (\tau_3, \tau_9), (\tau_4, \tau_{10}), (\tau_5, \tau_{11}), (\tau_6, \tau_{12})$. The 3 Sylow 2-subgroups of order 8 are nilpotent dihedral subgroups of nilpotency class $c = 2$. These are:

$$\{1, x^6, x^{12}, x^{18}, \tau_1, \tau_4, \tau_7, \tau_{10}\}$$

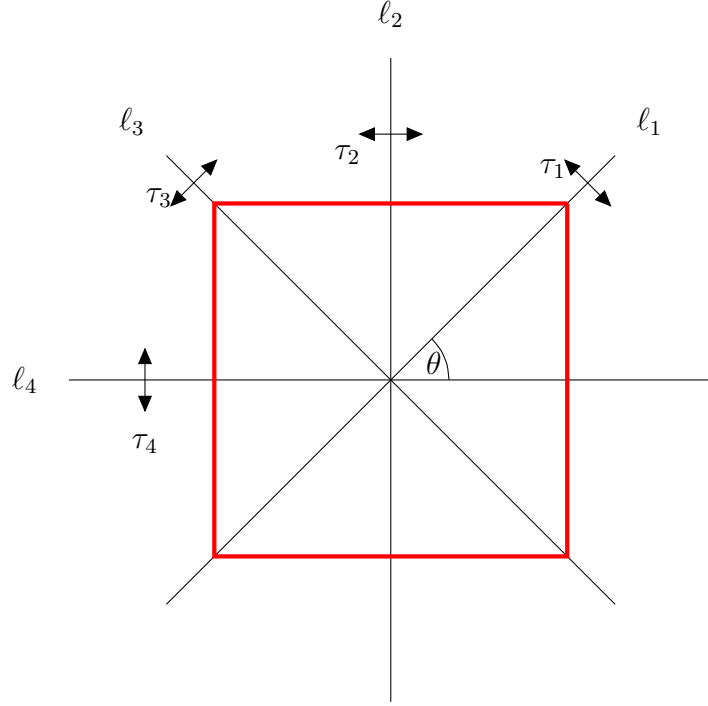
$$\{1, x^6, x^{12}, x^{18}, \tau_2, \tau_5, \tau_8, \tau_{11}\}$$

$$\{1, x^6, x^{12}, x^{18}, \tau_3, \tau_6, \tau_9, \tau_{12}\}.$$

The union of these 3 Sylow 2-subgroups and the cyclic subgroup is a 2-nilpotent cover of G of size $\frac{12}{2^2} + 1 = 4$.

There exists a non-2-nilpotent subset is of size 4. For instance, one such subset is $\{\tau_1, \tau_2, \tau_3, x_j\}$. We conclude that $\sigma_2(D_{24}) = \Sigma_2(D_{24}) = 4$.

A 1-nilpotent cover of D_{24} contains the cyclic subgroup and the 6 subgroups

Figure 4.6: Reflections in D_8 .

Source: Research.

isomorphic to $D_4 \cong K_4$, namely:

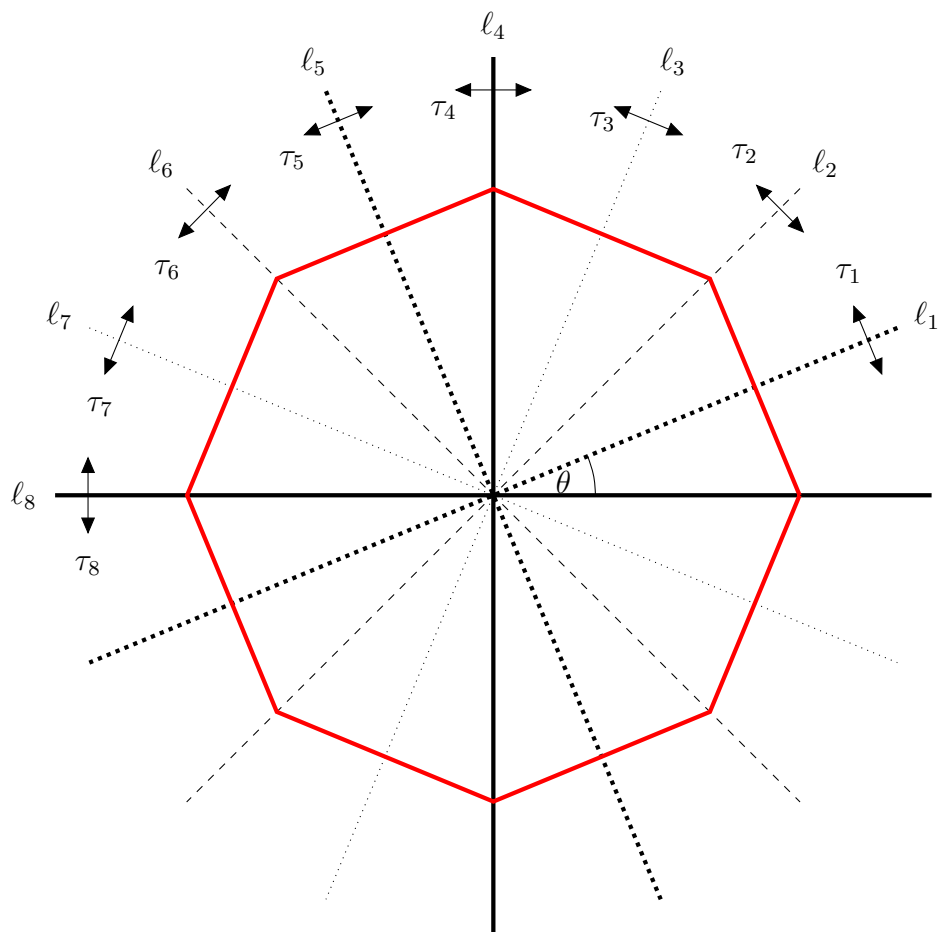
$$\{1, x^{12}, \tau_1, \tau_7\}, \{1, x^{12}, \tau_2, \tau_8\}, \{1, x^{12}, \tau_3, \tau_9\}$$

$$\{1, x^{12}, \tau_4, \tau_{10}\}, \{1, x^{12}, \tau_5, \tau_{11}\}, \{1, x^{12}, \tau_6, \tau_{12}\}$$

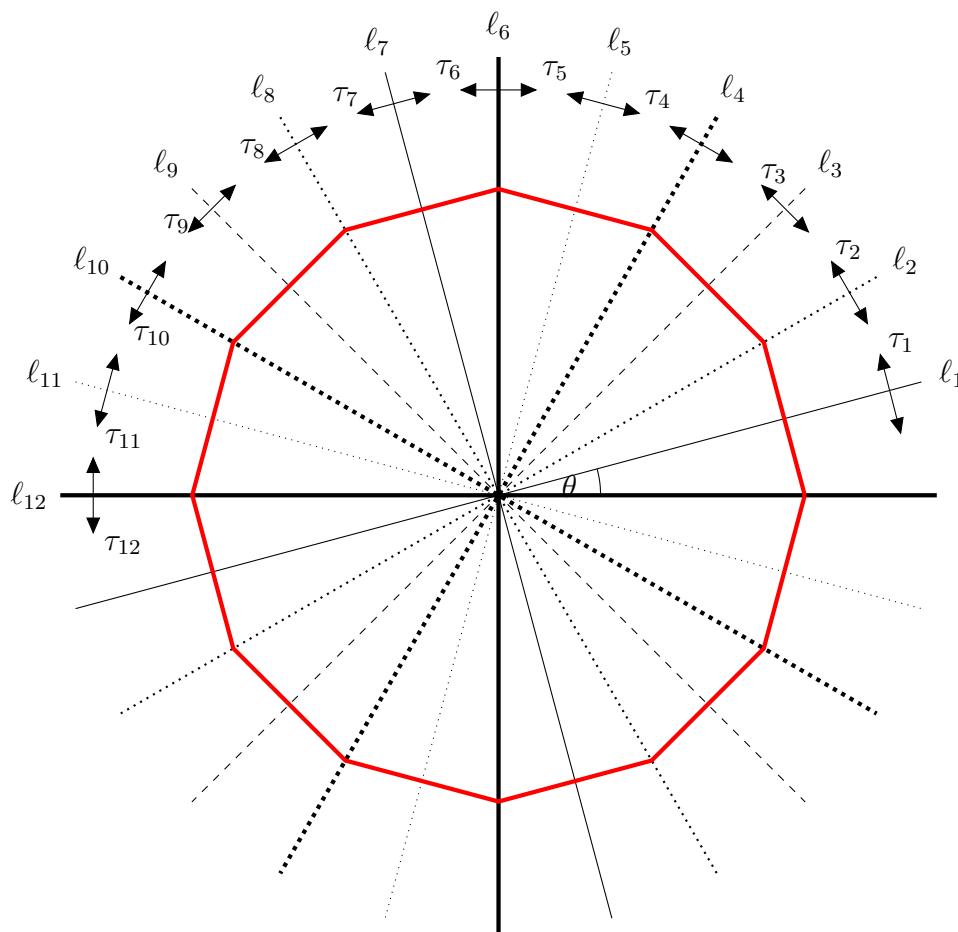
Each of these abelian subgroups contains at least one element that is not contained in the others. We see that this is a minimal 1-nilpotent cover whose size is equal to $\Sigma_1(D_{24}) = \frac{12}{2} + 1 = 7$. A non-1-nilpotent subset of D_{24} is $\{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, x\}$. It has size $\sigma_1(D_{24}) = 7$.

4.2.4 The general case

We are now ready to prove our main result. This result generalizes the special cases that came earlier in Propositions 4.2.1, 4.2.2 and 4.2.3. The statement we need is as follows.

Figure 4.7: Reflections in D_{16} .

Source: Research.

Figure 4.8: Reflections in D_{24} .

Source: Research.

Theorem 4.2.1. *Let $G = D_{2k}$ with $k \geq 2$, and let c be a positive integer.*

$$\sigma_c(G) = \Sigma_c(G) = \begin{cases} \frac{k}{\gcd(k, 2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \dots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \dots, 2^c\}. \end{cases} \quad (4.2.1)$$

In particular, writing $|k|_{2'}$ for the largest odd factor dividing k , we have

$$\sigma_\infty(G) = \Sigma_\infty(G) = \begin{cases} |k|_{2'} + 1, & \text{if } k \text{ is not a power of } 2. \\ 1, & \text{if } k \text{ is a power of } 2, \text{ i.e } G \text{ is nilpotent.} \end{cases}$$

For $k = 3, 4, \dots, 12$ we have constructed Table 4.1 to act as an example of the application of this Theorem. In addition, one notes that $\sigma_c(D_{2k}) = \Sigma_c(D_{2k})$ for a given c and a particular value of k .

Table 4.1: $\sigma_c(D_{2k})$ and $\Sigma_c(D_{2k})$ for $3 \leq k \leq 12$

k	nature of k	D_{2k}	$\sigma_1(D_{2k})$	$\sigma_2(D_{2k})$	$\sigma_3(D_{2k})$
3	odd	D_6	4	4	4
4	$4 \equiv 0 \pmod{4}$	D_8	1	1	1
5	odd	D_{10}	6	6	6
6	$6 \equiv 2 \pmod{4}$	D_{12}	4	4	4
7	odd	D_{14}	8	8	8
8	$8 \equiv 0 \pmod{4}$	D_{16}	1	1	1
9	odd	D_{18}	10	10	10
10	$10 \equiv 2 \pmod{4}$	D_{20}	6	6	6
11	odd	D_{22}	12	12	12
12	$12 \equiv 0 \pmod{4}$	D_{24}	7	4	4

Source: Research.

We will prove Theorem 4.2.1 in a series of lemmas. Our method will be analogous to the method that was used for the special cases.

Lemma 4.2.9. *Let $G = D_{2k}$ with $k \geq 2$ and let c be a positive integer. Then there exists a c -nilpotent cover of G of size t where*

$$t = \begin{cases} \frac{k}{\gcd(k, 2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \dots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \dots, 2^c\}. \end{cases}$$

Proof. If $k \in \{2^1, 2^2, \dots, 2^c\}$, then Lemma 4.1.8 implies that G is nilpotent of class $\leq c$, and so $\{G\}$ is a c -nilpotent cover of G , and the result holds. Assume, then, that $k \notin \{2^1, 2^2, \dots, 2^c\}$.

Write $\ell = \gcd(k, 2^c)$. Observe, first, that there is a cyclic subgroup containing all rotations. We need to show, therefore, that we can find $\frac{k}{\ell}$ c -nilpotent subgroups containing all reflections. Number these reflections $\tau_1, \dots, \tau_k$ moving sequentially around the origin, with π/k radians between the lines of reflection of τ_i and τ_{i+1} for $i = 1, \dots, k-1$.

The group $G = D_{2k}$ contains a pair of reflections that differ by an angle of $\frac{\pi}{\ell}$ radians. Lemma 4.1.5 says that such a pair of reflections, τ_i and τ_j , generate a dihedral subgroup D of G of order 2ℓ . But then Lemma 4.1.7 and Lemma 4.1.8 says that $D_{2\ell}$ is nilpotent of class $\leq c$. Lemma 4.1.5 implies that D contains a rotation of $2\pi/\ell$ radians; conjugating τ_i and τ_j by powers of this rotation we see that $D = \langle \tau_i, \tau_j \rangle$, where $j \equiv i + \frac{k}{\ell} \pmod{k}$. Now, D contains all ℓ reflections that differ from τ_i by a multiple of $\frac{\pi}{\ell}$ (including τ_i itself).

Now take $D_1 = \langle \tau_1, \tau_{\frac{k}{\ell}+1} \rangle$, $D_2 = \langle \tau_2, \tau_{\frac{k}{\ell}+2} \rangle$, and so on. We obtain $\frac{k}{\ell}$ distinct subgroups isomorphic to $D_{2\ell}$ that contain all of the reflections in G . The result follows. $\square$

Lemma 4.2.10. Let $G = D_{2k}$, with $k \geq 2$, and let c be a positive integer. Then there exists a non- c -nilpotent subset of G of size t where

$$t = \begin{cases} \frac{k}{\gcd(k, 2^c)} + 1, & \text{if } k \notin \{2^1, 2^2, \dots, 2^c\}. \\ 1, & \text{if } k \in \{2^1, 2^2, \dots, 2^c\}. \end{cases}$$

Proof. If $k \in \{2^1, 2^2, \dots, 2^c\}$, then we can take any element $x \in G$ and $\{x\}$ will be a non- c -nilpotent subset of G , and the result holds. Assume, then, that $k \notin \{2^1, 2^2, \dots, 2^c\}$.

Let $x, \tau_1, \dots, \tau_k$ be as in the previous proof. Note, first, that x is a rotation of order k . Since k does not divide 2^c , x does not lie in a nilpotent dihedral

subgroup of nilpotency class $\leq c$.

Write $\ell = \gcd(k, 2^c)$. We can find $\frac{k}{\ell}$ reflections, any pair of which do not generate a c -nilpotent subgroup. Consider the set $\tau_1, \dots, \tau_{k/\ell}$. Any pair of these differ by an angle *strictly less* than π/ℓ . Lemma 4.1.5 implies that they generate a dihedral group of order strictly greater than ℓ . Then, Lemma 4.1.8 implies that this group is not nilpotent of class at most c . The result follows. $\square$

We have all the components that we need to prove our main result.

Proof of Theorem 4.2.1. Write t for the quantity on the right hand side of (4.2.1). Lemma 4.2.9 implies that $\Sigma_c(G) \leq t$; Lemma 4.2.10 implies that $t \leq \sigma_c(G)$; finally Lemma 1.2.5 implies that $\sigma_c(G) \leq \Sigma_c(G)$. Hence

$$t \leq \sigma_c(G) \leq \Sigma_c(G) \leq t.$$

We conclude immediately that $\sigma_c(G) = \Sigma_c(G) = t$, as required. $\square$

4.3 S_n , the symmetric group on n symbols

Throughout this section, $G = S_n$ will be the finite symmetric group on the set $S = \{1, 2, \dots, n\}$. We study the quantities $\sigma_\infty(S_n), \Sigma_\infty(S_n)$ for small n . We calculate upper and lower bounds between which these quantities lie.

4.3.1 Background material on S_n

The set of all one-to-one and onto mappings from a set S to itself forms a group with respect to composition of maps. These maps are called permutations of S . If the set S has n elements then the group is called the symmetric group S_n of degree n whose order is $n!$. Clearly any subgroup of S_n is a permutation group. According to the well-known Cayley theorem (Fraleigh, 2002), every group is isomorphic to a subgroup of S_n .

The solution of many problems in algebra require knowledge on group actions. We first give a brief review on group actions which will be used throughout in this section.

Definition 4.3.1. Let G be a finite group and X a set. A left action of G on a set X is a map $G \times X \rightarrow X$ defined by $(g, x) \rightarrow gx$, such that the following conditions are met: $ex = x$ for all $x \in X$ and $(g_1g_2)x = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

Note that X is any set and is not necessarily related to the group G . Under these conditions the set X is called a G -set. It is important to see that every group G acts on every set X by the trivial action $(g, x) \rightarrow x$. When the set X is somehow related to the group G , group actions turn to be interesting.

Generally, if X is any set and G is a subgroup of S_X , where S_X is the group of all permutations acting on X , we say that X is a G -set under the group action $(\rho, x) \rightarrow \rho(x)$ for $\rho \in G$ and $x \in X$.

Example 4.3.1. Consider the symmetry group of a square, that is the dihedral group $G = D_4$. By labelling the vertices using 1, 2, 3, 4, we may take the set $X = \{1, 2, 3, 4\}$. We can then think of D_4 as acting on these vertices and hence consist of the following permutations: $\{(1), (13), (24), (1432), (1234), (12)(34), (14)(23), (13)(24)\}$. The elements of D_4 act on X as functions. The permutation $(12)(34)$ acts on vertex 1 by sending it to vertex 2, on vertex 3 by sending it to vertex 4, and so on. One can verify easily that the axioms of a group action hold.

Theorem 4.3.1. *Let X be a G -set. Then G -equivalence is an equivalence relation on X .*

Proof. The relation $\sim$ is reflexive since $ex = x$. Suppose that $x \sim y$ for $x, y \in X$. Then there exists a g such that $gx = y$. In this case $g^{-1}y = x$; hence, $y \sim x$. To show that the relation is transitive, suppose that $x \sim y$ and $y \sim z$. Then there must exist group elements g and h such that $gx = y$ and $hy = z$. So $z = hy = (hg)x$, and x is equivalent to z . $\square$

If X is a G -set, then each partition of X associated with G -equivalence is called an orbit of X under G . One can denote the orbit that contains an element x of X by $\text{Orb}(x)$. The orbits or transitivity classes of the action of G on the G -set are disjoint.

If G is a group acting on a set X and if we let g be an element of G , then the fixed point set of g in X , denoted by X_g , is the set of all $x \in X$ such that $gx = x$. One can also be interested in the group elements g that fix a given $x \in X$. This set is not merely a subset of G , it is a subgroup. This subgroup is called the *stabilizer* subgroup or *isotropy* subgroup of x . We may denote the stabilizer subgroup of x by G_x or $\text{Stab}_G(x)$.

Proposition 4.3.1. *Let G be a group acting on a set X and $x \in X$. The stabilizer group, $\text{Stab}_G(x) = G_x$, of x is a subgroup of G .*

Proof. Using the notation G_x for $\text{Stab}_G(x)$ we have that $e \in G_x$ because the identity element fixes every element in the set X . Let $g, h \in G_x$. Then $gx = x$ and $hx = x$. So $(gh)x = g(hx) = gx = x$; hence, the product of two elements in G_x is also in G_x . Finally, if $g \in G_x$, then $x = ex = (g^{-1}g)x = (g^{-1})gx = g^{-1}x$. So g^{-1} is in G_x . $\square$

We may denote the number of elements in the fixed point set of an element $g \in G$ by $\langle X_g \rangle$ and denote the number of elements in the orbit of $x \in X$ by $|\text{Orb}(x)|$.

Theorem 4.3.2. *Let G be a finite group and X a finite G -set. If $x \in X$, then $|\text{Orb}(x)| = [G : G_x]$.*

Proof. We know that $|G|/|G_x|$ is the number of left cosets of G_x in G by the well known Lagrange's Theorem. Define a bijective map π between the orbit $\text{Orb}(x)$ of X and the set of left cosets LG_x of G_x in G . Let $y \in \text{Orb}(x)$. Then there exists a g in G such that $gx = y$. Define π by $\pi(y) = gG_x$. First we must show that this map is well-defined and does not depend on our selection of g . Suppose that h is another element in G such that $hx = y$. Then $gx = hx$ or $x = g^{-1}hx$; hence,

$g^{-1}h$ is in the stabilizer subgroup of x . Therefore, $h \in gG_x$ or $gG_x = hG_x$. Thus, y gets mapped to the same coset regardless of the choice of the representative from that coset. To show that π is one-to-one, assume that $\pi(x_1) = \pi(x_2)$. Then there exist $g_1, g_2 \in G$ such that $x_1 = g_1x$ and $x_2 = g_2x$. Since there exists a $g \in Gx$ such that $g_2 = g_1gx_2 = g_2x = g_1gx = g_1x = x_1$. Consequently, the map π is one-to-one. Finally, to show that the map π is onto we let gG_x be a left coset. If $gx = y$, then $\pi(y) = gG_x$. This completes the proof. $\square$

Example 4.3.2. Let X be the set $1, 2, 3, 4, 5$. Consider the permutation group G on X , where G is defined by $G = \{(1), (123), (132), (45), (123)(45), (132)(45)\}$, then X is a G -set. The orbits are; $\text{Orb}(1) = (123)$, $\text{Orb}(2) = (123)$, $\text{Orb}(3) = (123)$, $\text{Orb}(4) = (45)$, $\text{Orb}(5) = (45)$

The fixed point sets of X under the action of G are; $X_{(1)} = X$, $X_{(123)} = \{4, 5\}$, $X_{(132)} = \{4, 5\}$, $X_{(45)} = \{1, 2, 3\}$, $X_{(123)(45)} = \emptyset$, $X_{(132)(45)} = \emptyset$, finally, the stabilizer subgroups are; $\text{Stab}_G(1) = G(1) = \{(1), (45)\}$, $\text{Stab}_G(2) = G(2) = \{(1), (45)\}$, $\text{Stab}_G(3) = G(3) = \{(1), (45)\}$, $\text{Stab}_G(4) = G(4) = \{(1), (123), (132)\}$, $\text{Stab}_G(5) = G(5) = \{(1), (123), (132)\}$. Observe that the number of elements in the orbit of x is equal to $[G : \text{Stab}_G(x)]$

If we let the set X to be the group G itself and pick a non trivial subgroup $H \leq G$ to act on G , then H acts on G (which we regard as a set) by right multiplication. Therefore to each $h \in H$ and each $g \in G$ there corresponds the element $gh \in G$. This defines an action of H on G which is immediate from the associative law for multiplication in G and the defining property of the identity element. For any $g \in G$ we have that $\text{Stab}_H(g) = \{h \in H : gh = g\} = 1$, of course this action is faithful(that is, the corresponding permutation representation of H is injective). The orbit of g is the set $\{gh : h \in H\} = gH$, which is the co set of H in G containing g . We know that orbits or transitivity classes of an action are disjoint, then the distinct left co sets of H in G are disjoint. Similarly if the multiplication is on the left on elements of G by elements of H , then we have a left action of H on G . It is clear in this case that the orbits of this action are the

right co sets of H in G .

Definition 4.3.2. Let G act on the set X . The action is said to be transitive if it has just one orbit. An action which is not transitive is called intransitive.

If n is any positive integer and if we let $X = \{1, 2, \dots, n\}$, then the natural action of the symmetry group of Σ_n on X is transitive. It is also clear that the natural action on X of the cyclic group $C = \langle (1, 2, \dots, n) \rangle$ of the symmetry group Σ_n is also transitive. The natural action on A_n on X is transitive if $n \geq 3$.

With the above review on group actions we now look at some background material on the symmetric group and finally study their minimal nilpotent cover as well as their maximal non nilpotent subsets.

Theorem 4.3.3. (Fraleigh, 2002)(*Cayley's Theorem*) Every group is isomorphic to a group of permutations.

Proof. Let G be a group. We show that G is isomorphic to a subgroup of S_G . We need to define a one-to-one function $\phi : G \rightarrow S_G$ such that $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. For $x \in G$, let $\alpha_x : G \rightarrow G$ be defined by $\alpha_x(g) = xg$ for all $g \in G$. We think of α_x as performing left multiplication by x . The equation $\alpha_x(x^{-1}c) = x(x^{-1}c) = c$ for all $c \in G$ shows that α_x maps G onto G . If $\alpha_x(a) = \alpha_x(b)$, then $xa = xb$ so $a = b$ by cancellation. Thus α_x is also one to one, and is a permutation of G . We now define $\phi : G \rightarrow S_G$ by defining $\phi(x) = \alpha_x$ for all $x \in G$. To show that ϕ is one to one, suppose that $\phi(x) = \phi(y)$. Then $\alpha_x = \alpha_y$ as functions mapping G into G . In particular $\alpha_x(e) = \alpha_y(e)$, so $xe = ye$ and $x = y$. Thus ϕ is one to one. It only remains to show that $\phi(xy) = \phi(x)\phi(y)$, that is, that $\alpha_{xy} = \alpha_x\alpha_y$. Now for any $g \in G$, we have $\alpha_{xy}(g) = (xy)g$. Permutation multiplication is function composition, so $(\alpha_x\alpha_y)(g) = \alpha_x(\alpha_y(g)) = \alpha_x(yg) = x(yg)$. Thus by associativity, $\alpha_{xy} = \alpha_x\alpha_y$. □

Let ρ be a permutation of a set T . Then ρ determines a natural partition of T into cells called orbits. If $x, y \in T$ are in the same orbit then $y = \rho^n(x)$ for

some $n \in \mathbb{Z}$. This partition is established by noting that if we let $x \sim y$ if and only if $y = \rho^n(x)$ for some $n \in \mathbb{Z}$ then this is an equivalence relation on the set T determined by ρ . The relation is reflexive since $x \sim x$ because $x = 1(x) = \rho^0(x)$. If $x \sim y$ then $y = \rho^n(x)$ for some $n \in \mathbb{Z}$, but one observes immediately that $x = \rho^{-n}(y)$ for some $n \in \mathbb{Z}$, therefore $y \sim x$ and the relation is symmetric. Next suppose $x \sim y$ and $y \sim z$ then $y = \rho^n(x)$ and $z = \rho^m(y)$ for some $m, n \in \mathbb{Z}$. This implies that $z = \rho^m(\rho^n(x)) = \rho^{n+m}(x)$ and we conclude $x \sim z$ and the relation is transitive.

Definition 4.3.3. Let ρ be a permutation of a set T . The equivalence classes in T determined by the equivalence relation $x \sim y$ if and only if $y = \rho^n(x)$ for some $n \in \mathbb{Z}$ are the orbits of ρ .

Theorem 4.3.4. (Fraleigh, 2002) *Every permutation ρ of a finite set is a product of disjoint cycles.*

Proof. Suppose $\Theta_1, \Theta_2, \dots, \Theta_n$ are the orbits of ρ , let δ_i be the cycle defined by $\delta_i(x) = \rho(x)$ for $x \in \Theta_i$ otherwise $\delta_i(x) = x$. Clearly $\rho = \delta_1 \delta_2 \dots \delta_n$. This is because the equivalence-class orbits $\Theta_1, \Theta_2, \dots, \Theta_n$ being distinct equivalence classes, are disjoint, the cycles $\delta_1, \delta_2, \dots, \delta_n$ are disjoint also. $\square$

Multiplication of permutations in general is not commutative, but it is clear that multiplication of disjoint cycles is commutative. Notice also that the orbits of a permutation are unique and the representation of a permutation as a product of disjoint cycles, none of which is the identity permutation, is unique up to the order of the factors.

Definition 4.3.4. Cycle of length 2 is a transposition.

A transposition leaves all elements but two fixed, and maps each of these onto the other. An investigation shows that: $(a_1, a_2, \dots, a_n) = (a_1, a_n)(a_1, a_{n-1}), \dots, (a_1, a_3)(a_1, a_2)$.

Theorem 4.3.5. (Fraleigh, 2002) *A permutation in S_n cannot be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.*

Proof. Let A and B be sets. If A and B have the same cardinality, then $S_A \simeq S_B$. Using our knowledge of linear algebra and on working with permutations of the n rows of the $n \times n$ identity matrix I_n , instead of using the set $\{1, 2, \dots, n\}$. The identity matrix I_n has determinant of 1. We know from the properties of determinants that interchanging any two rows of a square matrix changes the sign of the determinant. Suppose X is the matrix obtained by a permutation τ of the rows of I_n . Now if X could be obtained from I_n by both an even number and an odd number of transpositions of rows, or columns then its determinant would have to be both 1 and -1 , which is impossible. Thus τ cannot be expressed both as a product of an even number and an odd number of transpositions. $\square$

Theorem 4.3.6. (Frleigh, 2002) *Let $S_n, n \geq 2$ be the symmetric group on n letters, then the number of even permutations in S_n is equivalent to the number of odd permutations, that is, S_n is equally split and each number equals $\frac{n!}{2}$.*

Proof. For $n \geq 2$, let E_n be the set of even permutations in S_n and let D_n be the set of odd permutations. Define a function $\phi_\lambda E_n \rightarrow D_n$ by $\phi_\lambda(\rho) = \lambda\rho$, where ρ is any fixed transposition in S_n . Without loss of generality let $\rho = (1, 2)$. We see that under this mapping $\rho \in E_n$ is mapped into $(1, 2)\rho$ by ϕ_λ . Since ρ is even, the permutation $(1, 2)\rho$ can be expressed as a product of a $(1 + \text{even number})$, or odd number, of transpositions, so $(1, 2)\rho$ is indeed in D_n . Now suppose that $\rho, \tau \in E_n$ and let $\phi_\lambda(\rho) = \phi_\lambda(\tau)$, then $(1, 2)\rho = (1, 2)\tau$. But then S_n is a group so we have that $\rho = \tau$, hence ϕ_λ is an injective function. It is clear that $\lambda = (1, 2) = \lambda^{-1}$. So if $\rho \in D_n$, then $\lambda^{-1}\rho \in E_n$, and $\phi_\lambda(\lambda^{-1}\rho) = \lambda(\lambda^{-1}\rho) = \rho$. Therefore ϕ_λ is onto E_n . Hence the number of elements in E_n is the same as the number in D_n since there is a one-to-one correspondence between the elements of the two sets. $\square$

From above we observe that the product of any two even permutations is an even permutation. It is also true that the product of an even permutation and an odd permutation is an odd permutation. In a similar manner the product of an odd permutation and an even permutation is an odd permutation. Finally the product of two odd permutation is an even one.

If we let A_n be the subset of S_n consisting of all even permutations. We have seen that the product of any two even permutations is an even permutation, it follows that A_n is a subgroup of S_n consisting of all even permutations. Of course the odd permutations do not form a group. It can be shown easily that A_n is a normal subgroup of S_n . Indeed if we let $X = \{1, -1\}$ and define $\pi : S_n \rightarrow X$ by $\pi(\sigma) = 1$ if σ is an even permutation and $\pi(\sigma) = -1$ if σ is an odd permutation. Clearly by the facts mentioned above π is a homomorphism onto X . It is clear that A_n is the kernel of π , which implies that A_n is a normal subgroup of S_n of index 2. Therefore $X = S_n/A_n$. Now $|X| = 2 = \left(\frac{|S_n|}{|A_n|}\right) = \frac{|S_n|}{|A_n|}$ where we see that $|A_n| = \frac{1}{2}n!$. Therefore A_n is precisely the alternating group of degree n . We summarize this in the following lemma.

Lemma 4.3.1. (Fraleigh, 2002) Let S_n be the symmetric group on n symbols. S_n has a normal subgroup of index 2, the alternating group of A_n consisting of all even permutations.

Recall that each element of S_n can be written in cycle notation. We have seen that these permutations can either be even or odd. In particular, the identity permutation is an even permutation. The alternating group A_n of index 2 in S_n comprises of all the even permutations. It is one of the maximal subgroups of S_n .

The maximal subgroups help illuminate the structure of subgroups of a group. The O’Nan-Scott theorem (Liebeck et al., 1987) help classify maximal subgroups of S_n and A_n . Our major concern is whether we can get a glimpse of the minimal size of nilpotent covers and maximal size of non-nilpotent subsets by looking at these maximal subgroups. Of course, getting the exact size of such covers and subsets for large n is asking too much since this size grows fast as n increases. We use (GAP, 2013) code to do this.

We construct abelian covers and non-commuting subsets (Abdollahi, Azad, Mohammadi Hassanabadi, & Zarrin, 2010) which provide a starting point for hypothesizing our bounds for $\Sigma_\infty(S_n)$ and $\sigma_\infty(S_n)$. We begin this construction by looking at some cyclic subgroups of the symmetric group.

Lemma 4.3.2. S_n has $(n - 1)!$ different n -cycles.

Proof. Each n -cycle can be written in the form $(a_1, a_2 \dots a_{n-1}, n)$ where $a_1, a_2 \dots a_{n-1}$ is some permutation of $1, 2 \dots n - 1$. Since there are $(n - 1)!$ such permutations, we see that there are $(n - 1)!$ n -cycles. $\square$

Lemma 4.3.3. Let $G = S_n$, and let $g \in S_n$ be an n -cycle. Then $C_G(g) = \langle g \rangle \cong C_n$.

Proof. Since conjugacy structure in S_n is determined by cycle type, the previous lemma tells us that there are $(n - 1)!$ conjugates of an n -cycle. The Orbit-Stabilizer theorem tells us, therefore, that $|C_G(g)| = n!/(n - 1)! = n$. On the other hand, $\langle g \rangle \leq C_G(g)$. Since $|\langle g \rangle| = n$, the result follows. $\square$

Definition 4.3.5. The Euler totient function $\phi(n)$ where n is a positive integer gives number of integers relatively prime to n .

Lemma 4.3.4. Let a be an n -cycle in S_n . The number of elements b in S_n such that $\langle b \rangle = \langle a \rangle$ is $\phi(n)$.

Proof. If b is an element of S_n that satisfies $\langle b \rangle = \langle a \rangle$, then $b \in \langle a \rangle$.

Suppose then that $b = a^k$, where $k \in \{1, 2, \dots, n\}$. We claim that $\langle b \rangle = \langle a \rangle$ if and only if $\gcd(k, n) = 1$. The lemma follows from this claim, because there are precisely $\phi(n)$ values of k between 1 and n that satisfy $\gcd(k, n) = 1$.

It remains to prove the claim.

Let $m = \gcd(k, n)$. Suppose $m > 1$. Then $k = qm$, so $b = a^{qm} \in \langle a^m \rangle$. Hence $\langle b \rangle \subseteq \langle a^m \rangle$. Since $\langle a^m \rangle$ is contained in but not equal to $\langle a \rangle$, we see that $\langle b \rangle \neq \langle a \rangle$.

Next suppose that $m = 1$. Then there exists integers r and s such that $rk + sn = 1$, so $a = a^{rk+sn} = a^{rk}a^{sn} = (a^k)^r(a^n)^s = b^r$. This implies that $\langle a \rangle \subseteq \langle b \rangle$. Also, we know that $b = a^k$, which implies that $\langle b \rangle \subseteq \langle a \rangle$. Therefore $\langle b \rangle = \langle a \rangle$. This completes the proof of the claim. $\square$

Lemma 4.3.5. The number of cycles of length $k \leq n$ in S_n is equal to $\frac{n!}{k[(n-k)!]}$.

Proof. There are $\binom{n}{k}$ ways of selecting k points from n . Lemma 4.3.2 asserts that given k points there are $(k-1)!$ different cycles. The result follows by noting that $(k-1)!\binom{n}{k} = \frac{n!}{k[(n-k)!]}$. $\square$

Lemma 4.3.6. The number of distinct cyclic subgroups of S_n , generated by a k -cycle, $k \leq n$, is equal to $\frac{n!}{k[(n-k)!\phi(k)]}$.

Proof. The proof follows from Lemma 4.3.5. One notes that for such a k -cycle, the cyclic group of degree k has $\phi(k)$ generators. By Lemma 4.3.5, we obtain $\frac{n!}{k[(n-k)!\phi(k)]}$ cyclic subgroups. $\square$

Theorem 4.3.7. (Rose, 1994) Let β be a permutation in A_n . Then its conjugacy class in S_n splits into two equivalent classes in A_n if and only if the cycle type of β are distinct odd primes.

The following tables show a summary of conjugacy classes for some symmetric groups and their corresponding alternating groups.

Table 4.2: Conjugacy classes of S_3

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1	(1)	$\frac{3!}{1^3 3!} = 1$
2	2 + 1	(12)	$\frac{3!}{2^1 1! 1^1 1!} = 3$
3	3	(123)	$\frac{3!}{3^1 1!} = 2$

Source: Research.

Table 4.3: Conjugacy classes of A_3

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1	(1)	$\frac{3!}{1^3 3!} = 1$
2	3	(123)	$\frac{1}{2} \times \frac{3!}{3^1 1!} = 1$
3	3	(132)	$\frac{1}{2} \times \frac{3!}{3^1 1!} = 1$

Source: Research.

Table 4.4: Conjugacy classes of S_4

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1 + 1	(1)	$\frac{4!}{1^4 4!} = 1$
2	2 + 1 + 1	(12)	$\frac{4!}{2^1 1! 1! 2!} = 6$
3	3 + 1	(123)	$\frac{4!}{3^1 1! 1! 1!} = 8$
4	2 + 2	(12)(34)	$\frac{4!}{2^2 2!} = 3$
5	4	(1234)	$\frac{4!}{4^1 1!} = 6$

Source: Research.

Table 4.5: Conjugacy classes of A_4

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1 + 1	(1)	$\frac{4!}{1^4 4!} = 1$
2	3 + 1	(123)	$\frac{1}{2} \times \frac{4!}{3^1 1! 1! 1!} = 4$
3	3 + 1	(132)	$\frac{1}{2} \times \frac{4!}{3^1 1! 1! 1!} = 4$
4	2 + 2	(12)(34)	$\frac{4!}{2^2 2!} = 3$

Source: Research.

Table 4.6: Conjugacy classes of S_5

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1 + 1 + 1	(1)	$\frac{5!}{1^5 5!} = 1$
2	2 + 1 + 1 + 1	(12)	$\frac{5!}{2^1 1! 1! 1! 3!} = 10$
3	3 + 1 + 1	(123)	$\frac{5!}{3^1 1! 1! 2!} = 20$
4	3 + 2	(123)(45)	$\frac{5!}{3^1 1! 2! 1!} = 20$
5	2 + 2 + 1	(12)(34)	$\frac{5!}{2^2 2!} = 15$
6	4 + 1	(1234)	$\frac{5!}{4^1 1!} = 30$
7	5	(12345)	$\frac{5!}{5^1 1!} = 24$

Source: Research.

Table 4.7: Conjugacy classes of A_5

Count	Cycle Type	Class Representative	Class Size
1	1 + 1 + 1 + 1 + 1	(1)	$\frac{5!}{1^5 5!} = 1$
2	3 + 1 + 1	(123)	$\frac{5!}{3^1 1! 1! 2!} = 20$
3	2 + 2 + 1	(12)(34)	$\frac{5!}{2^2 2!} = 15$
4	5	(12345)	$\frac{1}{2} \times \frac{5!}{5^1 1!} = 12$
5	5	(54321)	$\frac{1}{2} \times \frac{5!}{5^1 1!} = 12$

Source: Research.

Table 4.8: Conjugacy classes of S_6

Count	Cycle Type	Class Representative	Class Size
1	$1 + 1 + 1 + 1 + 1 + 1$	(1)	$\frac{6!}{1^6 6!} = 1$
2	$2 + 1 + 1 + 1 + 1$	(12)	$\frac{6!}{2^1 1! 1! 1! 1! 4!} = 15$
3	$2 + 2 + 1 + 1$	(12)(34)	$\frac{6!}{2^2 2! 1! 2! 2!} = 45$
4	$2 + 2 + 2$	(12)(34)(56)	$\frac{6!}{2^3 3!} = 15$
5	$3 + 1 + 1 + 1$	(123)	$\frac{6!}{3^1 1! 1! 1! 3!} = 40$
6	$3 + 2 + 1$	(12)(345)	$\frac{6!}{3^1 1! 2^1 1! 1! 1!} = 120$
7	$3 + 3$	(123)(456)	$\frac{6!}{3^2 2!} = 40$
8	$4 + 1 + 1$	(1234)	$\frac{6!}{4^1 1! 1! 2! 2!} = 90$
9	$4 + 2$	(12)(3456)	$\frac{6!}{4^1 1! 2^1 1!} = 90$
10	$5 + 1$	(12345)	$\frac{6!}{5^1 1! 1! 1! 1!} = 144$
11	6	(123456)	$\frac{6!}{6^1 1!} = 120$

Source: Research.

Table 4.9: Conjugacy classes of A_6

Count	Cycle Type	Class Representative	Class Size
1	$1 + 1 + 1 + 1 + 1 + 1$	(1)	$\frac{6!}{1^6 6!} = 1$
2	$2 + 2 + 1 + 1$	(12)(34)	$\frac{6!}{2^2 2! 1! 2! 2!} = 45$
3	$3 + 1 + 1 + 1$	(123)	$\frac{6!}{3^1 1! 1! 1! 3!} = 40$
4	$3 + 3$	(123)(345)	$\frac{6!}{3^2 2!} = 40$
5	$4 + 2$	(12)(3456)	$\frac{6!}{4^1 1! 2^1 1!} = 90$
6	$1 + 5$	(12345)	$\frac{1}{2} \times \frac{6!}{5^1 1! 1! 1! 1!} = 72$
7	$1 + 5$	(23456)	$\frac{1}{2} \times \frac{6!}{5^1 1! 1! 1! 1!} = 72$

Source: Research.

4.3.2 Results for $\sigma_\infty(S_n)$

In this section we seek to prove and implement results concerning the value of $\sigma_\infty(S_n)$. Our method is to study elements which have a “large cycle” in their cycle structure. By “large” we mean, of size at least $\frac{n}{2}$. These elements are useful because they tend to lie in a unique maximal nilpotent subgroup of S_n – thus if we try and construct a nilpotent cover for S_n , we will need to include these particular maximal nilpotent subgroups.

Our analysis of elements with a large cycle proceeds in the following way: we write k for the size of the large cycle, and we assume that $\frac{n}{2} \leq k \leq n$.

1. We begin by assuming that $k = p$, a prime. This is the easiest case because then a Sylow p -subgroup of S_n will be isomorphic to C_p .

2. Next we assume that $k = p^a$, a prime power. In this case the Sylow p -structure may be more complicated.
3. Finally, we seek to drop all assumptions about the integer k , although we will focus primarily on the case $k = 6$.

We will use **GAP** to provide data about nilpotent groups containing elements with a large cycle; we will then follow up these **GAP** calculations with formal proofs.

At the end of these calculations, we provide a series of tables outlining specific calculations for $\sigma_\infty(S_n)$ and $\Sigma_\infty(S_n)$ with $n \leq 10$. Our main result will be stated at the end.

4.3.2.1 GAP code: centralizer calculations

In this subsection we assume that $g \in S_n$ is a k -cycle in with $k > n/2$. The following table clarifies the structure of $C_{S_n}(g)$ for $n \leq 15$. Our findings are summarised in Table 4.10. An examination of this table suggests that the $C_{S_n}(g) \cong C_p \times S_{n-p}$ – we will seek to prove this fact in the next section.

4.3.2.2 Proofs: the case when k is prime

In this section we give a series of results that end up with a description of the nilpotent subgroups of S_n that contain an element g whose cycle type includes a cycle of length k , where k is prime and $k \geq \frac{n}{2}$.

Our first three results do not require the hypothesis that k is prime – they will be useful later on.

Definition 4.3.6. Let $g \in S_n$ and $x \in \{1, 2, \dots, n\}$ then we define $\text{Orb}_g(x) = \{g^n(x) : n \in \mathbb{Z}\}$, the orbit of x under the group $\langle g \rangle$.

Lemma 4.3.7. Suppose that g and h are commuting elements of S_n . Let $x \in \{1, 2, \dots, n\}$. Then $\text{Orb}_g(h(x)) = h \text{Orb}_g(x)$ and hence $|\text{Orb}_g(h(x))| = |\text{Orb}_g(x)|$

Proof. Observe that $g^n h(x) = h g^n(x)$, for any integer n , because g and h commute. Hence;

$$\begin{aligned} \text{Orb}_g(h(x)) &= \{g^n h(x) : n \in \mathbb{Z}\} \\ &= \{h g^n(x) : n \in \mathbb{Z}\} \\ &= h\{g^n(x) : n \in \mathbb{Z}\} \\ &= h\{\text{Orb}_g(x)\}, \end{aligned}$$

as required. □

Lemma 4.3.8. Suppose that g is an element of S_n that contains the k -cycle $(1, 2, \dots, k)$ and contains no other k -cycles. Then

$$C_{S_n}(g) \leq \langle (1, 2, \dots, k) \rangle \times \text{Sym}(\{k+1, k+2, \dots, n\}),$$

with equality if $g = (1, 2, \dots, k)$.

Proof. Suppose that $h \in C_{S_n}(g)$, so h commutes with g . We now apply Lemma 4.3.7 with $x = 1$ to see that $|\text{Orb}_g(h(1))| = |\text{Orb}_g(1)| = |\{1, 2, \dots, k\}| = k$. Since g has a *unique* k -cycle it follows that $\text{Orb}_g h(1) = \text{Orb}_g(1) = \{1, 2, \dots, k\}$. Applying Lemma 4.3.7 again we see that

$$\begin{aligned} h(\{1, 2, \dots, k\}) &= h(\text{Orb}_g(1)) \\ &= \text{Orb}_g(h(1)) \\ &= \{1, 2, \dots, k\}. \end{aligned}$$

So h fixes $\{1, 2, \dots, k\}$ as a set and it also fixes the complement $\{k+1, k+2, \dots, n\}$ as a set. Consequently, $h \in \text{Sym}(\{1, 2, \dots, k\}) \times \text{Sym}(\{k+1, k+2, \dots, n\})$.

Next, Lemma 4.3.3 tells us that the only elements that commute with $(1, 2, \dots, k)$ inside $\text{Sym}(\{1, 2, \dots, k\})$ are powers of $(1, 2, \dots, k)$. Hence $h \in \langle (1, 2, \dots, k) \rangle \times$

$\text{Sym}(\{k+1, k+2, \dots, n\})$. Therefore

$$C_{S_n}(g) \leq \langle (1, 2, \dots, k) \rangle \times \text{Sym}(\{k+1, k+2, \dots, n\}).$$

For the last part of the lemma, observe that if $g = (1, 2, \dots, k)$, then $\langle (1, 2, \dots, k) \rangle \times \text{Sym}(\{k+1, k+2, \dots, n\})$ is contained in $C_{S_n}(g)$, so these two groups are equal. $\square$

Observe that if $\frac{n}{2} < p \leq n$, where p is a prime, then the only integer in the list $1, 2, \dots, n$ that is divisible by p is p itself. It follows that the highest power of p that is a factor of $n!$ is simply p . As a consequence, any Sylow p -subgroup of S_n is cyclic of order p .

We use this fact in the following corollary of Lemma 4.3.8.

Corollary 4.3.1. *Suppose that $g = (1, 2, \dots, p)$, where p is a prime satisfying $\frac{n}{2} < p \leq n$. If N is a maximal nilpotent subgroup of S_n containing g , then*

$$C_p \leq N \leq C_p \times N_0 \leq C_p \times S_{n-p},$$

where N_0 is a maximal nilpotent subgroup of S_{n-p} .

Proof. Since N is nilpotent we can express it as a direct product of Sylow subgroups. The order of N is divisible by p , so one of those Sylow subgroups is $\langle g \rangle$ itself, a cyclic group of order p . Thus $N = \langle g \rangle \times N_0$, where N_0 is a direct product of Sylow subgroups within $\text{Sym}(\{p+1, p+2, \dots, n\})$. The group N_0 is nilpotent (because it is a direct product of Sylow subgroups) and it is maximal in $\text{Sym}(\{p+1, p+2, \dots, n\})$ because N is maximal in S_n .

We have

$$\langle g \rangle \leq N \leq \langle g \rangle \times N_0 \leq \langle g \rangle \times \text{Sym}(\{p+1, p+2, \dots, n\}).$$

So

$$C_p \leq N \leq C_p \times N_0 \leq C_p \times S_{n-p},$$

as required. $\square$

The next corollary is a consequence of Corollary 4.3.1.

Corollary 4.3.2. *Let $g = (1, 2, \dots, p)$, where p is a prime satisfying $\frac{n}{2} < p \leq n$, and let N be any nilpotent subgroup containing g . Then the only k -cycles in N , where $\frac{n}{2} < k \leq n$, are powers of g .*

Proof. Suppose that h is a k -cycle in N with $\frac{n}{2} < k \leq n$. As we saw in Corollary 4.3.1, $h \in \langle (1, 2, \dots, p) \rangle \times \text{Sym}(\{p+1, p+2, \dots, n\})$. But h is a k -cycle, and $n-p < \frac{n}{2} < k$, so $h \in \langle (1, 2, \dots, p) \rangle$. That is, h is a power of $g = (1, 2, \dots, p)$. $\square$

We complete this section by giving a partial result about non-nilpotent subsets and nilpotent subgroups in S_n . To do this we need a definition:

Definition 4.3.7. Let

$$\mathcal{L}_{n,\text{prime}} = \left\{ g \in S_n : g \text{ contains a cycle of length a prime greater than } \frac{n}{2} \right\}.$$

Recall that if $X \subseteq S_n$, then we have the following statistics.

1. $\sigma_\infty(X)$ is the size of the largest subset of X such that any two distinct elements of the subset generate a non-nilpotent group in S_n .
2. $\Sigma_\infty(X)$ is the size of the smallest set of nilpotent subgroups of S_n whose union contains X .

We draw our results together in the following result.

Proposition 4.3.2. *Suppose that $g \in S_n$ and g contains a cycle of length $p > \frac{n}{2}$ with p prime. Suppose that N is a maximal nilpotent subgroup of S_n containing g . Then $N \cong \langle g \rangle \times N_0$, where N_0 is a maximal nilpotent subgroup of S_{n-p} .*

In particular, we conclude that

$$1. \sigma_\infty(\mathcal{L}_{n,\text{prime}}) = \sum_{\frac{n}{2} < p \leq n} \frac{n!}{p!(n-p)!} \times \frac{p!}{p(p-1)} \times \sigma_\infty(S_{n-p})$$

$$2. \Sigma_{\infty}(\mathcal{L}_{n,\text{prime}}) = \sum_{\frac{n}{2} < p \leq n} \frac{n!}{p!(n-p)!} \times \frac{p!}{p(p-1)} \times \Sigma_{\infty}(S_{n-p}).$$

Proof. Let p be a prime greater than $\frac{n}{2}$ and let $\mathcal{L}_p$ be the set of conjugacy classes of S_n associated to a partition containing the prime p .

We can construct a non-nilpotent subset of $\mathcal{L}_p$ using Corollary 4.3.1:

1. First we choose a subset X of size p in $\Omega = \{1, \dots, n\}$ – there are $\frac{n!}{p!(n-p)!}$ ways to do this.
2. Second we choose a Sylow p -subgroup of $\text{Sym}(X)$; this group is cyclic, so we choose τ , a single non-identity element of this Sylow p -subgroup. There are $\frac{p!}{p(p-1)}$ Sylow p -subgroups in S_p , hence the same number of choices for τ .
3. Finally we choose a non-nilpotent set in $\text{Sym}(\Omega \setminus X)$; there are $\sigma_{\infty}(S_{n-p})$ ways of doing this. We append each element of this set to τ thereby obtaining a non-nilpotent subset of $\mathcal{L}_p$ of the desired order.

It is clear that we cannot construct a non-nilpotent subset of $\mathcal{L}_p$ of larger size, since Corollary 4.3.1 and the pigeon-hole principle imply that we would then have, for fixed τ , a non-nilpotent subset that is larger than $\sigma_{\infty}(S_{n-p})$, which is impossible.

Now observe that if we do this for all primes p such that $\frac{n}{2} < p < n$, then we will obtain a non-nilpotent subset of $\mathcal{L}_{n,\text{prime}}$ – Corollary 4.3.1 implies that picking x in $\mathcal{L}_{p_1}$ and y in $\mathcal{L}_{p_2}$ must mean that $\langle x, y \rangle$ is non-nilpotent, since their long prime cycles must overlap.

A similar argument works for nilpotent covers. This time, instead of choosing individual elements, we choose subgroups. □

In the next two sections we will extend Proposition 4.3.2 to long cycles of length k , a prime power. We will work a little harder to allow for the possibility that $k = \frac{n}{2}$ (rather than always having $k > \frac{n}{2}$).

4.3.2.3 GAP code: Sylow calculations

We continue using **GAP** to analyse nilpotent subgroups containing our element g which has a long cycle.

We begin by calculating orders of Sylow p -subgroups. Let $P_{p,k}$ be a Sylow p -subgroup of S_k . In the code below we calculate:

1. $mp = |P_{p,n}|$
2. $mp_1 = |P_{p,k}|$
3. $mp_2 = |P_{p,(n-k)}|$

In every case we see that $mp = mp_1 \times mp_2$. This suggests that $P_{p,n} \cong P_{p,k} \times P_{p,(n-k)}$. Next we calculate;

1. np = number of Sylow p -subgroups in S_n containing g
2. np_2 = number of Sylow p -subgroups in $S_{(n-k)}$ containing g

In every case we see that $np = np_2$. This suggests that there is only one Sylow p -subgroup in S_n that contains g call this P^* . Finally we calculate;

1. l_1 = number of elements in P^* of order k .
2. l_2 = number of elements in $\langle g \rangle$ of order k .

In every case $l_2 = l_1$ suggesting that the only conjugates of g contained in P^* are powers of g .

All of these suggested facts will be discussed in the subsequent section. A summary table of the **GAP** code is in Table 4.11

4.3.2.4 Proofs: the case when k is a prime power

The code in the previous section suggests a number of facts which we will now seek to prove. Let us list these facts first: suppose that $g = (1, 2, \dots, k)$ where $k = p^a$, a prime power. Let $G = S_n$ where $\frac{n}{2} < k \leq n$. Then:

1. $C_G(g) = \langle g \rangle \times \text{Sym}(\{k+1, k+2, \dots, n\}) \cong C_k \times S_{n-k}$. In fact we saw this earlier, and it follows from Lemma 4.3.8. Note that this fact means that $C_G(g)$ lies in a unique subgroup isomorphic to $S_k \times S_{n-k}$ – let us call this subgroup M .
2. Any Sylow p -subgroup of G that contains g lies in M .
3. The two previous facts immediately imply that any nilpotent subgroup of G that contains g lies in M .
4. There is a unique Sylow p -subgroup of S_k that contains $(1, 2, \dots, k)$.

We need to prove Facts (2), (3) and (4). In fact we will seek to extend our results a little so that they apply to elements that contain a unique cycle of length k , with $k = p^a$, in their cycle structure. What is more we will only require that $k \geq \frac{n}{2}$ (rather than the strict inequality mentioned above).

Lemma 4.3.9. Let $g \in S_n$ be an element of order k where $k = p^a$, a prime power with $k \geq \frac{n}{2}$. Suppose that g contains the cycle $c = (1, 2, \dots, k)$ in its cycle structure. Then either $p = 2$ and $n = 2k$, or any Sylow p -subgroup of G that contains c lies in $\text{Sym}(\{1, \dots, k\}) \times \text{Sym}(\{k+1, \dots, n\})$.

Proof. Suppose that either p is odd or that $k \neq \frac{n}{2}$. Let P be a Sylow p -subgroup of G containing g . The orbit sizes of P correspond to the base p expansion of n . In particular, if $k > \frac{n}{2}$, there is precisely an orbit of length k . As P contains g , the orbit of c must be this orbit of length k . As this is a full orbit of P , the conclusion follows.

If $k = \frac{n}{2}$, then the same argument works, except that we might have an orbit of length n (if $p = 2$ and $n = 2k$ – this is excluded), or precisely two orbits of length k . But in that latter case, one of these must correspond to the orbit of c , and the same conclusion follows. $\square$

We have the following corollary.

Corollary 4.3.3. *Let $g \in G = S_n$ and suppose that g contains the cycle $c = (1, 2, \dots, k)$ in its cycle structure. Suppose, moreover, that $k = p^a$, a prime power with $k \geq \frac{n}{2}$. Then either $p = 2$ and $n = 2k$, or any nilpotent subgroup of G that contains g lies in $\text{Sym}(\{1, \dots, k\}) \times \text{Sym}(\{k+1, \dots, n\})$.*

Proof. If we raise g to an appropriate power we will obtain an element h that is an element of order k , and which still contains a unique cycle of length k in its cycle structure. Since any group containing g also contains h , we may assume without loss of generality, that g is of order k . Now let N be a nilpotent subgroup of G containing g . Recall that $N = N_0 N_1$ where N_0 centralizes g , and N_1 is a p -group. Hence the result will follow if we show, first, that $C_G(g) \leq \text{Sym}(\{1, \dots, k\}) \times \text{Sym}(\{k+1, \dots, n\})$ and, second, that any Sylow p -subgroup of G containing g is a subgroup of $\text{Sym}(\{1, \dots, k\}) \times \text{Sym}(\{k+1, \dots, n\})$.

Since c is the only cycle of length k in the cycle structure of g , Lemma 4.3.8 implies that $C_G(g) \leq \text{Sym}(\{1, \dots, k\}) \times \text{Sym}(\{k+1, \dots, n\})$. This is the first part done. The second part follows from Lemma 4.3.9. $\square$

Lemma 4.3.10. *Let $g = (1, 2, \dots, k)$ where $k = p^a$, a prime power. There is a unique Sylow p -subgroup of S_k that contains g .*

Proof. Let P be a Sylow p -subgroup of S_k that contains g . Note that $Z(P)$ is non-trivial and is a subgroup of $C_P(g) = \langle g \rangle$ (by Lemma 4.3.3; indeed it is the unique subgroup of $\langle g \rangle$ of size $|Z(P)|$). But this means that $Z(P)$ is determined by g , and hence so is P . The result follows. $\square$

Proposition 4.3.3. *Suppose that g contains a unique cycle of length $k \geq \frac{n}{2}$ with $k = p^a$ a prime power; if $k = \frac{n}{2}$, then assume that p is odd. Suppose that N is a maximal nilpotent subgroup of S_n containing g . Then $N \cong P_{p,k} \times N_0$, where N_0 is a maximal nilpotent subgroup of S_{n-k} .*

Proof. Raise g to an appropriate power ℓ so that g^ℓ still has a cycle of length k , and g^ℓ has order a power of p ; now replace g with g^ℓ , so we can assume that g has order a power of p .

Corollary 4.3.3 implies that $N \cong N_1 \times N_0$ where N_0 and N_1 are nilpotent with $N_1 < S_{n-k}$ and $N_0 < S_k$. We must show that $N_1 \cong P_{p,k}$. Clearly N_1 contains a cycle c of length k .

Now every element $n \in N_1$ can be written $n = n_1 n_2$ where $n_1 \in C_{S_k}(c)$ and $\langle c, n_2 \rangle$ is a p -group. But now Lemma 4.3.3 implies that n_1 is a power of c , and we conclude that every element of N_1 is a p -element, and we conclude that N_1 is a Sylow p -subgroup of S_k , as required. $\square$

We complete this section by improving Proposition 4.3.2, which was a partial result about non-nilpotent subsets and nilpotent subgroups in S_n . To do this we need a definition:

Definition 4.3.8. We define the following subset of S_n . If n is not a power of 2, then

$$\mathcal{L}_{n,\text{primepower}} := \left\{ g \in S_n \mid g \text{ contains a unique cycle of length } k \geq \frac{n}{2} \text{ with } k \text{ a prime power} \right\}.$$

If n is a power of 2, then

$$\mathcal{L}_{n,\text{primepower}} := \left\{ g \in S_n \mid g \text{ contains a unique cycle of length } k > \frac{n}{2} \text{ with } k \text{ a prime power} \right\}.$$

In general, for p a prime, we will write $P_{p,n}$ for a Sylow p -subgroup of S_n . We are now able to draw our results together:

Proposition 4.3.4. *If n is not a power of 2, then we can calculate the exact value of $\sigma_\infty(\mathcal{L}_{n,\text{primepower}})$ and $\Sigma_\infty(\mathcal{L}_{n,\text{primepower}})$, as follows:*

1. $\sigma_\infty(\mathcal{L}_{n,\text{primepower}}) = \sum_{\frac{n}{2} \leq k \leq n} \frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \sigma_\infty(S_{n-k});$
2. $\Sigma_\infty(\mathcal{L}_{n,\text{primepower}}) = \sum_{\frac{n}{2} \leq k \leq n} \frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \Sigma_\infty(S_{n-k});$ *if n is a power of 2, then the same formula applies but with $\frac{n}{2} < k < n$.*

Proof. Let k be a prime power such that $k \geq \frac{n}{2}$ (or $k > \frac{n}{2}$ if n is a power of 2). Let $\mathcal{L}_k$ be the set of conjugacy classes of S_n associated to a partition containing the prime p .

We can construct a non-nilpotent subset of $\mathcal{L}_k$ using Corollary 4.3.3:

1. First we choose a subset X of size k in $\Omega = \{1, \dots, n\}$ – there are $\frac{n!}{k!(n-k)!}$ ways to do this.
2. Second we choose a Sylow p -subgroup of $\text{Sym}(X)$; we choose τ , a single k -cycle inside this Sylow p -subgroup. There are $\frac{k!}{|N_{S_k}(P_{p,k})|}$ Sylow p -subgroups in S_p , hence the same number of choices for τ .
3. Finally we choose a non-nilpotent set in $\text{Sym}(\Omega \setminus X)$; there are $\sigma_\infty(S_{n-p})$ ways of doing this. We append each element of this set to τ thereby obtaining a non-nilpotent subset of $\mathcal{L}_k$ of the desired order.

It is clear that we cannot construct a non-nilpotent subset of $\mathcal{L}_p$ of larger size: Lemma 4.3.10 implies that we can't have more than one k -cycle from the same $P_{p,k}$, and Corollary 4.3.3 implies that for fixed τ , we can't pick a non-nilpotent subset that is larger than $\sigma_\infty(S_{n-p})$, which is impossible.

Now observe that if we do this for all prime powers k satisfying the given bounds, then we will obtain a non-nilpotent subset of $\mathcal{L}_{n, \text{primepower}}$ – Corollary 4.3.1 implies that picking x in $\mathcal{L}_{p_1}$ and y in $\mathcal{L}_{p_2}$ must mean that $\langle x, y \rangle$ is non-nilpotent, since their long primes must overlap.

A similar argument works for nilpotent covers. This time, instead of choosing individual elements, we choose subgroups. □

4.3.2.5 On powers

In this section we justify the exclusion of conjugacy classes that are powers of another. Recall, to start, that in S_n , conjugacy classes correspond to cycle-type.

We start by proving the following general lemma.

Lemma 4.3.11. Let X be a subset of a group G such that every element of G is a power of some element of X . Then

$$\Sigma_{\infty}(X) = \Sigma_{\infty}(G) \quad \text{and} \quad \sigma_{\infty}(X) = \sigma_{\infty}(G).$$

Proof. Suppose that $\mathcal{C}$ is a cover of G . Then $\mathcal{C}$ is also a cover of X , because $X \subseteq G$. Hence $\Sigma_{\infty}(X) \leq \Sigma_{\infty}(G)$.

Suppose now that $\mathcal{C}$ is a cover of X . Choose any element g of G . Then $g = h^n$, for some element h of X . Since $\mathcal{C}$ covers X , there is a subgroup H in $\mathcal{C}$ that contains h . And because H is a subgroup, it also contains h^n . Hence $g \in H$, so we see that $\mathcal{C}$ is a cover of G . Hence $\Sigma_{\infty}(G) \leq \Sigma_{\infty}(X)$.

It follows then that $\Sigma_{\infty}(X) = \Sigma_{\infty}(G)$, and this completes the first part of the proof.

For the second part of the proof, let us suppose first that $\mathcal{N}$ is a non-nilpotent subset of X . Then $\mathcal{N}$ is also a non-nilpotent subset of G , because $X \subseteq G$. Hence $\sigma_{\infty}(X) \leq \sigma_{\infty}(G)$.

Suppose now that $\mathcal{N}$ is a non-nilpotent subset of G . For each element g of $\mathcal{N}$, we can find an element h_g of X such that $g = h_g^n$, for some integer n . Let $\mathcal{N}_0 = \{h_g : g \in \mathcal{N}\}$. The two sets $\mathcal{N}$ and $\mathcal{N}_0$ have the same size. Moreover, $\mathcal{N}_0$ is a non-nilpotent subset of X , because if $h_{g_1}, h_{g_2} \in \mathcal{N}_0$, then $\langle g_1, g_2 \rangle \leq \langle h_{g_1}, h_{g_2} \rangle$, so $\langle h_{g_1}, h_{g_2} \rangle$ is not nilpotent because it contains the non-nilpotent group $\langle g_1, g_2 \rangle$. Hence $\sigma_{\infty}(G) \leq \sigma_{\infty}(X)$.

It follows then that $\sigma_{\infty}(X) = \sigma_{\infty}(G)$, and this completes the second part of the proof. $\square$

Our strategy for constructing nilpotent covers of S_n is to consider one conjugacy class of S_n at a time. Lemma 4.3.11 tells us that in constructing minimal nilpotent covers, we can exclude from consideration all conjugacy classes that are powers of another (and similar comments can be applied to non-nilpotent subsets of S_n instead of nilpotent covers). We have recorded this information in the tables

in Section 4.3.2.9.

Definition 4.3.9. We define the following subset of S_n .

$$\mathcal{L}_{n,\text{powers}} = \{g \in S_n \mid g = h^a \text{ where } a \in \mathbb{Z}^+ \text{ and } h \text{ is not conjugate to } g\}.$$

We have seen in Proposition 4.3.4 that there is effectively only one minimal nilpotent cover of the set $\mathcal{L}_{n,\text{primepower}}$. In particular, if $g \in \mathcal{L}_{n,\text{primepower}}$, then $g \in P_{p,k} \times S_{n-k}$ for some k , and a maximal nilpotent group containing g has the form $P_{p,k} \times N_0$ where N_0 is maximal nilpotent in S_n . This allows us to exclude from further considerations, not only all elements in $\mathcal{L}_{n,\text{primepower}}$, but also all element that are in a subgroup of the form $P_{p,k} \times S_{n-k}$.

Definition 4.3.10. We define the following subset of S_n . If n is not a power of 2, then

$$\mathcal{L}_{n,\text{primepower}X} := \left\{ g \in S_n \mid g \in P_{p,k} \times S_{n-k} \text{ where } k \geq \frac{n}{2} \text{ with } k \text{ a prime power} \right\}.$$

If n is a power of 2, then

$$\mathcal{L}_{n,\text{primepower}X} := \left\{ g \in S_n \mid g \in P_{p,k} \times S_{n-k} \text{ where } k > \frac{n}{2} \text{ with } k \text{ a prime power} \right\}.$$

In addition we now see in Lemma 4.3.11 that we do not need to worry about the elements in $\mathcal{L}_{n,\text{powers}}$. Thus, all that is left to worry about are the elements in

$$\mathcal{L}_{\text{remainder}} := S_n \setminus (\mathcal{L}_{n,\text{primepower}X} \cup \mathcal{L}_{n,\text{power}}).$$

Note that $\mathcal{L}_{\text{remainder}}$ is a union of conjugacy classes. For $n \leq 5$, $\mathcal{L}_{\text{remainder}}$ is empty. For $5 \leq n \leq 10$, the conjugacy classes in $\mathcal{L}_{\text{remainder}}$ are given in Table 4.12.

4.3.2.6 GAP code: k is not a prime power

Let g be a non-prime power k -cycle, then such a g does not exist in S_n for $n = 3, 4, 5$. For $n > 5$ it does exist. It is easy to see that for $n = 6, 7, 8, 9, 10$, g can only be a 6-cycle. We investigate whether the group $\langle g, h \rangle$ is nilpotent when g is a non-prime power k -cycle and h is a t -cycle such that:

$$(a) \quad k + t = n$$

$$(b) \quad k + t > n$$

The following **GAP** code experiments the two cases for $n = 6, 7, 8, 9$ and 10. We see that the only time we get nilpotent groups is when g and h are either powers of each other, or are disjoint. In particular, if $k + t > n$, then the latter possibility does not occur. In the next section, we will seek to prove this.

4.3.2.7 Proofs: the case when k is not a prime power

Referring to Table 4.12 we see that there are a number of conjugacy classes with cycles of length 6 and 10 that must be dealt with. The following lemma does the job.

Lemma 4.3.12. Suppose that g contains the cycle $c = (1, 2, \dots, k)$ in its cycle structure where $k = p_1 \cdot p_2$, where p_1, p_2 are distinct primes and $k > \frac{n}{2}$. Let N be a nilpotent subgroup of $G = S_n$ that contains g . Then

$$N \leq \langle c \rangle \times \text{Sym}(\{k + 1, \dots, n\}).$$

Proof. Let $h := g^{p_1^b}$ where b is chosen to be minimal so that the order of h is not divisible by p_1 ; then h has at least p_1 p_2 -cycles. Now observe that

$$C_G(h) \leq \left((C_{p_2})^\ell \rtimes S_\ell \right) \times S_{n-p_2 \cdot \ell},$$

where $p_1 < \ell < 2p_1$. Observe that g corresponds to a p_1 -cycle in S_ℓ . Then if

$H \leq C_G(h)$, H is nilpotent and $g, h \in H$, then $H \cap (C_{p_2})^\ell = \langle h \rangle$. Thus

$$H \leq \langle h \rangle \times H_0 \times S_{n-p_2 \cdot \ell}$$

where H_0 is a nilpotent subgroup of S_ℓ containing a p_1 -cycle corresponding to g . Now Lemma 4.3.8 implies that

$$H \leq \langle c \rangle \times S_{n-p_1 \cdot p_2}.$$

Note that if t is a prime distinct from p_2 , then the Sylow t -subgroups of N must lie in H .

Now we can run exactly the same argument with the role of p_1 and p_2 reversed. We obtain that all Sylow t -subgroups of N are subgroups of $\langle c \rangle \times S_{n-p_1 \cdot p_2}$, and the result follows. $\square$

Definition 4.3.11. Let $k \in \{6, 10\}$. We define the following subsets of S_n for $n \leq 11$:

$$\begin{aligned} \mathcal{L}_k &:= \{g \in S_n \mid g \text{ contains a unique cycle of length } k\}. \\ \mathcal{L}_{kX} &:= \{g \in S_n \mid g \in C_k \times S_{n-k}\}. \end{aligned}$$

Proposition 4.3.5. Let $k \in \{6, 10\}$ and $n \leq 11$. We can calculate the exact value of $\sigma_\infty(\mathcal{L}_k)$ and $\Sigma_\infty(\mathcal{L}_k)$, as follows:

1. $\sigma_\infty(\mathcal{L}_k) = \frac{n!}{k!(n-k)!} \times \frac{k!}{k\phi(k)} \times \sigma_\infty(S_{n-k})$.
2. $\Sigma_\infty(\mathcal{L}_k) = \frac{n!}{k!(n-k)!} \times \frac{k!}{k\phi(k)} \times \Sigma_\infty(S_{n-k})$.

Note, first, that $\phi(k)$ is the Euler totient function for k . Note second that the proof is exactly the same as for Proposition 4.3.4 so we do not repeat it here. Note, finally, that this proposition holds by necessity for $\sigma_\infty(\mathcal{L}_{kX})$ and $\Sigma_\infty(\mathcal{L}_{kX})$ as well.

We may now exclude $\mathcal{L}_{kX}$ from $\mathcal{L}_{\text{remainder}}$. This leaves us with the conjugacy classes listed in Table 4.13.

Proposition 4.3.6. *Assume that $n \in \{8, 9, 10\}$. If $n = 8$ assume that g is of type $4^1 3^1$, otherwise assume that g is of type $4^1 3^1 2^1$. Then g lies in a unique maximal nilpotent subgroup of S_n , N , and, moreover*

$$N \cong C_3 \times P_{2,4} \times P_{2,n-7}.$$

Proof. Write g so the unique 3-cycle in g is $h = (1, 2, 3)$ and observe that h is a power of g . Let N be a maximal nilpotent subgroup of S_n containing g . Then $C_{S_n}(g) < \text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, 5, \dots, n\})$. This implies that every Sylow t -subgroup of N is a subgroup of $\text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, 5, \dots, n\})$ except, possibly, when $t = 3$. On the other hand, if $f \in N$ is a 3-element, then f must centralize g^3 (which is an element of order 4) and must have the property that $\langle h, f \rangle$ is a 3-group. We conclude that $f \in \text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, 5, \dots, n\})$ and so $N < \text{Sym}(\{1, 2, 3\}) \times \text{Sym}(\{4, 5, \dots, n\})$. Indeed it is clear that

$$N < \langle (1, 2, 3) \times \text{Sym}(\{4, 5, \dots, n\}) \rangle.$$

Let $N_0 = N \cap \text{Sym}(\{4, 5, \dots, n\})$. Then N_0 contains a 4-cycle, and, since $n - 3 \leq 7$, Proposition 4.3.3 implies that $N_0 \cong P_{2,4} \times N_1$ where $N_1 \leq \text{Sym}\{8, \dots, n\}$.

If $n = 8$, the result follows immediately. If $n > 8$, then we use the fact that N_1 must contain a transposition and the result follows. $\square$

Proposition 4.3.7. *If $n \in \{8, 9, 10\}$, then let X be the union of the conjugacy classes listed in Table 4.13. Then*

$$\sigma_\infty(X) = \Sigma_\infty(X) = \frac{n!}{3!4!(n-7)!} \cdot 9 \cdot |(n-7)!|_{2'}.$$

Note that $|(n-7)!|_{2'}$ is the highest odd number dividing $(n-7)!$; it is equal to 1 if $n \leq 9$, and is equal to 3 if $n = 10$.

Proof. The group listed in Proposition 4.3.6 contains representatives from all conjugacy classes in X . Therefore the proposition implies that we just need to count the number of conjugates of this group in S_n . We obtain the listed number. $\square$

4.3.2.8 Final result

We can now put our results together.

Theorem 4.3.8. *Suppose that $5 \leq n \leq 10$. Then $\sigma_\infty(S_n) = \Sigma_\infty(S_n)$ and it is equal to:*

$$\begin{aligned} & \delta_{5,6,7,9,10} \sum_{\frac{n}{2} \leq k \leq n} \frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \sigma_\infty(S_{n-k}) \\ & + \delta_8 \sum_{\frac{n}{2} < k \leq n} \frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \sigma_\infty(S_{n-k}) \\ & + \delta_{6,7,8,9,10} \frac{n!}{6!(n-6)!} \times \frac{6!}{12} \times \sigma_\infty(S_{n-6}) \\ & + \delta_{8,9} \frac{n!}{3!4!(n-7)!} \cdot 9 \\ & + \delta_{10} \left(\frac{10!}{40} + \frac{10!}{3!4!3!} \cdot 9 \cdot 3 \right), \end{aligned}$$

where k is restricted to being a prime power, and $\delta_{x_1, \dots, x_k} = \begin{cases} 1, & \text{if } n \in \{x_1, \dots, x_k\}; \\ 0, & \text{otherwise.} \end{cases}$

What is more there is a unique cover of S_n by maximal nilpotent subgroups of size $\Sigma_\infty(S_n)$ and every element N of this cover contains an element g such that N is the unique maximal nilpotent subgroup containing g .

Proof. We start with the final statement which we prove by induction. Clearly there is a unique cover of minimal size of S_2 and S_3 by maximal nilpotent subgroups with the given uniqueness property.

Now let $n \geq 4$. Proposition 4.3.3 and Lemma 4.3.10 implies that there is a unique cover of $\mathcal{L}_{\text{primepower } X}$ with the given uniqueness property. Likewise Lemma 4.3.12 implies that there is a unique cover with the same property of $\mathcal{L}_{6X}$

and $\mathcal{L}_{10X}$; likewise Proposition 4.3.6 implies the same for the set Y . And thus the final statement holds.

Now, this uniqueness property implies that

$$\sigma_{\infty}(S_n) = \Sigma_{\infty}(S_n) = \Sigma_{\infty}(\mathcal{L}_{\text{primepower}X}) + \Sigma_{\infty}(\mathcal{L}_{6X}) + \Sigma_{\infty}(\mathcal{L}_{10X}) + \Sigma_{\infty}(Y).$$

The values of each of the four summands are given in Propositions 4.3.4, 4.3.5 and 4.3.7. □

4.3.2.9 Tables for $n \leq 10$

We conclude this chapter with tables giving details of non-nilpotent sets and nilpotent covers for S_n with $n \leq 10$. The calculations in the tables are consequences of the code and proof presented in the preceding sections.

We need to explain how to read the tables. There are two tables for each group. The first table identifies the nature of elements of a non-nilpotent set and that of subgroups of a nilpotent cover. The second table gives the number of non-nilpotent sets and that of subgroups in a nilpotent cover for S_n . The same format will be used for A_n .

Each row corresponds to a conjugacy class in S_n . Let us consider what the column entries mean for some conjugacy class C :

1. “Label”: this column gives a name to the conjugacy class C ;
2. “conj.class”: this column gives an element from the conjugacy class C ;
3. “power of”: this column contains an entry if the elements in C are powers of some other conjugacy class – in this case, the label of such a conjugacy class is given. Using Lemma 4.3.11, we know that any conjugacy class which has an entry in this column can be ignored when it comes to calculating non-nilpotent sets and nilpotent covers. If there is an entry in this column, there will be no entries in all further columns.

4. “subgroup”: this column contains a variety of entries, explained as follows:

- (a) $P_{p,k} \times S_{n-k}$: an entry of this type occurs when C is a subset of $\mathcal{L}_{n,\text{primepower}}$ (see Definition 4.3.8). In this case elements of C lie in a subgroup of the form $P_{p,k} \times S_{n-k}$, and this group is listed and will be used to form a minimal cover, using Proposition 4.3.4. In addition, we will construct a non-nilpotent subset using elements from C , using Proposition 4.3.4.
- (b) $(P_{p,k} \times S_{n-k})$: this column contains an entry when C is not in $\mathcal{L}_{n,\text{primepower}}$ but elements of C are contained in a subgroup of the form $P_{p,k} \times S_{n-k}$. In this case, elements of C will not be needed to construct a maximal non-nilpotent subset, and they can be safely ignored when we calculate minimal covers.
- (c) “ $C_k \times S_{n-k}$ ”: an entry of this type occurs when C is a subset of the group $C_k \times S_{n-k}$ which is a direct product of a cyclic subgroup C_k and S_{n-k} . Note that $P_{p,p} \times S_0 \cong C_p \times S_0 \equiv C_p$

5. “ $|N_p|$ ”: This entry gives the number of elements in a conjugacy class C ;

6. “nil.cover”: This entry identifies the kind of nilpotent subgroup to be included in the nilpotent cover.

To help the reader we also give some more detailed analysis for $n \leq 5$ in a series of examples.

Example 4.3.3. In S_3 , we consider $k = n = 3$ and $k = 2$. This corresponds respectively to the Sylow 3-subgroups and Sylow 2-subgroups of S_3 . Table 4.10 says that these are the cyclic subgroups C_3 and C_2 . Now using Proposition 4.3.2 we have:

$$\begin{aligned}
\sigma_\infty(S_3) = |N| &= \sum_{\frac{n}{2} \leq k \leq n} \left(\frac{n!}{p!(n-p)!} \times \frac{p!}{p(p-1)} \times \sigma_\infty(S_{n-p}) \right) \\
&= \left(\frac{3!}{3!(3-3)!} \times \frac{3!}{3(3-1)} \times \sigma_\infty(S_{3-3}) \right) \\
&\quad + \left(\frac{3!}{2!(3-2)!} \times \frac{2!}{2(2-1)} \times \sigma_\infty(S_{3-2}) \right) \\
&= 1 + 3 = 4
\end{aligned}$$

This agrees with the calculations in Example 1.2.25. The same calculation yields $\Sigma_\infty(S_3) = 7$.

Example 4.3.4. In S_4 we consider $k = n = 4$, since 4 is a prime power and $k = 3$. First note that from Table 4.10 the cyclic subgroup C_4 is contained in a Sylow 2-subgroup, this corresponds to $k = 4$. For $k = 3$ we have the Sylow 3-subgroups. In particular these are the three conjugate subgroups of order 8 namely:

$$G_1 = \{1, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$$

$$G_2 = \{1, (13), (24), (13)(24), (12)(34), (14)(23), (1234), (1432)\}$$

$$G_3 = \{1, (14), (23), (12)(34), (13)(24), (14)(23), (1243), (1342)\}$$

and the 4 Sylow 3-subgroups of order 3 which are:

$$H_1 = \{1, (123), (132)\}$$

$$H_2 = \{1, (124), (142)\}$$

$$H_3 = \{1, (134), (143)\}$$

$$H_4 = \{1, (234), (243)\}$$

Using Propositions 4.3.2 and 4.3.4 we see that:

$$\begin{aligned}
\sigma_\infty(S_4) &= |N| \\
&= \sum_{\frac{n}{2} \leq k \leq n} \left(\frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \sigma_\infty(S_{n-k}) \right) \\
&= \left(\frac{4!}{4!(4-4)!} \times \frac{4!}{|N_{S_4}(P_{2,4})|} \times \sigma_\infty(S_{4-4}) \right) \\
&\quad + \left(\frac{4!}{3!(4-3)!} \times \frac{3!}{3(3-1)} \times \sigma_\infty(S_{4-3}) \right) \\
&= \left(1 \times \frac{4!}{8} \times 1 \right) + (4 \times 1) \\
&= 3 + 4 = 7
\end{aligned}$$

Notice also that $\Sigma_\infty(S_4) \geq 7$.

Example 4.3.5. A similar argument, can be advanced in S_5 . In this case $k = 3, 4, 5$. Note that the centralizer of a 3-cycle is the nilpotent subgroup $C_3 \times C_2$. The number of such distinct nilpotent subgroup $C_3 \times C_2$ is equal to the number of distinct C_3 cyclic subgroups. So we estimate that:

$$\begin{aligned}
\sigma_\infty(S_5) &= |N| \\
&= \sum_{\frac{n}{2} \leq k \leq n} \left(\frac{n!}{k!(n-k)!} \times \frac{k!}{|N_{S_k}(P_{p,k})|} \times \sigma_\infty(S_{n-k}) \right) \\
&= \left(\frac{5!}{5!(5-5)!} \times \frac{5!}{5(5-1)} \times \sigma_\infty(S_{5-5}) \right) \\
&\quad + \left(\frac{5!}{4!(5-4)!} \times \frac{4!}{|N_{S_4}(P_{2,4})|} \times \sigma_\infty(S_{5-4}) \right) \\
&\quad + \left(\frac{5!}{3!(5-3)!} \times \frac{3!}{3(3-1)} \times \sigma_\infty(S_{5-3}) \right) \\
&= (1 \times 6 \times 1) + \left(5 \times \frac{4!}{8} \times 1 \right) + (10 \times 1) \\
&= 6 + 15 + 10 = 31
\end{aligned}$$

In particular we note that S_5 contains 15 Sylow 2-subgroups of order 8,

namely:

$$\begin{aligned}
&\langle (12), (34), (13)(24) \rangle, \langle (12), (35), (13)(25) \rangle, \langle (12), (45), (14)(25) \rangle \\
&\langle (13), (24), (12)(34) \rangle, \langle (13), (45), (14)(35) \rangle, \langle (13), (25), (12)(35) \rangle \\
&\langle (14), (23), (12)(43) \rangle, \langle (14), (25), (12)(45) \rangle, \langle (14), (35), (13)(45) \rangle \\
&\langle (15), (23), (12)(35) \rangle, \langle (15), (24), (12)(45) \rangle, \langle (15), (34), (13)(45) \rangle \\
&\langle (23), (45), (24)(35) \rangle, \langle (24), (35), (23)(45) \rangle, \langle (25), (34), (23)(45) \rangle
\end{aligned}$$

Next, S_5 contains 10 finitely generated abelian subgroups of order 6 which are a direct product of p -groups, for $p = 3$ and 2 i.e $C_3 \times C_2$. These contain the Sylow 3-subgroups which are:

$$\begin{aligned}
&\langle (12), (345) \rangle, \langle (13), (245) \rangle, \langle (14), (235) \rangle, \langle (15), (234) \rangle, \langle (23), (145) \rangle, \\
&\langle (24), (135) \rangle, \langle (25), (134) \rangle, \langle (34), (125) \rangle, \langle (35), (124) \rangle, \langle (45), (123) \rangle
\end{aligned}$$

Finally there are 6 cyclic subgroups of order 5. These are the Sylow 5-subgroups, namely:

$$\langle (12345) \rangle, \langle (21345) \rangle, \langle (13245) \rangle, \langle (12435) \rangle, \langle (12354) \rangle, \langle (52341) \rangle$$

All these subgroups are nilpotent and their union is equal to S_5 . So we have a nilpotent cover for S_5 . Now, each of these subgroups contains at least one element not present in the others. So we have obtained a minimal nilpotent cover of size $\Sigma_\infty(S_5) = 31$. On the other hand a maximal non- nilpotent subset N of S_5 contains a non-trivial element from each of the 31 subgroups and so $\Sigma_\infty(S_5) = \sigma_\infty(S_5) = |N| = 31$. This result agrees with the findings in Example 4.3.5.

Table 4.10: Centralizer of a k -cycle in S_n , for $k > n/2$

n	k -cycle	Order of centralizer	Order of centralizer/ $(n - k)!$	Centralizer
3	(1, 2)	2	2	C_2
	(1, 2, 3)	3	3	C_3
4	(1, 2, 3)	3	3	C_3
	(1, ..., 4)	4	4	C_4
5	(1, 2, 3)	6	3	$C_3 \times C_2$
	(1, ..., 4)	4	4	C_4
	(1, ..., 5)	5	5	C_5
6	(1, ..., 4)	8	4	$C_4 \times C_2$
	(1, ..., 5)	5	5	C_5
7	(1, ..., 4)	24	4	$C_4 \times S_3$
	(1, ..., 5)	10	5	$C_5 \times C_2$
	(1, ..., 7)	7	7	C_7
8	(1, ..., 5)	30	5	$C_5 \times S_3$
	(1, ..., 7)	7	7	C_7
	(1, ..., 8)	8	8	C_8
9	(1, ..., 5)	120	5	$C_5 \times S_4$
	(1, ..., 7)	14	7	$C_7 \times C_2$
	(1, ..., 8)	8	8	C_8
	(1, ..., 9)	9	9	C_9
10	(1, ..., 7)	42	7	$C_7 \times S_3$
	(1, ..., 8)	16	8	$C_8 \times C_2$
	(1, ..., 9)	9	9	C_9
11	(1, ..., 7)	168	7	$C_7 \times S_4$
	(1, ..., 8)	48	8	$C_8 \times S_3$
	(1, ..., 9)	18	9	$C_9 \times C_2$
	(1, ..., 11)	11	11	C_{11}
12	(1, ..., 7)	840	7	$C_7 \times S_5$
	(1, ..., 8)	192	8	$C_8 \times S_4$
	(1, ..., 9)	54	9	$C_9 \times S_3$
	(1, ..., 11)	11	11	C_{11}
13	(1, ..., 7)	5040	7	$C_7 \times S_6$
	(1, ..., 8)	960	8	$C_8 \times S_5$
	(1, ..., 9)	216	9	$C_9 \times S_4$
	(1, ..., 11)	22	11	$C_{11} \times C_2$
	(1, ..., 13)	13	13	C_{13}
14	(1, ..., 8)	5760	8	$C_8 \times S_6$
	(1, ..., 9)	1080	9	$C_9 \times S_5$
	(1, ..., 11)	66	11	$C_{11} \times S_3$
	(1, ..., 13)	13	13	C_{13}
15	(1, ..., 8)	40320	8	$C_8 \times S_7$
	(1, ..., 9)	6480	9	$C_9 \times S_6$
	(1, ..., 11)	264	11	$C_{11} \times S_4$
	(1, ..., 13)	26	13	$C_{13} \times C_2$

Source: Research.

Table 4.11: Number of Sylow p -subgroups in S_n and S_{n-k}

S_n	g	p	mp	mp_1	mp_2	n_p	np_2	l_1	l_2
S_5	(1, 2, 3, 4)	2	8	8	1	1	1	2	2
S_6	(1, 2, 3, 4)	2	16	8	2	1	1	2	2
S_7	(1, 2, 3, 4)	2	16	8	2	3	3	2	2
S_8	(1, 2, 3, 4, 5, 6, 7, 8)	2	128	128	1	1	1	16	4
S_9	(1, 2, 3, 4, 5, 6, 7, 8)	2	128	128	1	1	1	16	4
S_9	(1, 2, 3, 4, 5, 6, 7, 8, 9)	3	81	81	1	1	1	36	6
S_{10}	(1, 2, 3, 4, 5, 6, 7, 8)	2	256	128	2	1	1	64	4
S_{10}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	3	81	81	1	1	1	36	6
S_{11}	(1, 2, 3, 4, 5, 6, 7, 8)	2	256	128	2	3	3	16	4
S_{11}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	3	81	81	1	1	1	36	6

Source: Research.

Key: Table 4.11.

1. n_p = number of Sylow p -subgroups in S_n containing g
2. np_2 = number of Sylow p -subgroups in S_{n-k} containing g
3. mp = order of a Sylow p -subgroup in S_n containing g
4. mp_1 = order of a Sylow p -subgroup in S_k containing g
5. mp_2 = order of a Sylow p -subgroup in S_{n-k} containing g
6. l_1 = number of elements in P^* -(The Sylow p -subgroup in S_n).
7. l_2 = number of elements in $\langle g \rangle$ of order k .

Table 4.12: Conjugacy classes in $\mathcal{L}_{remainder}$

n	Class	Element
6	6^1	$(1, 2, 3, 4, 5, 6)$
7	6^1	$(1, 2, 3, 4, 5, 6)$
8	$3^1 2^1$	$(1, 2, 3)(4, 5)$
8	$3^2 2^1$	$(1, 2, 3)(4, 5, 6)(7, 8)$
8	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
8	$6^1 2^1$	$(1, 2, 3, 4, 5, 6)(7, 8)$
9	$3^1 2^3$	$(1, 2, 3)(4, 5)(6, 7)(8, 9)$
9	$3^2 2^1$	$(1, 2, 3)(4, 5, 6)(7, 8)$
9	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
9	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$
9	6^1	$(1, 2, 3, 4, 5, 6)$
9	$6^1 2^1$	$(1, 2, 3, 4, 5, 6)(7, 8)$
9	$6^1 3^1$	$(1, 2, 3, 4, 5, 6)(7, 8, 9)$
10	$3^1 2^3$	$(1, 2, 3)(4, 5)(6, 7)(8, 9)$
10	$3^2 2^1$	$(1, 2, 3)(4, 5, 6)(7, 8)$
10	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
10	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$
10	$4^1 3^2$	$(1, 2, 3, 4)(5, 6, 7)(8, 9, 10)$
10	6^1	$(1, 2, 3, 4, 5, 6)$
10	$6^1 2^1$	$(1, 2, 3, 4, 5, 6)(7, 8)$
10	$6^1 2^2$	$(1, 2, 3, 4, 5, 6)(7, 8)(9, 10)$
10	$6^1 3^1$	$(1, 2, 3, 4, 5, 6)(7, 8, 9)$
10	$6^1 4^1$	$(1, 2, 3, 4, 5, 6)(7, 8, 9, 10)$
10	10^1	$(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)$

Source: Research.

Table 4.13: Conjugacy classes in $\mathcal{L}_{remainder} \setminus \mathcal{L}_{kX}$

n	Class	Element
6	None	None
7	None	None
8	$3^1 2^1$	$(1, 2, 3)(4, 5)$
8	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
9	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
9	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$
10	$4^1 3^1$	$(1, 2, 3, 4)(5, 6, 7)$
10	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$

Source: Research.

Table 4.14: Generating non-nilpotent sets and nilpotent covers of S_3

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$P_{2,2} \times S_1$	3	C_2
Cl_3	(1, 2, 3)	-	$P_{3,3} \times S_0$	1	C_3
Total				4	

Source: Research.

Table 4.15: Size of non-nilpotent set and nilpotent cover of S_3

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	3	3	C_2	3
Cl_3	(1, 2, 3)	2	1	C_3	1
Total			4		4

Source: Research.

Table 4.16: Generating non-nilpotent sets and nilpotent cover of S_4

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$P_{2,4} \times S_0$	-	-
Cl_3	(1, 2)(3, 4)	Cl_5	-	-	-
Cl_4	(1, 2, 3)	-	$P_{3,3} \times S_1$	4	C_3
Cl_5	(1, 2, 3, 4)	-	$P_{2,4} \times S_0$	3	D_8
Total				7	

Source: Research.

Table 4.17: Size of non-nilpotent set and nilpotent cover of S_4

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	6	-	-	-
Cl_3	(1, 2)(3, 4)	3	-	-	-
Cl_4	(1, 2, 3)	8	4	C_3	4
Cl_5	(1, 2, 3, 4)	6	3	D_8	3
Total			7		7

Source: Research.

Table 4.18: Generating non-nilpotent sets and nilpotent covers of S_5

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$(P_{2,4} \times S_1)$	-	-
Cl_3	(1, 2)(3, 4)	Cl_6	-	-	-
Cl_4	(1, 2, 3)	Cl_5	-	-	-
Cl_5	(1, 2, 3)(4, 5)	-	$P_{3,3} \times S_2$	10	$C_3 \times C_2$
Cl_6	(1, 2, 3, 4)	-	$P_{2,4} \times S_1$	15	D_8
Cl_7	(1, 2, 3, 4, 5)	-	$P_{5,5} \times S_0$	6	C_5
Total				31	

Source: Research.

Table 4.19: Size of non-nilpotent set and nilpotent cover of S_5

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	10	-	-	-
Cl_3	(1, 2)(3, 4)	15	-	-	-
Cl_4	(1, 2, 3)	20	-	-	-
Cl_5	(1, 2, 3)(4, 5)	20	10	$C_3 \times C_2$	10
Cl_6	(1, 2, 3, 4)	30	15	D_8	15
Cl_7	(1, 2, 3, 4, 5)	24	6	C_5	6
Total			31		31

Source: Research.

Table 4.20: Generating non-nilpotent sets and nilpotent covers of S_6

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	Cl_7	-	-	-
Cl_3	(1, 2)(3, 4)	Cl_9	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	Cl_{11}	-	-	-
Cl_5	(1, 2, 3)(4, 5, 6)	Cl_{11}	-	-	-
Cl_6	(1, 2, 3)	Cl_7	-	-	-
Cl_7	(1, 2, 3)(4, 5)	-	$P_{3,3} \times S_3$	60	$C_3 \times C_2$
Cl_8	(1, 2, 3, 4)	-	$P_{2,4} \times S_2$	-	-
Cl_9	(1, 2, 3, 4)(5, 6)	-	$P_{2,4} \times S_2$	45	$D_8 \times C_2$
Cl_{10}	(1, 2, 3, 4, 5)	-	$P_{5,5} \times S_1$	36	C_5
Cl_{11}	(1, 2, 3, 4, 5, 6)	-	$C_6 \times S_0$	60	C_6
Total				201	

Source: Research.

Table 4.21: Size of non-nilpotent set and nilpotent cover of S_6

Label	conj.class	class Size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	15	-	-	-
Cl_3	(1, 2)(3, 4)	45	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	15	-	-	-
Cl_5	(1, 2, 3)(4, 5, 6)	40	-	-	-
Cl_6	(1, 2, 3)	40	-	-	-
Cl_7	(1, 2, 3)(4, 5)	120	60	$C_3 \times C_2$	60
Cl_8	(1, 2, 3, 4)	90	-	-	-
Cl_9	(1, 2, 3, 4)(5, 6)	90	45	$D_8 \times C_2$	45
Cl_{10}	(1, 2, 3, 4, 5)	144	36	C_5	36
Cl_{11}	(1, 2, 3, 4, 5, 6)	120	60	C_6	60
Total			201		201

Source: Research.

Table 4.22: Generating non-nilpotent sets and nilpotent covers of S_7

Label	conj.class	"power of"	subgroup	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$(P_{2,4} \times S_3)$	-	-
Cl_3	(1, 2)(3, 4)	Cl_{10}	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	Cl_{14}	-	-	-
Cl_5	(1, 2, 3)(4, 5, 6)	Cl_{14}	-	-	-
Cl_6	(1, 2, 3)	Cl_7	-	-	-
Cl_7	(1, 2, 3)(4, 5)	-	$(P_{2,4} \times S_3)$	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	Cl_{11}	-	-	-
Cl_9	(1, 2, 3, 4)	Cl_{11}	-	-	-
Cl_{10}	(1, 2, 3, 4)(5, 6)	-	$P_{2,4} \times S_3$	315	$D_8 \times C_2$
Cl_{11}	(1, 2, 3, 4)(5, 6, 7)	-	$P_{2,4} \times S_3$	105	$D_8 \times C_3$
Cl_{12}	(1, 2, 3, 4, 5)	Cl_{13}	-	-	-
Cl_{13}	(1, 2, 3, 4, 5)(6, 7)	-	$P_{5,5} \times S_2$	126	$C_5 \times C_2$
Cl_{14}	(1, 2, 3, 4, 5, 6)	-	$C_6 \times S_1$	420	C_6
Cl_{15}	(1, 2, 3, 4, 5, 6, 7)	-	$P_{7,7} \times S_0$	120	C_7
Total				1086	

Source: Research.

Table 4.23: Size of non-nilpotent set and nilpotent cover of S_7

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	21	-	-	-
Cl_3	(1, 2)(3, 4)	105	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	105	-	-	-
Cl_5	(1, 2, 3)(4, 5, 6)	280	-	-	-
Cl_6	(1, 2, 3)	70	-	-	-
Cl_7	(1, 2, 3)(4, 5)	420	-	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	210	-	-	-
Cl_9	(1, 2, 3, 4)	210	-	-	-
Cl_{10}	(1, 2, 3, 4)(5, 6)	630	315	$D_8 \times C_2$	315
Cl_{11}	(1, 2, 3, 4)(5, 6, 7)	420	105	$D_8 \times C_3$	105
Cl_{12}	(1, 2, 3, 4, 5)	504	-	-	-
Cl_{13}	(1, 2, 3, 4, 5)(6, 7)	504	126	$C_5 \times C_2$	126
Cl_{14}	(1, 2, 3, 4, 5, 6)	840	420	C_6	420
Cl_{15}	(1, 2, 3, 4, 5, 6, 7)	720	120	C_7	120
Total			1086		1086

Source: Research.

Table 4.24: Generating non-nilpotent sets and nilpotent covers of S_8

Label	conj.class	“power of”	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$(P_{2,8} \times S_0)$	-	-
Cl_3	(1, 2)(3, 4)	Cl_{11}	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	Cl_{19}	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_{22}	-	-	-
Cl_6	(1, 2, 3)	Cl_{18}	-	-	-
Cl_7	(1, 2, 3)(4, 5)	-	$P_{2,4} \times S_4$	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	Cl_{14}	-	-	-
Cl_9	(1, 2, 3)(4, 5, 6)	Cl_{20}	-	-	-
Cl_{10}	(1, 2, 3)(4, 5, 6)(7, 8)	-	$C_6 \times S_2$	-	-
Cl_{11}	(1, 2, 3, 4)	-	$(P_{2,8} \times S_0)$	-	-
Cl_{12}	(1, 2, 3, 4)(5, 6)	-	$(P_{2,8} \times S_0)$	-	-
Cl_{13}	(1, 2, 3, 4)(5, 6)(7, 8)	-	$(P_{2,8} \times S_0)$	-	-
Cl_{14}	(1, 2, 3, 4)(5, 6, 7)	-	$P_{2,4} \times S_4$	840	$D_8 \times C_3$
Cl_{15}	(1, 2, 3, 4)(5, 6, 7, 8)	Cl_{22}	-	-	-
Cl_{16}	(1, 2, 3, 4, 5)	Cl_{17}/Cl_{18}	-	-	-
$\left\{ \begin{array}{l} Cl_{17} \\ Cl_{18} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5)(6, 7) \\ (1, 2, 3, 4, 5)(6, 7, 8) \end{array} \right.$	-	$\left\{ \begin{array}{l} P_{5,5} \times S_3 \end{array} \right.$	1344	$\left\{ \begin{array}{l} C_5 \times C_2 \\ C_5 \times C_3 \end{array} \right.$
Cl_{19}	(1, 2, 3, 4, 5, 6)	Cl_{20}	-	-	-
Cl_{20}	(1, 2, 3, 4, 5, 6)(7, 8)	-	$C_6 \times S_2$	1680	$C_6 \times C_2$
Cl_{21}	(1, 2, 3, 4, 5, 6, 7)	-	$P_{7,7} \times S_1$	960	C_7
Cl_{22}	(1, 2, 3, 4, 5, 6, 7, 8)	-	$P_{2,8}$	315	$P_{2,8}$
Total				5139	

Source: Research.

 $P_{2,8} = (C_2 \wr C_2) \wr C_2$ where $\wr$ means ‘wreath product’.

Table 4.25: Size of non-nilpotent set and nilpotent cover of S_8

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	28	-	-	-
Cl_3	(1, 2)(3, 4)	210	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	420	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	105	-	-	-
Cl_6	(1, 2, 3)	112	-	-	-
Cl_7	(1, 2, 3)(4, 5)	1120	-	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	1680	-	-	-
Cl_9	(1, 2, 3)(4, 5, 6)	1120	-	-	-
Cl_{10}	(1, 2, 3)(4, 5, 6)(7, 8)	1120	-	-	-
Cl_{11}	(1, 2, 3, 4)	210	-	-	-
Cl_{12}	(1, 2, 3, 4)(5, 6)	2520	-	-	-
Cl_{13}	(1, 2, 3, 4)(5, 6)(7, 8)	1260	-	-	-
Cl_{14}	(1, 2, 3, 4)(5, 6, 7)	3360	840	$D_8 \times C_3$	840
Cl_{15}	(1, 2, 3, 4)(5, 6, 7, 8)	1260	-	-	-
Cl_{16}	(1, 2, 3, 4, 5)	1344	-	-	-
$\left\{ \begin{array}{l} Cl_{17} \\ Cl_{18} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5)(6, 7) \\ (1, 2, 3, 4, 5)(6, 7, 8) \end{array} \right.$	$\left\{ \begin{array}{l} 4032 \\ 2688 \end{array} \right.$	1344	$\left\{ \begin{array}{l} C_5 \times C_2 \\ C_5 \times C_3 \end{array} \right.$	1344
Cl_{19}	(1, 2, 3, 4, 5, 6)	3360	-	-	-
Cl_{20}	(1, 2, 3, 4, 5, 6)(7, 8)	3360	1680	$C_6 \times C_2$	1680
Cl_{21}	(1, 2, 3, 4, 5, 6, 7)	5760	960	C_7	960
Cl_{22}	(1, 2, 3, 4, 5, 6, 7, 8)	5040	315	$P_{2,8}$	315
Total			5139		5139

Source: Research.

 $P_{2,8} = (C_2 \wr C_2) \wr C_2$ where $\wr$ means 'wreath product'.

Table 4.26: Generating non-nilpotent sets and nilpotent covers of S_9

Label	conj.class	“power of”	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$(P_{2,8} \times S_1)$	-	-
Cl_3	(1, 2)(3, 4)	-	$(P_{2,8} \times S_1)$	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	Cl_{24}	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_{29}	-	-	-
Cl_6	(1, 2, 3)	-	$(P_{3,9} \times S_0)$	-	-
Cl_7	(1, 2, 3)(4, 5)	-	$(P_{2,6} \times S_3)$	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	Cl_{16}	-	-	-
Cl_9	(1, 2, 3)(4, 5)(6, 7)(8, 9)	-	$(C_6 \times S_3)$	-	-
Cl_{10}	(1, 2, 3)(4, 5, 6)	Cl_{25}	-	-	-
Cl_{11}	(1, 2, 3)(4, 5, 6)(7, 8)	-	$C_6 \times S_3$	-	-
Cl_{12}	(1, 2, 3)(4, 5, 6)(7, 8, 9)	Cl_{30}	-	-	-
Cl_{13}	(1, 2, 3, 4)	-	$(P_{2,8} \times S_1)$	-	-
Cl_{14}	(1, 2, 3, 4)(5, 6)	-	$(P_{2,8} \times S_1)$	-	-
Cl_{15}	(1, 2, 3, 4)(5, 6)(7, 8)	-	$(P_{2,8} \times S_1)$	-	-
Cl_{16}	(1, 2, 3, 4)(5, 6, 7)	-	$(P_{2,6} \times S_3)$	-	-
Cl_{17}	(1, 2, 3, 4)(5, 6, 7)(8, 9)	-	$P_{2,6} \times S_3$	3780	$D_8 \times C_2 \times C_3$
Cl_{18}	(1, 2, 3, 4)(5, 6, 7, 8)	Cl_{29}	-	-	-
Cl_{19}	(1, 2, 3, 4, 5)	-	$(P_{5,5} \times S_4)$	-	-
Cl_{20}	(1, 2, 3, 4, 5)(6, 7)	-	$(P_{5,5} \times S_4)$	-	-
Cl_{21}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	-	$(P_{5,5} \times S_4)$	-	-
$\left\{ \begin{array}{l} Cl_{22} \\ Cl_{23} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5)(6, 7, 8, 9) \end{array} \right.$	-	$\left\{ \begin{array}{l} (P_{5,5} \times S_4) \\ P_{5,5} \times S_4 \end{array} \right.$	5292	$\left\{ \begin{array}{l} C_5 \times C_3 \\ C_5 \times D_8 \end{array} \right.$
$\left\{ \begin{array}{l} Cl_{24} \\ Cl_{25} \\ Cl_{26} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6) \\ (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5, 6)(7, 8, 9) \end{array} \right.$	-	$\left\{ \begin{array}{l} C_6 \times S_3 \end{array} \right.$	20160	$\left\{ \begin{array}{l} C_6 \times C_3 \\ C_6 \times C_2 \end{array} \right.$
Cl_{27}	(1, 2, 3, 4, 5, 6, 7)	Cl_{28}	-	-	-
Cl_{28}	(1, 2, 3, 4, 5, 6, 7)(8, 9)	-	$P_{7,7} \times S_2$	4320	$C_7 \times C_2$
Cl_{29}	(1, 2, 3, 4, 5, 6, 7, 8)	-	$P_{2,8} \times S_1$	2835	$P_{2,8}$
Cl_{30}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	-	$P_{3,9} \times S_0$	1120	$P_{3,9}$
Total				37507	

Source: Research.

 $P_{2,8} = (C_2 \wr C_2) \wr C_2$, $P_{3,9} = C_3 \wr C_3$ where $\wr$ means ‘wreath product’.

Table 4.27: Size of non-nilpotent set and nilpotent cover of S_9

Label	conj.class	class size	size of non-nil.set	nil.cover	size of nil.cover
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	36	-	-	-
Cl_3	(1, 2)(3, 4)	378	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	1260	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	945	-	-	-
Cl_6	(1, 2, 3)	168	-	-	-
Cl_7	(1, 2, 3)(4, 5)	2520	-	-	-
Cl_8	(1, 2, 3)(4, 5)(6, 7)	7560	-	-	-
Cl_9	(1, 2, 3)(4, 5)(6, 7)(8, 9)	2520	-	-	-
Cl_{10}	(1, 2, 3)(4, 5, 6)	3360	-	-	-
Cl_{11}	(1, 2, 3)(4, 5, 6)(7, 8)	10080	-	-	-
Cl_{12}	(1, 2, 3)(4, 5, 6)(7, 8, 9)	2240	-	-	-
Cl_{13}	(1, 2, 3, 4)	756	-	-	-
Cl_{14}	(1, 2, 3, 4)(5, 6)	756	-	-	-
Cl_{15}	(1, 2, 3, 4)(5, 6)(7, 8)	11340	-	-	-
Cl_{16}	(1, 2, 3, 4)(5, 6, 7)	15210	-	-	-
Cl_{17}	(1, 2, 3, 4)(5, 6, 7)(8, 9)	15210	3780	$D_8 \times C_3 \times C_2$	3780
Cl_{18}	(1, 2, 3, 4)(5, 6, 7, 8)	11340	-	-	-
Cl_{19}	(1, 2, 3, 4, 5)	3024	-	-	-
Cl_{20}	(1, 2, 3, 4, 5)(6, 7)	18144	-	-	-
Cl_{21}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	9072	-	-	-
$\left\{ \begin{array}{l} Cl_{22} \\ Cl_{23} \end{array} \right\}$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5)(6, 7, 8, 9) \end{array} \right\}$	$\left\{ \begin{array}{l} 24192 \\ 18144 \end{array} \right\}$	5292	$\left\{ \begin{array}{l} C_5 \times C_3 \\ C_5 \times D_8 \end{array} \right\}$	5292
$\left\{ \begin{array}{l} Cl_{24} \\ Cl_{25} \\ Cl_{26} \end{array} \right\}$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6) \\ (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5, 6)(7, 8, 9) \end{array} \right\}$	$\left\{ \begin{array}{l} 10080 \\ 30240 \\ 20160 \end{array} \right\}$	20160	$\left\{ C_6 \times S_3 \right\}$	20160
Cl_{27}	(1, 2, 3, 4, 5, 6, 7)	25920	-	-	-
Cl_{28}	(1, 2, 3, 4, 5, 6, 7)(8, 9)	25920	4320	$C_7 \times C_2$	4320
Cl_{29}	(1, 2, 3, 4, 5, 6, 7, 8)	45360	2835	$P_{2,8}$	2835
Cl_{30}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	40320	1120	$P_{3,9}$	1120
Total			37507		37507

Source: Research.

 $P_{2,8} = (C_2 \wr C_2) \wr C_2$, $P_{3,9} = C_3 \wr C_3$ where $\wr$ means 'wreath product'.

Table 4.28: Generating non-nilpotent sets and nilpotent covers of S_{10}

Label	conj.class	"power of"	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)	-	$(P_{2,8} \times S_2)$	-	-
Cl_3	(1, 2)(3, 4)	-	$(P_{2,8} \times S_2)$	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	Cl_{35}	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_{40}	-	-	-
Cl_6	(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)	Cl_{42}	-	-	-
Cl_7	(1, 2, 3)	-	$(P_{3,9} \times S_0)$	-	-
Cl_8	(1, 2, 3)(4, 5)	-	$(P_{2,7} \times S_3)$	-	-
Cl_9	(1, 2, 3)(4, 5)(6, 7)	-	-	-	-
Cl_{10}	(1, 2, 3)(4, 5)(6, 7)(8, 9)	-	$(C_6 \times S_4)$	-	-
Cl_{11}	(1, 2, 3)(4, 5, 6)	Cl_{35}	-	-	-
Cl_{12}	(1, 2, 3)(4, 5, 6)(7, 8)	-	$(C_6 \times S_4)$	-	-
Cl_{13}	(1, 2, 3)(4, 5, 6)(7, 8)(9, 10)	Cl_{35}	-	-	-
Cl_{14}	(1, 2, 3)(4, 5, 6)(7, 8, 9)	Cl_{41}	-	-	-
Cl_{15}	(1, 2, 3, 4)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{16}	(1, 2, 3, 4)(5, 6)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{17}	(1, 2, 3, 4)(5, 6)(7, 8)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{18}	(1, 2, 3, 4)(5, 6)(7, 8)(9, 10)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{19}	(1, 2, 3, 4)(5, 6, 7)	-	$(P_{2,7} \times S_3)$	-	-
Cl_{20}	(1, 2, 3, 4)(5, 6, 7)(8, 9)	-	$P_{2,7} \times S_3$	37800	$D_8 \times C_3 \times C_2$
Cl_{21}	(1, 2, 3, 4)(5, 6, 7)(8, 9, 10)	-	$(C_6 \times S_4)$	-	-
Cl_{22}	(1, 2, 3, 4)(5, 6, 7, 8)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{23}	(1, 2, 3, 4)(5, 6, 7, 8)(9, 10)	-	$(P_{2,8} \times S_2)$	-	-
Cl_{24}	(1, 2, 3, 4, 5)	Cl_{42}	-	-	-
Cl_{25}	(1, 2, 3, 4, 5)(6, 7)	Cl_{28}	-	-	-
Cl_{26}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	Cl_{29}	-	-	-
Cl_{27}	(1, 2, 3, 4, 5)(6, 7, 8)	Cl_{28}	-	-	-
$\left\{ \begin{array}{l} Cl_{28} \\ Cl_{29} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5)(6, 7, 8)(9, 10) \\ (1, 2, 3, 4, 5)(6, 7, 8, 9) \end{array} \right.$	-	$\left\{ \begin{array}{l} P_{5,5} \times S_5 \end{array} \right.$	37200	$\left\{ \begin{array}{l} C_5 \times C_3 \times C_2 \\ C_5 \times D_8 \end{array} \right.$
Cl_{30}	(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)	Cl_{42}	-	-	-
Cl_{31}	(1, 2, 3, 4, 5, 6)	Cl_{32}	-	-	-
$\left\{ \begin{array}{l} Cl_{32} \\ Cl_{33} \\ Cl_{34} \\ Cl_{35} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5, 6)(7, 8)(9, 10) \\ (1, 2, 3, 4, 5, 6)(7, 8, 9) \\ (1, 2, 3, 4, 5, 6)(7, 8, 9, 10) \end{array} \right.$	-	$\left\{ \begin{array}{l} C_6 \times S_4 \end{array} \right.$	88200	$\left\{ \begin{array}{l} C_6 \times D_8 \\ C_6 \times C_3 \end{array} \right.$
Cl_{36}	(1, 2, 3, 4, 5, 6, 7)	-	$(P_{7,7} \times S_3)$	-	-
$\left\{ \begin{array}{l} Cl_{37} \\ Cl_{38} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6, 7)(8, 9) \\ (1, 2, 3, 4, 5, 6, 7)(8, 9, 10) \end{array} \right.$	-	$\left\{ \begin{array}{l} P_{7,7} \times S_3 \end{array} \right.$	57600	$\left\{ \begin{array}{l} C_7 \times C_2 \\ C_7 \times C_3 \end{array} \right.$
Cl_{39}	(1, 2, 3, 4, 5, 6, 7, 8)	Cl_{40}	-	-	-
Cl_{40}	(1, 2, 3, 4, 5, 6, 7, 8)(9, 10)	-	$P_{2,8} \times S_2$	14175	$P_{2,8} \times C_2$
Cl_{41}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	-	$P_{3,9} \times S_1$	11200	$P_{3,9}$
Cl_{42}	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)	-	$C_{10} \times S_0$	90720	C_{10}
Total				336895	

Source: Research.

$$P_{2,8} = (C_2 \wr C_2) \wr C_2, \quad P_{3,9} = C_3 \wr C_3 \text{ where } \wr \text{ means 'wreath product'}.$$

Table 4.29: Size of non-nilpotent set and nilpotent cover of S_{10}

Label	conj.class	class size	size(non-nil.set)	nil.cover	size(nil.cover)
Cl_1	(1)	1	-	-	-
Cl_2	(1, 2)	45	-	-	-
Cl_3	(1, 2)(3, 4)	630	-	-	-
Cl_4	(1, 2)(3, 4)(5, 6)	3150	-	-	-
Cl_5	(1, 2)(3, 4)(5, 6)(7, 8)	4725	-	-	-
Cl_6	(1, 2)(3, 4)(5, 6)(7, 8)(9, 10)	945	-	-	-
Cl_7	(1, 2, 3)	240	-	-	-
Cl_8	(1, 2, 3)(4, 5)	5040	-	-	-
Cl_9	(1, 2, 3)(4, 5)(6, 7)	25200	-	-	-
Cl_{10}	(1, 2, 3)(4, 5)(6, 7)(8, 9)	25200	-	-	-
Cl_{11}	(1, 2, 3)(4, 5, 6)	8400	-	-	-
Cl_{12}	(1, 2, 3)(4, 5, 6)(7, 8)	50400	-	-	-
Cl_{13}	(1, 2, 3)(4, 5, 6)(7, 8)(9, 10)	25200	-	-	-
Cl_{14}	(1, 2, 3)(4, 5, 6)(7, 8, 9)	22400	-	-	-
Cl_{15}	(1, 2, 3, 4)	1260	-	-	-
Cl_{16}	(1, 2, 3, 4)(5, 6)	18900	-	-	-
Cl_{17}	(1, 2, 3, 4)(5, 6)(7, 8)	56700	-	-	-
Cl_{18}	(1, 2, 3, 4)(5, 6)(7, 8)(9, 10)	18900	-	-	-
Cl_{19}	(1, 2, 3, 4)(5, 6, 7)	50400	-	-	-
Cl_{20}	(1, 2, 3, 4)(5, 6, 7)(8, 9)	151200	37800	$D_8 \times C_3 \times C_2$	37800
Cl_{21}	(1, 2, 3, 4)(5, 6, 7)(8, 9, 10)	50400	-	-	-
Cl_{22}	(1, 2, 3, 4)(5, 6, 7, 8)	56700	-	-	-
Cl_{23}	(1, 2, 3, 4)(5, 6, 7, 8)(9, 10)	56700	-	-	-
Cl_{24}	(1, 2, 3, 4, 5)	6048	-	-	-
Cl_{25}	(1, 2, 3, 4, 5)(6, 7)	60480	-	-	-
Cl_{26}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	90720	-	-	-
Cl_{27}	(1, 2, 3, 4, 5)(6, 7, 8)	120960	-	-	-
$\left\{ \begin{array}{l} Cl_{28} \\ Cl_{29} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5)(6, 7, 8)(9, 10) \\ (1, 2, 3, 4, 5)(6, 7, 8, 9) \end{array} \right.$	$\left\{ \begin{array}{l} 120960 \\ 181440 \end{array} \right.$	$\left\{ \begin{array}{l} 37200 \\ 37200 \end{array} \right.$	$\left\{ \begin{array}{l} C_5 \times C_3 \times C_2 \\ C_5 \times D_8 \end{array} \right.$	$\left\{ \begin{array}{l} 37200 \\ 37200 \end{array} \right.$
Cl_{30}	(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)	72576	-	-	-
Cl_{31}	(1, 2, 3, 4, 5, 6)	25200	-	-	-
$\left\{ \begin{array}{l} Cl_{32} \\ Cl_{33} \\ Cl_{34} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6)(7, 8) \\ (1, 2, 3, 4, 5, 6)(7, 8)(9, 10) \\ (1, 2, 3, 4, 5, 6)(7, 8, 9) \end{array} \right.$	$\left\{ \begin{array}{l} 151200 \\ 75600 \\ 201600 \end{array} \right.$	$\left\{ \begin{array}{l} 88200 \\ 88200 \\ 88200 \end{array} \right.$	$\left\{ \begin{array}{l} C_6 \times D_8 \\ C_6 \times C_3 \end{array} \right.$	$\left\{ \begin{array}{l} 88200 \\ 88200 \\ 88200 \end{array} \right.$
Cl_{35}	(1, 2, 3, 4, 5, 6)(7, 8, 9, 10)	151200	-	-	-
Cl_{36}	(1, 2, 3, 4, 5, 6, 7)	86400	-	-	-
$\left\{ \begin{array}{l} Cl_{37} \\ Cl_{38} \end{array} \right.$	$\left\{ \begin{array}{l} (1, 2, 3, 4, 5, 6, 7)(8, 9) \\ (1, 2, 3, 4, 5, 6, 7)(8, 9, 10) \end{array} \right.$	$\left\{ \begin{array}{l} 259200 \\ 172800 \end{array} \right.$	$\left\{ \begin{array}{l} 57600 \\ 57600 \end{array} \right.$	$\left\{ \begin{array}{l} C_7 \times C_2 \\ C_7 \times C_3 \end{array} \right.$	$\left\{ \begin{array}{l} 57600 \\ 57600 \end{array} \right.$
Cl_{39}	(1, 2, 3, 4, 5, 6, 7, 8)	226800	-	-	-
Cl_{40}	(1, 2, 3, 4, 5, 6, 7, 8)(9, 10)	226800	14175	$P_{2,8} \times C_2$	14175
Cl_{41}	(1, 2, 3, 4, 5, 6, 7, 8, 9)	403200	11200	$P_{3,9}$	11200
Cl_{42}	(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)	362880	90720	C_{10}	90720
Total			336895		336895

Source: Research.

$$P_{2,8} = (C_2 \wr C_2) \wr C_2, P_{3,9} = C_3 \wr C_3 \text{ where } \wr \text{ means 'wreath product'}.$$

4.4 Results for $\sigma_\infty(A_n)$

In this section we deduce $\sigma_\infty(A_n)$ and $\Sigma_\infty(A_n)$ for $n = 3, \dots, 10$, using the tables on generating non-nilpotent sets and nilpotent covers of S_3 upto S_{10} in section §§4.3.2.9. We will recall Theorem 4.3.7. This Theorem tells us that given a permutation in A_n , its conjugacy class in S_n splits into two equivalent classes in A_n if and only if its cycle type are distinct odd primes. The summary is given in the tables that follow.

Our notation is consistent with the work on S_n . Note that we write $Q_{2,n}$ to mean “a Sylow 2-subgroup of A_n ”.

4.4.1 Proofs

The argument to justify the tables that follow is very similar to that presented for S_n , so we do not give full details. In what follows, g is always an element of A_n .

First, we note that Lemma 4.3.11 still applies, so we can exclude from consideration all conjugacy classes whose elements are powers of elements from another conjugacy class.

Next, let us consider elements containing a long cycle of prime power order:

Proposition 4.4.1. *Suppose that g contains a unique cycle of length $k \geq \frac{n}{2}$ with $k = p^a$ a prime power; if $k = \frac{n}{2}$, then assume that p is odd. Suppose that N is a maximal nilpotent subgroup of A_n containing g .*

1. *If p is odd, then $N \cong P_{p,k} \times N_0$, where N_0 is a maximal nilpotent subgroup of A_{n-k} .*
2. *If p is even, then N contains $Q_{2,k} \times N_0$ as a subgroup of index 1 or 2, where $Q_{2,k}$ is a Sylow 2-subgroup of A_k and N_0 is a maximal nilpotent subgroup of A_{n-k} .*

Proof. This result follows directly from Proposition 4.3.3 – the subgroup N here is simply the intersection of A_n with the subgroup N in Proposition 4.3.3. $\square$

Proposition 4.4.1 (along with Lemma 4.3.11) means that the only conjugacy classes we need to consider are given in Table 4.30.

Lemma 4.4.1. Suppose that $n \leq 10$, $G = A_n$, and that g is in a conjugacy class from Table 4.30. Let N be a maximal nilpotent subgroup of G containing g . Then N is isomorphic to the group listed in the final column of Table 4.30.

Note that our notation is the same as for S_n . In particular, if the group in the final column is enclosed in parentheses, then we are asserting only that the given group is maximal nilpotent and contains g , but may not be the only such maximal nilpotent group.

Proof. Let $G = A_n$. Suppose that g is of type k^2 with $n = 2k$ or $n = 2k + 1$. Since $6 \leq n \leq 10$, k is a prime power. If p is odd, then $C_G(g) \cong C_k \times C_k$ and we conclude immediately that $N = C_G(g)$. If $p = 2$, then $C_G(g)$ is a 2-group and, again, $N = C_G(g)$. Note that if $n = 8$, then the group N also contains elements of type $4^1 2^1$.

Now suppose that g is one of the remaining conjugacy classes that contains a unique cycle of length 3; write this cycle as $h = (1, 2, 3)$. Then $g^a = h$ for some integer a . Then $C_G(h) \cong \langle h \rangle \times \text{Sym}(\{4, \dots, n\})$, and this group must contain all Sylow t -subgroups of N unless, possibly, when $t = 3$. But now choose b minimal so that g^b does not have order divisible by 3; it is easy to check that the only 3-elements commuting with g^b are powers of h . We conclude that $N < C_3 \times A_{n-3}$. If $n \geq 9$, Proposition 4.4.1 implies that $N = C_3 \times P_{2,4}$, as required. If $n < 9$, then we can take $g^b = (4, 5)(6, 7)$ and it is an easy matter to check that $C_{A_{n-3}}(g^b)$ is a 2-group, hence $N = C_3 \times Q_{2,4}$, as required.

All remaining g occur when $n = 9$ or 10 and contain the cycle $(1, 2, 3, 4, 5, 6)$. Let $h_1 = g^a$, with a chosen so that h_1 is an element of order 3, and let $h_2 = g^3$, an element of order 2 or 4. Then $C_G(h_1) \leq S_6 \times S_{n-6}$ and so all Sylow t -subgroups

of N must be contained in $C_G(h_1)$ except, possibly, when $t = 3$. Now suppose that f is a 3-element such that $\langle f, h_1 \rangle$ is a 3-group and f commutes with h_2 ; then f is a power of h_1 , and we conclude that $N < S_6 \times S_{n-6}$. Then Lemma 4.3.12 implies that $N < C_6 \times S_{n-6}$. If $n = 10$, then $N = C_3 \times D_8$. $\square$

Now the counts given in the table are simply obtained by calculating $|G : N_G(N)|$ for every group N listed in the column “nilp. cover”.

Table 4.30: Conjugacy classes in A_n not covered by Proposition 4.4.1

n	Class	Element	nilp. cover
6	3^2	$(1, 2, 3)(4, 5, 6)$	$C_3 \times C_3$
7	3^2	$(1, 2, 3)(4, 5, 6)$	$C_3 \times C_3$
7	$3^1 2^2$	$(1, 2, 3)(4, 5)(6, 7)$	$C_3 \times C_2 \times C_2$
8	$3^1 2^2$	$(1, 2, 3)(4, 5)(6, 7)$	$C_3 \times C_2 \times C_2$
8	$4^1 2^1$	$(1, 2, 3, 4)(5, 6)$	$(Q_{2,8})$
8	4^2	$(1, 2, 3, 4)(5, 6, 7, 8)$	$Q_{2,8}$
9	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$	$C_3 \times D_8$
9	4^2	$(1, 2, 3, 4)(5, 6, 7, 8)$	$Q_{2,8}$
9	$6^1 2^1$	$(1, 2, 3, 4, 5, 6)(7, 8)$	C_6
10	$4^1 3^1 2^1$	$(1, 2, 3, 4)(5, 6, 7)(8, 9)$	$C_3 \times D_8$
10	5^2	$(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)$	$C_5 \times C_5$
10	$6^1 2^1$	$(1, 2, 3, 4, 5, 6)(7, 8)$	$C_3 \times D_8$
10	$6^1 4^1$	$(1, 2, 3, 4, 5, 6)(7, 8, 9, 10)$	$C_3 \times D_8$

Source: Research.

Table 4.31: Generating non-nilpotent sets and nilpotent covers of A_3

Label	conj.class	“power of”	subgroup	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2, 3)	-	$P_{3,3} \times S_0$	1	C_3
Total				1	

Source: Research.

Table 4.32: Generating non-nilpotent sets and nilpotent cover of A_4

Label	conj.class	“power of”	subgroup	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	-	$P_{2,4} \times S_0$	1	$C_2 \times C_2$
$Cl_{3a,3b}$	(1, 2, 3)	-	$P_{3,3} \times S_1$	4	C_3
Total				5	

Source: Research.

Table 4.33: Generating non-nilpotent sets and nilpotent covers of A_5

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	
Cl_2	(1, 2)(3, 4)	-	$P_{2,4} \times S_1$	5	$C_2 \times C_2$
Cl_3	(1, 2, 3)	-	$P_{3,3} \times S_2$	10	C_3
$Cl_{4a,4b}$	(1, 2, 3, 4, 5)	-	$P_{5,5} \times S_0$	6	C_5
Total				21	

Source: Research.

Table 4.34: Generating non-nilpotent sets and nilpotent covers of A_6

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	Cl_5	-	-	
Cl_3	(1, 2, 3)	-	$(P_{3,6})$	-	-
Cl_4	(1, 2, 3)(4, 5, 6)	-	$P_{3,6}$	10	$C_3 \times C_3$
Cl_5	(1, 2, 3, 4)(5, 6)	-	$P_{2,4} \times S_2$	45	$D_8 \times C_2$
$Cl_{6a,6b}$	(1, 2, 3, 4, 5)	-	$P_{5,5} \times S_1$	36	C_5
Total				91	

Source: Research.

Table 4.35: Generating non-nilpotent sets and nilpotent covers of A_7

Label	conj.class	“power of”	<i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	Cl_6	$(P_{2,4} \times S_3)$	-	-
Cl_3	(1, 2, 3)	Cl_5	-	-	-
Cl_4	(1, 2, 3)(4, 5, 6)	-	$P_{3,6}$	70	$C_3 \times C_3$
Cl_5	(1, 2, 3)(4, 5)(6, 7)	-	$P_{2,4} \times S_3$	35	$C_2 \times C_2 \times C_3$
Cl_6	(1, 2, 3, 4)(5, 6)	-	$P_{2,4} \times S_3$	315	D_8
Cl_7	(1, 2, 3, 4, 5)	-	$P_{5,5} \times S_2$	126	C_5
$Cl_{8a,8b}$	(1, 2, 3, 4, 5, 6, 7)	-	$P_{7,7} \times S_0$	120	C_7
Total				666	

Source: Research.

Table 4.36: Generating non-nilpotent sets and nilpotent covers of A_8

Label	conj.class	"power of"	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	Cl_5	-	-	-
Cl_3	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_8	-	-	-
Cl_4	(1, 2, 3)	Cl_{10}	-	-	-
Cl_5	(1, 2, 3)(4, 5)(6, 7)	-	$P_{2,4} \times S_4$	280	$C_3 \times C_2 \times C_2$
Cl_6	(1, 2, 3)(4, 5, 6)	Cl_{11}	-	-	-
Cl_7	(1, 2, 3, 4)(5, 6)	-	$(P_{2,8} \times S_0)$	-	-
Cl_8	(1, 2, 3, 4)(5, 6, 7, 8)	-	$P_{2,8} \times S_0$	315	$Q_{2,8}$
Cl_9	(1, 2, 3, 4, 5)	Cl_{10}	-	-	-
$Cl_{10a,10b}$	(1, 2, 3, 4, 5)(6, 7, 8)	-	$P_{5,5} \times S_3$	336	$C_5 \times C_3$
Cl_{11}	(1, 2, 3, 4, 5, 6)(7, 8)	-	$C_6 \times S_2$	1680	$C_6 \times C_2$
$Cl_{12a,12b}$	(1, 2, 3, 4, 5, 6, 7)	-	$P_{7,7} \times S_1$	960	C_7
Total				3571	

Source: Research.

$$P_{2,8} = (C_2 \wr C_2) \wr C_2, \text{ where } \wr \text{ means 'wreath product'}$$

Table 4.37: Generating non-nilpotent sets and nilpotent covers of A_9

Label	conj.class	“power of”	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	Cl_8	-	-	-
Cl_3	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_{10}	-	-	-
Cl_4	(1, 2, 3)	Cl_5	-	-	-
Cl_5	(1, 2, 3)(4, 5)(6, 7)	Cl_9	-	-	-
Cl_6	(1, 2, 3)(4, 5, 6)	Cl_{14}	-	-	-
Cl_7	(1, 2, 3)(4, 5, 6)(7, 8, 9)	Cl_{16}	-	-	-
Cl_8	(1, 2, 3, 4)(5, 6)	Cl_9	-	-	-
Cl_9	(1, 2, 3, 4)(5, 6, 7)(8, 9)	-	$P_{2,6} \times S_3$	3780	$D_8 \times C_3$
Cl_{10}	(1, 2, 3, 4)(5, 6, 7, 8)	-	$P_{2,8} \times S_1$	???	$Q_{2,9}$
Cl_{11}	(1, 2, 3, 4, 5)	Cl_{12}	-	-	-
Cl_{12}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	-	$P_{5,5} \times S_4$	756	$C_5 \times C_2 \times C_2$
$Cl_{13a,13b}$	(1, 2, 3, 4, 5)(6, 7, 8)	-	$P_{5,5} \times S_4$	3024	$C_5 \times C_3$
Cl_{14}	(1, 2, 3, 4, 5, 6)(7, 8)	-	$C_6 \times S_3$	15120	C_6
Cl_{15}	(1, 2, 3, 4, 5, 6, 7)	-	$P_{7,7} \times S_2$	4320	C_7
$Cl_{16a,16b}$	(1, 2, 3, 4, 5, 6, 7, 8, 9)	-	$P_{3,9} \times S_0$	1120	$P_{3,9}$
Total				28120+?	

Source: Research.

$$P_{2,8} = (C_2 \wr C_2) \wr C_2, P_{3,9} = C_3 \wr C_3 \text{ where } \wr \text{ means 'wreath product'.$$

Remark 1. We should say a word regarding the missing entry for A_9 in Table 4.37 and Table 5.2. Note that elements of the cycle type $4 - 4 - 1$ e.g. $(1, 2, 3, 4)(5, 6, 7, 8)$ of A_9 are contained in maximal nilpotent subgroups which are the Sylow 2-subgroups of A_9 (See conjugacy classes Cl_{18} and Cl_{29} of Table 4.27). In order for us to calculate $\Sigma_\infty(A_9)$ we need to know how many of the 2835 Sylow 2-subgroups of A_9 are needed in order to cover the 11340 elements of cycle type $4 - 4 - 1$. Our (GAP, 2013) algorithm always hangs after working for many hours!. This problem does not arise in A_{10} . We conjecture that $28120 \leq \Sigma_\infty(A_9) \leq 30955$ and $28120 \leq \sigma_\infty(A_9) \leq 30955$

Table 4.38: Generating non-nilpotent sets and nilpotent covers of A_{10}

Label	conj.class	"power of"	in <i>subgroup</i>	$ N_p $	nil.cover
Cl_1	(1)	<i>All</i>	-	-	-
Cl_2	(1, 2)(3, 4)	Cl_9	-	-	-
Cl_3	(1, 2)(3, 4)(5, 6)(7, 8)	Cl_{21}	-	-	-
Cl_4	(1, 2, 3)	Cl_5	-	-	-
Cl_5	(1, 2, 3)(4, 5)(6, 7)	Cl_{11}	-	-	-
Cl_6	(1, 2, 3)(4, 5, 6)	Cl_{18}	-	-	-
Cl_7	(1, 2, 3)(4, 5, 6)(7, 8)(9, 10)	Cl_{17}	-	-	-
Cl_8	(1, 2, 3)(4, 5, 6)(7, 8, 9)	Cl_{22}	-	-	-
Cl_9	(1, 2, 3, 4)(5, 6)	Cl_{11}	-	-	-
Cl_{10}	(1, 2, 3, 4)(5, 6)(7, 8)(9, 10)	Cl_{18}	-	-	-
Cl_{11}	(1, 2, 3, 4)(5, 6, 7)(8, 9)	-	$P_{2,7} \times S_3$	37800	$C_4 \times C_3 \times C_2$
Cl_{12}	(1, 2, 3, 4)(5, 6, 7, 8)	Cl_{21}	-	-	-
Cl_{13}	(1, 2, 3, 4, 5)	Cl_{14}	-	-	-
Cl_{14}	(1, 2, 3, 4, 5)(6, 7)(8, 9)	-	$P_{5,5} \times S_5$	7560	$C_5 \times C_2 \times C_2$
Cl_{15}	(1, 2, 3, 4, 5)(6, 7, 8)	-	$P_{5,5} \times S_5$	15120	$C_5 \times C_3$
Cl_{16}	(1, 2, 3, 4, 5)(6, 7, 8, 9, 10)	-	$P_{5,5} \times S_5$	4536	$C_5 \times C_5$
Cl_{17}	(1, 2, 3, 4, 5, 6)(7, 8)	-	$(C_6 \times S_4)$	-	-
Cl_{18}	(1, 2, 3, 4, 5, 6)(7, 8, 9, 10)	-	$C_6 \times S_4$	37800	$C_3 \times D_8$
Cl_{19}	(1, 2, 3, 4, 5, 6, 7)	Cl_{20}	-	-	-
$Cl_{20a,20b}$	(1, 2, 3, 4, 5, 6, 7)(8, 9, 10)	-	$P_{7,7} \times S_3$	14400	$C_7 \times C_3$
Cl_{21}	(1, 2, 3, 4, 5, 6, 7, 8)(9, 10)	-	$P_{2,8} \times S_2$	14175	$Q_{2,10}$
$Cl_{22a,22b}$	(1, 2, 3, 4, 5, 6, 7, 8, 9)	-	$P_{3,9} \times S_1$	11200	$P_{3,9}$
Total				142591	

Source: Research.

$$P_{2,8} = (C_2 \wr C_2) \wr C_2, P_{3,9} = C_3 \wr C_3 \text{ where } \wr \text{ means 'wreath product'.$$

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

This chapter presents a summary of the results for our study based on the research objectives and draws conclusions out of these results. It also presents recommendations of the study and suggests areas exposed by the study on which further investigations can be done.

5.1 Summary

In this section we summarise our results. This is done in three subsections as follows.

1. Results on dihedral groups.
2. Results on symmetric groups.
3. Results on alternating groups.

Recall that for each of these three families of groups our results were primarily on:

1. $\sigma_c(G)$, the size of a maximal non- c -nilpotent subset;
2. $\Sigma_c(G)$, the size of a minimal c -nilpotent cover;
3. A statement describing the conditions under which $\sigma_c(G) = \Sigma_c(G)$.

5.1.1 Results on dihedral groups

In this section we summarise the results on dihedral groups. We have produced a closed formula that works for all k . This is summarized by Theorem 4.2.1 and the example in Table 4.1

5.1.2 Results on symmetric groups

In this section we summarise the results on symmetric groups for small values of n where $n = 3, 4, 5, 6, 7, 8, 9, 10$. We have calculated values for the statistics $\sigma_\infty(S_n)$ and $\Sigma_\infty(S_n)$. Obtaining a formula that works for all n remains an open question. These values are summarized in Table 5.1.

Table 5.1: Values for $\sigma_\infty(S_n)$ and $\Sigma_\infty(S_n)$ for $n = 3, 4, 5, 6, 7, 8, 9, 10$

n	S_n	$\sigma_\infty(S_n)$	$\Sigma_\infty(S_n)$
3	S_3	4	4
4	S_4	7	7
5	S_5	31	31
6	S_6	201	201
7	S_7	1086	1086
8	S_8	5139	5139
9	S_9	37507	37507
10	S_{10}	336895	336895

Source: Research.

5.1.3 Results on alternating groups

In this section we summarise the results on alternating groups for small values of n where $n = 3, 4, 5, 6, 7, 8, 10$. We have calculated values for the statistics $\sigma_\infty(A_n)$ and $\Sigma_\infty(A_n)$. Obtaining a formula that works for all n remains an open question. These values are summarized in Table 5.2.

It turns out that for S_n , the two values coincide but not for A_n . Note that it is not true for σ_1 and Σ_1 (for S_n this is a result of Brown (Brown, 1988); for A_n this is a result of Barrantes *et al* (Barrantes et al., 2016)).

For $\sigma_\infty(A_9)$ and $\Sigma_\infty(A_9)$ in Table 5.2 a comment is given in section §4.4 as Remark 1.

5.2 Conclusion

The following conclusions were made based on the objectives of this study. Recall that we looked at minimal nilpotent cover and maximal non-nilpotent subset for

Table 5.2: Values of $\sigma_\infty(A_n)$ and $\Sigma_\infty(A_n)$ for $n = 3, 4, 5, 6, 7, 8, 10$

n	A_n	$\sigma_\infty(A_n)$	$\Sigma_\infty(A_n)$
3	A_3	1	1
4	A_4	5	5
5	A_5	21	21
6	A_6	91	91
7	A_7	666	666
8	A_8	3571	3571
9	A_9	28120+?	28120+?
10	A_{10}	142591	142591

Source: Research.

each of D_{2k} for all k and S_n, A_n for $n = 2, 3, \dots, 10$.

- (i) The size of minimal nilpotent cover and that of maximal non-nilpotent subset for D_{2k} coincide for all k , that is:

$$\sigma_\infty(D_{2k}) = \Sigma_\infty(D_{2k})$$

- (ii) The general formula for calculating the size of minimal nilpotent cover and that of maximal non-nilpotent subset for D_{2k} is give by Theorem 4.2.1.

- (iii) The size of minimal nilpotent cover and that of maximal non-nilpotent subset for S_n coincide for $n = 3, 4, \dots, 10$, that is:

$$\sigma_\infty(S_n) = \Sigma_\infty(S_n)$$

- (iv) Except for A_9 , the size of minimal nilpotent cover and that of maximal non-nilpotent subset for A_n coincide for $n = 3, 4, 5, 6, 7, 8, 10$, that is:

$$\sigma_\infty(A_n) = \Sigma_\infty(A_n), n \neq 9$$

5.3 Recommendation

- (i) This research work was on minimal nilpotent cover and maximal non-nilpotent subset for each of the three families of groups; finite symmetric, alternating and dihedral groups. We produced a general formula for the size of the minimal nilpotent cover and maximal non-nilpotent subset for the dihedral group D_{2k} .

On minimal nilpotent cover and maximal non-nilpotent subset for the finite symmetric group S_n and alternating group A_n , we considered $n = 3, 4, \dots, 10$. We were not able to produce a general formula. We therefore recommend an investigation on minimal nilpotent cover and maximal non-nilpotent subset of the finite symmetric group S_n as well as finite alternating group A_n for large $n \geq 11$ and produce a closed formula that works for general n . As n grows large, the computer algebra package **GAP** fails to generate results for S_n and A_n . We therefore recommend an investigation and incorporation of advanced programs into **GAP** that can deal with large values of n and sophisticated calculations.

- (ii) This study suggests that similar studies can be conducted for normal nilpotent covers for dihedral group D_{2k} , finite symmetric group S_n and finite alternating group A_n .
- (iii) Similar studies can also be done on minimal nilpotent cover for other finite simple and dicyclic groups.

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APPENDICES

Appendix A: (GAP code for section §§4.3.2.1)

```

gap> s3:=SymmetricGroup(3);
Sym( [ 1 .. 3 ] )
gap> AsGroup(s3);
Sym( [ 1 .. 3 ] )

gap> A:=Centralizer(s3,(1,2)); Size(A);Size(A)/Factorial(1);
      B:=Centralizer(s3,(1,2,3)); Size(B);Size(B)/Factorial(0);

Group([ (1,2) ])
2
2
Group([ (1,2,3) ])
3
3

gap> s4:=SymmetricGroup(4);
Sym( [ 1 .. 4 ] )
gap> C:=Centralizer(s4,(1,2,3)); Size(C);Size(C)/Factorial(1);
      D:=Centralizer(s4,(1,2,3,4)); Size(D);Size(D)/Factorial(0);

Group([ (1,2,3) ])
3
3
Group([ (1,2,3,4) ])
4
4

gap>s5:=SymmetricGroup(5);
Sym( [ 1 .. 5 ] )
gap> F:=Centralizer(s5,(1,2,3)); Size(F);Size(F)/Factorial(2);
      G:=Centralizer(s5,(1,2,3,4)); Size(G);Size(G)/Factorial(1);
      H:=Centralizer(s5,(1,2,3,4,5)); Size(H);Size(H)/Factorial(0);

Group([ (1,2,3), (4,5) ])
6
3
Group([ (1,2,3,4) ])
4
4
Group([ (1,2,3,4,5) ])
5
5

```

```

gap> s6:=SymmetricGroup(6);
Sym( [ 1 .. 6 ] )

gap> K:=Centralizer(s6,(1,2,3,4)); Size(K);Size(K)/Factorial(2);
      L:=Centralizer(s6,(1,2,3,4,5)); Size(L);Size(L)/Factorial(1);

Group([ (1,2,3,4), (5,6) ])
8
4
Group([ (1,2,3,4,5) ])
5
5

gap> s7:=SymmetricGroup(7);
Sym( [ 1 .. 7 ] )

gap> M:=Centralizer(s7,(1,2,3,4)); Size(M);Size(M)/Factorial(3);
      N:=Centralizer(s7,(1,2,3,4,5)); Size(N);Size(N)/Factorial(2);
      P:=Centralizer(s7,(1,2,3,4,5,6,7)); Size(P);Size(P)/Factorial(0);

Group([ (1,2,3,4), (5,7), (6,7) ])
24
4
Group([ (1,2,3,4,5), (6,7) ])
10
5
Group([ (1,2,3,4,5,6,7) ])
7
7

gap> s8:=SymmetricGroup(8);
Sym( [ 1 .. 8 ] )
gap> M:=Centralizer(s8,(1,2,3,4,5)); Size(M);Size(M)/Factorial(3);
      N:=Centralizer(s8,(1,2,3,4,5,6,7)); Size(N);Size(N)/Factorial(1);
      P:=Centralizer(s8,(1,2,3,4,5,6,7,8)); Size(P);Size(P)/Factorial(0);

Group([ (1,2,3,4,5), (6,8), (7,8) ])
30
5
Group([ (1,2,3,4,5,6,7) ])
7
7
Group([ (1,2,3,4,5,6,7,8) ])
8
8

gap> s9:=SymmetricGroup(9);
Sym( [ 1 .. 9 ] )

```

```

gap> M:=Centralizer(s9,(1,2,3,4,5));
      Size(M);Size(M)/Factorial(4);
      N:=Centralizer(s9,(1,2,3,4,5,6,7));
      Size(N);Size(N)/Factorial(2);
      P:=Centralizer(s9,(1,2,3,4,5,6,7,8));
      Size(P);Size(P)/Factorial(1);
      Q:=Centralizer(s9,(1,2,3,4,5,6,7,8,9));
      Size(Q);Size(Q)/Factorial(0);

Group([ (1,2,3,4,5), (6,9), (7,9), (8,9) ])
120
5
Group([ (1,2,3,4,5,6,7), (8,9) ])
14
7
Group([ (1,2,3,4,5,6,7,8) ])
8
8
Group([ (1,2,3,4,5,6,7,8,9) ])
9
9

gap> s10:=SymmetricGroup(10);
Sym( [ 1 .. 10 ] )
gap> M:=Centralizer(s10,(1,2,3,4,5,6,7));
      Size(M);Size(M)/Factorial(3);
      N:=Centralizer(s10,(1,2,3,4,5,6,7,8));
      Size(N);Size(N)/Factorial(2);
      P:=Centralizer(s10,(1,2,3,4,5,6,7,8,9));
      Size(P);Size(P)/Factorial(1);

Group([ (1,2,3,4,5,6,7), (8,10), (9,10) ])
42
7
Group([ (1,2,3,4,5,6,7,8), (9,10) ])
16
8
Group([ (1,2,3,4,5,6,7,8,9) ])
9
9

gap> s11:=SymmetricGroup(11);
Sym( [ 1 .. 11 ] )
gap> M:=Centralizer(s11,(1,2,3,4,5,6,7));
      Size(M);Size(M)/Factorial(4);
      N:=Centralizer(s11,(1,2,3,4,5,6,7,8));
      Size(N);Size(N)/Factorial(3);
      P:=Centralizer(s11,(1,2,3,4,5,6,7,8,9));

```

```

Size(P);Size(P)/Factorial(2);
Q:=Centralizer(s11,(1,2,3,4,5,6,7,8,9,10,11));
Size(Q);Size(Q)/Factorial(0);

Group([ (1,2,3,4,5,6,7), (8,11), (9,11), (10,11) ])
168
7
Group([ (1,2,3,4,5,6,7,8), (9,11), (10,11) ])
48
8
Group([ (1,2,3,4,5,6,7,8,9), (10,11) ])
18
9
Group([ (1,2,3,4,5,6,7,8,9,10,11) ])
11
11

gap> s12:=SymmetricGroup(12);
Sym( [ 1 .. 12 ] )
gap> M:=Centralizer(s12,(1,2,3,4,5,6,7)); Size(M);Size(M)/Factorial(5);
N:=Centralizer(s12,(1,2,3,4,5,6,7,8)); Size(N);Size(N)/Factorial(4);
P:=Centralizer(s12,(1,2,3,4,5,6,7,8,9)); Size(P);Size(P)/Factorial(3);
Q:=Centralizer(s12,(1,2,3,4,5,6,7,8,9,10,11)); Size(Q);
Size(Q)/Factorial(1);

Group([ (1,2,3,4,5,6,7), (8,12), (9,12), (10,12), (11,12) ])
840
7
Group([ (1,2,3,4,5,6,7,8), (9,12), (10,12), (11,12) ])
192
8
Group([ (1,2,3,4,5,6,7,8,9), (10,12), (11,12) ])
54
9
Group([ (1,2,3,4,5,6,7,8,9,10,11) ])
11
11

gap> s13:=SymmetricGroup(13);
Sym( [ 1 .. 13 ] )
gap> M:=Centralizer(s13,(1,2,3,4,5,6,7));
Size(M);Size(M)/Factorial(6);
N:=Centralizer(s13,(1,2,3,4,5,6,7,8));
Size(N);Size(N)/Factorial(5);
P:=Centralizer(s13,(1,2,3,4,5,6,7,8,9));
Size(P);Size(P)/Factorial(4);
Q:=Centralizer(s13,(1,2,3,4,5,6,7,8,9,10,11));
Size(Q);Size(Q)/Factorial(2);

```

```

R:=Centralizer(s13,(1,2,3,4,5,6,7,8,9,10,11,12,13));
Size(R);Size(R)/Factorial(0);

Group([ (1,2,3,4,5,6,7), (8,13), (9,13), (10,13), (11,13), (12,13) ])
5040
7
Group([ (1,2,3,4,5,6,7,8), (9,13), (10,13), (11,13), (12,13) ])
960
8
Group([ (1,2,3,4,5,6,7,8,9), (10,13), (11,13), (12,13) ])
216
9
Group([ (1,2,3,4,5,6,7,8,9,10,11), (12,13) ])
22
11
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13) ])
13
13

gap> s14:=SymmetricGroup(14);
Sym( [ 1 .. 14 ] )
gap> M:=Centralizer(s14,(1,2,3,4,5,6,7,8));
Size(M);Size(M)/Factorial(6);
N:=Centralizer(s14,(1,2,3,4,5,6,7,8,9));
Size(N);Size(N)/Factorial(5);
P:=Centralizer(s14,(1,2,3,4,5,6,7,8,9,10,11));
Size(P);Size(P)/Factorial(3);
Q:=Centralizer(s14,(1,2,3,4,5,6,7,8,9,10,11,12,13)); Size(Q);
Size(Q)/Factorial(1);

Group([ (1,2,3,4,5,6,7,8), (9,14), (10,14), (11,14), (12,14), (13,14) ])
5760
8
Group([ (1,2,3,4,5,6,7,8,9), (10,14), (11,14), (12,14), (13,14) ])
1080
9
Group([ (1,2,3,4,5,6,7,8,9,10,11), (12,14), (13,14) ])
66
11
Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13) ])
13
13

gap> s15:=SymmetricGroup(15);
Sym( [ 1 .. 15 ] )
gap> M:=Centralizer(s15,(1,2,3,4,5,6,7,8)); Size(M);Size(M)/Factorial(7);
N:=Centralizer(s15,(1,2,3,4,5,6,7,8,9)); Size(N);Size(N)/Factorial(6);

```

```

P:=Centralizer(s15,(1,2,3,4,5,6,7,8,9,10,11)); Size(P);
Size(P)/Factorial(4);Q:=Centralizer(s15,(1,2,3,4,5,6,7,8,9,10,11,12,13));
Size(Q);Size(Q)/Factorial(2);

```

```

<permutation group with 7 generators>

```

```

40320

```

```

8

```

```

Group([(1,2,3,4,5,6,7,8,9), (10,15), (11,15), (12,15), (13,15), (14,15)])

```

```

6480

```

```

9

```

```

Group([ (1,2,3,4,5,6,7,8,9,10,11), (12,15), (13,15), (14,15) ])

```

```

264

```

```

11

```

```

Group([ (1,2,3,4,5,6,7,8,9,10,11,12,13), (14,15) ])

```

```

26

```

```

13

```

Appendix B:(GAP code for section §§4.3.2.3)

```

G:=SymmetricGroup(5); g:=(1,2,3,4); P:=SylowSubgroup(G,2);;
  SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(4);;
G3:=SymmetricGroup(1);; mp1:= Order(SylowSubgroup(G4,2));;
P3:=SylowSubgroup(G3,2));; mp2:= Order(P3);;
np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=2;;

(1,2,3,4)
gap> Print(p," ",mp, " ",mp1," ",mp2," ",np," ", np2," ", l1," ", l2, "\n");
2 8 8 1 1 1 2 2

G:=SymmetricGroup(6); g:=(1,2,3,4); P:=SylowSubgroup(G,2);;
SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(4);;
G3:=SymmetricGroup(2);; mp1:= Order(SylowSubgroup(G4,2));;
P3:=SylowSubgroup(G3,2));; mp2:= Order(P3);;
np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=2;;

Sym( [ 1 .. 6 ] )
(1,2,3,4)

gap> Print(p," ", mp, " ",mp1, " ",mp2, " ",np," ", np2," ", l1," ", l2, "\n");
2 16 8 2 1 1 2 2

Sym( [ 1 .. 7 ] )
(1,2,3,4)
G:=SymmetricGroup(7); g:=(1,2,3,4); P:=SylowSubgroup(G,2);;
SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(4);;
G3:=SymmetricGroup(3);; mp1:= Order(SylowSubgroup(G4,2));;
P3:=SylowSubgroup(G3,2));; mp2:= Order(P3);;
np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=2;;

```

```
gap> Print(p, " ", mp, " ", mp1, " ", mp2, " ", np, " ", np2, " ", l1, " ", l2, "\n")
2 16 8 2 3 3 2 2
```

```
Sym( [ 1 .. 8 ] )
(1,2,3,4,5,6,7,8)
G:=SymmetricGroup(8); g:=(1,2,3,4,5,6,7,8); P:=SylowSubgroup(G,2);;
SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(8);; G3:=SymmetricGroup(0);;
mp1:= Order(SylowSubgroup(G4,2));; P3:=SylowSubgroup(G3,2));;
mp2:= Order(P3);; np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));; Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=2;;
```

```
gap> Print(p, " ", mp, " ", mp1, " ", mp2, " ", np, " ", np2, " ", l1, " ", l2, "\n")
2 128 128 1 1 1 16 4
```

```
Sym( [ 1 .. 9 ] )
(1,2,3,4,5,6,7,8)
G:=SymmetricGroup(9); g:=(1,2,3,4,5,6,7,8);
P:=SylowSubgroup(G,2);; SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(8);; G3:=SymmetricGroup(1);;
mp1:= Order(SylowSubgroup(G4,2));; P3:=SylowSubgroup(G3,2));;
mp2:= Order(P3);; np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));; Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=2;;
```

```
gap> Print(p, " ", mp, " ", mp1, " ", mp2, " ", np, " ", np2, " ", l1, " ", l2, "\n")
2 128 128 1 1 1 16 4
```

```
Sym( [ 1 .. 9 ] )
(1,2,3,4,5,6,7,8,9)
G:=SymmetricGroup(9); g:=(1,2,3,4,5,6,7,8,9);
P:=SylowSubgroup(G,3);; SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);; mp:= Order(fs[1]);;
G4:=SymmetricGroup(9);; G3:=SymmetricGroup(0);;
mp1:= Order(SylowSubgroup(G4,3));; P3:=SylowSubgroup(G3,3));;
mp2:=Order(P3);; np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,2));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=3;;
```

```
gap> Print(p, " ", mp, " ", mp1, " ", mp2, " ", np, " ", np2, " ", l1, " ", l2, "\n")
3 81 81 1 1 1 36 6
```

```
Sym( [ 1 .. 10 ] )
(1,2,3,4,5,6,7,8)
```

```
2 256 128 2 1 1 16 4
```

```
G:=SymmetricGroup(10); g:=(1,2,3,4,5,6,7,8,9); P:=SylowSubgroup(G,3);;
fs:=Filtered(SylowpofG, x-> g in x);;np:= Length(fs);;
mp:= Order(fs[1]);; G4:=SymmetricGroup(9);;G3:=SymmetricGroup(3);;
mp1:= Order(SylowSubgroup(G4,3));; P3:=SylowSubgroup(G3,1));;
mp2:= Order(P3);; np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,3));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=3;;
```

```
Sym( [ 1 .. 10 ] )
(1,2,3,4,5,6,7,8,9)
```

```
3 81 81 1 1 1 36 6
```

```
G:=SymmetricGroup(11); g:=(1,2,3,4,5,6,7,8);
P:=SylowSubgroup(G,2);;
SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);;
np:= Length(fs);; mp:= Order(fs[1]);; G4:=SymmetricGroup(8);;
G3:=SymmetricGroup(3);; mp1:= Order(SylowSubgroup(G4,2));;
P3:=SylowSubgroup(G3,2));; mp2:= Order(P3);;
np2:=Index(G3, Normalizer(G3,P3));; Pk:=Elements(SylowSubgroup(G4,2));;
Pkk:=Filtered(Pk, x -> Order(x)=Order(g));; l1:=Length(Pkk);;
l2:=Phi(Order(g));; p:=2;;
```

```
Sym( [ 1 .. 11 ] )
(1,2,3,4,5,6,7,8)
```

```
gap> Print(p, " ", mp, " ", mp1, " ", mp2, " ", np, " ", np2, " ", l1, " ", l2, "\n")
2 256 128 2 3 3 16 4
```

```
G:=SymmetricGroup(11); g:=(1,2,3,4,5,6,7,8,9);
P:=SylowSubgroup(G,3);; SylowpofG:=Elements(ConjugacyClassSubgroups(G,P));;
fs:=Filtered(SylowpofG, x-> g in x);; np:= Length(fs);;
mp:= Order(fs[1]);;G4:=SymmetricGroup(9);; G3:=SymmetricGroup(2);;
mp1:= Order(SylowSubgroup(G4,3));; P3:=SylowSubgroup(G3,3));;
mp2:= Order(P3);; np2:=Index(G3, Normalizer(G3,P3));;
Pk:=Elements(SylowSubgroup(G4,3));; Pkk:=Filtered(Pk, x -> Order(x)=Order(g));;
```

```
l1:=Length(Pkk);; l2:=Phi(Order(g));; p:=3;;
```

```
Sym( [ 1 .. 11 ] )  
(1,2,3,4,5,6,7,8,9)
```

```
Print(p," ", mp, " ",mp1, " ",mp2, " ",np," ", np2," ", l1," ", l2, "\n");  
3 81 81 1 1 1 36 6
```

Appendix C:(GAP code for section §§4.3.2.6)

```

gap> G:=SymmetricGroup(6);
Sym( [ 1 .. 6 ] )
gap> g:=(1,2,3,4,5,6);
(1,2,3,4,5,6)
gap> T:=[ (1,2,3,4), (1,2,3,4,5),(1,2,3,4,5,6)];
[ (1,2,3,4), (1,2,3,4,5), (1,2,3,4,5,6) ]
gap> for y in T do
> C:=ConjugacyClass(G,y);
> for x in C do
> H:=Group([g,x]);
> if IsNilpotent(H) then Print(g, " ", x, " ", Order(H), "\n");
> fi;od;od;
(1,2,3,4,5,6) (1,2,3,4,5,6) 6
(1,2,3,4,5,6) (1,6,5,4,3,2) 6
gap>

gap> G:=SymmetricGroup(7);
Sym( [ 1 .. 7 ] )
gap> g:=(1,2,3,4,5,6);
(1,2,3,4,5,6)
gap> T:=[ (1,2,3,4), (1,2,3,4,5),(1,2,3,4,5,6),(1,2,3,4,5,6,7)];
[ (1,2,3,4), (1,2,3,4,5), (1,2,3,4,5,6), (1,2,3,4,5,6,7) ]
gap> for y in T do
> C:=ConjugacyClass(G,y);
> for x in C do
> H:=Group([g,x]);
> if IsNilpotent(H) then Print(g, " ", x, " ", Order(H), "\n");
> fi;od;od;
(1,2,3,4,5,6) (1,2,3,4,5,6) 6
(1,2,3,4,5,6) (1,6,5,4,3,2) 6
gap>

gap> G:=SymmetricGroup(9);
Sym( [ 1 .. 9 ] )
gap> g:=(1,2,3,4,5,6);
(1,2,3,4,5,6)
gap> T:=[(7,8,9),(1,2,3,4,5), (1,2,3,4,5,6),(1,2,3,4,5,6,7),
(1,2,3,4,5,6,7,8),(1,2,3,4,5,6,7,8,9)];
[ (7,8,9), (1,2,3,4,5), (1,2,3,4,5,6), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7,8),
(1,2,3,4,5,6,7,8,9) ]
gap> for y in T do
> C:=ConjugacyClass(G,y);
> for x in C do
> H:=Group([g,x]);
> if IsNilpotent(H) then Print(g, " ", x, " ", Order(H), "\n");
> fi;od;od;

```

```

(1,2,3,4,5,6) (7,8,9) 18
(1,2,3,4,5,6) (7,9,8) 18
(1,2,3,4,5,6) (1,2,3,4,5,6) 6
(1,2,3,4,5,6) (1,6,5,4,3,2) 6
gap>

gap> G:=SymmetricGroup(10);
Sym( [ 1 .. 10 ] )
gap> g:=(1,2,3,4,5,6);
(1,2,3,4,5,6)
gap> T:=[(7,8,9,10),(1,2,3,4,5,6),(1,2,3,4,5,6,7),(1,2,3,4,5,6,7,8),
(1,2,3,4,5,6,7,8,9),(1,2,3,4,5,6,7,8,9,10)];
[ (7,8,9,10), (1,2,3,4,5,6), (1,2,3,4,5,6,7), (1,2,3,4,5,6,7,8),
  (1,2,3,4,5,6,7,8,9), (1,2,3,4,5,6,7,8,9,10) ]
gap> for y in T do
> C:=ConjugacyClass(G,y);
> for x in C do
>   H:=Group([g,x]);
>   if IsNilpotent(H) then Print(g, " ", x, " ", Order(H), "\n");
> fi;od;od;
(1,2,3,4,5,6) (7, 8, 9,10) 24
(1,2,3,4,5,6) (7, 8,10, 9) 24
(1,2,3,4,5,6) (7, 9, 8,10) 24
(1,2,3,4,5,6) (7, 9,10, 8) 24
(1,2,3,4,5,6) (7,10, 8, 9) 24
(1,2,3,4,5,6) (7,10, 9, 8) 24
(1,2,3,4,5,6) (1,2,3,4,5,6) 6
(1,2,3,4,5,6) (1,6,5,4,3,2) 6

```