RELATIVE EFFICIENCY AND DT- OPTIMALITY CRITERIA FOR THE EXISTING SIX SPECIFIC SECOND ORDER ROTATABLE DESIGNS IN THREE DIMENSIONS

BY

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DECLARATION

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DEDICATION

I would like to dedicate this thesis to my fiancée Vibian and my lovely daughter Chumbaa.

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ABSTRACT

Experimentation plays an important role in Science, Engineering and Industry. This is an application of treatments to experimental units, and then measurement of one or more responses. It is part of scientific method which requires observing and gathering information about how process and system work where some input variable *x*'s transform into an output that has one or more observable response variables *y*. Therefore, useful results and conclusions can be drawn. In order to obtain an objective conclusion there is need to plan and design an experiment and analyze the results. The approximation of the response function y is called Response Surface Methodology. This study focused on the existing six specific second order rotatable designs in three dimensions. These designs were denoted by M_1, M_2, M_3, M_4, M_5 and M_6. A design matrix X was developed from the designs, further their information matrices C_1, C_2, C_3, C_4, C_5 and C_6were obtained from which the alphabetic optimal values for these designs were evaluated, the optimum values obtained were used to calculate the A-, D-, E- and T- relative efficiencies for both calculus optimum and unit value designs, for instance to evaluate E- relative efficiency the formula $\frac{\lambda_{min}(\varepsilon)}{\lambda_{min}(\varepsilon^{¿})}$ where $\lambda_{min}(\varepsilon)$ is the least E- optimum valuewhile $\lambda_{min}(\varepsilon^{¿})$ is the respective least design optimal value. Finally the compound optimality criterion (DT-) for all the six designs was also evaluated. In this study optimal values already evaluated were used to evaluate efficiencies and the DT- optimality criterion. From the results Calculus optimum values designs are generally more efficient than Unit Value Designs, on checking both the calculus optimum designs and unit value designs D- efficiency was found to be the best as it gave a higher efficiency than the rest relative efficiency criteria. On comparison of all designs $M_1$was found to be most efficient as compared to the rest. For DT- optimality M_2 is DT- Optimal.

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LIST OF ABBREVIATIONS AND DEFINITIONS

RSM	Response Surface Methodology
DOE	Design of experiments
A_{eff}	A-Efficiency
D_{eff}	D-Efficiency
E_{eff}	E-Efficiency
T_{eff}	T-Efficiency
X^T	Transpose of a matrix X
N_D	Number of Runs in a design
P	Number of parameters in a design
RSORD	robust second order rotatable designs
NND(k)	Non negative definite matrix of k dimension

CHAPTER ONE: INTRODUCTION

In this chapter the background information is given , then statement problem is also stated, next justification is given, then further on the purpose of the study is given followed by the objectives of the study both general objective and specific objectives and finally the significance of the study is given.

1.0 Background of the Study.

Response surface methodology (RSM) consists of a group of mathematical and statistical techniques used in the development of an adequate functional relationship between a response of interest, *y*, and a number of associated control (or input) variables denoted by $x_1, x_2, \ldots, x_k$. In general, such a relationship is unknown but can be approximated by a low-degree polynomial model of the form

$$y = f'(\boldsymbol{x})\boldsymbol{\beta} + \varepsilon \tag{1.1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_k)$, $\boldsymbol{f}(\boldsymbol{x})$ is a vector function of *p* elements that consists of powers and cross- products of powers of $x_1, x_2, \ldots, x_k$ up to a certain degree denoted by d (≥ 1), $\boldsymbol{\beta}$ is a vector of *p* unknown constant coefficients referred to as parameters, and ε is a random experimental error assumed to have a zero mean. This is conditioned on the belief that model (1.1) provides an adequate representation of the response. In this case, the quantity $f'(\boldsymbol{x})\boldsymbol{\beta}$ represents the mean response, that is, the expected value of *y*, and is denoted by $\mu(\boldsymbol{x})$. Two important models are commonly used in RSM. These are special cases of model (1.1) and include the first-degree model ($d = 1$),

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \varepsilon \tag{1.2}$$

and the second-degree model ($d = 2$)with *k* factors is represented as follows

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{ij,i\angle j}^{k} \beta_{ij} x_i x_j + \varepsilon \tag{1.3}$$

where

β_0 is the intercept

$\beta_i$is the linear coefficient for the i^{th} factor

$\beta_{ii}$is the quadratic coefficient for the i^{th} factor

$\beta_{ij}$is the cross product coefficient for the i^{th} and j^{th} factor

x_i is the level of the i^{th} factor

x_{ij} is the level of the i^{th} and j^{th} factor

ε is the error term.

The purpose of considering a model such as (1.1) is threefold:

1. To establish a relationship, albeit approximate, between y and $x_1, x_2, \ldots, x_k$ that can be used to predict response values for given settings of the control variables.

2. To determine, through hypothesis testing, significance of the factors whose levels are represented by $x_1, x_2, \ldots, x_k$

3. To determine the optimum settings of $x_1, x_2, \ldots, x_k$. That result in the maximum (or minimum) response over a certain region of interest.

In order to achieve the above three objectives, a series of n experiments should first be carried out, in each of which the response y is measured (or observed) for specified settings of the control variables. The totality of these settings constitutes the response surface design. For this study the existing six specific second order rotatable designs in three dimensions developed by draper were utilized.

1.1 Problem Statement.

The study was focused on determination of the D–, A–, T– and E–relative efficiencies. Also the DT– optimality criterion for existing specific six calculus optimum and unit values specific second order rotatable designs in three dimensions was evaluated.

1.2 Justification.

For the existing six specific calculus value optimum and unit value designs their optimality criterion values have already been obtained in previous studies so for this study the relative efficiencies for both calculus optimum value designs and unit value designs were obtained. Also the compound DT- optimality criteria was evaluated.

1.3 Purpose of the Study.

The result will foster the development and dissemination of the theory and applications of statistics as efficient designs will result in improved production of goods and services.

1.4 Objectives of the Study

1.4.1 General Objective

To determine D–, A–, E– and T–Relative Efficiencies criteria and calculate the compound DT- optimality criterion for the existing six calculus optimum and unit values specific second order rotatable designs in three dimensions.

1.4.2 Specific Objectives

i) To determine D–, A–, E– and T–Relative Efficiencies for the six Specific second order rotatable designs in three dimensions for calculus optimum values and for unit value designs.

ii) To compare relative efficiencies of calculus optimum values designs with relative efficiencies of unit value designs.

iii) To determine DT– optimality Criterion for both calculus optimum and unit value designs.

1.5 Significance of the study

There is need for more efficient experimental designs to get maximum yields at a reduced cost and greater speed. This can be achieved when a researcher uses more efficient experimental designs. In this study the focus was on second order rotatable designs in three dimensions. Efficient designs minimize the inputs and maximize the outputs. The study culminated in the identification of more efficient designs between calculus optimum value designs and unit value designs and determination of DT-compound optimality.

CHAPTER TWO: LITERATURE REVIEW.

2.0 Introduction.

In this chapter the relevant literature is reviewed. First the Literature on D–, A–, E– and T– Optimality Criteria from which Alphabetic Relative Efficiencies Criteria Are Evaluated is given and finally the literature on DT- optimality Criterion is also given.

2.1 Optimality Criteria.

2.1.1The Design.

A well-defined experiment is an efficient method of learning about the world. Because experiments in the world, and even in the carefully controlled conditions of laboratories, cannot avoid random errors, statistical methods are essential for their efficient design and analyses (Atkinson and Donev, 1992).

The choice of design depends on the properties it is required, or desired, to have. Some of the design properties considered in the early development of RSM include; orthogonality, Uniform precision and rotatability.

Statistical design is about understanding where the variance comes from and making sure that is where the replication is (Casella, 2008). Fisher (1947) compared a database to a sample of gold ore. The finest analysis could only extract the proportion of gold contained in the ore. But a good design could produce a sample with more gold (Casella, 2008).

By "***design***", we mean the synthesis of a suitable experiment to test, estimate and develop a current conjectured model (Box and Draper, 1987). There are many statistical issues to consider in the design of an empirical study. Among the problems are the control of unwanted variation and the internal validity of the study. How can we be sure that a study is internally valid? In other words, how can we be sure that the treatment effect is attributed to the variables that are manipulated and not mainly influenced by unwanted variation? (Cox, 1958 and Cox and Reid, 2000).

The fundamental idea is the importance of the model related to the responses obtained in the experiment to the experimental factors. The purpose of the experiment is to find out about the model, including its adequacy. Experiments can be designed to answer a variety of questions. Often, estimates of the parameters of interest together with the predictions of the response from the fitted model. The variances of the parameter estimates and predictions depend on the particular experimental design used and should be as small as possible. Poorly designed experiments waste resources by yielding unnecessarily large variances and imprecise predictions.

In the past, statistical procedures were applied to data collected without a definite design. However, even in the 19^{th} century, many researchers felt the importance of rational choice of experimental design (Viatcheslav, 2006).

Fisher was the first to consider design problems systematically. His popular book (Fisher, 1935) passed through many editions and affected the development and applications of experimental designs. Fisher's approach is still developing (Viatcheslav, 2006). The paper by Box and Wilson (1951) offers an approach to finding the conditions for some output variable to be of maximal value. The approach is called *Response Surface Methodology* and is outlined in the paper by Box and Draper (1987).

2.1.2 Rotatable designs

A design D is said to be rotatable if the prediction variance in Var[ˆy(x)] = $\sigma^2 f'$ (x) $(X' X)^{-1}$ f (x).is constant at all points that are equidistant from the design center, which, by a proper coding of the control variables, can be chosen to be the point at the origin of the k-dimensional coordinates system. It follows that

Var[$\hat{y}$ (x)] is constant at all points that fall on the surface of a hyper sphere centered at the origin if the design is rotatable. The advantage of this property is that the prediction variance remains unchanged under any rotation of the coordinate axes. In addition, if optimization of $\hat{y}$ (x) is desired on concentric hyper spheres, as in the application of ridge analysis, which will be discussed later, then it would be desirable for the design to be rotatable. This makes it easier to compare the values of $\hat{y}$ (x) on a given hyper sphere as all such values have the same variance.

The necessary and sufficient condition for a design to be rotatable was given by Box and Hunter (1957). More recently, Khuri (1996) introduced a measure of rotatability as a function of moments of the design under consideration. The function is expressible as a percentage taking large values for a high degree of rotatability. The value 100 is attained when the design is rotatable. The advantages of this measure are:

1. The ability to compare designs on the basis of rotatability.

2. The assessment of the extent of departure from rotatability of a design whose rotatability may be ‘sacrificed’ to satisfy another desirable design property.

3. The ability to improve rotatability by a proper augmentation of a non-rotatable design.

A rotatable design is said to have the additional uniform precision property if Var[ˆy(x)] at the origin is equal to its value at a distance of one from the origin. This property, which was also introduced by Box and Hunter (1957), provides for an approximate uniform distribution of the prediction variance inside a hyper sphere of radius one. This helps in producing some stability in the prediction variance in the vicinity of the design center.

Box and Hunter (1957) introduced rotatable designs for the exploration of response surface designs. Panda and Das (1994) introduced first order rotatable designs with correlated errors. Das (1997) introduced robust second order rotatable designs (RSORD). Das (1999, 2003b) studied RSORD. In response surface methodology, good estimation of the derivatives of the response function may be as important or perhaps more important than estimation of mean response. Estimation of differences in responses at two different points in the factor space will often be of great importance. If difference in responses at two points close together is of interest then estimation of local slope (rate of change) of the response is required. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses and rate of disintegration of radioactive material in animal.

2.1.3Response Surface Methodology and Second Order Designs

2.1.3.1 Response Surface Methodology

Response Surface Methodology is a basic tool in statistical analysis of experiments where the yield is believed to be influenced by one or more controllable factors. To cut on costs, an experimenter has to make a choice of the experimental design prior to experimentation.

Box and Wilson (1951) discussed experimental designs whose purpose is found using smallest number of observations, the point on a response surface at which the maximum output or yield is achieved. They compared the performances of some experimental designs and introduced the concept of composite designs for the first time. The dominant assumption of Box and Wilson's (1951) paper is that the response can be approximated by a polynomial in the levels of the various treatment factors involved. Different experimental designs are then compared in terms of variance – covariance matrix of the parameter estimates.

According to Dean and Voss (1999), Response Surface Methodology was developed by Box and Wilson (1951) to aid the improvement of manufacturing processes in the chemical industry. The purpose was to optimize chemical reactions to obtain, for example, high yield and purity at low cost. This was accomplished through the use of sequential experimentation involving factors such as temperature, pressure, duration of reaction, and proportion of reactants. The same methodology can be used to model or optimize any response that is affected by the levels of one or more quantitative factors. The general scenario is as follows. The response is a quantitative continuous variable (e.g., yield, purity, cost), and the mean response is a smooth but unknown function of the levels of *p* factors (e.g., temperature, pressure), and the levels are real-valued and accurately controllable. The mean response, when plotted as a function of the treatment combinations, is a surface in *p* + 1 dimensions, called the response surface. Response Surface Methodology comprises a group of techniques for empirical model building and model exploitation. By careful design and analysis of experiments, it seeks to relate a response, or output variable to the level of a number of predictors or output variables that affect it (Box and Draper, 1987).

The objective of obtaining a response surface is two-fold:

(i) To locate a feasible treatment combination ***x*** for which the mean response is maximized (or minimized, or equal to a specific target value); and

(ii) To estimate the response surface in the vicinity of this good location or region, in order to better understand the “local” effects of the factors on the mean response (Dean and Voss, 1999).

2.1.3.2 Second Order Designs

Consider the functional relationship

$y_u = \eta(x_u, \beta) + \varepsilon_u; u = 1, 2, ..., N$ where y_u is the u^{th} observed response value, $\eta(x_u, \beta)$ is a given function, with unknown parameter vector $\beta = (\beta_1, \beta_2, ..., \beta_k)'$, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_N$ are random values corresponding to the observed error and $x_1, x_2, ..., x_N$ are experimental conditions belonging to the set χ usually called the design region. The opportunity to represent the result of real experiments in form $y_u = \eta(x_u \beta) + \varepsilon_u$ has been shown in many examples (Pukelsheim, 2006 and Fedorov, 1972). The function η is usually unknown and its choice is central to the model building process (Box and Draper, 1987). In this thesis, a second – degree polynomial was considered. According to Box and Draper (1987), experiments in which all factors are quantitative frequently take place at or near the maximum or minimum of the response, that is, in the neighborhood of conditions which are optimum according to some criteria. In order to model the curvature present, a full second order model is required.

2.1.4Optimal design theory

The tool that is used to design experiments is the theory of optimum experimental design. The ideas of optimum experimental design are introduced through the comparison of the variances of parameter estimates and the variance of the predicted responses from a variety of designs and models. The relationship between these two sets of variances leads to the general equivalence theorem which, in turn, leads to algorithms for designs and models. The *General Equivalence Theorem* is the central result on which the optimal design of experiments depend (Atkinson and Donev, 1992). The theorem applies to a wide variety of design criteria. It provides methods for the construction and checking of optimum designs. The general equivalence theorem states that the equivalence of the following three conditions on design ξ

(i) The design ξ minimizes $\psi\{M(\xi)\}$

(ii) The design ξ maximizes the minimum over χ of $\phi(x, \xi)$

(iii) The minimum over χ of $\phi(x,\xi)=0$, this minimum occurring at the points of support of the design.

The general equivalence theorem provides necessary and sufficient conditions for a moment matrix to be ϕ- optimal for the parameter system of interest in a compact and convex set of competing moment matrices where ϕ is an information function.

Optimal design theory was initiated by Kiefer (1985). According to him, the experimental design is a discrete probability measure defined by the set of various experimental conditions and weight coefficients corresponding to them. The coefficients show how many experiments (with respect to their total amount) should be performed under the condition. Here, the optimality criteria are represented as various functions defined on the set of information matrices and possessing some statistical sense. A design at which such a functional attains its extremum is called the optimal one.

2.1.5 The Design Efficiency.

Kuhfeld (2010) referred to design efficiency as design goodness. The goodness of an experimental design (efficiency) can be quantified as a function of the variances and covariances of the parameter estimates. Efficiency increases as the variances decrease. Designs should not be thought of in terms of the dichotomy between orthogonal versus non-orthogonal but rather as varying along the continuous attribute of efficiency. Some orthogonal designs are less efficient than other (orthogonal and non-orthogonal) alternatives. Orthogonality is not the primary goal in design creation. It is a secondary goal, associated with the primary goal of minimizing the variances of the parameter estimates. Degree of orthogonality is an important consideration, but other factors should not be ignored.

The goodness or efficiency of an experimental design can be quantified. Common measures of the efficiency of an (ND × p) design matrix X are based on the information matrix X^TX. The variance covariance matrix of the vector of parameter estimates β in a least-squares analysis is proportional to$(X^TX)^{-1}$. More precisely, it equals ∂ 2$(X^TX)^{-1}$. The variance parameter, ∂ 2, is an unknown constant.

2.2 DT- compound optimality Criterion.

2.2.1 Optimality Criteria.

Optimality criteria is based on how well parameters or a response are estimated or researched. Design optimality criteria are primarily concerned with optimal properties of the $X^{/}X$ matrix for the model matrix X. By studying the optimality criteria, the experimenter can determine the adequacy of a proposed experimental design prior to running it. If several alternative designs are proposed, the optimality properties can be compared to aid in the choice of design. The most common empirical model used as an approximation of the true model over the experimental region is a polynomial. The use of the $X^{/}X$ matrix in design evaluation stresses the importance of the assumption that the empirical model is adequate. This implies that the $X^{/}X$ optimality criteria are highly model dependent. Although a design may be best among several designs by one optimality criterion, it may perform poorly when evaluated by a different optimality criterion. Hence, the choice of a design will itself depend upon the choice of the evaluation criteria.

According to Pukelsheim (2006), real optimality criteria are functions with such properties as are appropriate to measure largeness of information matrices. These functions have properties as discussed, that is, positively homogenous, superadditive, non-negative, non-constant and upper semi continuous. Such criteria are called information functions. The most prominent information functions are matrix means $\phi_p, P \in \left[-\infty;1\right]$. They comprise the classical optimality *D*-, *A*-, *E*- and *T*- criteria as special matrix means where our interest is. These criteria are as stated in the methodology. Mutiso (1998) developed theory for the optimum estimation of the free letter parameters in the rotatable design point sets first considered by Draper (1960) for which Kosgei (2002) obtained alphabetic optimality criteria.

CHAPTER THREE: METHODOLOGY.

3.0 Introduction.

In this chapter explanations on determination of D–, A–, E– and T–Relative Efficiencies for the specific second order rotatable designs in three dimensions both for unit values and calculus optimum values designs are given furthermore comparison of the relative efficiencies of calculus optimum values designs with unit value designs given is undertaken thereby demonstrating preferred Relative efficiency criterion. Finally, the DT– optimality Criterion for both calculus optimum and unit value designs is determined. The designs were developed by Draper (1960), Mutiso (1998) used differential calculus and the general equivalence theorem to estimate the free or arbitrary letter parameters with substitution of value one in the free letter parameters being demonstrated by Kosgei (2002) in evaluating the optimality criteria. The methods of evaluation of the particular criteria as given by Pukelsheim (1993) were utilized.

3.1Determination of D–, A–, E– and T–Relative Efficiencies for the Specific Second Order Rotatable designs.

3.1.1The Calculus Optimum values and unit value designs Second Order Rotatable Designs.

This section gives the second order rotatable designs both Calculus optimum values and unit value designs that were used to obtain alphabetic optimality criteria which were used in calculation of relative efficiencies.

3.1.2 Calculus optimum designs.

The Twenty Four Points Three Dimensional Specific Rotatable Design of Order Two

We consider the design

$M_1 = S(f,f,0) + S(c_1,0,0) + S(c_2,0,0)$

This gives the following set of points

(f,f,0)	(f,0,f)	(0,f,f)
(-f,f,0)	(-f,0,f)	(0,-f,f)
(f,-f,0)	(f,0,-f)	(0,f,-f)
(-f,-f,0)	(-f,0,-f)	(0,-f,-f)
$(c_1,0,0)$	$(0,0,c_1)$	$(0,c_1,0)$
$(-c_1,0,0)$	$(0,0,-c_1)$	$(0,-c_1,0)$
$(c_2,0,0)$	$(0,0,c_2)$	$(0,c_2,0)$
$(-c_2,0,0)$	$(0,0,-c_2)$	$(0,-c_2,0)$

The moment conditions that the set of twenty-four points should satisfy to form a rotatable arrangement of order two are:

$$\sum_{u=i}^{24} x_{iu}^2 = 2(c_1^2+c_2^2+4f^2) = 24\lambda_2 \qquad (3.1)$$

$$\sum_{u=i}^{24} x_{iu}^4 = 2(c_1^4+c_2^4+4f^4) = 72\lambda_4 \qquad (3.2)$$

$$\sum_{u=i}^{24} x_{iu}^2 x_{ju}^2 = 4f^4 = 24\lambda_4 \qquad (3.3)$$

Resulting into the following design;

$$M1=\left(S\left(f,f,0\right)+S\left(0.70711067f,0,0\right)+S\left(1.1501633f,0,0\right)\right) .$$

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into $\dot{f}$ =1.1072569 which makes the design M_1 calculus optimal whence;

M_1=(S(1.1072569, 1.1072569, 0)+S(0.7829487,0,0)+S(1.2735263,0,0)).

The Thirty-Two Points Three Dimensional Specific Rotatable Design of Order Two

We consider the design,

M_2= S(p,q,q) + S(a,a,a)

This gives the following set of points

(p,q,q)	(q,q,p)	(q,p,q)	(a,a,a)
(-p,q,q)	(-q,q,p)	(-q,p,q)	(-a,a,a)
(p,-q,q)	(q,-q,p)	(q,-p,q)	(a,-a,a)
(p,q,-q)	(q,q,-p)	(q,p,-q)	(a,a,-a)
(-p,-q,q)	(-q,-q,p)	(-q,-p,q)	(-a,-a,a)
(-p,q,-q)	(-q,q,-p)	(-q,p,-q)	(-a,a,-a)
(p,-q,-q)	(q,-q,-p)	(q,-p,-q)	(a,-a,-a)
(-p,-q,-q)	(-q,-q,-p)	(-q,-p,-q)	(-a,-a,-a)

The moment conditions that the set of thirty-two points should satisfy to form a rotatable arrangement of order two are:

$$\sum_{u=1}^{32} x_{iu}^2 = 8p^2+16q^2+8a^2 = 32\lambda_2 \quad (3.4)$$

$$\sum_{u=1}^{32} x_{iu}^4 = 8p^4+16q^4+8a^4 = 96\lambda_4 \quad (3.5)$$

$$\sum_{u=1}^{32} x_{iu}^2 x_{ju}^2 = 8q^4+16q^2p^2+8a^4 = 32\lambda_4 \quad (3.6)$$

Resulting into the following design;

M2= $\left(S\left(4.472136a, 1.7971477a, 1.7971477a\right) + S\left(a,a,a\right)\right)$.

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into â = 0.2982861which makes the design M_2calculus optimal whence;

M_2 = (S(1.3338955, 0.5360318) + S(0.2982681, 0.2982681, 0.2982681)).

The Twenty-Two Points Three Dimensional Specific Rotatable Design of Order Two

We consider the design,

$M_3=S(a_1,a_1,a_1) + S(a_2,a_2,a_2) + S(c,0,0)$

This gives the following set of twenty-two points

(a_1,a_1,a_1)	(a_2,a_2,a_2)	
$(-a_1,a_1,a_1)$	$(-a_2,a_2,a_2)$	$(c,0,0)$
$(a_1,-a_1,a_1)$	$(a_2,-a_2,a_2)$	$(-c,0,0)$
$(a_1,a_1,-a_1)$	$(a_2,a_2,-a_2)$	$(0,c,0)$
$(-a_1,-a_1,a_1)$	$(-a_2,-a_2,a_2)$	$(0,-c,0)$
$(-a_1,a_1,-a_1)$	$(-a_2,a_2,-a_2)$	$(0,0,c)$
$(a_1,-a_1,-a_1)$	$(a_2,-a_2,-a_2)$	$(0,0,-c)$
$(-a_1,-a_1,-a_1)$	$(-a_2,-a_2,-a_2)$	

The moment conditions, which this set of twenty-two points should satisfy to form a rotatable arrangement of order two are;

$$\sum_{u=i}^{22} x_{iu}^2 = 8a_1^2+8a_2^2+2c^2= 22\lambda_2 \qquad (3.7)$$

$$\sum_{u=i}^{22} x_{iu}^4 = 8a_1^4+8a_2^4+2c^4= 66\lambda_4 \qquad (3.8)$$

$$\sum_{u=i}^{22} x_{iu}^2 x_{ju}^2 = 8a_1^4+8a_2^4= 22\lambda_4 \qquad (3.9)$$

Resulting into the following design;

$$M3=\left(S\left(0.03162277c,0.03162277c,0.03162277c\right)+S\left(0.5823371c,0.5823371c,0.5823371c\right)+S\left(c,0,0\right)\right).$$

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into $\hat{c}$ = 1.5494481 making the design M_3calculus optimal whence**;**

M_3=(S(0.4899784, 0.4899784, 0.4899784) + S(0.9023011, 0.9023011, .9023011)+

S(1.5494481, 0, 0)).

The Twenty Points Three Dimensional Specific Rotatable Design of Order Two

We consider the design,

$M_4=S(a_3,a_3,a_3)+S(c_3,0,0)+S(c_4,0,0)$

This design gives the following set of points

$(c_3,0,0)$	
$(-c_3,0,0)$	
$(0,0,c_3)$	(a_3,a_3,a_3)
$(0,0,-c_3)$	$(-a_3,a_3,a_3)$
$(0,c_3,0)$	$(a_3,-a_3,a_3)$
$(0,-c_3,0)$	$(a_3,a_3,-a_3)$
$(c_4,0,0)$	$(-a_3,-a_3,a_3)$
$(-c_4,0,0)$	$(-a_3,a_3,-a_3)$
$(0,0,c_4)$	$(a_3,-a_3,-a_3)$
$(0,0,-c_4)$	$(-a_3,-a_3,-a_3)$
$(0,c_4,0)$	
$(0,-c_4,0)$	

The moment conditions that the set of points should satisfy to form a rotatable arrangement of order two are:

$$\sum_{u=1}^{20} x_{iu}^2 = 8a_3^2+2c_3^2+2c_4^2=20\lambda_2 \quad (3.10)$$

$$\sum_{u=1}^{20} x_{iu}^4 = 8a_3^4+2c_3^4+2c_4^4=60\lambda_4 \quad (3.11)$$

$$\sum_{u=1}^{20} x_{iu}^2 x_{ju}^2 = 8a_3^4=20\lambda_4 \quad (3.12)$$

Resulting into the following design;

$$M4=\left(S\left(a_3,a_3,a_3\right)+S\left(0.8944271a_3,0,0\right)+S\left(1.6470981,0,0\right)\right).$$

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into$\hat{a}_3$ = 0.6277576 which makes the design M_4 calculus optimal whence;

M_4=(s(0.6277576, 0.6277576, 0.6277576)+s(0.5614834, 0,0)+s(1.0339784,0,0)).

Twenty Six Points Three Dimensional Specific Rotatable Design of Order Two

We consider the design.

M_5= S(f,f,0)+S(a_4,a_4,a_4)+S(c_5,0,0)

The design gives the following set of points

(f,f,0)	(f,0,f)	(0,f,f)
(-f,f,0)	(-f,0,f)	(0,-f,f)
(f,-f,0)	(f,0,-f)	(0,f,-f)
(-f,-f,0)	(-f,0,-f)	(0,-f,-f)
(a_4,a_4,a_4)	(-a_4,-a_4,-a_4)	(c_5,0,0)
(-a_4,a_4,a_4)	(-a_4,-a_4,a_4)	(-c_5,0,0)
(a_4,-a_4,a_4)	(-a_4,a_4,-a_4)	(0,0,c_5)
(a_4,a_4,-a_4)	(a_4,-a_4,-a_4)	(0,0,-c_5)
(0,c_5,0)	(0,-c_5,0)	

The moment conditions which the set of twenty six points should satisfy to form a rotatable arrangement of order two are:

$$\sum_{u=1}^{26} x_{iu}^2 = 8f^2+8a_4^2+2c_5^2=26\lambda_2 \tag{3.13}$$

$$\sum_{u=1}^{26} x_{iu}^4 = 8f^4+8a_4^4+2c_5^4=78\lambda_4 \tag{3.14}$$

$$\sum_{u=1}^{26} x_{iu}^2 x_{ju}^2 = 4f^4+8a_4^4=26\lambda_4 \tag{3.15}$$

Resulting into the following design;

$M5=(S(0.7162612a_4, 0.7162612a_4, 0) + S(a_4, a_4, a_4) + S(1.6470981a_4, 0, 0))$.

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into $\hat{a}_4$ =0.9359294which makes the design M_5calculus optimal whence;

$M_5=(S(0.6703699, 0.6703699, 0) + S(0.9359294, 0.9359294, 0.9359294) + S(1.5993168, 0, 0))$.

Thirty points three dimensional specific rotatable design of order two

We consider the design,

$M_6=S(p,q,q)+S(c_6,0,0)$

This gives the following set of points

$(c_6,0,0)$ $(0,0,c_6)$ $(0,c_6,0)$

$(-c_6,0,0)$ $(0,0,-c_6)$ $(0,-c_6,0)$

$(p_1\ q_1\ q_1)$ $(q_1 q_1\ p_1)$ $(q_1\ p_1\ q_1)$

$(-p_1\ q_1\ q_1)$ $(-q_1 q_1\ p_1)$ $(-q_1\ p_1\ q_1)$

$(p_1\ -q_1\ q_1)$ $(q_1\ -q_1\ p_1)$ $(q_1\ -p_1\ q_1)$

$(p_1\ q_1\ -q_1)$ $(q_1 q_1\ -p_1)$ $(q_1\ p_1\ -q_1)$

$(-p_1\ -q_1\ q_1)$ $(-q_1\ -q_1\ p_1)$ $(-q_1\ -p_1\ q_1)$

$(-p_1\ q_1\ -q_1)$ $(-q_1 q_1\ -p_1)$ $(-q_1\ p_1\ -q_1)$

$(p_1\ -q_1\ -q_1)$ $(q_1\ -q_1\ -p_1)$ $(q_1\ -p_1\ -q_1)$

$(-p_1\ -q_1\ -q_1)$ $(-q_1\ -q_1\ -p_1)$ $(-q_1\ -p_1\ -q_1)$

The moment conditions for the set of the thirty points to form a rotatable arrangement of order two are:

$$\sum_{u=1}^{30} x_{iu}^2 = 8p_1^2+16q_1^2+2c_6^2=30\lambda_2 (4.7) \qquad (3.16)$$

$$\sum_{u=1}^{30} x_{iu}^4 \quad =3= 8p_1^4+16q_1^4+2c_6^2=90\lambda_4$$

(3.17)

$$\sum_{u=1}^{30} x_{iu}^2 x_{ju}^2 \quad =8q_1^4+16p_1^2q_1^2= 30\lambda_4$$

(3.18)

For i≠j=1,2,3 with all sums of powers and products up to and including power four being zero.

The excess of $\sum_{u=1}^{30} x_{iu}^4 \; =3 \; \sum_{u=1}^{30} x_{iu}^2 x_{ju}^2$

is given by

Resulting into the following design;

$$M6=\left(S\left(3.8729833c_6,1.5610253c_6,\ 1.5610253c_6\right) +S\left(c_6,0,0\right)\right).$$

The free letter parameter in the twenty four points design was estimated using the differential calculus and general equivalence theorem resulting into $\hat{c}_6$ =0.3357566which makes the design M_6calculus optimal whence;

M_6=(S(1.3003797, 0.5241245, 0.5241245) +S(0.3357566,0,0)).

3.1.3 Unit Value Designs.

When the unit values were substituted in the free or arbitrary letter parameters the following designs were obtained from Mutiso (1998).

$$M1 = \left(S\left(1,1,0\right) + S\left(0.7071067,0,0\right) + S\left(1.1501633,0,0\right)\right)$$
$$M2 = \left(S\left(4.472136,\ 1.7971477,1.7971477\right) + S\left(1,1,1\right)\right)$$
$$M3 = \left(S\left(0.3162277,0.3162277,0.3162277\right) + S\left(0.5823371,0.5823371,0.5823371\right)+S\left(1,0,0\right)\right)$$
$$M4 =\left(S\left(1,1,1\right)+S\left(0.8944271,0,0\right)+S\left(1.6470981,0,0\right)\right)$$
$$M5=\left(S\left(0.7162612,\ 0.\ 7162612,\ 0\right) +S\left(1,1,1\right) + S\left(1.7088007,\ 0,\ 0\right)\right)$$
$$M6=\left(S\left(3.8729833,\ 1.5610253,\ 1.5610253\right) +S\left(1,0,0\right)\right).$$

3.1.4 The Relative Efficiencies Method.

Here the methods used in evaluating the relative efficiencies of the designs are given. The calculus optimum and unit values obtained in Mutiso (1998) for the free letter parameters are used in evaluating the relative efficiencies.

3.1.4.1 D-efficiency.

This measure is related to the D-optimality criterion, the D- efficiency is evaluated by the formula;

$$\mathbf{D}(\xi) = \left[\frac{M(\xi)}{M^{¿}(\xi)} \right]$$

Where $M(\xi)$ is the determinant of the information matrix and $\xi_D^{¿}$ is D optimal

3.1.4.2 A-efficiency.

This measure is related to the A-optimality criterion and to calculate this efficiency the formula below is used

$$\mathbf{A}(\xi) = \frac{tr\left(m^{-1}\left(\xi_A^{¿}\right)\right)}{tr\left(m^{-1}(\xi)\right)}$$

Where A(ξ) is the average variance of the information matrix and $\xi_A^{¿}$ is A optimal.

3.1.4.3 T-efficiency.

This measure is related to the T-optimality criterion, the formula below is used in calculating T- efficiency

$$\mathbf{T}(\xi) = \frac{\Delta_1(\xi)}{\Delta_1(\xi_T^{¿})}$$

where $T(\xi)$ is the trace of the information matrix and $\xi_T^{¿}$ is T-optimal

3.1.4.4 E-efficiency.

This measure is related to the E-optimality criterion and its efficiency is given by;

$$E(\xi) = \frac{\lambda_{min}(\xi)}{\lambda_{min}(\xi^{¿})}$$

Where $\lambda_{min}(\xi)$ is the Eigenvalue of the information matrix and $\xi_E^{¿}$ is E-optimal

3.2 Comparison of the relative efficiencies of calculus optimum values designs with Unit Value Designs.

After the evaluation of the D–, A–, E– and T–Relative Efficiencies for the Specific Second Order rotatable designs both calculus optimum and unit values, the values obtained are expressed as a percentage and the best relative efficiency criterion demonstrated, the higher the percentage the more efficient a design is. Designs that produce average relative efficiency value close to 100% are the most efficient designs.

3.3 The DT- Optimality Criterion.

Atkinson (2008) introduced DT–optimality which is a combination of D– optimality and T–optimality for discriminating between models. It provides a specified balance between model discrimination and parameter estimation. The criterion to be maximized is

$$\Delta_2(¿\xi)+\left(\frac{K}{p}\right)\log\left|M_1(\xi)\right|$$
$$\phi^{DT}(\xi)=(1-k)\log¿$$

Where $\phi^{DT}(\xi)$ a convex combination of two design criteria, the first criterion is

$\Delta_2(¿\xi)$
$\log¿$, the logarithm of the T–optimality and the second is D–optimality. Then the designs which maximize the above criterion are called DT–optimum and are denoted by $\xi_{DT}^{¿}$. K is the degree of the design. In this research K=3 and p is the number of parameters and for this research P = 7.

CHAPTER FOUR: RESULTS AND DISCUSSIONS.

4.0 Introduction.

In this chapter the D–, A–, E– and T–Relative Efficiencies for the six Specific second order rotatable designs in three dimensions both for calculus optimum values and unit value designs was determined, the relative efficiencies of calculus optimum values designs were compared with relative efficiencies of unit value designs and finally to determine the DT– optimality Criterion for both calculus optimum and unit value designs is determined.

4.1 Determination of D–, A–, E– and T–Relative Efficiencies for the Specific Second Order.

In this section the relative efficiencies for both unit values and calculus optimum values are evaluated using the alphabetic optimum criterion values earlier evaluated

4.1.1The Calculus optimum values Designs.

4.1.1.1The particular optimality for the Twenty Four Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_1=(S(1.1072569, 1.1072569, 0)+S(0.7829487,0,0)+S(1.2735263,0,0))

The design matrix X is given by

1.0000	1.1100	1.1100	0	1.2300	1.2300	0
1.0000	-1.1100	1.1100	0	1.2300	1.2300	0
1.0000	1.1100	-1.1100	0	1.2300	1.2300	0
1.0000	-1.1100	-1.1100	0	1.2300	1.2300	0
1.0000	0.7800	0	0	0.6100	0	0
1.0000	-0.7800	0	0	0.6100	0	0
1.0000	1.2700	0	0	1.6100	0	0
1.0000	-1.2700	0	0	1.6100	0	0
1.0000	1.1100	0	1.1100	1.2300	0	1.2300
1.0000	-1.1100	0	1.1100	1.2300	0	1.2300
1.0000	1.1100	0	-1.1100	1.2300	0	1.2300
1.0000	-1.1100	0	-1.1100	1.2300	0	1.2300
1.0000	0	0	0.7800	0	0	0.6100
1.0000	0	0	-0.7800	0	0	0.6100
1.0000	0	0	1.2700	0	0	1.6100
1.0000	0	0	-1.2700	0	0	1.6100
1.0000	0	1.1100	1.1100	0	1.2300	1.2300
1.0000	0	-1.1100	1.1100	0	1.2300	1.2300
1.0000	0	1.1100	-1.1100	0	1.2300	1.2300
1.0000	0	-1.1100	-1.1100	0	1.2300	1.2300
1.0000	0	0.7800	0	0	0.6100	0
1.0000	0	- 0.7800	0	0	0.6100	0
1.0000	0	1.2700	0	0	1.6100	0
1.0000	0	-1.2700	0	0	1.6100	0

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The information matrix c_1 for the design M_1 is given by

$c_1 =$ ¿

1.0000	0.5949	0.5949	0.5949	0	0	0
0.5949	0.7516	0.2505	0.2505	0	0	0
0.5949	0.2505	0.7516	0.2505	0	0	0
0.5949	0.2505	0.2505	0.7516	0	0	0
0	0	0	0	0.5949	0	0
0	0	0	0	0	0.5949	0
0	0	0	0	0	0	0.5949

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D-optimality

This criterion is obtained by finding the determinant of the information matrix c_1 given by the formula $\emptyset_O$ (C) = (det C) $^{1/7}$andfor this design the D-Criterion gives 0.5184684.

T-optimality

This criterion is obtained by finding the trace of the information matrix c_1 the trace is given by the formula $\varphi_1(C)=\frac{1}{s}trace(C)$ and for this design the T-Criterion gives 0.71991756.

E-optimality

This criterion is obtained by finding the smallest eigen value of the information matrix c_1 given by the formula Ø-∞(C)= λmin(C) and for this design the E-Criterion gives 0.08817.

A-optimality

This criterion is obtained by finding the maximum average- variance of the information matrix c_1 given by the formula Ø-1 (C)=1/s(trace c-1)-1 if C is positive definite and for this design the A-Criterion gives 0.00685555.

4.1.1.2 The particular optimality for the Thirty-Two Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_2 = (S(1.3338955, 0.5360318) + S(0.2982681, 0.2982681, 0.2982681))

The design matrix X is given by

1.0000	1.3300	0.5400	0.5400	1.7700	0.2900	0.2900
1.0000	-1.3300	0.5400	0.5400	1.7700	0.2900	0.2900
1.0000	1.3300	-0.5400	0.5400	1.7700	0.2900	0.2900
1.0000	1.3300	0.5400	-0.5400	1.7700	0.2900	0.2900
1.0000	-1.3300	-0.5400	0.5400	1.7700	0.2900	0.2900
1.0000	-1.3300	0.5400	-0.5400	1.7700	0.2900	0.2900
1.0000	1.3300	-0.5400	-0.5400	1.7700	0.2900	0.2900
1.0000	-1.3300	-0.5400	-0.5400	1.7700	0.2900	0.2900
1.0000	0.5400	0.5400	1.3300	0.2900	0.2900	1.7700
1.0000	-0.5400	0.5400	1.3300	0.2900	0.2900	1.7700
1.0000	0.5400	-0.5400	1.3300	0.2900	0.2900	1.7700
1.0000	0.5400	0.5400	-1.3300	0.2900	0.2900	1.7700
1.0000	-0.5400	-0.5400	1.3300	0.2900	0.2900	1.7700
1.0000	-0.5400	0.5400	-1.3300	0.2900	0.2900	1.7700
1.0000	0.5400	-0.5400	-1.3300	0.2900	0.2900	1.7700
1.0000	-0.5400	-0.5400	-1.3300	0.2900	0.2900	1.7700
1.0000	0.5400	1.3300	0.5400	0.2900	1.7700	0.2900
1.0000	-0.5400	1.3300	0.5400	0.2900	1.7700	0.2900
1.0000	0.5400	-1.3300	0.5400	0.2900	1.7700	0.2900
1.0000	0.5400	1.3300	-0.5400	0.2900	1.7700	0.2900
1.0000	-0.5400	-1.3300	0.5400	0.2900	1.7700	0.2900
1.0000	-0.5400	1.3300	-0.5400	0.2900	1.7700	0.2900
1.0000	0.5400	-1.3300	-0.5400	0.2900	1.7700	0.2900
1.0000	-0.5400	-1.3300	-0.5400	0.2900	1.7700	0.2900
1.0000	0.3000	0.3000	0.3000	0.0900	0.0900	0.0900
1.0000	-0.3000	0.3000	0.3000	0.0900	0.0900	0.0900
1.0000	0.3000	-0.3000	0.3000	0.0900	0.0900	0.0900
1.0000	0.3000	0.3000	-0.3000	0.0900	0.0900	0.0900
1.0000	-0.3000	-0.3000	0.3000	0.0900	0.0900	0.0900
1.0000	-0.3000	0.3000	-0.3000	0.0900	0.0900	0.0900
1.0000	0.3000	-0.3000	-0.3000	0.0900	0.0900	0.0900
1.0000	-0.3000	-0.3000	-0.3000	0.0900	0.0900	0.0900

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The information matrix c_2 for the design M_2 is given by

$$c_2 = ¿ \begin{matrix} 1.0000 & 0.6107 & 0.6107 & 0.6107 & 0 & 0 & 0 \\ 0.6107 & 0.8347 & 0.2782 & 0.2782 & 0 & 0 & 0 \\ 0.6107 & 0.2782 & 0.8347 & 0.2782 & 0 & 0 & 0 \\ 0.6107 & 0.2782 & 0.2782 & 0.8347 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6107 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6107 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6107 \end{matrix}$$

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The optimum values for this design are;

D-optimality = 0.588535

T-optimality= 0.7623315.

E-optimality =0.119857.

A-optimality = 0.00826251.

4.1.1.3 The particular optimality criteria for the Twenty-Two Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_3=(S(0.4899784, 0.4899784, 0.4899784) + S(0.9023011, 0.9023011, .9023011)+ S(1.5494481, 0, 0))

The design matrix X of M_3 is given by

1.0000	0.4900	0.4900	0.4900	0.2400	0.2400	0.2400
1.0000	-0.4900	0.4900	0.4900	0.2400	0.2400	0.2400
1.0000	0.4900	-0.4900	0.4900	0.2400	0.2400	0.2400
1.0000	0.4900	0.4900	-0.4900	0.2400	0.2400	0.2400
1.0000	-0.4900	-0.4900	0.4900	0.2400	0.2400	0.2400
1.0000	-0.4900	0.4900	-0.4900	0.2400	0.2400	0.2400
1.0000	0.4900	-0.4900	-0.4900	0.2400	0.2400	0.2400
1.0000	-0.4900	-0.4900	-0.4900	0.2400	0.2400	0.2400
1.0000	0.9000	0.9000	0.9000	0.8100	0.8100	0.8100
1.0000	-0.9000	0.9000	0.9000	0.8100	0.8100	0.8100
1.0000	0.9000	-0.9000	0.9000	0.8100	0.8100	0.8100
1.0000	0.9000	0.9000	-0.9000	0.8100	0.8100	0.8100
1.0000	-0.9000	-0.9000	0.9000	0.8100	0.8100	0.8100
1.0000	-0.9000	0.9000	-0.9000	0.8100	0.8100	0.8100
1.0000	0.9000	-0.9000	-0.9000	0.8100	0.8100	0.8100
1.0000	-0.9000	-0.9000	-0.9000	0.8100	0.8100	0.8100
1.0000	1.5500	0	0	2.4000	0	0
1.0000	-1.5500	0	0	2.4000	0	0
1.0000	0	1.5500	0	0	2.4000	0
1.0000	0	-1.5500	0	0	2.4000	0
1.0000	0	0	1.5500	0	0	2.4000
1.0000	0	0	-1.5500	0	0	2.4000

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The information matrix c_3 for the design M_3 is given by

$c_3=¿$

1.0000	0.6016	0.6016	0.6016	0	0	0
0.6016	0.7860	0.2620	0.2620	0	0	0
0.6016	0.2620	0.7860	0.2620	0	0	0
0.6016	0.2620	0.2620	0.7860	0	0	0
0	0	0	0	0.6016	0	0
0	0	0	0	0	0.6016	0
0	0	0	0	0	0	0.6016

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After evaluating the various criteria the following values were obtained

D-optimality = 0.540062.

T-optimality = 0.737534.

E-optimality = 0.101498.

A-optimality = 0.36672210.

4.1.1.4 The particular optimality criteria for the Twenty Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_4=(s(0.6277576, 0.6277576, 0.6277576)+s(0.5614834, 0,0)+s(1.0339784,0,0))

The design matrix X of M_4 is given by

1.0000	0.5600	0	0	0.3100	0	0
1.0000	-0.5600	0	0	0.3100	0	0
1.0000	0	0	0.5600	0	0	0.3100
1.0000	0	0	-0.5600	0	0	0.3100
1.0000	0	0.5600	0	0	0.3100	0
1.0000	0	-0.5600	0	0	0.3100	0
1.0000	1.0300	0	0	1.0600	0	0
1.0000	-1.0300	0	0	1.0600	0	0
1.0000	0	0	1.0300	0	0	1.0600
1.0000	0	0	-1.0300	0	0	1.0600
1.0000	0	1.0300	0	0	1.0600	0
1.0000	0	-1.0300	0	0	1.0600	0
1.0000	0.6300	0.6300	0.6300	0.4000	0.4000	0.4000
1.0000	-0.6300	0.6300	0.6300	0.4000	0.4000	0.4000
1.0000	0.6300	-0.6300	0.6300	0.4000	0.4000	0.4000
1.0000	0.6300	0.6300	-0.6300	0.4000	0.4000	0.4000
1.0000	-0.6300	-0.6300	0.6300	0.4000	0.4000	0.4000
1.0000	-0.6300	0.6300	-0.6300	0.4000	0.4000	0.4000
1.0000	0.6300	-0.6300	-0.6300	0.4000	0.4000	0.4000
1.0000	-0.6300	-0.6300	-0.6300	0.4000	0.4000	0.4000

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The information matrix c_4 for the design M_4 is given by

$$c_4 = \begin{bmatrix} 1.0000 & 0.2961 & 0.2961 & 0.2961 & 0 & 0 & 0 \\ 0.2961 & 0.1864 & 0.0621 & 0.0621 & 0 & 0 & 0 \\ 0.2961 & 0.0621 & 0.1864 & 0.0621 & 0 & 0 & 0 \\ 0.2961 & 0.0621 & 0.0621 & 0.1864 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2961 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2961 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.2961 \end{bmatrix}$$

The following are the various optimum values for this design

D-optimality = 0.787020.

T-optimality= 0.3496119.

E-optimality = 0.037407.

A-optimality = 0.13023463.

4.1.1.5 The particular optimality criteria for the Twenty Six Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_5=(S(0.6703699, 0.6703699, 0) +S(0.9359294, 0.9359294, 0.9359294) + S(1.5993168, 0, 0))

The design matrix X of M_5 is given by

1.0000	0.6700	0.6700	0	0.4500	0.4500	0
1.0000	-0.6700	0.6700	0	0.4500	0.4500	0
1.0000	0.6700	-0.6700	0	0.4500	0.4500	0
1.0000	-0.6700	-0.6700	0	0.4500	0.4500	0
1.0000	0.9400	0.9400	0.9400	0.8800	0.8800	0.8800
1.0000	-0.9400	0.9400	0.9400	0.8800	0.8800	0.8800
1.0000	0.9400	-0.9400	0.9400	0.8800	0.8800	0.8800
1.0000	0.9400	0.9400	-0.9400	0.8800	0.8800	0.8800
1.0000	0	1.6000	0	0	2.5600	0
1.0000	0.6700	0	0.6700	0.4500	0	0.4500
1.0000	-0.6700	0	0.6700	0.4500	0	0.4500
1.0000	0.6700	0	-0.6700	0.4500	0	0.4500
1.0000	-0.6700	0	-0.6700	0.4500	0	0.4500
1.0000	-0.9400	-0.9400	-0.9400	0.8800	0.8800	0.8800
1.0000	-0.9400	-0.9400	0.9400	0.8800	0.8800	0.8800
1.0000	-0.9400	0.9400	-0.9400	0.8800	0.8800	0.8800
1.0000	0.9400	-0.9400	-0.9400	0.8800	0.8800	0.8800
1.0000	0	-1.6700	0	0	2.5600	0
1.0000	0	0.6700	0.6700	0	0.4500	0.4500
1.0000	0	-0.6700	0.6700	0	0.4500	0.4500
1.0000	0	0.6700	-0.6700	0	0.4500	0.4500
1.0000	0	-0.6700	-0.6700	0	0.4500	0.4500
1.0000	1.6000	0	0	2.5600	0	0
1.0000	-1.6000	0	0	2.5600	0	0
1.0000	0	0	1.6000	0	0	2.5600
1.0000	0	0	-1.6000	0	0	2.5600

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The information matrix c_5 for the design M_5 is given by

$$c_5 = \begin{bmatrix} 1.0000 & 0.6046 & 0.6046 & 0.6046 & 0 & 0 & 0 \\ 0.6046 & 0.8015 & 0.2672 & 0.2672 & 0 & 0 & 0 \\ 0.6046 & 0.2672 & 0.8015 & 0.2672 & 0 & 0 & 0 \\ 0.6046 & 0.2672 & 0.2672 & 0.8015 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6046 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6046 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6046 \end{bmatrix}$$

The various optimum criteria for this design yields the following values

D-optimality= 0.549356.

T-optimality = 0.7454531.

E-optimality = 0.107413.

A-optimality = 0.13023463.

4.1.1.6 The particular optimality criteria for the Thirty Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_6=(S (1.3003797, 0.5241245, 0.5241245) +S(0.3357566,0,0))

The design matrix X is given by

1.0000	0.3400	0	0	0.1200	0	0
1.0000	-0.3400	0	0	0.1200	0	0
1.0000	1.3000	0.5200	0.5200	1.6900	0.2700	0.2700
1.0000	-1.3000	0.5200	0.5200	1.6900	0.2700	0.2700
1.0000	1.3000	-0.5200	0.5200	1.6900	0.2700	0.2700
1.0000	1.3000	0.5200	-0.5200	1.6900	0.2700	0.2700
1.0000	-1.3000	-0.5200	0.5200	1.6900	0.2700	0.2700
1.0000	-1.3000	0.5200	-0.5200	1.6900	0.2700	0.2700
1.0000	1.3000	-0.5200	-0.5200	1.6900	0.2700	0.2700
1.0000	-1.3000	-0.5200	-0.5200	1.6900	0.2700	0.2700
1.0000	0	0	0.3400	0	0	0.1200
1.0000	0	0	-0.3400	0	0	0.1200
1.0000	0.5200	0.5200	1.3000	0.2700	0.2700	1.6900
1.0000	-0.5200	0.5200	1.3000	0.2700	0.2700	1.6900
1.0000	0.5200	-0.5200	1.3000	0.2700	0.2700	1.6900
1.0000	0.5200	0.5200	-1.3000	0.2700	0.2700	1.6900
1.0000	-0.5200	-0.5200	1.3000	0.2700	0.2700	1.6900
1.0000	-0.5200	0.5200	-1.3000	0.2700	0.2700	1.6900
1.0000	0.5200	-0.5200	-1.3000	0.2700	0.2700	1.6900
1.0000	-0.5200	-0.5200	-1.3000	0.2700	0.2700	1.6900
1.0000	0	0.3400	0	0	0.1200	0
1.0000	0	-0.3400	0	0	0.1200	0
1.0000	0.5200	1.3000	0.5200	0.2700	1.6900	0.2700
1.0000	-0.5200	1.3000	0.5200	0.2700	1.6900	0.2700
1.0000	0.5200	-1.3000	0.5200	0.2700	1.6900	0.2700
1.0000	0.5200	1.3000	-0.5200	0.2700	1.6900	0.2700
1.0000	-0.5200	-1.3000	0.5200	0.2700	1.6900	0.2700
1.0000	-0.5200	1.3000	-0.5200	0.2700	1.6900	0.2700
1.0000	0.5200	-1.3000	-0.5200	0.2700	1.6900	0.2700
1.0000	-0.5200	-1.3000	-0.5200	0.2700	1.6900	0.2700

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The information matrix c_6 for the design M_6 is given by

$$c_6 = \begin{bmatrix} 1.0000 & 0.6050 & 0.6050 & 0.6050 & 0 & 0 & 0 \\ 0.6050 & 0.8036 & 0.2679 & 0.2679 & 0 & 0 & 0 \\ 0.6050 & 0.2679 & 0.8036 & 0.2679 & 0 & 0 & 0 \\ 0.6050 & 0.2679 & 0.2679 & 0.8036 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6050 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6050 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6050 \end{bmatrix}$$

This design gives the following optimum values

D-optimality= 0.550390.

T-optimality= 0.746528.

E-optimality = 0.108414.

A-optimality=0.38081157.

4.1.2 Relative Efficiency for Calculus values optimum designs.

Table 4.1: A summary of calculus Optimum Values from Calculus Optimum Designs.

	A	D	E	T
M_1	0.00685555	0.518684	0.08817	0.719916756
M_2	0.00826251	0.588535	0.119857	0.7623315
M_3	0.36672210	0.540062	0.101498	0.737534
M_4	0.13023463	0.536882	0.037407	0.3496119
M_5	0.37913341	0.549356	0.107413	0.7454531
M_6	0.380881157	0.550390	0.108214	0.746528

4.1.2.1 A-efficiency.

This measure is related to the A-optimality criterion and evaluated as:

$$A(\xi) = \frac{tr\left(m^{-1}\left(\varepsilon_A^{¿}\right)\right)}{tr\left(m^{-1}(\varepsilon)\right)}$$

Table 4.2 Calculus optimum values A-efficiencies.

M_1	$= \frac{0.00685555}{0.00685555}$ = 100%
M_2	$= \frac{0.00685555}{0.00826251}$ = 82.972%
M_3	$= \frac{0.00685555}{0.36672210}$ = 1.869%
M_4	$= \frac{0.00685555}{0.13023463}$ = 5.264%
M_5	$= \frac{0.00685555}{0.37913341}$ = 1.808%
M_6	$= \frac{0.00685555}{0.380881157}$ = 1.800%

From the table above M_1 is the most efficient design

4.1.2.2 D-efficiency.

This measure is related to the D-optimality criterion:

$$D(\xi) = \left[\frac{M(\varepsilon)}{M^{¿}(\varepsilon)} \right]$$

Table 4.3 Calculus optimum values D-efficiencies.

M_1	$= \frac{0.51684}{0.51684}$ = 100%
M_2	$= \frac{0.51684}{0.568535}$ = 90.907%
M_3	$= \frac{0.51684}{0.540062}$ = 95.700%
M_4	$= \frac{0.51684}{0.536882}$ = 96.267%
M_5	$= \frac{0.51684}{0.549356}$ = 99.516%
M_6	$= \frac{0.51684}{0.550390}$ = 93.904%

From the table above M_1 is the most efficient design

4.1.2.3 T-efficiency.

This measure is related to the T-optimality criterion:

$$T(\xi) = \frac{\Delta_1(\varepsilon)}{\Delta_1(\varepsilon_T^{*})}$$

Table 4.4 Calculus optimum T-efficiencies.

M_1	$= \frac{0.3496119}{0.719916756} = 47.563\%$
M_2	$= \frac{0.3496119}{0.7623315} = 45.861\%$
M_3	$= \frac{0.3496119}{0.737534} = 47.403\%$
M_4	$= \frac{0.3496119}{0.3496119} = 100\%$
M_5	$= \frac{0.3496119}{0.7454531} = 46.899\%$
M_6	$= \frac{0.3496119}{0.746528} = 46.832\%$

4.1.2.4 E-efficiency.

This measure is related to the E-optimality criterion:

$$E(\xi) = \frac{\lambda_{min}(\varepsilon)}{\lambda_{min}(\varepsilon^{*})}$$

where $\lambda_{min}(\varepsilon)$ is the Eigen value of the information matrix.

Table 4.5 Calculus optimum values E-efficiencies.

M_1	$= \frac{0.037407}{0.08817} = 42.426\%$
M_2	$= \frac{0.037407}{0.119857} = 31.210\%$
M_3	$= \frac{0.037407}{0.101498} = 36.855\%$
M_4	$= \frac{0.037407}{0.037407} = 100\%$
M_5	$= \frac{0.037407}{0.107413} = 34.825\%$
M_6	$= \frac{0.037407}{0.108214} = 34.568\%$

From the table above M_4 is the most efficient design

4.1.3The Unit Value Designs.

4.1.3.1 The particular optimality for the Twenty Four Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_1=(S(1, 1, 0)+S(0.7071067,0,0)+S(1.1501633,0,0))

The design matrix for M_1 is given by;

1	1	1	0	1	1	0
1	-1	1	0	1	1	0
1	1	-1	0	1	1	0
1	-1	-1	0	1	1	0
1	0.71	0	0	0.5	0	0
1	-0.71	0	0	0.5	0	0
1	1.15	0	0	1.32	0	0
1	-1.15	0	0	1.32	0	0
1	1	0	1	1	0	1
1	-1	0	1	1	0	1
1	1	0	-1	1	0	1
1	-1	0	-1	1	0	1
1	0	0	0.71	0	0	0.5
1	0	0	-0.71	0	0	0.5
1	0	0	1.15	0	0	1.32
1	0	0	-1.15	0	0	1.32
1	0	1	1	0	1	1
1	0	-1	1	0	1	1
1	0	1	-1	0	1	1
1	0	-1	-1	0	1	1
1	0	0.71	0	0	0.5	0
1	0	-0.71	0	0	0.5	0
1	0	1.15	0	0	1.32	0
1	0	-1.15	0	0	1.32	0

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$$C_1 = \begin{bmatrix} 1 & 0.4852 & 0.4852 & 0.4852 & 0 & 0 & 0 \\ 0.4852 & 0.5 & 0.1667 & 0.1667 & 0 & 0 & 0 \\ 0.4852 & 0.1667 & 0.5 & 0.1667 & 0 & 0 & 0 \\ 0.4852 & 0.1667 & 0.1667 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4852 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4852 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4852 \end{bmatrix}$$

D-optimality

This criterion is obtained by finding the determinant of the information matrix c_1 given by the formula $\emptyset_O$ (C) = (det C) $^{1/7}$ and for this design the D-Criterion gives0.398647.

T-optimality

This criterion is obtained by finding the trace of the information matrix c_1 the trace is given by the formula $\varphi_1(C) = \frac{1}{s} trace(C)$ and for this design the T-Criterion gives0.565103.

E-optimality

This criterion is obtained by finding the smallest eigen value of the information matrix c_1 given by the formula $\emptyset_{-\infty}(C) = \lambda_{min}(C)$ and for this design the E-Criterion gives0.072085.

A-optimality

This criterion is obtained by finding the maximum average- variance of the information matrix c_1 given by the formula $\emptyset_{-1}$ (C)=1/s(trace $c^{-1})^{-1}$ if C is positive definite and for this design the A-Criterion gives0.005366.

4.1.3.2 The particular optimality For the Thirty-Two Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_2 = (S(4.472136, 1.7971477,1.7971477) + S(1,1,1))

The design matrix X for M_2isgiven by

1	1.33	0.54	0.54	1.77	0.29	0.29
1	-1.33	0.54	0.54	1.77	0.29	0.29
1	1.33	-0.54	0.54	1.77	0.29	0.29
1	1.33	0.54	-0.54	1.77	0.29	0.29
1	-1.33	-0.54	0.54	1.77	0.29	0.29
1	-1.33	0.54	-0.54	1.77	0.29	0.29
1	1.33	-0.54	-0.54	1.77	0.29	0.29
1	-1.33	-0.54	-0.54	1.77	0.29	0.29
1	0.54	0.54	1.33	0.29	0.29	1.77
1	-0.54	0.54	1.33	0.29	0.29	1.77
1	0.54	-0.54	1.33	0.29	0.29	1.77
1	0.54	0.54	-1.33	0.29	0.29	1.77
1	-0.54	-0.54	1.33	0.29	0.29	1.77
1	-0.54	0.54	-1.33	0.29	0.29	1.77
1	0.54	-0.54	-1.33	0.29	0.29	1.77
1	-0.54	-0.54	-1.33	0.29	0.29	1.77
1	0.54	1.33	0.54	0.29	1.77	0.29
1	-0.54	1.33	0.54	0.29	1.77	0.29
1	0.54	-1.33	0.54	0.29	1.77	0.29
1	0.54	1.33	-0.54	0.29	1.77	0.29
1	-0.54	-1.33	0.54	0.29	1.77	0.29
1	-0.54	1.33	-0.54	0.29	1.77	0.29
1	0.54	-1.33	-0.54	0.29	1.77	0.29
1	-0.54	-1.33	-0.54	0.29	1.77	0.29
1	0.30	0.30	0.30	0.09	0.09	0.09
1	-0.30	0.30	0.30	0.09	0.09	0.09
1	0.30	-0.30	0.30	0.09	0.09	0.09
1	0.30	0.30	-0.30	0.09	0.09	0.09
1	-0.30	-0.30	0.30	0.09	0.09	0.09
1	-0.30	0.30	-0.30	0.09	0.09	0.09
1	0.30	-0.30	-0.30	0.09	0.09	0.09
1	-0.30	-0.30	-0.30	0.09	0.09	0.09

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and its information matrix C_2 is;

$$\begin{bmatrix} 1 & 6.8649 & 6.8649 & 6.8649 & 0 & 0 & 0 \\ 6.8649 & 105.4656 & 35.1552 & 35.1552 & 0 & 0 & 0 \\ 6.8649 & 35.1552 & 105.4656 & 35.1552 & 0 & 0 & 0 \\ 6.8649 & 35.1552 & 35.1552 & 105.4656 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6.8649 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6.8649 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6.8649 \end{bmatrix}$$

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the resultant optimum values for this design are;

D-optimality= 3.467942.

T-optimality = 48.284494.

E-optimality =0.194797.

A-optimality= 0.025489.

4.1.3.3 The particular optimality criteria for the Twenty-Two Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_3 = (S(0.3162277, 0.3162277, 0.3162277) +S(0.5823371, 0.5823371, 0.5823371)+ S(1, 0,0))

The design matrix for M_3 is given by;

1	0.32	0.32	0.32	0.1	0.1	0.1
1	-0.32	0.32	0.32	0.1	0.1	0.1
1	0.32	-0.32	0.32	0.1	0.1	0.1
1	0.32	0.32	-0.32	0.1	0.1	0.1
1	-0.32	-0.32	0.32	0.1	0.1	0.1
1	-0.32	0.32	-0.32	0.1	0.1	0.1
1	0.32	-0.32	-0.32	0.1	0.1	0.1
1	-0.32	-0.32	-0.32	0.1	0.1	0.1
1	0.58	0.58	0.58	0.34	0.34	0.34
1	-0.58	0.58	0.58	0.34	0.34	0.34
1	0.58	-0.58	0.58	0.34	0.34	0.34
1	0.58	0.58	-0.58	0.34	0.34	0.34
1	-0.58	-0.58	0.58	0.34	0.34	0.34
1	-0.58	0.58	-0.58	0.34	0.34	0.34
1	0.58	-0.58	-0.58	0.34	0.34	0.34
1	-0.58	-0.58	-0.58	0.34	0.34	0.34
1	1	0	0	1	0	0
1	-1	0	0	1	0	0
1	0	1	0	0	1	0
1	0	-1	0	0	1	0
1	0	0	1	0	0	1
1	0	0	-1	0	0	1

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and its information matrix

1	0.2506	0.2506	0.2506	0	0	0
0.2506	0.1365	0.0455	0.0455	0	0	0
0.2506	0.0455	0.1365	0.0455	0	0	0
0.2506	0.0455	0.2620	0.1365	0	0	0
0	0	0	0	0.2506	0	0
0	0	0	0	0	0.2506	0
0	0	0	0	0	0	0.2506

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C$_3$ =
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For this design the following optimum values were obtained

D-optimality= 0.535518.

T-optimality= 0.308693.

E-optimality = 0.032551.

A-optimality=0.106874.

4.1.3.4 The particular optimality criteria for the Twenty Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_4=(s(1,1,1)+s(0.8944271, 0,0)+s(1.6470981,0,0))

The design Matrix X forM_4 is given by;

1	0.89	0	0	0.79	0	0
1	-0.89	0	0	0.79	0	0
1	0	0	0.89	0	0	0.79
1	0	0	-0.89	0	0	0.79
1	0	0.89	0	0	0.79	0
1	0	-0.89	0	0	0.79	0
1	1.65	0	0	2.72	0	0
1	-1.65	0	0	2.72	0	0
1	0	0	1.65	0	0	2.72
1	0	0	-1.65	0	0	2.72
1	0	1.65	0	0	2.72	0
1	0	-1.65	0	0	2.72	0
1	1	1	1	1	1	1
1	-1	1	1	1	1	1
1	1	-1	1	1	1	1
1	1	1	-1	1	1	1
1	-1	-1	1	1	1	1
1	-1	1	-1	1	1	1
1	1	-1	-1	1	1	1
1	-1	-1	-1	1	1	1

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and its information matrix

$$C_4=\begin{bmatrix} 1 & 0.4771 & 0.4771 & 0.4771 & 0 & 0 & 0 \\ 0.4771 & 0.4320 & 0.1440 & 0.1440 & 0 & 0 & 0 \\ 0.4771 & 0.1440 & 0.4320 & 0.1440 & 0 & 0 & 0 \\ 0.4771 & 0.1440 & 0.1440 & 0.4320 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4771 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4771 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4771 \end{bmatrix}$$

This design yields the following optimum values;

D-optimality=0.536882.

T-optimality=0.532477.

E-optimality = 0.021841.

A-optimality=0.117436.

4.1.3.5 The particular optimality criteria for the Twenty Six Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_5=(S(0.7162612, 0. 7162612, 0) +S(1,1,1) + S(1.7088007, 0, 0))

1	0.72	0.72	0	0.52	0.52	0
1	-0.72	0.72	0	0.52	0.52	0
1	0.72	-0.72	0	0.52	0.52	0
1	-0.72	-0.72	0	0.52	0.52	0
1	1	1	1	1	1	1
1	-1	1	1	1	1	1
1	1	-1	1	1	1	1
1	1	1	-1	1	1	1
1	0	1.71	0	0	2.92	0
1	0.72	0	0.72	0.45	0	0.45
1	-0.72	0	0.72	0.45	0	0.45
1	0.72	0	-0.72	0.45	0	0.45
1	-0.72	0	-0.72	0.45	0	0.45
1	-1	-1	-1	1	1	1
1	-1	-1	1	1	1	1
1	-1	1	-1	1	1	1
1	1	-1	-1	1	1	1
1	0	-1.71	0	0	2.92	0
1	0	0.72	0.72	0	0.45	0.45
1	0	-0.72	0.72	0	0.45	0.45
1	0	0.72	-0.72	0	0.45	0.45
1	0	-0.72	-0.72	0	0.45	0.45
1	1.71	0	0	2.92	0	0
1	-1.71	0	0	2.92	0	0
1	0	0	1.71	0	0	2.92
1	0	0	-1.71	0	0	2.92

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and its information matrix

$$C_5=\begin{bmatrix} 1 & 0.6902 & 0.6902 & 0.6902 & 0 & 0 & 0 \\ 0.6902 & 1.0446 & 0.3482 & 0.3482 & 0 & 0 & 0 \\ 0.6902 & 0.3482 & 1.0446 & 0.3482 & 0 & 0 & 0 \\ 0.6902 & 0.3482 & 0.3482 & 1.0446 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6902 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6902 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6902 \end{bmatrix}$$

D-optimality= 0.651276.

T-optimality= 0.886307

E-optimality = 0.118976.

A-optimality = 0.437355.

4.1.3.6 The particular optimality criteria for the Thirty Points Three Dimensional Second Rotatable Design of Order Two.

Consider the design;

M_6=(S(3.8729833, 1.5610253, 1.5610253) +S(1,0,0))

The design matrix X for M_6 is given by;

1	1	0	0	1	0	0
1	-1	0	0	1	0	0
1	3.87	1.56	1.56	14.98	2.43	2.43
1	-3.87	1.56	1.56	14.98	2.43	2.43
1	3.87	-1.56	1.56	14.98	2.43	2.43
1	3.87	1.56	-1.56	14.98	2.43	2.43
1	-3.87	-1.56	1.56	14.98	2.43	2.43
1	-3.87	1.56	-1.56	14.98	2.43	2.43
1	3.87	-1.56	-1.56	14.98	2.43	2.43
1	-3.87	-1.56	-1.56	14.98	2.43	2.43
1	0	0	1	0	0	1
1	0	0	-1	0	0	1
1	1.56	1.56	3.87	2.43	2.43	14.98
1	-1.56	1.56	3.87	2.43	2.43	14.98
1	1.56	-1.56	3.87	2.43	2.43	14.98
1	1.56	1.56	-3.87	2.43	2.43	14.98
1	-1.56	-1.56	3.87	2.43	2.43	14.98
1	-1.56	1.56	-3.87	2.43	2.43	14.98
1	1.56	-1.56	-3.87	2.43	2.43	14.98
1	-1.56	-1.56	-3.87	2.43	2.43	14.98
1	0	1	0	0	1	0
1	0	-1	0	0	1	0
1	1.56	1.30	1.56	2.43	14.98	2.43
1	-1.56	1.30	1.56	2.43	14.98	2.43
1	1.56	-1.30	1.56	2.43	14.98	2.43
1	1.56	1.30	-1.56	2.43	14.98	2.43
1	-1.56	-1.30	1.56	2.43	14.98	2.43
1	-1.56	1.30	-1.56	2.43	14.98	2.43
1	1.56	-1.30	-1.56	2.43	14.98	2.43
1	-1.56	-1.30	-1.56	2.43	14.98	2.43

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and its information matrix **C_6 is;**

C_6=

15.3663 5.3663 5.3663 000
5.3663 63.2337 21.0779 21.0779 000
5.3663 21.0779 63.2337 21.0779 000
5.3663 21.0779 21.0779 63.2337 000
0000 5.3663 0 0
00000 0 5.3663 0
000000 0 5.3663

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This design gives the following optimum values;

D-optimality=2.983628.

T-optimality=29.542809.

E-optimality= 0.178871.

A-optimality=1.127848.

4.1.4 Relative Efficiency for Unit Value Designs.

Table 4.6: A Summary of Unit Values from Calculus Optimum Designs

	A	**D**	**E**	**T**
M_1	0.005366	0.398647	0.072085	0.565103
M_2	0.025489	3.467942	0.194797	48.284494
M_3	0.106874	0.535518	0.032551	0.308693
M_4	0.117436	0.536882	0.021841	0.532477
M_5	0.437355	0.651276	0.118976	0.886307
M_6	1.127848	2.983628	0.178871	29.542809

4.1.4.1 A-efficiency.

This measure is related to the A-optimality criterion and is evaluated by the following formula:

$$A(\xi) = \frac{tr\left(m^{-1}\left(\varepsilon_A^{\ast}\right)\right)}{tr\left(m^{-1}(\varepsilon)\right)}$$

Table 4.7: Unit Values A-Efficiencies.

M_1	$= \frac{0.005366}{0.005366} = 100\%$
M_2	$= \frac{0.005366}{0.025489} = 21.052\%$
M_3	$= \frac{0.005366}{0.106874} = 5.021\%$
M_4	$= \frac{0.005366}{0.117436} = 4.570\%$
M_5	$= \frac{0.005366}{0.437355} = 1.227\%$
M_6	$= \frac{0.005366}{1.127848} = 0.476\%$

From the table above M_1 is the most efficient design

4.1.4.2 D-efficiency.

This measure is related to the D-optimality criterion:

$$D(\xi) = \left[\frac{M(\varepsilon)}{M^{¿}(\varepsilon)}\right]$$

Table 4.8: Unit values D- efficiencies.

M_1	$= \frac{0.398647}{0.398647}$ = 100%
M_2	$= \frac{0.398647}{3.467942}$ = 11.495%
M_3	$= \frac{0.398647}{0.535518}$ = 74.441%
M_4	$= \frac{0.398647}{0.536882}$ =74.252%
M_5	$= \frac{0.398647}{0.651276}$ =61.210%
M_6	$= \frac{0.398647}{2.983628}$ = 13.361%

From the table above M_1 is the most efficient design

4.1.4.3 T-efficiency.

This measure is related to the T-optimality criterion: $T(\xi) = \frac{\Delta_1(\varepsilon)}{\Delta_1(\varepsilon_T^{¿})}$

Table 4.9: Unit Values T-Efficiencies.

M_1	$= \frac{0.308693}{0.565103}$ = 54.626%
M_2	$= \frac{0.308693}{48.284494}$ = 0.633%
M_3	$= \frac{0.308693}{0.308693}$ = 100%
M_4	$= \frac{0.308693}{0.532477}$ = 57.973%
M_5	$= \frac{0.308693}{0.886307}$ =34.829%
M_6	$= \frac{0.308693}{29.542809}$ = 1.045%

From the table above M_3 is the most efficient design

4.1.4.4E-efficiency.

This measure is related to the E-optimality criterion:

$$E(\xi) = \frac{\lambda_{min}(\varepsilon)}{\lambda_{min}(\varepsilon^{¿})}$$

Where $\lambda_{min}(\varepsilon)$ is the Eigen value of the information matrix.

Table 4.10: Unit Values E- Efficiencies.

M_1	$= \frac{0.021841}{0.072085}$ = 30.299%
M_2	$= \frac{0.021841}{0.194797}$ = 48.047%
M_3	$= \frac{0.021841}{0.032551}$ = 67.098%
M_4	$= \frac{0.021841}{0.021841}$ = 100%
M_5	$= \frac{0.021841}{0.118976}$ = 18.357%
M_6	$= \frac{0.021841}{0.178871}$ = 12.210%

From the table above M_4 is the most efficient design

4.2 Comparison of the Relative Efficiencies of Calculus Optimum Values Designs with Unit Value Designs.

	A_{eff}		D_{eff}		E_{eff}		T_{eff}	
	Calc opt	**Unit Values**	**Calc opt**	**Unit Values**	**Calc opt**	**Unit Values**	**Calc opt**	**Unit Values**
M_1	100%	100%	100%	100%	42.426%	30.299%	47.563%	54.626 %
M_2	82.972%	21.052%	90.907%	11.495%	31.210%	48.047%	45.861%	0.633%
M_3	1.869%	5.021%	95.700%	74.441%	36.855%	67.098%	47.403%	100%
M_4	5.264%	4.570%	96.267%	74.252%	100%	100%	100%	57.973 %
M_5	1.808%	1.227%	99.516%	61.210%	34.825%	18.357%	46.899%	34.829 %
M_6	1.800%	0.476%	93.904%	13.361%	34.568%	12.210%	46.832%	1.045%
Ave	**32.289**	**22.058**	**96.049**	**55.793**	**46.647**	**46.002**	**55.760**	**41.518**

Following from the values obtained in section 4.1 then the comparisons of the relative efficiencies for both calculus optimum and unit values is done here.

Table 4.11: A summary of the Relative efficiencies of the six specific second order rotatable designs of order two in three dimensions both calculus optimum and unit values

From the table above it is seen from the averages that calculus optimum values designs obtain a higher average as compared to unit value designs therefore more efficient than unit value designs, furthermore it is still evident from the individual efficiencies of the designs that calculus optimum values generally obtain higher relative efficiency values.

4.3 DT-Optimality

This criterion is obtained by finding the DT–optimality which is a combination of D–optimality and T–optimality given by:

$$\Delta_2(¿\xi)+\left(\frac{K}{p}\right)\log\left|M_1(\xi)\right|$$
$$\phi^{DT}(\xi)=(1-k)\log ¿$$

4.3.1 The DT-optimality Criterion for the Calculus optimum Designs.

M_1 (The twenty four points)

$$\phi^{DT}(\xi)=(1-3)\log 0.719916756+\left(\frac{3}{7}\right)\log|0.51684|$$

= 0.162588

M_2 (The thirty-two points)

$$\phi^{DT}(\xi)=(1-3)\log 0.7623315+\left(\frac{3}{7}\right)\log|0.568535|$$

= 0.130608

M_3 (The twenty-two points)

$$\phi^{DT}(\xi)=(1-3)\log 0.737534+\left(\frac{3}{7}\right)\log|0.540062|$$

$$= 0.149769$$

M_4 (The twenty points)

$$\phi^{DT}(\xi) = (1-3)\log 0.3496119 + \left(\frac{3}{7}\right)\log|0.787020|$$

$$= 0.868250$$

M_5 (The twenty-six points)

$$\phi^{DT}(\xi) = (1-3)\log 0.7454531 + \left(\frac{3}{7}\right)\log|0.549356|$$

$$= 0.143668$$

M_6 (The thirty points)

$$\phi^{DT}(\xi) = (1-3)\log 0.746528 + \left(\frac{3}{7}\right)\log|0.550390|$$

$$= 0.142767$$

The results above for designs M_1, M_2, M_3, M_4, M_5 and M_6 represent the compound DT- optimum criterion values for calculus optimum designs.

4.3.2 The DT-optimality Criterion for the unit value Designs.

M_1 (The twenty four points)

$$\phi^{DT}(\xi) = (1-3)\log 0.5702211 + \left(\frac{3}{7}\right)\log|0.3849673|$$

$$= 0.310238$$

M_2 (The thirty-two points)

$$\phi^{DT}(\xi) = (1-3)\log 1.5401845 + \left(\frac{3}{7}\right)\log|0.8734885|$$

$$= -0.400320$$

M_3 (The twenty-two points)

$$\phi^{DT}(\xi) = (1-3)\log 0.8641483 + \left(\frac{3}{7}\right)\log|0.5472958|$$

$= 0.014636$

M_4 (The twenty points)

$$\phi^{DT}(\xi) = (1-3)\log 0.3592492 + \left(\frac{3}{7}\right)\log\left|0.200834435\right|$$

$= 0.590424$

M_5 (The twenty-six points)

$$\phi^{DT}(\xi) = (1-3)\log 0.3621662 + \left(\frac{3}{7}\right)\log\left|0.194937713\right|$$

$= 0.577854$

M_6 (The thirty points)

$$\phi^{DT}(\xi) = (1-3)\log 0.4677112 + \left(\frac{3}{7}\right)\log\left|0.319476573\right|$$

$= 0.447661$

The results above for designs M_1, M_2, M_3, M_4, M_5 and M_6 represent the compound DT- optimum criterion values for unit value designs.

Table 4.12 A Summary of the DT- Optimality Criterion for Both Calculus Optimum and Unit Value Designs.

	M_1	M_2	M_3	M_4	M_5	M_6
Calc optimum	0.162588	0.130608	0.149769	0.868250	0.143668	0.142767
Unit Values	0.310238	-0.400320	0.014636	0.590424	0.577854	0.447661

From the table above calculus optimum values obtain less optimum values making them optimum

CHAPTER FIVE: SUMMARY, CONCLUSIONS AND RECOMMENDATIONS.

5.0 Introduction.

In this chapter the conclusions and recommendations and further work are outlined here. The general comments on the study are also given.

5.1Summary.

It was found that the designs which were calculus optimum were generally more efficient as compared to the Unit value designs as it was evident from the average values calculated in table 4.3. From the Calculus optimum Designs it was found that the D-efficiency was the highest amongst all the other relative efficiency criteria. The Unit value designs were found to be more efficient when D- relative efficiency Criterion was utilized. Amongst all the calculus optimum designs the D- efficiency Criterion is found to be the best. When the designs were considered the design M_1 was found to be the most efficient because it has higher efficiency values than the rest of the designs.

It was observed that increasing or reducing the number of design points does not necessarily increase or reduce efficiency.

For the DT- optimality Criterion the design with the least value was optimal and in this regard M_2 was found to be the most optimal design when the individual designs were taken into account. For calculus optimum values designs $M_2$was found to be the most optimal design and for unit value designs M_3 was found to be optimal.

In general it was found that calculus optimum values designs were found to be more optimal under the DT- compound optimality criterion.

5.2 Conclusions.

For Relative efficiency criteria, Calculus optimum designs are more efficient as compared with unit value designs. And when the compound DT- optimality criterion is analyzed the again Calculus optimum designs are DT- optimal as compared with unit value designs.

5.3 Recommendation.

It is observed that the calculus optimum values designs were more efficient as compared with unit value designs. Calculus optimum values designs are also optimal therefore they are the best designs to be chosen for experimentation because they will minimize the inputs but maximize the outputs and at the same time saving on time and money.

5.4 Further Work

It's recommended that there is need to calculate the relative efficiencies of higher order designs both calculus optimum and unit values. More compound optimality criteria for the designs be calculated and the relationship between alphabetic optimality and compound optimality criteria analyzed.

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