Construction of a four dimensional third order rotatable design through balanced incomplete block designs

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CONSTRUCTION OF THIRD ORDER ROTATABLE DESIGNS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract

In this paper, a method of constructing third order rotatable designs in k-dimensions from a third order rotatable designs in $k - \iota$ dimensions is provided. The result of the experiment in lower-dimensional design need not be discarded. In Section 3, third order rotatable designs in five and six dimensions are constructed through balanced incomplete block designs.

1. Introduction

Draper [6] provided a method of constructing second order rotatable designs in k-dimensions from the second order rotatable designs in $(k - l)$-dimensions. Herzberg [10] provided an alternative method which always works and for which the results of the experiments performed according to

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the –dimensional $(k - l)$-dimensional design need not be discarded. The method of construction of a third order rotatable designs presented here shares some of the features of Herzberg's method and is analogous to the method for second order rotatable designs suggested in Huda [11]. For example, the experimenter may start with a $(k - l)$-dimensional design. If after performing the design, either singular or non-singular, the experimenter feels that one more factors should be included, then he can proceed by the method described. For some new sequential methods the reader is referred to Huda [13-14], and Mutiso and Koske [17-18].

Box and Hunter [2] introduced rotatable designs for the exploration of response surfaces. For theses designs, the variance of the estimated response is constant at points equidistant from the centre of the designs. They called these designs *rotatable designs* when the relationship between the response variable and several input variables is a quadratic or cubic or higher order polynomial. Das and Narasimham [3] constructed rotatable designs through balanced incomplete block designs (BIBD).

The k-dimensional point set $(x_{iu}, \ldots, x_{ku})$ $(u = 1, \ldots, N)$ is a third order rotatable arrangement if,

$$\sum_{u=1}^{N} x_{iu}^2 = A \quad (i = 1, \ldots, k), \tag{1}$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{iu}^2 = 3B \quad (i \neq j; \;\; i,\; j = 1, \ldots, k), \tag{2}$$

$$\sum_{u=1}^{N} x_{iu}^6 = 5\sum_{u=1}^{N} x_{iu}^4 x_{ju}^2 = 15\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{lu}^2 = 15C \; (i \neq j \neq l; \;\; i,\; j,\; l = 1, \ldots, k) \tag{3}$$

and all other sums of powers and product up to order six are zero.

From Gardiner et al. [8] it is known that if (1), (2) and (3) are satisfied, then the design points also satisfy

$$\frac{NB}{A^2} \geq \frac{k}{(k+2)}, \tag{4}$$

$$\frac{AC}{B^2} \geq \frac{(k+2)}{(k+4)} \tag{5}$$

and the arrangement forms a non-singular third order design if and only if strict inequality is achieved in (4) and (5). The strict inequality in (4) can always be achieved, if necessary by the addition of centre points. It is known (Draper [4]) that the strict inequality in (4) is also achieved by a third order rotatable arrangement if and only if the points lie on two more distinct spheres centered at the origin and then the strict inequality in (5) is also automatically obtained.

2. Definition of a Third Order Rotatable Designs (TORD)

A third order response surface design D is said to be a *TORD*, if in this design all the conditions (1) to (5) hold and hence the variance of the estimated response of Y_u from the fitted surface is only a function of the distance $\left(d^2 = \sum_{i=1}^{k} x_{iu}^2\right)$ of the point $(x_{1u}, x_{2u}, ..., x_{ku})$ from the origin of the design (Gardiner et al. [8]).

3. Construction of the Design

Huda [15] gave a method of constructing designs for $k - l$ dimensional points $\{x_{iu}, ..., x_{k-l,u}\}(1, 2, ..., N^l)$ which satisfy (1) in $k - l$ dimensional space to k-dimensional points. Koske et al. [16] extended the method of construction to obtain a third order rotatable design in five dimensions.

The problem is to derive a k-dimensional third order rotatable design such that the points $(x_{1u}, ..., x_{k-1,u}, 0, ..., 0)$ $(u = 1, 2, ..., N)$ are included in the design. This method entails that a B. I. B is utilized in the construction of the designs. Let B. I. B $(t, b, r, s, \lambda, \mu)$ denote a doubly balanced incomplete design of t treatments, b blocks each of size s with r, λ and μ representing the number of occurrences of each treatment, each pair of treatments and each triplet of treatments, respectively. Let $S(z_1, ..., z_k)$ denote all permutations of $(\pm z_1, ..., \pm z_k)$. Then $S(a_1, 0, ..., 0)$, $S(a_2, a_2, 0, ..., 0), \cdots$ may be used to denote the symmetric point sets associated with the appropriate irreducible balanced incomplete block designs.

3.1. Design in five dimensions

Now consider the double balanced incomplete block designs with $t = k = 5$ and $s = k - l = 2.$ It is assumed that the first block $h_i = a(i = 1, 2)$ and $h_i = 0(i = 3, 4, 5).$ Then for each block of B. I. B (5, 10, 4, 2, 1, 0) and each $u(u = 1, 2, ..., N')$ there is an associated point generated by replacing the $(k - l)$ nonzero entries of $(h_1, ..., h_k)$ by $x_{1u}, ..., x_{k-l,u}$ in any order without replacing the x_{iu}'$s(i = 1, ..., k - \iota).$ Let the x_{iu}'s be placed in the ascending order of the i's in the first block. Let ABIB (5, 10, 4, 2, 1, 0) denote the set of all such points generated from the corresponding block design and the given $(k - l)$-dimensional arrangement. Then by combining ABIB (5, 10, 4, 2, 1, 0) with symmetric point sets it is possible to obtain a design of five dimensions satisfying (1), (2), (3) and containing $(x_{1u}, x_{2u}, 0, 0, 0)(u = 1, 2, ..., N').$

3.2. Design in six dimensions

Similarly as in (3.1) we consider the double balanced incomplete block designs with $6t = k = 5$ and $s = k - l = 2.$ It is assumed that the first block $h_i = a(i = 1, 2)$ and $h_i = 0(i = 3, 4, 5, 6).$ Then for each block of B. I. B (6, 15, 5, 2, 1, 0) and each $u(u = 1, 2, ..., N')$ there is an associated point generated by replacing the $(k - l)$ nonzero entries of $(h_1, ..., h_k)$ by $x_{1u}, ..., x_{k-l,u}$ in any order without replacing the x_{iu}'$s(i = 1, ..., k - \iota).$ Let the x_{iu}'s be placed in the ascending order of the i's in the first block. Let ABIB (6, 15, 5, 2, 1, 0) denote the set of all such points generated from the corresponding block design and the given $(k - l)$-dimensional arrangement. Then by combining ABIB (6, 15, 5, 2, 1, 0) with symmetric point sets it is possible to obtain a design of six dimensions satisfying (1), (2), (3) and containing $(x_{1u}, x_{2u}, 0, 0, 0, 0)(u = 1, 2, ..., N')$.

4. The Design

Consider a third order rotatable design in two dimensions consisting of N' points equally spaced on a circle of radius ρ. Then from Bose and Carter [1] and Gardiner et al. [8] it is known that for this arrangement

$$A = \frac{N'}{2}\rho^2, \; B = \frac{N'}{8}\rho^4 \; \text{ and } \; C = \frac{N'}{48}\rho^6, \tag{6}$$

A, B, C are as defined in (1), (2) and (3).

4.1. The design in five dimensions

As stated in (3.1) we consider the five-dimensional point set BIB (5, 10, 4, 2, 1, 0) generated from this arrangement. Combine the $5N'$ points of this set with the 168 points of $S(a, a, 0, 0, 0)$, $S(b, b, b, b, b)$ and $S(d, d, d, d, d)$ with $d^2 = b^2t, \; t \geq 0.$ This arrangement forms a third rotatable design in five dimensions if

$$a^2 = \left(\frac{13}{4}C\right)^{\frac{1}{3}}, \quad b^2 = \left[\left(\frac{16C}{64}\right)\frac{1}{1 + 3t^3}\right]^{\frac{1}{3}} \tag{7}$$

and if t is such that

$$\frac{(1 + 3t^2)^3}{(1 + 3t^3)^2} = \frac{\left(9B + 4\left(\frac{13C}{4}\right)^{\frac{2}{3}}\right)^3}{64(16C)^2}. \tag{8}$$

If $N' = 8$, then t is needed such that

$$\frac{(1 + 3t^2)^3}{(1 + 3t^3)^2} = 3.481363 \tag{9}$$

which gives $t = 0.643374.$

Hence, a third order rotatable design in five dimension with 248 points exists.

The design is given by

$$D_1 = S(a, a, 0, 0, 0) + S(b, b, b, b, b) + S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0, 0, 0\right)$$

$$+ 3S(d, d, d, d, d) + 4S(\rho, 0, 0, 0, 0), \tag{10}$$

where

$$a^2 = 0.81516\rho^2, \; b^2 = 0.17957\rho^2 \text{ and } d^2 = 0.11553\rho^2.$$

4.2. The design in six dimensions

Similarly as explain in (3.2) a third order rotatable design is obtained by considering the six-dimensional point set BIB (6, 15, 5, 2, 1, 0) generated from this arrangement. Combine the $5N'$ points of this set with the 188 points of $S(a, a, 0, 0, 0, 0)$, $S(b, b, b, b, b, b)$ and $S(d, d, d, d, d, d)$ with $d^2 = b^2 t$, $t \geq 0$. This arrangement forms a third rotatable design in six dimensions if

$$a^2 = \left(\frac{27}{4}C\right)^{\frac{1}{3}}, \quad b^2 = \left[\left(\frac{30C}{128}\right)\frac{1}{1+2t^3}\right]^{\frac{1}{3}} \tag{11}$$

and if t is such that

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = \frac{\left(12B + 8\left(\frac{27C}{4}\right)^{\frac{2}{3}}\right)^3}{128(30C)^2}. \tag{12}$$

If $N' = 8$, then t is needed such that

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = 2.753157 \tag{13}$$

which gives $t = 0.71113$.

Hence, a third order rotatable design in six dimension with 372 points exists.

The design is given by

$$D_2 = S(a, a, 0, 0, 0, 0) + S(b, b, b, b, b, b) + S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0, 0, 0, 0\right)$$

$$+ 2S(d, d, d, d, d, d) + 5S(\rho, 0, 0, 0, 0, 0), \tag{14}$$

where

$$a^2 = 1.04\rho^2, \;\; b^2 = 0.17843\rho^2 \;\; \text{and} \;\; d^2 = 0.126885\rho^2.$$

5. Results and Conclusions

It is concluded that third order rotatable designs can be obtained through balanced incomplete block designs. Many third order rotatable designs have been described in Gardiner et al. [8], Draper [4, 5, 7], Thaker and Das [20], Das and Narasimham [3], Herzberg [9]. These designs would usually require many more points than the available minimal point designs and hence may not always be desirable. For example, the experimenter might also be interested in some of the $(k - l)$ subsets of the factors. These subsets may be identified with the blocks generating ABIB $(k, b, r, K - L, \lambda, \mu)$ so that the k-dimensions involving the subsets of factors the experimenter is interested in.

During the last few decades, rotatable designs using BIBDs have been introduced by a number of workers. This introduces a new method of constructing higher level of third order rotatable designs using BIBDs.

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