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A NEW METHOD OF CONSTRUCTING THIRD ORDER ROTATABLE DESIGN

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Abstract

This paper provides a new method of constructing third order rotatable designs in k dimensions from third order rotatable designs in $k - l$ dimensions. The method given shows how the incidence matrices of balance incomplete block designs are used in construction of these designs. The results of the experiments performed according to the lower-dimensional designs need not be discarded when this method is used.

1. Introduction

The idea of making a design rotatable is a useful one, in practice. It enables the experimental information to be obtained equally in all directions, at the same distance from the origin, in the space of real design variables

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$x_1, x_2, ..., x_m$. Third order rotatability assumes that it is desired to fit a cubic polynomial in $x_1, x_2, ..., x_m$ to the available experimental data.

Rotatability is a natural and desirable property, which requires that the variance of a predicted response at a point remains constant at all such points that are equidistant from the design centre. To achieve the stability in prediction variance, this important property of rotatability was evolved and was formerly developed by Box and Hunter [1], assuming the errors in the observations are uncorrelated. Herzberg [10] presented two other 72 points third order rotatable designs in four dimensions among the designs which contain 72 points listed by Das and Narasimham [7]. Huda [12] constructed three-dimensional designs from two-dimensional designs from what is known from Gardiner et al. [9] that a set of $N' (\geq 7)$ points equally spaced on a circle centred at the origin satisfies the moment requirements of a third order rotatable design to be rotatable. Koske et al. [16] extended the method and constructed a third order rotatable design in 5 dimensions with 320 points while Koske et al. [17] generalized the construction for units of size $s = k - 1 = 2$. Here the incidence matrix corresponding to the BIBD used to generate units of size $s = k - 1 = 2$ is utilized.

Draper [4] stated that this set of points forms a rotatable arrangement of third order in *k* factors if the following relations hold:

$$\sum_{u=1}^{N} x_{iu}^2 = A \quad (i = 1, 2, ..., k), \tag{1}$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 3B, \tag{2}$$

$$\sum_{u=1}^{N} x_{iu}^6 = 5\sum_{u=1}^{N} x_{iu}^2 x_{ju}^4 = 15\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{lu}^2 = 15C,$$

$$i \neq j \neq l = 1, 2, ..., k, \quad u = 0, 1, ..., N, \tag{3}$$

and all other sums of powers and products up to order six are zero, where

$$A = N\lambda_2, \quad B = N\lambda_4 \quad \text{and} \quad C = N\lambda_6. \tag{4}$$

The arrangement of points is said to form a *rotatable design of third order only* if it forms a non-singular third order design (if the points give rise to a non-singular matrix).

Gardiner et al. [9] derived the non-singularity conditions as:

$$\begin{aligned} D &: \frac{NB}{A^2} > \frac{k}{(k+2)}, \\ E &: \frac{AC}{B^2} > \frac{(k+2)}{(k+4)}. \end{aligned} \tag{5}$$

These are the conditions required for a set of points to form a third order rotatable design.

Draper [4] proved that it is impossible for the inequalities of (5) to be reversed. He further showed that in order to get usable third order rotatable designs, at least two spherical sets of points with different positive radii but centred at the origin of the design must be combined.

2. Definition of a Third Order Rotatable Design (TORD)

A third order response surface design D is said to be a *TORD*, if in this design all the conditions (1) to (5) hold and hence the variance of the estimated response of Y_u from the fitted surface is only a function of the distance $\left(d^2 = \sum_{i=1}^{k} x_{iu}^2\right)$ of the point $(x_{1u}, x_{2u}, ..., x_{ku})$ from the origin of the design (Gardiner et al. [9]).

3. Construction of the Design

Huda [12] gave a method of constructing designs for $k - l$ dimensional points $\{x_{iu}, ..., x_{k-l,u}\}$ which satisfy (1) to (5) in $k - l$ dimensional space to

k dimensional points. Koske et al. [16] extended the method of construction to obtain a third order rotatable design in five dimensions.

The problem is to derive a v-dimensional third order rotatable design such that the points $(x_{1u}, ..., x_{k-l,u}, 0, ..., 0)$ $(u = 1, 2, ..., N)$ are included in the design. This method entails that a BIB is utilized in the construction of the designs. Let $B.I.B(t, b, r, s, \lambda, \mu)$ denote a doubly balanced incomplete design of v treatments, b blocks each of size k with r, λ and μ representing the number of occurrences of each treatment, each pair of treatments and each triplet of treatments, respectively. Let $S(z_1, ..., z_k)$ denote all the permutations of $(\pm z_1, ..., \pm z_k)$. Then $S(a_1, 0, ..., 0)$, $S(a_2, a_2, 0, ..., 0), ...$ may be used to denote the symmetric point sets associated with the appropriate irreducible balanced incomplete block designs.

3.1. Design in v dimensions

Now consider the double balanced incomplete block designs with $t = k = v$ and $s = k - l = 2$. It is assumed that the first block $h_i = a$ for $i = 1, 2,$ and $h_i = 0$ for $i = 3, 4, ..., v$. Then for each block of $B.I.B(v, b, k, r, \lambda, \mu)$ and each u $(u = 1, 2, ..., N')$, there is an associated point generated by replacing the $(k - l)$ nonzero entries of $(h_1, ..., h_k)$ by $x_{1u}, ..., x_{k-l,u}$ in any order without replacing the x_{iu}'s $(i = 1, ..., k - l)$. Let the x_{iu}'s be placed in the ascending order of the i's in the first block. Let $B.I.B(v, b, k, r, \lambda, \mu)$ denote the set of all such points generated from the corresponding block design and the given $(k - l)$-dimensional arrangement. Then by combining $B.I.B(v, b, k, r, \lambda, \mu)$ with the corresponding incidence matrix and symmetric point sets, it is possible to obtain a design of v dimensions satisfying (1), (2), (3) and containing $(x_{1u}, x_{2u}, 0, 0, 0)$ $(u = 1, 2, ..., N')$.

Considering a third order rotatable design in two dimensions consisting of N points equally spaced on a circle of radius ρ, from Gardiner et al. [9] it is known that for this arrangement,

$$A = \frac{N'}{2}\rho^2, \quad B = \frac{N'}{8}\rho^4 \quad \text{and} \quad C = \frac{N'}{48}\rho^6,$$

where A, B and C are defined in equations (1), (2) and (3).

Consider k-dimensional point set $B.I.B(v, b, k-l, s, \lambda, \mu)$ generated from this arrangement.

Theorem. *The union of the points set* $B.I.B(v, b, k-l, s, \lambda, \mu)$ *with points of the incidence matrix of* $B.I.B(k, b, k-l, r, \lambda, \mu)f$, $S(b, b, ..., b)$ *and* $(v-1)S(d, d, ..., d)$ *with* $b^2 = td^2$, $t \geq 0$ *forms a third order rotatable design in k dimensions if*

$$f^2 = \left(\frac{3c(5v-12)}{2^k(7\lambda - r)}\right)^{1/3}, \quad d^2 = \frac{3B(v-2) + (r-3\lambda)2^k f^2}{2^{v+1}(t^2 + v - 2)}, \quad b = td^2 \tag{6}$$

and if t is such that

$$\frac{(t^2+v-1)^3}{(t^3+v-1)^2} = \frac{\left[3B(v-2) + 2^k(r-3\lambda)\left(\frac{3c(5v-12)}{2^k(7\lambda-r)}\right)^{2/3}\right]^3 (7\lambda - r)^2}{2^{v+1}(3c(5\lambda(v-1)-r))^2}. \tag{7}$$

Proof. The conditions given in (1), (2) and (3) are applied to the design points given in the theorem to obtain the following equations:

$$3B(v-2) + 2^k f^4(r-3\lambda) - 2^{v+1}(b^4 + (\nu-2)d^4) = 0, \tag{8}$$

$$15c(v-1) + 2^k rf^6 - 7(2^{v+1})(b^6 + (\nu-2)d^6) = 0, \tag{9}$$

$$3c + 2^k \lambda f^6 - 2^{v+1}(b^6 + (\nu-2)d^6) = 0. \tag{10}$$

The solution of these equations completes the proof of the theorem and hence the values given in (6) and (7) are obtained.

Hence a third order rotatable design in k dimension with $4[k(k+1) + 2^k]$ points exists for N'. The design is given by

$$S(b, b, ..., b),\ S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0, ..., 0\right),\ 3S(d, d, ..., d) \text{ and } 4S(\rho, 0, ..., 0),$$

where b^2 and d^2 are as given in (6).

4. Results and Conclusion

Third order rotatable designs in k dimensions through (BIB) design exist with $4[k(k+1) + 2^k]$ points.

In situations where, for example, the experimenter might be interested in some of the $(k - l)$ subsets of factors, these designs may be desirable, since these subsets may be identified with the blocks generating a $B.I.B(k, b, r, k - l, \lambda)$ so that the k-dimensional third order rotatable design contains third order rotatable designs in $(k - l)$ dimensions involving the subsets of factors the experimenter is interested in.

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