SOME SECOND ORDER REDUCED PARAMETER SUBSYSTEM SPECIFIC OPTIMUM ROTATABLE DESIGNS IN THREE, FOUR AND FIVE DIMENSIONS

BY

JACOB MASAI KISAA

A THESIS SUBMITTED TO THE SCHOOL OF SCIENCE AND AEROSPACE STUDIES IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN STATISTICS

MOI UNIVERSITY

2022

DECLARATION

Declaration by Candidate

This thesis is my original work and has not been presented for a degree in any other university or institution of higher learning. No part of this work may be reproduced without prior permission of the author and/or Moi University.

Sign:______________________ Date:______________________

Jacob Masai Kisaa

Reg. No: MSC/BS/02/14

Declaration by the Supervisors

This thesis has been submitted for examination with our approval as University Supervisors.

Sign:______________________ Date:______________________

Prof. John M. Mutiso

Department of Mathematics, Physics and Computing

School of Science and Aerospace Studies

Moi University,

Eldoret, Kenya.

Sign:______________________ Date:______________________

Dr. Silver K. Rambaei

Department of Mathematics, Physics and Computing

School of Science and Aerospace Studies

Moi University,

Eldoret, Kenya.

DEDICATION

To my late Dad Paul Kisaa, Mum Francisca, brothers Maingi, Manasseh & Sister in law Mercy.

.

ACKNOWLEDGEMENT

First and foremost I am grateful to the Lord who is my Shepherd and has graciously capacitated me to get this far. Secondly, I am deeply indebted to my University Supervisors Prof. J.M. Mutiso and Dr. S.J.K. Rambaei for the inestimable supervision they accorded me to make this academic dream a reality. I also extend my sincere gratitude to the members of the Statistics and Computer Science Department of Moi University and to the university at large for granting me the opportunity to join the M.Sc. Programme.

I will not forget to show my appreciation to my fellow course mates Albert, Anne, Cruyff, Dickson, James, Jeremy, Juliet, Kenneth, Maraswa, Moses, Mwan, Noela, Rhoda, Sylvia and not forgetting my friends and predecessors Frank, Nicholas and Nyakundi for the invaluable support they gave me in this academic journey.

Finally I would like to sincerely thank my parents, brothers and sister in law for the prayers, love, financial and moral support.

May the Lord bless you all.

ABSTRACT

Response surface methodology is a tool used to optimize output variables by systematically altering the input variables in a given process to obtain the optimum yield. This is facilitated by the use of experimental designs otherwise known as optimal designs which allow for parameters to be estimated without bias and with minimum variance. Optimal designs are selected using existing statistical criteria where the alphabetic optimality criteria are widely used. Optimality criteria for some specific Second Order Rotatable Designs (SORDs) in three factors have been obtained using the full parameter system. However, optimality criteria for the specific SORDs in three and higher dimensions have not been determined using a reduced parameter system. The purpose of this study therefore was to evaluate the optimality criteria for specific SORDs in three, four and five dimensions using calculus optimum values and a reduced parameter system. The objectives of the study were: to evaluate the alphabetic optimality criteria for three specific SORDs in three, four and five dimensions; to compute the Determinant-Trace (DT) compound optimality criteria for the designs and to determine the efficiency of the specific designs. The alphabetic optimality criteria were computed using specific alphabetic criteria generalized methods for SORDs in k dimensions, the DT- optimality criteria was obtained by logarithmic combination of the Determinant (D) criteria and the Trace (T) criteria of the specific SORDs whereas the designs' efficiencies were determined by comparing the specific designs' optimality criteria to the optimal design criterion. The study found that three and four dimensional designs were Smallest Eigen Value (E-) Optimal with values 0.14040422, 0.1477622, 0.1487512 for the designs M_3, M_5, M_6 in three dimensions and 0.1354509, 0.1336771, 0.5671459 for designs M_3^1, M_5^1, M_6^1 in four dimensions. The five dimensional designs were Average Variance (A-) Optimal with values 0.4344145, 0.4283984, 0.4122856 for designs M_3^2, M_5^2, M_6^2 respectively. Designs M_3, M_5^1, and M_6^2 were found to be the most efficient designs in three, four and five dimensions respectively. Additionally, the DT- compound optimality efficiencies were comparably higher in respect to their alphabetic optimality equivalents. In conclusion, the more homogenous the design is, the more optimal it became, and thus the designs obtained provides very essential tools for use in various fields such as in medicine, agriculture and industry. The study recommends the application of the designs in the planning of field experiments particularly in agricultural experiments.

TABLE OF CONTENTS

LIST OF TABLES

ABBREVIATIONS AND ACRONYMS

DOE : Designs of Experiments

RSM : Response Surface Methodology

SORD : Second Order Rotatable Designs.

OPERATIONAL DEFINITION OF TERMS

M_3: Twenty two points calculus optimum value second order rotatable design in three dimensions.

$M_3^{(1)}$: Forty eight points calculus optimum value second order rotatable design in four dimensions.

$M_3^{(2)}$: Hundred points calculus optimum value second order rotatable design in five dimensions.

M_5: Twenty six points calculus optimum value second order rotatable design in three dimensions.

$M_5^{(1)}$: Fifty six points calculus optimum value second order rotatable design in four dimensions.

$M_5^{(2)}$: Hundred and sixteen points calculus optimum value second order rotatable design in five dimensions.

M_6: Thirty points calculus optimum value second order rotatable design in three dimensions.

$M_6^{(1)}$: Sixty four points calculus optimum value second order rotatable design in four dimensions.

$M_6^{(2)}$: Hundred and thirty two points calculus optimum value second order rotatable design in five dimensions.

Calculus Optimum Values; A free parameter can be given a unit value or a calculus optimum value which arises from differentiation and the general equivalence theorem. This is the letter parameter in a general design given in terms of letters in which all other letters are evaluated and expressed.

Optimal Designs; are those that are constructed on the basis of a certain optimality criterion that pertains to the 'closeness' of the predicted response, $\hat{\boldsymbol{y}}(\boldsymbol{x})$ to the mean response, $\boldsymbol{\mu}(\boldsymbol{x})$, over a certain region of interest denoted by *R*.

Rotatability A design $\boldsymbol{D}$ is said to be rotatable if the prediction variance in $\boldsymbol{Var}\,[\hat{y}(\boldsymbol{x})] = \boldsymbol{\sigma^2}\,\boldsymbol{f}'\,(\boldsymbol{x})\,(\boldsymbol{X}'\boldsymbol{X})^{-\mathbf{1}}\boldsymbol{f}(\boldsymbol{x})$ is constant at all points that are equidistant from the design center, which, by a proper coding of the control variables, can be chosen to be the point at the origin of the *k*-dimensional coordinates.

CHAPTER ONE

INTRODUCTION

1.1 Overview

This chapter covers the background to the study, statement of the problem, justification, purpose of the study, objectives of the study, research questions, study significance, scope of the study and the study limitations.

1.2 Background of the Study

In a world of limited resources and an ever growing population, several processes have to be optimized so as to make maximum use of the available resources. For example farmers have limited land but they are under constant, ever-piling pressure to produce higher yields on the same piece of land, therefore an urgent solution is needed.

To determine the most appropriate and efficient approach (method with optimum yield); field experiments have to be performed. This is achieved through formulation of hypotheses, conducting experiments and applying statistical procedures and tests to check if the hypotheses are valid.

Response surface methodology (RSM) consists of a group of mathematical and statistical techniques used in the development of an adequate functional relationship between a response of interest, y, and a number of associated control (or input) variables denoted by x_1, $x_2,\ldots$, x_k . It is concerned with the modeling of one or more responses to the settings of several input variables. The nature of the function relating the responses to the variables is assumed to be unknown and the function or surface is modeled empirically using a first- or a second-order polynomial model. It is acknowledged that this model is only an approximation, but it is used because such a model is easy to estimate and apply, even when little is known about the process. The broad aims of RSM are to investigate the nature of the response surface over a region of interest and to identify operating conditions associated with maximum or minimum responses. RSM is generally conducted in three phases, as emphasized in Myers and Montgomery (2002). Phase 0 involves the screening of explanatory variables to identify those which have a significant effect on the responses, phase 1 is concerned with the location of optimum operating conditions by conducting a sequence of suitable experiments and phase 2 involves the fitting of an appropriate empirical model, usually a second-order polynomial model, in order to examine the nature of the response surface in the vicinity of the optimum. The method was introduced by Box and Wilson (1951) and the fundamentals of RSM are set out in the papers of Box and Wilson (1951) where they discussed a way of obtaining an optimum response surface using the least number of observations in experimental designs.

The main objective of the experimenter is customarily to estimate the absolute response or the parameters of a model providing the relationship between the response and the factors. Rotatable designs were introduced by Box and Hunter (1957) in order to explore the response surface.

Rotatable designs have the variance of the estimated response being constant at points equidistant from the center of the design, befittingly taken to be the origin of the factor space, after transformations if need arises. This study is geared towards exploring some existing three specific second order rotatable designs in three, four and five dimension to determine their suitability for real life use through analysis of their optimality criteria and relative efficiencies.

1.2.1 Optimality Criteria

An optimality criterion is a function ϕ from the closed cone of nonnegative definite s×s matrices (C_{sxs}) into the real line ϕ: NND(s) → R, with the properties that capture the idea of whether an information matrix is either large or small. In order to inform the decision on the best model selection, there are a set of criteria employed. The alphabetic optimality criteria are widely used in statistics and comprise of; The Determinant criterion (D- Criterion) where a D – Optimum design minimizes the content of the ellipsoidal confidence region for the parameters of the linear model, the Average Variance Criterion (A- Criterion) which minimizes the sum (or average) of the variances of the estimates, the Trace Criterion (T- Criterion) which considers the linearity of the matrix means and the Smallest Eigen Value Criterion (E- Criterion) which reduces the variance of each individual parameter estimate as presented by Pukelsheim (1993).

1.2.2 Compound Optimality Criteria

A compound optimality criterion combines two or more optimality criteria to give a criterion that carries the combined properties of each of the individual criterion. An experimenter may wish to get the combined properties of two or more optimality criteria during design selection hence the method of compound optimality criteria becomes instrumental in creating a balance between the adequacies and inadequacies of the selected optimality criteria in regard to a specific design.

1.2.3 Design Efficiency

The efficiency of an experimental design is a function of the variances and covariances of the parameter estimates. The efficiency is usually determined by comparing a specific optimality criterion of a design to the optimal design criterion.

1.3 Statement of the Problem

Response surface methodology plays a major role in experimentation and research. Most developed and developing countries have embraced research as an improvement tool on their production processes especially in design and analysis of experiments. As a result of this, researchers/statisticians come up with sophisticated experimental designs which are geared towards optimizing production processes. The current study endeavors to fill a knowledge gap by evaluating the optimality criteria and relative efficiencies for existing specific calculus optimum designs in three, four and five dimensions using a parameter of interest system. This will go a long way in providing researchers with evaluation criteria on choosing an appropriate design for a particular purpose.

1.4 Justification

There is need to evaluate the optimality criteria of some specific second order rotatable designs in higher dimensions to establish their suitability for experimental use.

1.5 Objectives of the Study

1.5.1 General Objective

The main objective of this study is to evaluate the optimality criteria and relative efficiencies for some second order rotatable designs in three, four and five dimensions using a parameter subsystem.

1.5.2 Specific Objectives

1. To calculate A-, D-, E- and T- optimality criteria for three specific calculus optimum values second order rotatable designs in three, four and five dimensions.

2. To obtain DT- compound optimality criteria for three specific calculus optimum values second order rotatable designs in three, four and five dimensions.
3. To evaluate the relative efficiencies for three specific calculus optimum values second order rotatable designs in three, four and five dimensions.

1.6 Significance of the Study

Findings from this study will help to inform decisions on model selection before experiments are conducted which involve second order rotatable designs in three, four or five dimensions.

1.7 Scope of the Study

This study will focus on three existing second order rotatable designs in three, four and five dimensions namely; $\mathbf{M_3, M_5}$ **&** $\mathbf{M_6}$ in three dimensions, $\mathbf{M_3^{(1)}, M_5^{(1)}}$ **&** $\mathbf{M_6^{(1)}}$ in four dimensions and $\mathbf{M_3^{(2)}, M_5^{(2)}}$ **&** $\mathbf{M_6^{(2)}}$ in five dimensions.

1.8 Study Limitations

These designs are theoretical and require a lot of time to put them into practice so as to establish a steady trend. However, by using the most optimal designs, one would save a big fraction of time that they would have used in experimentation of all the designs.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter reviews the literature on attaining alphabetic optimality criteria, DT-compound optimality criteria and design relative efficiencies for $\mathbf{M_3}$, $\mathbf{M_5}$, $\mathbf{M_6}$, $\mathbf{M_3^{(1)}}$, $\mathbf{M_5^{(1)}}$, $\mathbf{M_6^{(1)}}$, $\mathbf{M_3^{(2)}}$, $\mathbf{M_5^{(2)}}$ **and** $\mathbf{M_6^{(2)}}$

2.2 Optimality Criteria

2.2.1 Optimal Design Theory

Optimal Design Theory entails the use of experimental designs that are generated based on a particular optimality criterion and is generally optimal only for a specific statistical model. Optimal design theory was introduced by Kiefer and Wolfowitz (1959). Later Kiefer (1985) building on the foundation that had been laid stated that the experimental design is a discrete probability measure defined by the set of various experimental conditions and weight coefficients corresponding to them where the coefficients indicate how many experiments should be performed under the condition. In this case, the optimality criteria are represented as various functions defined on the set of information matrices and possessing some statistical sense thus, a design is said to be optimal if such a function attains its extreme.

Mead and Pike (1975) additionally stated that the theory of optimal design produced very strong reactions and that the division between theoretical statisticians researching into the theory of optimal designs and practical statisticians designing experiments for applied research workers was still very wide because the assumptions in the theory of optimal design had been restrictive as linear models were assumed almost exclusively and optimality criterion was based on the generalized variance of the parameter estimates. However, this restrictiveness undoubtedly explains some of the reluctance of practical statisticians to try to produce optimal designs for practical problems.

The ideas of optimum experimental design are also explained through the comparison of the variances of parameter estimates and those of the predicted responses from a variety of designs and models. The existing association between these two sets of variances leads to the general equivalence theorem which, in turn, leads to algorithms for designs and models. The General Equivalence Theorem is the central result on which the optimal design of experiments depend (Atkinson and Donev, 1992). The theorem extensively applies to a variety of design criteria where it presents methods for the construction and checking of optimum designs.

Response Surface Methodology is an efficient mathematical approach that is widely applied in the optimization of experimental responses. Prior work in response surface methodology began with the problem of fitting a curve to the relationship between concentration of a stimulus and the proportion of individuals responding, Fechner (1860). Smith (1918) obtained optimal experimental designs for regression problems. This was followed up by experimental designs whose purpose was to find using the smallest possible number of observations, the point on a response surface at which the optimum output or yield is achieved then Fisher (1935) went on to expound on the development and applications of experimental designs in response surface methodology. Box (1955) published another paper discussing the exploration and exploitation of response surfaces: An example of the link between the fitted surface and the basic mechanism of the system. Box and Hunter (1957) discussed the problem of choosing the N sets of levels at which the observations are to be made and the rotatability conditions for second order designs. Gardiner, Grandage and Hader (1959) gave the rotatability conditions for third order designs. However, Box and Draper

(1959) explained that the nature of the variables whose levels are represented by actual values ξ_{ik} will change from one application of a design to another. In one case, for example ξ_i may refer to a temperature reading and in another to the dosage of a drug. Therefore it is useful to define the general design in terms of "standardized" variables x_i which in any particular application are related linearly to the ξ_i. They also gave the considerations that influence the choice of the design matrix emphasizing two particular cases that arise as; one when the form of the true functional relationship n = $g(\xi,\theta)$ (Which is of course $\underline{\theta}$ not necessarily linear in the parameters $\underline{\theta}$ in the variables ξ) being assumed $\theta_1, \theta_2,\ldots, \theta_p$; and two when the form of the true functional relationship being unknown, the object is to approximate within a given region R of the k dimensions $g(\xi,\underline{\theta})$ by some graduating function $f(\xi_1,\xi_2,\ldots,\xi_k;\beta_1, \beta_2,\ldots,\beta_l)$. The function $f(\underline{\xi},\underline{\beta})$ would often be a polynomial in which case $\underline{\beta}$ would be lx1 vector of polynomial coefficients.

The two different objectives lead to different types of design. Designs for number one may be called designs for estimating parameters; designs for number two may be called designs for exploring a response surface, the surface involved being that defined by a function $\eta = g(\xi_1,\xi_2,\ldots,\xi_k;\ \theta_1,\ \theta_2,\ldots,\ \theta_p) = g(\xi,\theta)$ in the (k+1) dimensional space of the response η, and the variables $\xi_1,\xi_2,\ldots,\xi_k$. Bose and Draper (1959) wrote on a transformation group in three dimensions and its generated point sets which aids the generation of design points in cyclical groups and the formation of rotatable designs by combination of several generated points sets. Draper (1960b) gave six second order rotatable design classes designated D_1 (20 points), D_2 (22 points), D_3 (24 points), D_4 (26 points), D_5 (32 points), and D_6 (30 points) which he combined some into third order rotatable designs. Mutiso (1998) further developed the theory for the optimum estimation of the free letter parameters in the rotatable design point sets using the same designs that had been developed.

2.2.2 Alphabetic Optimality Criteria

An optimality criterion shows how good a design is on either a set of statistical properties or on a particular property. The theory of optimal designs was necessitated by the fact that an experimental design must be selected prior to the experimentation. Pulling together formerly separate entities to build a greater community will always face opponents who fear an assault on their way of thinking Pukelsheim (1993). He went on to explain that his intention was constructive to generate a framework for those design problems that share a common goal. Here the goal of investigating optimal theoretical designs being to provide a gauge for identifying efficient, practical designs. This was influenced by the fact that resources for conducting field experiments have always remained limited hence making it prudent to find the best possible way desired results could be achieved using minimum resources. Smith (1918) gave a criterion and obtained optimal experimental designs for a set of regression problems. This criterion was later called G- optimality by Keifer and Wolfowitz (1959). Later, Wald (1943) brought about the criteria of maximizing the determinant of matrix X'X and Keifer and Wolfowitz called it the D- optimality criterion. Design optimality criteria are basically concerned with optimal properties of the design matrix for the model matrix X. The D- Criterion is the most commonly used and together with A- and E- criteria which were later developed are parameter estimation criteria. Further developments in generating optimality criteria can be found in the works of Elfving (1952) and Chernoff (1953) who minimized the trace of $(X'X)^{-1}$ to obtain regression designs. Ehrenfeld (1955) also suggested that maximizing the minimum eigenvalue of X'X could be used as a criterion. Hoel (1958) by using both the determinant criterion of Wald and the min max criterion of Smith obtained the optimal allocation for a polynomial of degree p-1.Fedorov (1972) discussed the mathematical theory and algorithms for applying the Kiefer-type theory and Box approaches. Atkinson and Fedorov (1975) introduced experimental designs for discriminating between two models and also between several models. There are two choices for defining T–optimality criterion according to the number of models under discrimination. One of the choices is by discriminating between two models and discriminating between several models. Silvey (1980) discussed the theory of parameter estimation and deduced that the theory of design for parameter estimation may well have to be combined with a theory of model validation before its practical potential is fully realized but the discussion was limited to the theory of design optimal for parameter estimation. Pázman (1986) and Mandal (2000) dwelt much on the D- optimality then Yang (2008) gave an algebraic approach for constructing A-optimal design under generalized linear models. Pukelsheim (1993) gave a detailed mathematical discussion that presented a method of computing the optimality criteria where he discussed the D-, A-, E- and T- optimality criteria.

Mutiso (1998) constructed second and third order specific and sequential rotatable designs in 3, 4 and 5 dimensions, later Kosgei (2002) stated that in the analysis of the designs, all the criteria are evaluated with respect to a particular design and the one with the least value is taken as the optimality criterion of that design. He then obtained the optimality criteria for the six specific second order rotatable designs in three dimensions that had been developed by Mutiso.

Rambaei (2014) in the Thesis, “optimal designs for second order rotatability” presented generalized D-, A-, E- and T-optimality criteria for second order rotatable designs in k dimensions. Kiplagat (2016) obtained I- and G- optimal second order rotatable designs in three dimensions and Nyakundi (2016) obtained the optimality criteria of some specific third order rotatable designs. Later, Koech (2017) discussed relative efficiency criteria for specific second order rotatable designs in three dimensions. Mwan (2017) focused on compound optimality criteria which combined two or more alphabetic optimality criteria and considered second order rotatable designs constructed using Balanced Incomplete Block designs (BIBDs). Magangi (2018) constructed modified optimal second order rotatable designs. This study shall use some three specific designs constructed by Mutiso (1998) to obtain their optimality criteria using methods presented by Rambaei (2014).

2.3 DT- Compound Optimality Criteria

By studying an optimality criterion, the experimenter can determine the adequacy of a proposed experimental design prior to running it. There are essentially two ways for the construction of design criteria in Design of Experiments which assimilate different purposes of the experiment. One approach is the construction of new optimality criteria by averaging several competitive design criteria. Alternatively one could try to maximize one primary optimality criteria subject to constraints for specific minimum efficiencies of other criteria, (Dette and Franke, 2000). Again, designs which are efficient for parameter estimation may not provide suitable power to discriminate between the models (Waterhouse et al, 2004). This has posed a great challenge to a statistician who may want to utilize the two properties in one design. Atkinson (2008) stated that the goal of an experiment should be dual: to obtain an adequate model and to estimate the parameters of the selected model efficiently. Unfortunately, this has never been the case a design which is optimum for parameter estimation may be inadequate for model discrimination and vice versa. A common strategy to solve this problem is through combining two alphabetic optimality criteria for model validation with another for parameter estimation in one design.

2.4 Design Efficiency

Efficiency has to do with how well an experimental design answers the research question and numerically it can also be quantified as the value which reflects the ability of a specific design to detect the effect of interest. Kuhfeld (2010) referred to design efficiency as design goodness. The efficiency of an experimental design can be quantified as a function of the variances and covariances of the parameter estimates where the efficiency increases as the variances decrease.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

This chapter discusses the methods and techniques that are used to obtain the alphabetic optimality criteria, DT- compound optimality criterion and design relative efficiencies for the specific second order rotatable designs.

3.2 Optimality Criteria for the Specific Second Order Rotatable Designs

Three specific second order rotatable designs were considered in their three, four and five dimensions.

3.2.1 The Specific Second Order Rotatable Designs in Three, Four and Five Dimensions under Study

This study considered the following designs from Mutiso (1998);

$\mathbf{M_3}$ = (S(0.4899784, 0.4899784, 0.4899784) + S(0.9023011, 0.9023011, 0.9023011) + S(1.5494481,0,0)), is a twenty two points three dimensional SORD.

$\mathbf{M_5}$ = (S(0.6703699, 0. 6703699, 0) + S(0.9359294, 0.9359294, 0.9359294) + S(1.5993168,0,0), is a twenty six points three dimensional SORD.

$\mathbf{M_6}$ = (S(1.3003797, 0.5241245, 0.5241245) + S(0.3357566, 0, 0)), is a thirty points three dimensional SORD.

$\mathbf{M_3^{(1)}}$ = ((S(0.4899784, 0.4899784, 0.4899784), ±0.6599115) + (S(0.9023011, 0.9023011, 0.9023011), ±0.6599115) + (S(0,0,0),±1.9031264) + (0,0,0,±0.1812953)), is a forty eight points four dimensional SORD

$\mathbf{M}_{\mathbf{5}}^{(\mathbf{1})}$= ((S(0.6703699, 0. 6703699, 0), ±0.6647708) +

(S(0.9359294, 0.9359294, 0.9359294), ±0.6647708) +

(S(1.5993168,0,0),±0.6647708) + (0,0,0,±1.990234) + (0,0,0,±0.5172493)), is a fifty six points four dimensional SORD.

$\mathbf{M}_{\mathbf{6}}^{(\mathbf{1})}$= ((S(1.3003797, 0.5241245, 0.5241245), ±0.6654273) +

(S(0.3357566, 0, 0), ±0.6654273) + (0, 0, 0, ±2.0543357) + (0, 0, 0, ±0.8028484)), is a sixty four points four dimensional SORD.

$\mathbf{M}_{\mathbf{3}}^{(\mathbf{2})}$= (((S(0.4899784, 0.4899784, 0.4899784),±0.6599115) ±0.6599115) +

((S(0.9023011, 0.9023011, 0.9023011), ±0.6599115), ±0.6599115) +

((S(1.5494481,0,0),±0.6599115) ±0.6599115) + ((0,0,0,±1.9031264), ±0.6599115) + ((0,0,0,±0.1812953), ±0.6599115) + (0,0,0,0,±2.2400238) + (0,0,0,0,±0.741532)), is a hundred points five dimensional SORD.

$\mathbf{M}_{\mathbf{5}}^{(\mathbf{2})}$ = (((S(0.6703699, 0.6703699, 0),±0.6647708), ±0.6647708) +

((S(0.9359294, 0.9359294, 0.9359294), ±0.6647708), ±0.6647708) +

((S(1.5993168,0,0), ±0.6647708), ±0.6647708) + ((0,0,0,±1.990234) ±0.6647708) + ((0,0,0,±0.5172493),±0.6647708) + (0,0,0,0,±2.3216581) + (0,0,0,0,±1.399032)), is a hundred and sixteen points five dimensional SORD.

$\mathbf{M}_{\mathbf{6}}^{(\mathbf{2})}$ = (((S(1.3003797, 0.5241245, 0.5241245), ±0.6654273) ±0.6654273) +

(S(0.3357566, 0, 0), ±0.6654273) ±0.6654273) + ((0,0,0,±2.0543357) ±0.6654273) + ((0,0,0,±0.8028484) ±0.6654273) + (0,0,0,0,±2.3223592) + (0,0,0,0,±1.6016223)), is a hundred and thirty two points five dimensional SORD.

3.2.2 Second Degree Response Model

A full parameter second degree response model with k factors is represented as follows;

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{ij,i\angle j}^{k} \beta_{ij} x_i x_j + \varepsilon$$

where

β_0 is the intercept

β_i is the linear coefficient for the i^{th} factor

β_{ii} is the quadratic coefficient for the i^{th} factor

β_{ij} is the cross product coefficient for the i^{th} and j^{th} factor

x_i is the level of the i^{th} factor

x_{ij} is the level of the i^{th} and j^{th} factor

ε is the error term.

y is the response value.

In matrix notation, it can be expressed as;

$$\underline{Y} = X^{/} \underline{\beta} + \underline{\varepsilon}$$

Where

$\underline{y} = (y_1, y_2, \ldots, y_N)$ is an $N \times 1$ vector of response values.

$\underline{x} = (x_1, x_2, \ldots, x_N)$ is an $N \times k$ model matrix or matrix of observation

$\underline{\beta} = \beta = (\beta_1, \beta_2, \ldots \beta_k)$ is a $k \times 1$ vector of parameter and

$\underline{\varepsilon} = (\varepsilon_1, \varepsilon_2 \ldots, \varepsilon_N)$ is an $N \times 1$ vector of errors and where the random errors ε_u's are independently and identically distributed with mean 0 and variance σ^2, that is,

$E(\varepsilon_u) = 0$, var $(\varepsilon_u) = \sigma^2$ and cov $(\varepsilon_u, \varepsilon_u{}^{/}) = 0$.

This study shall consider the parameter system of interest where;

$$y = \sum_{i=1}^{k} B_i x_i + \frac{1}{k}\sum_{i=1}^{k} B_{ii}x_{ii} + \frac{1}{\binom{k}{2}}\sum_{i=1}^{k} B_{ij}x_{ij} + \varepsilon_{ij}$$

3.2.3 Moment, Design and Information Matrices

Matrix $\sum_{i=1}^{l} n_i x_i x_i^{/} = X^{/}X$ is called the moment matrix of the design ξ_n and is denoted by $M(\xi_n)$, that is, the moment matrix of a design $\xi \in \equiv$ is the $k \times k$ matrix defined by

$M(\xi) = \sum \xi(x) xx^{/} = \int_x xx^{/} d\xi$ where $x_i^{/}$ is the *i*th regression vector.

In simple notation, the moment matrix is given by

$$M = \frac{X^{/}X}{N}$$

where *X* is the design matrix.

For a design ξ with moment matrix *M*, the information matrix for $K^{/}\theta$ with $k \times s$ coefficient matrix *K* of full column rank *s* is defined as

$$C_K(M) = \left(K^{/}M^{-1}K\right)^{-1}$$

3.2.3.1 Moment Matrix for SORD in Three Factors

The moment matrix for second degree rotatable design in three factors is

$$\frac{X'X}{N}=M=\begin{bmatrix} 1 & \lambda_2 & \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_4 & 3\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_4 & \lambda_4 & 3\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

3.2.3.2 Moment Matrix for SORD in Four Factors

The moment matrix for second degree rotatable design in four factors from Rambaei (2014) is

$$\frac{X'X}{N}=M=\begin{bmatrix} 1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_4 & 3\lambda_4 & \lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_4 & \lambda_4 & 3\lambda_4 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_2 & \lambda_4 & \lambda_4 & \lambda_4 & 3\lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

3.2.3.3 Moment Matrix for SORD in Five Factors

The moment matrix for a second order rotatable design with 5 factors as given by Rambaei (2014) is;

$$\frac{X'X}{N} = M = \begin{bmatrix} B & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix}$$

$$\text{Where } B = \begin{bmatrix} 1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\ \lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_4 & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_4 & \lambda_4 & 3\lambda_4 & \lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_4 & \lambda_4 & \lambda_4 & 3\lambda_4 & \lambda_4 \\ \lambda_2 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & 3\lambda_4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\text{And } A_2 = \begin{bmatrix} \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

3.2.3.4 Information Matrix

The generalized information matrix as given by Pukelsheim (1993) is

$C_k\left(M\right)=\left[K_k{}^{/}M_k{}^{-1}K_k\right]^{-1}$

Where k is the number of factors.

$$
=\begin{bmatrix}
1 & k\lambda_2 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
k\lambda_2 & k\left(k+2\right)\lambda_4 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & \lambda_2 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \lambda_2 & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \binom{k}{2}\lambda_4
\end{bmatrix}_{(k+3)(k+3)}
$$

The optimality criteria are based on how well parameters or a response are estimated or researched. Design optimality criteria are primarily concerned with optimal properties of the $X^{/}X$ matrix for the design matrix X.

The methods of evaluation of the particular criteria given by Pukelsheim (1993) and Rambaei (2014) provided a generalized formula for working out the D-, A-, T- and E-criteria.

3.2.4 D- Criterion

The most extensively used criterion is the D-criterion for which

$\phi(M)=$log det M, if M is non-singular

=$-\infty$ otherwise

The determinant criterion ϕ_o(C) differs from the determinant det(C) by taking the s^{th} root whence both functions induce the same preordering among information matrices. For comparing different criteria, and applying the theory of information functions, the version

$$\phi_o(C) = (detC)^{1/s}$$

Where s is the number of model parameters.

is appropriate.

Maximizing the determinant of information matrices is the same as minimizing the determinant of dispersion matrices, because of the formula

$$(\det C)^{-1}=\det(C^{-1})$$

The Generalized Determinant Criterion as given by Rambaei (2014) is;

$$\phi_0 C_k\left(M\right)=\left[k\begin{bmatrix}k\\2\end{bmatrix}\lambda_2{}^k\lambda_4\left[\left[k+2\right]\lambda_4-k\lambda_2{}^2\right]\right]^{\frac{1}{k+3}}. \qquad (3.1)$$

Where

$$\begin{bmatrix}k\\2\end{bmatrix} = C_{(k,2)} = \frac{k!}{2!(k-2)!} \quad ,$$

k is the dimension of the specific design, hence k=3 for SORD in three dimensions, k=4 for SORD in four dimensions and k=5 for SORD in five dimensions.

3.2.5 A- Criterion

This is the average of the standardized variances of the optimal estimators for the scalar parameter systems $c_1\theta, \ldots, c_s\theta$ formed from the columns of K The average variance criterion ϕ_{-1}(C) is given by

$$\phi_{-1}(\text{C})=\left(\frac{1}{s}traceC^{-1}\right)^{-1}$$

if C is positive definite

The generalized Average Variance Criterion for 2nd Degree Design as given by Rambaei (2014) is;

$$\phi_{-1}C_k\left(M\right)=\left[\frac{1}{k+3}\left[\frac{\left[k+2\right]\lambda_4}{\left[k+2\right]\lambda_4-k\lambda_2{}^2}+\frac{1}{k\left[\left[k+2\right]\lambda_4-k\lambda_2{}^2\right]}+\frac{k}{\lambda_2}+\frac{1}{\begin{bmatrix}k\\2\end{bmatrix}\lambda_4}\right]\right]^{-1}. \qquad (3.2)$$

Where

$\begin{bmatrix}k\\2\end{bmatrix} = C_{(k,2)} = \frac{k!}{2!(k-2)!}$,

k is the dimension of the specific design, hence k=3 for SORD in three dimensions, k=4 for SORD in four dimensions and k=5 for SORD in five dimensions.

3.2.6 T- Criterion

The trace-criterion is the extreme members of the ϕ_1-family. According to Pukelsheim (1993) the trace criterion by itself is rather meaningless because of its linearity property which makes it susceptible to interpolation.

The evaluation of the trace criterion is given by;

$\phi_1(C) = \frac{1}{s} trace(C)$

The generalized Trace or T – criterion as given by Rambaei (2014) is;

$$\phi_1 C_k(M) = \frac{1}{k+3}\left[1+(k+2)k\lambda_4 + k\lambda_2 + \binom{k}{2}\lambda_4\right]. \quad (3.3)$$

Where

$\begin{bmatrix} k \\ 2 \end{bmatrix} = C_{(k,2)} = \frac{k!}{2!(k-2)!}$,

k is the dimension of the specific design, hence k=3 for SORD in three dimensions, k=4 for SORD in four dimensions and k=5 for SORD in five dimensions.

3.2.7 E- Criterion

The smallest eigenvalue criterion, $\phi_{-\infty}$(C) is one extreme member of the matrix means family $\phi_p(C)$, corresponding to the parameter p=-∞. This criterion involves the evaluation of the smallest eigenvalue.

The smallest eigenvalue criterion

$\phi_{-\infty}$(C)=$\lambda_{(\min)}(C)$

It is the same as minimizing the largest eigenvalue of the information matrix. In terms of variance, it is a minimax approach, in terms of information a maxmin approach. This criterion plays a crucial role in the admissibility investigations.

The Generalized Smallest Eigenvalue Criterion for 2nd Degree Design as given by Rambaei (2014) is;

$\phi_{-\infty} C_k(M) = \min(\gamma)$

Given $[(1-\gamma)[(k+2)k\lambda_4 - \gamma] - (k\lambda_2)^2]\,(\lambda_2 - \gamma)^k \left[\binom{k}{2}\lambda_4 - \gamma\right] = 0$ (3.4)

Where

$\begin{bmatrix} k \\ 2 \end{bmatrix} = C_{(k,2)} = \frac{k!}{2!(k-2)!}$,

k is the dimension of the specific design, hence k=3 for SORD in three dimensions, k=4 for SORD in four dimensions and k=5 for SORD in five dimensions.

3.3 DT- Compound Optimality Criteria for $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

This study combines two alphabetic optimality criteria D- and T- by using the concept that was introduced by Atkinson (2008), where DT- optimality criterion is a combination of D-optimality criterion for parameter estimation with the T-optimality criterion for discriminating between models. The DT- criterion provides a specified balance between model discrimination and parameter estimation.

The Generalized Determinant and Trace Criteria were given by Rambaei (2014) respectively as;

$$\phi_0 C_k(M) = \left[k \begin{bmatrix} k \\ 2 \end{bmatrix} \lambda_2^{\,k} \lambda_4 \left[[k+2]\lambda_4 - k\lambda_2^{\,2} \right] \right]^{\frac{1}{k+3}}$$

(3.1)

$$\phi_1 C_k(M) = \frac{1}{k+3}\left[1 + (k+2)k\lambda_4 + k\lambda_2 + \binom{k}{2}\lambda_4 \right] \tag{3.3}$$

From (Rambaei, 2014)

The criterion to be maximized according to Atkinson (2008) will be given by the formula;

$$\emptyset_2^{DT}(\varepsilon) = (1\text{-k}) \log \Delta_1(\varepsilon) + (\frac{k}{p_1}) \log|m_1(\varepsilon)|. \tag{3.5}$$

where $\emptyset_2^{DT}(\varepsilon)$ is a combination of two design criteria, the first criterion is log $\Delta_1(\varepsilon)$ which is the logarithm of T- optimality and the second log $|m_1(\varepsilon)|$ is also the logarithm of D- optimality.

Designs maximizing (3.3) are called DT-optimum designs. The quantities in (3.1) and (3.4) will be substituted in (3.5) to obtain the DT-optimality criterion.

3.3 Relative Efficiency of M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

3.3.1 Relative D-efficiency

The Relative D-efficiency is closely associated with the D- Optimality Criteria and is defined by Burgess (2004) as;

$$D_{eff} = \left|\frac{M(\varepsilon^*)}{M(\varepsilon)}\right| \tag{3.6}$$

Where

$$M(\varepsilon^*) \tag{3.7}$$

is the value of the Determinant Criterion of a set of designs under study and

$$M(\varepsilon) \tag{3.8}$$

is the numerical value of a specific D-optimal design.

3.3.2 Relative A- Efficiency

This measure is related to the A- Optimality Criteria and is given by;

$$A(\xi) = \frac{tr(m^{-1}(\varepsilon_A^*))}{tr(m^{-1}(\varepsilon))} \tag{3.9}$$

Where

$$tr(m^{-1}(\varepsilon_A^*)) \tag{3.10}$$

is the A- optimal numerical value of a particular set of designs and

$$tr(m^{-1}(\varepsilon)) \tag{3.11}$$

is the computed value of a specific A- Optimal design.

3.3.3 Relative T-efficiency

The Relative T- efficiency is associated with the T- Optimality Criterion and is given by;

$$\frac{\Delta_1(\varepsilon_T^*)}{\Delta_1(\varepsilon)}. \tag{3.12}$$

Where

$$\Delta_1(\varepsilon_T^*) \tag{3.13}$$

is the value of the T- Optimal design and

$$\Delta_1(\varepsilon) \tag{3.14}$$

is the value of a specific T- Optimal design.

3.3.4 Relative E- Efficiency

This measure as defined by is related to the E- Optimality Criterion and is given by;

$$E(\xi) = \frac{\lambda_{min}(\varepsilon)}{\lambda_{min}(\varepsilon^*)} \tag{3.15}$$

Where

$$\lambda_{min}(\varepsilon) \tag{3.16}$$

is the E- optimal numerical value of a particular set of designs and

$$\lambda_{min}(\varepsilon^*) \tag{3.17}$$

is the value of a specific E- Optimal design.

3.3.5 Relative DT-efficiency

The Relative DT-efficiency of any design is given by;

$$\left[\frac{(1-k)\log \Delta_1(\varepsilon) + (\frac{k}{p_1})\log |m_1(\varepsilon)|.}{(1-k)\log \Delta_1(\varepsilon^*) + (\frac{k}{p_1})\log |m_1(\varepsilon^*)|.}\right]. \tag{3.18}$$

Where

p is the number of parameters,

$$(1-\mathrm{k})\log\Delta_1(\varepsilon)\ +\ (\frac{k}{p_1})\log|m_1(\varepsilon)| \tag{3.19}$$

is the value of a specific DT- design,

and $$(1-\mathrm{k})\log\Delta_1(\varepsilon^*)\ +\ (\frac{k}{p_1})\log|m_1(\varepsilon^*)| \tag{3.20}$$

is the value of the DT- optimal design and k is the dimension of the design.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.1 Introduction

This chapter entails the evaluation of the alphabetic optimality criteria for three specific SORDs in three, four and five dimensions. The compound DT- optimality criteria of the specific designs are obtained and the relative efficiencies are also determined.

4.2 Evaluation of the Alphabetic Optimality Criteria for the Six Specific Second Order Rotatable Designs in Three, Four and Five Dimensions.

The optimality criteria of the designs were computed as follows;

4.2.1 Particular Criteria for the Twenty Two Points Three Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

Consider the design from Mutiso (1998);

M_3 = (S(0.4899784, 0.4899784, 0.4899784) + S(0.9023011, 0.9023011, 0.9023011) + S(1.5494481,0,0))

Where $\lambda_2^2 = 0.0627942c^4$

$\lambda_4 = 0.0454545c^4$

The estimate of the free parameter c in M_3 from Mutiso (1998) is also given as;

$$c = 1.5494481 \tag{4.1}$$

Thus,

$$\lambda_2 = 0.6016085 \tag{4.2}$$

$$\lambda_4 = 0.26199018 \tag{4.3}$$

4.2.1.1 D- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.2) and (4.3) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[3\begin{bmatrix}3\\2\end{bmatrix}(0.6016085)^3(0.26199018)\left[[3+2](0.26199018) - 3(0.6016085)^2\right]\right]^{\frac{1}{3+3}}$$

=0.69744186

4.2.1.2 A- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.2) and (4.3) in equation (3.2) we have;

$$\emptyset_{-1} C_k(M) = \left[\frac{1}{3+3}\left[\frac{[3+2](0.2619902)}{[3+2](0.2619902) - 3(0.6016085)^2} + \frac{1}{3\left[[3+2](0.2619902) - 3((0.6016085)^2\right]} + \frac{3}{0.6016085} + \frac{1}{\begin{bmatrix}3\\2\end{bmatrix}(0.2619902)}\right]\right]^{-1}$$

=0.443092

4.2.1.3 T- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.2) and (4.3) in equation (3.3) we have;

$$\emptyset_1 C_k(M) = \frac{1}{3+3}\left[1 + (3+2)(3)(0.2619902) + 3(0.6016085) + \binom{3}{2}(0.2619902)\right]$$

=1.2534415

4.2.1.4 E- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.2) and (4.3) in equation (3.4) we have;

$$\emptyset_{-\infty} C_k(M) = \left[(1-\gamma)\,[(3+2)\,3(0.2619902) - \gamma] - [3(0.6016085)]^2\right](0.6016085 - \gamma)^3\left[\binom{3}{2}(0.2619902) - \gamma\right]$$

= 0

Solving for γ gives;

Either γ = 4.78944898, **0.14040422**, 0.6016085 or 0.7859706

The smallest Eigen value from the results above is 0.14040422

4.2.2 Particular Criteria for the Twenty Six points Three Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the design is given as;

M_5 = (S(0.6703699, 0. 6703699, 0) + S(0.9359294, 0.9359294, 0.9359294) +

S(1.5993168,0,0)

Where $\lambda_2^2 = 0.4763251 a_4^4$

$\lambda_4 = 0.3481846 a_4^4$

The estimate of the free parameter a_4 in M_5 from Mutiso (1998) is also given as;

$$a_4 = 0.9359294 \tag{4.4}$$

Thus,

$$\lambda_2 = 0.6045578 \tag{4.5}$$

$$\lambda_4 = 0.2671664 \tag{4.6}$$

4.2.2.1 D- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.5) and (4.6) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (0.6045578)^3 (0.2671664) \left[[3+2](0.2671664) - 3(0.6045578)^2 \right] \right]^{\frac{1}{3+3}}$$

$$= 0.7091401$$

4.2.2.2 A- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.5) and (4.6) in equation (3.2) we have;

$$\emptyset_{-1}C_k(M) = \left[\frac{1}{3+3}\left[\frac{[3+2](0.2671664)}{[3+2](0.2671664)-3(0.6045578)^2} + \frac{1}{3\left[[3+2](0.2671664)-3((0.6045578)^2\right]} + \frac{3}{0.6045578} + \frac{1}{\begin{bmatrix}3\\2\end{bmatrix}(0.2671664)}\right]\right]^{-1}$$

$=0.3757384$

4.2.2.3 T- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.5) and (4.6) in equation (3.3) we have;

$$\emptyset_1C_k(M) = \frac{1}{3+3}\left[1 + (3+2)(3)(0.2671664) + 3(0.6045578) + \binom{3}{2}(0.2671664)\right]$$

$=1.270445$

4.2.2.4 E- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.5) and (4.6) in equation (3.4) we have;

$\emptyset_{-\infty}C_k(M) =$

$\left[(1 - \gamma)\ [(3+2)\ 3(0.2671664) - \gamma] - [3(0.6045578)]^2\right](0.6045578 - \gamma)^3\ \left[\binom{3}{2}(0.2671664) - \gamma\right]$

$=0$

Solving for γ gives;

Either γ = 4.859733791, **0.147762209**, 0.6045578 or 0.8014992

The smallest Eigen value from the results above is 0.147762209

4.2.3 Particular Criteria for the Thirty Points Three Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the specific design is given as;

M_6 = (S(1.3003797, 0.5241245, 0.5241245) + S(0.3357566, 0, 0))

Where $\lambda_2^2 = 28.797104c_6^4$

$\lambda_4 = 21.077865c_6^4$

The estimate of the free parameter c_6 in M_6 from Mutiso (1998) is also given as;

$c_6 = 0.3357566$ (4.7)

Thus,

$\lambda_2 = 0.6049556$ (4.8)

$\lambda_4 = 0.2678705$ (4.9)

4.2.3.1 D- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.8) and (4.9) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} (0.6049556)^3 (0.2678705) \left[[3+2](0.2678705) - 3(0.6049556)^2 \right] \right]^{\frac{1}{3+3}}$$

=0.7107073

4.2.3.2 A- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.8) and (4.9) in equation (3.2) we have;

$$\emptyset_{-1} C_k(M) = \left[\frac{1}{3+3} \left[\frac{[3+2](0.2678705)}{[3+2](0.2678705) - 3(0.6049556)^2} + \frac{1}{3\left[[3+2](0.2678705) - 3((0.6049556)^2\right]} + \frac{3}{0.6045578} + \frac{1}{\begin{bmatrix} 3 \\ 2 \end{bmatrix}(0.2678705)} \right] \right]^{-1}$$

=0.4569194

4.2.3.3 T- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.8) and (4.9) in equation (3.3) we have;

$$\emptyset_1 C_k(M) = \frac{1}{3+3}\left[1 + (3+2)(3)(0.2678705) + 3(0.6049556) + \binom{3}{2}(0.2678705)\right]$$

=1.272756

4.2.3.4 E- Criterion

Substituting for the values of λ_2 and λ_4 given by (4.8) and (4.9) in equation (3.4) we have;

$$\emptyset_{-\infty} C_k(M) = \left[(1 - \gamma)\,[(3+2)\,3(0.2678705) - \gamma] - [3(0.6049556)]^2\right](0.6049556 - \gamma)^3\left[\binom{3}{2}(0.2678705) - \gamma\right]$$

=0

Solving for γ gives;

Either γ = 4.869306227, **0.148751273**, 0.6049556 or 0.8036115

The smallest Eigen value from the results above is 0.148751273

4.2.4 Particular Criteria for the Forty Eight Points Four Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

Consider the design from Mutiso (1998);

$M_3^{(1)}$ = ((S(0.4899784, 0.4899784, 0.4899784), ±0.6599115) +

(S(0.9023011, 0.9023011, 0.9023011), ±0.6599115) +

(S(0,0,0),±1.9031264) + (0,0,0,±0.1812953))

Where $\lambda_2^{(1)2} = 0.0527645c^4$

$\lambda_4^{(1)} = 0.020833c^4$

The estimate of the free parameter c in $M_3^{(1)}$ is as given in (4.1)

Thus,

$$\lambda_2^{(1)} = 0.551473922 \qquad (4.10)$$

$$\lambda_4^{(1)} = 0.240157908 \qquad (4.11)$$

4.2.4.1 D- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.10) and (4.11) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[4\begin{bmatrix}4\\2\end{bmatrix}(0.5514739)^4(0.2401579)\left[[4+2](0.2401579) - 4(0.5514739)^2\right]\right]^{\frac{1}{4+3}}$$

=0.7383739

4.2.4.2 A- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.10) and (4.11) in equation (3.2) we have;

$$\emptyset_{-1}C_k(\text{ M }) = \left[\frac{1}{4+3}\left[\frac{[4+2](0.2401579)}{[4+2](0.2401579)\ -4(0.5514739)^2} + \frac{1}{4\left[[4+2](0.2401579)-4((0.5514739)^2\right]} + \frac{4}{0.5514739} + \frac{1}{\begin{bmatrix}4\\2\end{bmatrix}(0.2401579)}\right]\right]^{-1}$$

=0.4521702

4.2.4.3 T- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.10) and (4.11) in equation (3.3) we have;

$$\emptyset_1C_k(\text{ M }) = \frac{1}{4+3}\left[1 + (4+2)(4)(0.2401579) + 4(0.5514739) + \binom{4}{2}(0.2401579)\right]$$

=1.487233229

4.2.4.4 E- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.10) and (4.11) in equation (3.4) we have;

$$\emptyset_{-\infty}C_k(M) = \left[(1 - \gamma)\,[(4+2)\,4(0.2401579) - \gamma] - [4(0.5514739)]^2\right](0.5514739 - \gamma)^4 \left[\binom{4}{2}(0.2401579) - \gamma\right]$$

$= 0$

Solving for γ gives;

Either γ = 6.6283387, **0.1354509**, 0.5514739 or 1.4409474

The smallest Eigen value from the results above is **0.1354509**

4.2.5 Particular Criteria for the Fifty Six Points Four Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the design is given as;

$M_5^{(1)}$= ((S(0.6703699, 0. 6703699, 0), ±0.6647708) +

(S(0.9359294, 0.9359294, 0.9359294), ±0.6647708) +

(S(1.5993168,0,0),±0.6647708) + (0,0,0,±1.990234) + (0,0,0,±0.5172493))

Where $\lambda_2^{(1)2}$ = 0.4107089 a_4^4

$\lambda_4^{(1)}$= 0.3233142 a_4^4

The estimate of the free parameter a_4 in $M_5^{(1)}$ is as given in (4.4)

Thus,

$$\lambda_2^{(1)} = 0.5613752 \tag{4.12}$$

$$\lambda_4^{(1)} = 0.2480831 \tag{4.13}$$

4.2.5.1 D- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.12) and (4.13) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[4\begin{bmatrix}4\\2\end{bmatrix}(0.5613752)^4(0.2480831)\left[[4+2](0.2480831) - 4(0.5613752)^2\right]\right]^{\frac{1}{4+3}}$$

=0.7510409

4.2.5.2 A- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.12) and (4.13) in equation (3.2) we have;

$$\emptyset_{-1} C_k(M) = \left[\frac{1}{4+3}\left[\frac{[4+2](0.2480831)}{[4+2](0.2480831) - 4(0.5613752)^2} + \frac{1}{4\left[[4+2](0.2480831) - 4((0.5613752)^2\right]} + \frac{4}{0.5613752} + \frac{1}{\begin{bmatrix}4\\2\end{bmatrix}(0.2480831)}\right]\right]^{-1}$$

=0.4538232

4.2.5.3 T- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.12) and (4.13) in equation (3.3) we have;

$$\emptyset_1 C_k(M) = \frac{1}{4+3}\left[1+(4+2)(4)(0.2480831)+4(0.5613752)+\binom{4}{2}(0.2480831)\right]$$

=1.5268563

4.2.5.4 E- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.12) and (4.13) in equation (3.4) we have;

$$\emptyset_{-\infty} C_k(M) = \left[(1-\gamma)\,[(4+2)\,4(0.2480831)-\gamma]-[4(0.5613752)]^2\right](0.5613752-\gamma)^4\left[\binom{4}{2}(0.2480831)-\gamma\right]$$

=0

Solving for γ gives;

Either γ = 6.8203173, **0.1336771**, 0.5613752 or 1.4884986

The smallest Eigen value from the results above is **0.1336771**

4.2.6 Particular Criteria for the Sixty Four Points Four Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the specific design is given as;

$M_6^{(1)}$= ((S(1.3003797, 0.5241245, 0.5241245), ±0.6654273) +

(S(0.3357566, 0, 0), ±0.6654273) + (0, 0, 0, ±2.0543357) + (0, 0, 0, ±0.8028484))

Where $\lambda_2^{(1)2}$ = 25.309955c_6^4

$\lambda_4^{(1)}$= 19.760498c_6^4

The estimate of the free parameter c_6 in $M_6^{(1)}$ is as given in (4.7)

Thus,

$$\lambda_2^{(1)} = 0.5671459 \tag{4.14}$$

$$\lambda_4^{(1)} = 0.2511286 \tag{4.15}$$

4.2.6.1 D- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.14) and (4.15) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[4\begin{bmatrix}4\\2\end{bmatrix}(0.5671459)^4(0.2511286)\left[[4+2](0.2511286) - 4(0.5671459)^2\right]\right]^{\frac{1}{4+3}}$$

$$=0.7530118$$

4.2.6.2 A- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.14) and (4.15) in equation (3.2) we have;

$$\emptyset_{-1}C_k(M) = \left[\frac{1}{4+3}\left[\frac{[4+2](0.2511286)}{[4+2](0.2511286)-4(0.5671459)^2} + \frac{1}{4\left[[4+2](0.2511286)-4((0.5671459)^2\right]} + \frac{4}{0.5671459} + \frac{1}{\begin{bmatrix}4\\2\end{bmatrix}(0.2511286)}\right]\right]^{-1}$$

=0.4459656

4.2.6.3 T- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.14) and (4.15) in equation (3.3) we have;

$$\emptyset_{1}C_k(M) = \frac{1}{4+3}\left[1+(4+2)(4)(0.2511286)+4(0.5671459)+\binom{4}{2}(0.2511286)\right]$$

=1.5432059

4.2.6.4 E- Criterion

Substituting for the values of $\lambda_2^{(1)}$ and $\lambda_4^{(1)}$ given by (4.14) and (4.15) in equation (3.4) we have;

$$\emptyset_{-\infty}C_k(M) = \left[(1-\gamma)\left[(4+2)\,4(0.2511286)-\gamma\right]-\left[4(0.5671459)\right]^2\right](0.5671459-\gamma)^4\left[\binom{4}{2}(0.2511286)-\gamma\right]$$

$$=0$$

Solving for γ gives;

Either γ = 6.4438133, 0.5832731, **0.5671459** or 1.5067716

The smallest Eigen value from the results above is **0.5671459**

4.2.7 Particular Criteria for the Hundred Points Five Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the specific Design is given as;

$M_3^{(2)}$= (((S(0.4899784, 0.4899784, 0.4899784),±0.6599115) ±0.6599115) +

((S(0.9023011, 0.9023011, 0.9023011), ±0.6599115), ±0.6599115) +

((S(1.5494481,0,0),±0.6599115) ±0.6599115) + ((0,0,0,±1.9031264), ±0.6599115) + ((0,0,0,±0.1812953), ±0.6599115) + (0,0,0,0,±2.2400238) + (0,0,0,0,±0.741532))

Where $\lambda_2^{(2)2}$ = 0.0486278c^4

$\lambda_4^{(2)}$= 1.5494481c^4

The estimate of the free parameter c in $M_3^{(2)}$ is as given in (4.1)

Thus,

$$\lambda_2^{(2)} = 0.5294152 \tag{4.16}$$

$$\lambda_4^{(2)} = 0.2305516 \tag{4.17}$$

4.2.7.1 D- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.16) and (4.17) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[5\begin{bmatrix}5\\2\end{bmatrix}(0.5294152)^5(0.2305516)\left[[5+2](0.2305516) - 5(0.5294152)^2\right]\right]^{\frac{1}{5+3}}$$

=0.7516218

4.2.7.2 A- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.16) and (4.17) in equation (3.2) we have;

$$\emptyset_{-1} C_k(M) = \left[\frac{1}{5+3}\left[\frac{[5+2](0.2305516)}{[5+2](0.2305516) - 5(0.5294152)^2} + \frac{1}{5\left[[5+2](0.2305516) - 5((0.5294152)^2\right]} + \frac{5}{0.5294152} + \frac{1}{\begin{bmatrix}5\\2\end{bmatrix}(0.2305516)}\right]\right]^{-1}$$

=0.4344145

4.2.7.3 T- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.16) and (4.17) in equation (3.3) we have;

$$\emptyset_1 C_k(M) = \frac{1}{5+3}\left[1 + (5+2)(5)(0.2305516) + 5(0.5294152) + \binom{5}{2}(0.2305516)\right]$$

=1.7527373

4.2.7.4 E- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.16) and (4.17) in equation (3.4) we have;

$$\emptyset_{-\infty} C_k(M) = \left[(1-\gamma)\,[(5+2)\,5(0.2305516)-\gamma] - [5(0.5294152)]^2\right](0.5294152 - \gamma)^4\left[\binom{5}{2}(0.2305516) - \gamma\right]$$

= 0

Solving for γ gives;

Either γ = 8.2622761, 0.8070299, **0.5294152** or 2.305516

The smallest Eigen value from the results above is 0.5294152

4.2.8 Particular Criteria for the Hundred Sixteen Points Five Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the specific design is given as;

$M_5^{(2)}$ = (((S(0.6703699, 0.6703699, 0),±0.6647708), ±0.6647708) +

((S(0.9359294, 0.9359294, 0.9359294), ±0.6647708), ±0.6647708) +

((S(1.5993168,0,0), ±0.6647708), ±0.6647708) + ((0,0,0,±1.990234) ±0.6647708) + ((0,0,0,±0.5172493),±0.6647708) + (0,0,0,0,±2.3216581) + (0,0,0,0,±1.399032))

Where $\lambda_2^{(2)2} = 0.3828724a_4^4$

$\lambda_4^{(2)} = 0.3121654a_4^4$

The estimate of the free parameter a_4 in $M_5^{(2)}$ is as given in (4.4)

Thus,

$$\lambda_2^{(2)} = 0.5420174 \tag{4.18}$$

$$\lambda_4^{(2)} = 0.2395285 \tag{4.19}$$

4.2.8.1 D- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.18) and (4.19) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[5\begin{bmatrix}5\\2\end{bmatrix}(0.5420174)^5(0.2395285)\left[[5+2](0.2395285) - 5(0.5420174)^2\right]\right]^{\frac{1}{5+3}}$$

$$=0.7642774$$

4.2.8.2 A- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.18) and (4.19) in equation (3.2) we have;

$\emptyset_{-1}$Ck(M) =

$$\left[\frac{1}{5+3}\left[\frac{[5+2](0.2395285)}{[5+2](0.2395285)\ -5(0.5420174)^2} + \frac{1}{5\left[[5+2](0.2395285)-5((0.5420174)^2\right]} + \frac{5}{0.5420174} + \frac{1}{\left[\begin{smallmatrix}5\\2\end{smallmatrix}\right](0.2395285)}\right]\right]^{-1}$$

=0.4283984

4.2.8.3 T- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.18) and (4.19) in equation (3.3) we have;

$\emptyset_1$Ck(M) = $\frac{1}{5+3}$

$$\left[1 + (5+2)(5)(0.2395285) + 5(0.5420174) + \binom{5}{2}(0.2395285)\right]$$

=1.8123587

4.2.8.4 E- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.18) and (4.19) in equation (3.4) we have;

$\emptyset_{-\infty}C_k(M) =$
$$\left[(1 - \gamma)\ [(5+2)\ 5(0.2395285) - \gamma] - [5(0.5420174)]^2\right](0.5420174 - \gamma)^5\ \left[\binom{5}{2}(0.2395285) - \gamma\right]$$
$=0$

Solving for γ gives;

Either γ = 8.5773534, 0.8061441, **0.5420174** or 2.3952850

The smallest Eigen value from the results above is **0.5420174**

4.2.9 Particular Criteria for the Hundred Thirty Two Points Five Dimensional Second Order Rotatable Design Using Free Letter Parameter Estimates.

From Mutiso (1998) the specific Design is given as;

$M_6^{(2)}$ = (((S(1.3003797, 0.5241245, 0.5241245), ±0.6654273) ±0.6654273) + (S(0.3357566, 0, 0), ±0.6654273) ±0.6654273) + ((0,0,0,±2.0543357) ±0.6654273) + ((0,0,0,±0.8028484) ±0.6654273) + (0,0,0,0,±2.3223592) + (0,0,0,0,±1.6016223))

Where $\lambda_2^{(2)2}$ = $23.79926c_6^4$

$\lambda_4^{(2)}$= $19.161696c_6^4$

The estimate of the free parameter c_6 in $M_6^{(2)}$ is as given in (4.7)

Thus,

$$\lambda_2^{(2)} = 0.5499597 \tag{4.20}$$

$$\lambda_4^{(2)} = 0.2435186 \tag{4.21}$$

4.2.9.1 D- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.20) and (4.21) in equation (3.1) we have;

$$\emptyset_0 C_k(M) = \left[5\begin{bmatrix}5\\2\end{bmatrix}(0.5499597)^5(0.2435186)\left[[5+2](0.2435186) - 5(0.5499597)^2\right]\right]^{\frac{1}{5+3}}$$

=0.7654320

4.2.9.2 A- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.20) and (4.21) in equation (3.2) we have;

$$\emptyset_{-1} C_k(M) = \left[\frac{1}{5+3}\left[\frac{[5+2](0.2435186)}{[5+2](0.2435186) - 5(0.5499597)^2} + \frac{1}{5\left[[5+2](0.2435186) - 5((0.5499597)^2\right]} + \frac{5}{0.5499597} + \frac{1}{\begin{bmatrix}5\\2\end{bmatrix}(0.2435186)}\right]\right]^{-1}$$

=0.4122856

4.2.9.3 T- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.20) and (4.21) in equation (3.3) we have;

$$\emptyset_1 C_k(M) = \frac{1}{5+3}\left[1 + (5 + 2)(5)(0.2435186) + 5(0.5499597) + \binom{5}{2}(0.2435186)\right]$$

$=1.8385169$

4.2.9.4 E- Criterion

Substituting for the values of $\lambda_2^{(2)}$ and $\lambda_4^{(2)}$ given by (4.20) and (4.21) in equation (3.4) we have;

$$\emptyset_{-\infty} C_k(M) = \left[(1 - \gamma)\ [(5 + 2)\ 5(0.2435186) - \gamma] - [5(0.5499597)]^2\right](0.5499597 - \gamma)^4\ \left[\binom{5}{2}(0.2435186) - \gamma\right]$$

$=0$

Solving for γ gives;

Either γ = 8.7190657, 0.80408531, **0.5499597** or 2.435186

The smallest Eigen value from the results above is **0.5499597**

Table 4.1: Summary of the Alphabetic Optimality Criteria For M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Design Dimension	Design	Optimality Criteria			
		D-	A-	T-	E-
Three	M_3	0.6974419	0.443092	1.2534415	**0.14040422**
	M_5	0.7091401	0.3757384	1.270445	0.1477622
	M_6	0.7107073	0.4569194	1.272756	0.1487512
Four	$M_3^{(1)}$	0.7383739	0.4521702	1.4872332	0.1354509
	$M_5^{(1)}$	0.7510409	0.4538232	1.5268563	**0.1336771**
	$M_6^{(1)}$	0.7530118	0.4459656	1.5432059	0.5671459
Five	$M_3^{(2)}$	0.7516218	0.4344145	1.7527373	0.5294152
	$M_5^{(2)}$	0.7642774	0.4283984	1.8123589	0.5420174
	$M_6^{(2)}$	0.7654320	**0.4122856**	1.8385169	0.5499597

All three and four dimensional designs are E- optimal whereas the five dimensional designs are A- optimal.

4.3 DT- Compound Optimality

The DT- Compound Optimality Criterion which combines D- optimality and T- optimality is as given in (3.5)

4.3.1 DT- Compound Optimality Criteria for M_3 (Twenty Two Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-3) log 1.2534415 + ($\frac{3}{6}$) log 0.6974419

= -0.274454

4.3.2 DT- Compound Optimality Criteria for M_5 (Twenty Six Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-3) log 1.270445 + ($\frac{3}{6}$) log 0.7091401

= -0.282546

4.3.3 DT- Compound Optimality Criteria for M_6 (Thirty Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-3) log 1.272756 + ($\frac{3}{6}$) log 0.7107073

= -0.283645

4.3.4 DT- Compound Optimality Criteria for $M_3^{(1)}$ (Forty Eight Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-4) log 1.4872332 + ($\frac{4}{7}$) log 0.7383739

= -0.5924079

4.3.5 DT- Compound Optimality Criteria for $M_5^{(1)}$ (Fifty Six Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-4) log 1.5268563 + ($\frac{4}{7}$) log 0.7510409

= -0.6224439

4.3.6 DT- Compound Optimality Criteria for $M_6^{(1)}$ (Sixty Four Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-4) log 1.5432059 + ($\frac{4}{7}$) log 0.7530118

= -0.6356706

4.3.7 DT- Compound Optimality Criterion for $M_3^{(2)}$ (Hundred Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-5) log 1.7527373 + ($\frac{5}{9}$) log 0.7516218

= -1.0437566

4.3.8 DT- Compound Optimality Criterion for M5 (2) (Hundred and Sixteen Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-5) log 1.8123589 + ($\frac{5}{9}$) log 0.7642774

= -1.0978374

4.3.9 DT- Compound Optimality Criterion for M6 (2) (Hundred and Thirty Two Points)

$\emptyset_2^{DT}(\varepsilon)$ = (1-5) log 1.8385169 + ($\frac{5}{9}$) log 0.7654320

= -1.1223668

4.4 Relative Efficiency

4.4.1 D- Efficiency

Substituting for (3.7) and (3.8) in (3.6) we obtain;

Table 4.2: Summary of the respective Relative D- Efficiencies for M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Dimension of Design	Design	Relative Efficiency
Three	M_3	$= \frac{0.6974419}{0.6974419} * 100$ = 100%
	M_5	$= \frac{0.6974419}{0.7091401} * 100$ = 98.3504%
	M_6	$= \frac{0.6974419}{0.7107073} * 100$ = 98.1335%
Four	$M_3^{(1)}$	$= \frac{0.7383739}{0.7383739} * 100$ = 100%
	$M_5^{(1)}$	$= \frac{0.7383739}{0.7510409} * 100$ = 98.3134%
	$M_6^{(1)}$	$= \frac{0.7383739}{0.7530118} * 100$ = 98.0561%
Five	$M_3^{(2)}$	$= \frac{0.7516218}{0.7516218} * 100$ = 100%
	$M_5^{(2)}$	$= \frac{0.7516218}{0.7642774} * 100$ = 98.3441%
	$M_6^{(2)}$	$= \frac{0.7516218}{0.7654320} * 100$ = 98.1958%

4.4.2 A- Efficiency

Substituting for (3.10) and (3.11) in (3.9) we have;

Table 4.3: Summary of the respective Relative A- Efficiencies for M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Dimension of Design	Design	Relative Efficiency
	M_3	$= \frac{0.3757384}{0.443092} * 100$ = 84.7992%
Three	**M_5**	$= \frac{0.3757384}{0.3757384} * 100$ = 100%
	M_6	$= \frac{0.3757384}{0.4569194} * 100$ = 82.2330%
	$M_3^{(1)}$	$= \frac{0.4459656}{0.4521702} * 100$ = 98.6278%
Four	**$M_5^{(1)}$**	$= \frac{0.4459656}{0.4538232} * 100$ = 98.2686%
	$M_6^{(1)}$	$= \frac{0.4459656}{0.4459656} * 100$ = 100%
	$M_3^{(2)}$	$= \frac{0.4122856}{0.4344145} * 100$ = 94.9060%
Five	**$M_5^{(2)}$**	$= \frac{0.4122856}{0.4283984} * 100$ = 96.2388%
	$M_6^{(2)}$	$= \frac{0.4122856}{0.4122856} * 100$ = 100%

4.4.3 T- Efficiency

Substituting for (3.13) and (3.14) in (3.12) we get;

Table 4.4: Summary of the respective Relative T- Efficiencies for M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Design	Relative Efficiency
$\mathbf{M_3}$	$= \frac{1.2534415}{1.2534415} * 100 = 100\%$
$\mathbf{M_5}$	$= \frac{1.2534415}{1.270445} * 100 = 98.6616\%$
$\mathbf{M_6}$	$= \frac{1.2534415}{1.272756} * 100 = 98.4825\%$
$\mathbf{M_3^{(1)}}$	$= \frac{1.4872332}{1.4872332} * 100 = 100\%$
$\mathbf{M_5^{(1)}}$	$= \frac{1.4872332}{1.5268563} * 100 = 97.4049\%$
$\mathbf{M_6^{(1)}}$	$= \frac{1.4872332}{1.5432059} * 100 = 96.3730\%$
$\mathbf{M_3^{(2)}}$	$= \frac{1.7527373}{1.7527373} * 100 = 100\%$
$\mathbf{M_5^{(2)}}$	$= \frac{1.7527373}{1.8123589} * 100 = 96.7103\%$
$\mathbf{M_6^{(2)}}$	$= \frac{1.7527373}{1.8385169} * 100 = 95.3343\%$

4.4.4 E- Efficiency

Substituting for (3.16) and (3.17) in (3.15) we obtain;

Table 4.5: Summary of the respective Relative E- Efficiencies for M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Design	Relative Efficiency
M_3	$= \frac{0.1404042}{0.404042} * 100$ = 100%
M_5	$= \frac{0.1404042}{0.1477622} * 100$ = 95.0204%
M_6	$= \frac{0.1404042}{0.1487512} * 100$ = 94.3886%
$M_3^{(1)}$	$= \frac{0.1336771}{0.1354509} * 100$ = 98.6904%
$M_5^{(1)}$	$= \frac{0.1336771}{0.1336771} * 100$ =100%
$M_6^{(1)}$	$= \frac{0.1336771}{0.5671459} * 100$ = 23.5701%
$M_3^{(2)}$	$= \frac{0.5294152}{0.5294152} * 100$ = 100%
$M_5^{(2)}$	$= \frac{0.5294152}{0.5420174} * 100$ = 97.6749%
$M_6^{(2)}$	$= \frac{0.5294152}{0.5499597} * 100$ = 96.2644%

4.4.5 DT- Efficiency

Substituting for (3.19) and (3.20) in (3.18) we have;

Table 4.6: Summary of the respective Relative DT- Efficiencies for M_3, M_5, M_6, $M_3^{(1)}$, $M_5^{(1)}$, $M_6^{(1)}$, $M_3^{(2)}$, $M_5^{(2)}$ & $M_6^{(2)}$

Design	Relative Efficiency
M_3	$= \frac{-0.274454}{-0.274454} * 100 = 100\%$
M_5	$= \frac{-0.274454}{-0.282546} * 100 = 97.1360\%$
M_6	$= \frac{-0.274454}{-0.283645} * 100 = 96.7597\%$
$M_3^{(1)}$	$= \frac{-0.5924079}{-0.5924079} * 100 = 100\%$
$M_5^{(1)}$	$= \frac{-0.5924079}{-0.6224439} * 100 = 95.1745\%$
$M_6^{(1)}$	$= \frac{-0.5924079}{-0.6356706} * 100 = 93.1942\%$
$M_3^{(2)}$	$= \frac{-1.0437566}{-1.0437566} * 100 = 100\%$
$M_5^{(2)}$	$= \frac{-1.0437566}{-1.0978374} * 100 = 95.0739\%$
$M_6^{(2)}$	$= \frac{-1.0437566}{-1.1223668} * 100 = 92.9960\%$

Table 4.7: Comparison of the Relative Efficiencies

Dimension of Design		D_{Eff}	A_{Eff}	T_{Eff}	E_{Eff}	DT_{Eff}
	M_3	100%	84.7992%	100%	100%	100%
Three	M_5	98.3504%	100%	98.6616%	95.0204%	97.1360%
	M_6	98.1335%	82.2330%	98.4825%	94.3886%	96.7597%
	$M_3^{(1)}$	100%	98.6278%	100%	98.6904%	100%
	$M_5^{(1)}$	98.3134%	98.2686%	97.4049%	100%	95.1745%
Four	$M_6^{(1)}$	98.0561%	100%	96.3730%	23.5701%	93.1942%
	$M_3^{(2)}$	100%	94.9060%	100%	100%	100%
	$M_5^{(2)}$	98.3441%	96.2388%	96.7103%	97.6749%	95.0739%
Five	$M_6^{(2)}$	98.1958%	100%	95.3343%	96.2644%	92.9960%

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter contains the conclusions and recommendations for further work.

5.2 Conclusions

The objectives of the study were to; calculate the A-,D-,E-, and T-Optimality criteria for three specific calculus optimum values second order rotatable designs in three, four and five dimensions, obtain DT-compound optimality criteria for the three specific calculus optimum values second order rotatable designs in three, four and five dimensions, and evaluate the relative efficiencies for the three specific calculus optimum values second order rotatable designs in three, four and five dimensions.

With regard to evaluation of the A-, D-, E- and T- optimality criteria, the specific three and four dimensional SORDs were all found to be E- optimal and the five dimensional designs were A- Optimal. Considering the D- optimality criterion, the designs M_3 in three dimensions, $M_3^{(1)}$ in four dimensions and $M_3^{(2)}$ in five dimensions were the most optimal among the specific SORDs considered. The Average Variance Criterion results indicated that designs M_5 in three dimensions, $M_6^{(1)}$ in four dimensions and $M_6^{(2)}$ in five dimensions were most optimal in that criterion. Further regarding the Trace Criterion, designs M_3 in three dimensions, $M_3^{(1)}$ in four dimensions and $M_3^{(2)}$ in five dimensions were found to be the most optimal, additionally M_3 in three dimensions, $M_5^{(1)}$ in four dimensions and $M_3^{(2)}$ in five dimensions were most optimal in respect to the Smallest Eigen Value Criterion.

With regard to DT- compound optimality criteria for the three specific calculus optimum values second order rotatable designs in three, four and five dimensions, M_3 in three dimensions, $\mathrm{M}_3^{(1)}$ in four dimensions, and $\mathrm{M}_3^{(2)}$ in five dimensions were most optimal.

With regard to evaluation of relative efficiencies for the three specific calculus optimum values second order rotatable designs in three, four and five dimensions,M_5 was found to be most efficient among the three dimensional SORDs considered, where $\mathrm{M}_3^{(1)}$and $\mathrm{M}_3^{(2)}$ were the most efficient in regard to four and five dimensional designs respectively as shown in table 4.7 above.

If one considers undertaking an experiment with three factors, the twenty two points design denoted by M_5 is most suitable; for an experiment with four factors an experimenter should consider the fifty six points design denoted by $\mathrm{M}_3^{(1)}$ and for five dimensions the a hundred and thirty two points design denoted by $\mathrm{M}_3^{(2)}$will be the best to employ.

5.3 Recommendations

These specific second order rotatable designs can be extended beyond five factors to cater for experiments with more factors.

REFERENCES

Atkinson, A.C. (2008). DT-optimum designs for model discrimination and parameter estimation. *Journal of statistical planning and inference. 138, 56-64.*

Atkinson, A.C., & Donev, A.N (1992). *Optimum Experimental Designs.* Oxford: Clarendon Press.

Atkinson, A.C., & Federov, V. (1975). Optimal design experiments for discriminating between several models. *Biometrika, 62, 289-303*.

Bose, R.C, and Draper, N.R. (1959). Second Order Rotatable Designs in Three Dimensions. *Annals of Mathematical Statistics*, 30(4), 1097-1112.

Box, G.E.P. and Hunter, J.S. (1957). Multifactor Experimental Designs for Exploring Response Surfaces. *Annals of Mathematical Statistics*, 28(1), 195-241.

Box, G.E.P and Wilson, K.B. (1951). On Attaintment of Optimal Conditions. *J.R. Statist. Soc.*, B13,1-45.

Chernoff, H. (1953). Locally optimal designs for estimating parameters. *Annals of Mathematical Statistics*, 24(4), 586–602.

Cornelius, N.O. (2016). Sequential Third Order Rotatable Designs in Three, Four and Five Dimensions. *M.Sc. Thesis, Moi University Eldoret Kenya.*

Dette, H., & Franke, T. (2000). Constrained D- and D1- optimal designs for polynomial regression. *The Annals of Statistics*, *28*(6), 1702-1727.

Ehrenfeld, E. (1955). On the efficiency of experimental design. *Annals of Mathematical Statistics*, 26(2) , 247–255.

Elfving, G. (1952). Optimum allocation in linear regression theory. *The Annals of Mathematical Statistics*, *23*(2), 255-262.

Fechner, G.T. (1860). Elementstder Psychophysik. Breitkopf and Hartel. Leipzig.

Fedorov V.V (1972). *Theory of Optimal Experiments.* New York: Academic Press.

Fisher, R. (1935). *The Design of Experiments.* Edinburgh: Oliver & Boyd.

Gardiner, D.A.; Grandage A.H.E. and Harder R.J. (1959). Third Order Rotatable Design for Exploring Response Surface. *Annals of Mathematical Statistics*, 1082-1096

Hoel, P.G. (1958). Efficiency problems in polynomial estimation. Ann. Math. Statist.,29, 1134-1146.

Kiefer, J., & Wolfowitz, J. (1959). Optimum designs in regression problems. *The Annals of Mathematical Statistics*, 271-294.

Kiefer, J. (1985). The *Design of Experiments.* New York: Springer.

Kiplagat, N.K. (2016). I- and G- Optimal Second Order Rotatable Designs in Three Dimensions. *M.Sc. Thesis, Moi University Eldoret Kenya.*

Koech, F.K. (2017). Relative Efficiency and DT- Optimality Criteria For The Existing Six Specific Second Order Rotatable Designs In Three Dimensions. *M.Sc. Thesis, Moi University Eldoret Kenya.*

Kosgei, M.S.K. (2002). Optimality Criteria for the Specific Second Order Rotatable Designs in Three Dimensions. *M.Sc. Thesis, Moi University Eldoret Kenya*.

Kuhfeld, Warren F. (2010). Marketing Research Methods in SAS. Analytical Solutions Manager at SAS. *Sas Publishing.*

Magangi, O. J. (2018). *Construction of modified optimal second order rotatable designs*. M.Sc. Thesis, Moi University Eldoret Kenya.

Mandal, S. (2000). Construction of optimizing distributions with applications in estimation and optimal design. *PhD Thesis, University of Glasgow*.

Mead, R., & Pike, D. J. (1975). A biometrics invited paper. A review of response surface methodology from a biometric viewpoint. *Biometrics*, *31*(4), 803-851.

Mutiso J.M. (1998). Second and third order Specific and Sequential Rotatable Designs in k Dimensions. *PhD. Thesis. Moi University Eldoret Kenya.*

Mwan, M. D. (2017). Evaluation of compound optimality criteria for second order rotatable designs constructed using balanced incomplete block design, *M.Sc. Thesis Moi University Kenya*.

Pázman, A. (1986). *Foundations of optimum experimental design* (Vol. 14). Springer.

Pukelsheim, F (1993). *Optimal design of experiments*. John Wiley and Sons.

Rambaei S.K. (2014). Optimal designs for second order rotatability. *Ph.D. Thesis. Moi University Eldoret Kenya.*

Silvey, S. D. (1980). *Optimal design:An introduction to the theory of paarameter estimation.* Chapman and Hall, London.

Smith, K.(1918). On the standard deviations of adjusted and interpolated values of an observed polynomial function and its constant and the guidance, they give towards a proper choice of the distribution of observations. *Biometrika*, *12,1-85.*

Wald, A. (1943). On the efficient design of statistical investigations. *The annals of mathematical statistics*, *14*(2), 134-140.

Waterhouse, T. H., Eccleston, J. A., & Duffull, S. B. (2004). On optimal design for discrimination and estimation. In *COMPSTAT 2004: Proceedings in Computational Statistics* (pp. 1963-1970).

Yang, M. (2008). A-optimal designs for generalized linear models with two parameters. *Journal of Statistical Planning and Inference*, *138*(3), 624-641.