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USING MATHEMATICAL EFFICIENCY CRITERION TO OPTIMIZE A TRIANGULAR OPEN CHANNEL FOR STABLE VELOCITY DURING STORMS

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Abstract

During heavy rains, open channels are prone to being overrun with storm water. Since the velocity of a flowing fluid increases with depth, sudden overflows may cause velocities to exceed certain limits. This damages the channel by scouring. On the other hand, siltation of suspended matter occurs in sluggish flow. Channel dimensions and

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shapes must minimize cost and maximize discharge in normal seasons, while regulating the discharge to minimize velocity fluctuations during overflow. Depending on the designer's objectives, channel design involves numerous parameters, including the characteristics of construction materials and earthwork. Traditional methods such as Lagrange multipliers, sequential quadratic programming (SQP), differential evolution algorithm (DEA), genetic algorithms, ant-colony optimization, and lately, meta-heuristic algorithms are often used to minimize a cost function subject to channel cross-section. In this paper, using only the mathematical hydraulic efficiency criterion (other factors assumed optimum), a direct integro-differential technique is applied to determine the optimum triangular channel design that additionally minimizes velocity fluctuations during excessive discharge. The triangular channel is treated as a special case of a trapezoidal channel.

1. Introduction

In open channel flow, one surface of the flow is exposed to the atmosphere. Partially full closed pipes are often categorized as open channels. This flow differs from *full bore* flow in various aspects especially in that the flow cross-section for open channels is not determined entirely by the solid boundaries, but is free to change without restraint, depending on other parameters of the flow. The free surface is usually subjected to constant atmospheric pressure and therefore, the flow is caused by the component of the weight of the liquid. This brings about a drop in the piezometric pressure, $p + \rho gz$, but not a drop in the pressure at the free surface. Wahome [4] gives a synoptic comparison of the two flows.

The commonest examples of natural channel flow are rivers and streams, while irrigation canals, flumes and spillways are artificial open channels.

In practical channel hydrodynamics where conservation of resources is of prime importance, it is crucial to interrogate the issue of the most hydraulically optimum shape. This means the channel shape which allows maximum discharge for a fixed area, surface roughness and bed slope. Common open channel formulae, including those of Chézy and Manning predict that for uniform flow with a given bed gradient, roughness and cross-sectional area, the mean velocity and discharge, Q - parameters which influence channel efficiency - all depend on the hydraulic mean depth m, Massey and Smith [10]. m is defined as the ratio of the flow area A to the wetted perimeter P. This implies the hydraulic efficacy of a channel will be predicated on the channel-shape. Maximizing m, which is equivalent to minimizing the wetted perimeter, will therefore maximize the discharge. Cost of lining material will also be minimized.

Among the common channel designs, the semicircular shape has been found to possess the maximum hydraulic mean depth, Doughlas et al. [8]. However, the consideration of other factors, such as *angle of repose* for loose granular banking material, cost of excavation and relative ease of construction make the semicircular shapes applicable only to small channels, leaving the trapezoidal shape more preferable in practice, Massey and Smith [10]. Notably, rectangular and triangular channels are special cases of the trapezoidal channel. Hameed [1] has studied triangular canals with round bottoms and found them more efficient than circular channels.

On the hypothesis that different triangular bases give different efficiencies, it becomes important to determine the most economical triangle, that is, the one giving the maximum discharge for a given amount of excavation, based on dimensional ratios and side slopes.

In this paper, the dimensional characteristics of the most economical triangular cross-section with minimum velocity fluctuations (to minimize scouring or siltation, Stephenson [15] and Chow [12]) are derived progressively. It is assumed that the fluid flow is steady, inviscid, incompressible and irrotational. Surface tension and viscosity are neglected. It is further assumed that the mathematical hydraulic efficiency is the only consideration, and that the triangular shape is an inverted right pyramid. The condition for the turning points, and hence the optimal dimensions, are obtained by maximizing the hydraulic mean depth function using integro-differential calculus.

2. Mathematical Formulation

The most economical (or efficient) section of a channel is defined as one *which gives the maximum discharge for a given amount of excavation*, Rajput [6]. The general expression for the mean velocity in an open channel flow is

$$V = km^a S^b \tag{2.1}$$

with m, S and k being the hydraulic mean depth, channel slope and flow resistance factor, respectively, and a and b are the so called hydraulic components. Based on the continuity equation, the discharge Q may be expressed as product of the flow area and average velocity, so that

$$Q = Akm^a S^b. \tag{2.2}$$

Equation (2.2) is the general form of several uniform flow equations for open channels, among them the Chézy and Manning equations, and clearly shows that with the slope and surface roughness held constant, flow discharge maximizes with the hydraulic mean depth m. We optimize the efficiency of the channel by optimizing m.

2.1. The triangle as a special trapezium

We consider the triangular cross-section of flow as a special case of trapezoidal cross-section (see Figure 1). In Figure 1, $AOFC$ represents the symmetric trapezoidal cross-section of the open channel. We use an integration approach to determine the area of the cross-section of the flow (this method is versatile for other non-regular shapes, when need arises). We take the origin to be at O to have

$$\text{Area } A = \int_0^{\eta} \int_{y\cot(\pi-\theta)}^{y\cot\theta+\zeta} dxdy \tag{2.3}$$

with

$$y\cot(\pi-\theta) \leq x \leq y\cot\theta + \zeta \tag{2.4}$$

between the opposite banks of the channel. Since $\cot(\pi - \theta) = -\cot\theta,$ the area evaluates to

$$A = \zeta\eta + \eta^2 \cot\theta. \tag{2.5}$$

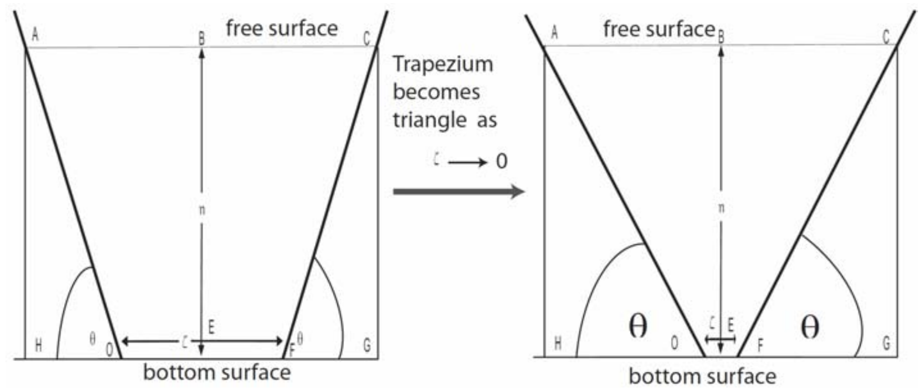

Figure 1. Triangle as a trapezium with no base.

The wetted perimeter in the channel is

$$P = \zeta + 2\eta \csc\theta. \tag{2.6}$$

The hydraulic mean depth m which we need to maximize is

$$m = \frac{A}{P} = \frac{\zeta\eta + \eta^2 \cot\theta}{\zeta + 2\csc\theta} = \frac{A}{\zeta + 2\eta\csc\theta}. \tag{2.7}$$

When the channel shape is triangular, we have the wetted perimeter as

$$P = \lim_{\zeta\to 0}[\zeta + 2\eta\csc\theta] = 2\eta\csc\theta, \tag{2.8}$$

area as

$$A = \lim_{\zeta\to 0} \zeta\eta + \eta^2 \cot\theta = \eta^2 \cot\theta \tag{2.9}$$

and hydraulic mean depth m as

$$m = \lim_{\zeta \to 0} \frac{A}{P} = \frac{\zeta\eta + \eta^2 \cot\theta}{\zeta + 2\csc\theta} = \frac{A}{2\eta\csc\theta} = \frac{A}{2\left[\sqrt{\dfrac{A}{\cot\theta}}\right]\csc\theta} \tag{2.10}$$

by (2.8).

2.1.1. Optimum channel slope

With area A constant, we minimize the denominator in (2.10) (equivalent to maximizing m) by differentiating w.r.t. θ, that is, we set

$$\frac{\partial}{\partial\theta}\left[2\left[\sqrt{\frac{A}{\cot\theta}}\right]\csc\theta\right] = 2\sqrt{A}\,\frac{\partial}{\partial\theta}\left[\frac{\csc\theta}{\sqrt{\cot\theta}}\right] = 0 \Rightarrow \frac{\partial}{\partial\theta}\left[\frac{\csc\theta}{\sqrt{\cot\theta}}\right] = 0. \tag{2.11}$$

By setting $\dfrac{\csc\theta}{\sqrt{\cot\theta}} = I$, we have $\dfrac{\csc^2\theta}{\cot\theta} = \cot\theta + \tan\theta = I^2$, which we differentiate on both the sides to have $(-\csc^2\theta + \sec^2\theta) = 2I\dfrac{dI}{d\theta}$. We want $\dfrac{dI}{d\theta}$ which is $\dfrac{-\csc^2\theta + \sec^2\theta}{\dfrac{\csc\theta}{\sqrt{\cot\theta}}}$. Equation (2.11) now reduces to

$$-\csc^2\theta + \sec^2\theta = 0. \tag{2.12}$$

This means

$$\cos\theta = \sin\theta \Rightarrow \theta = \frac{\pi}{4}. \tag{2.13}$$

We have shown that the most economical triangular channel needs to have its sloping sides making an angle of 45° with the vertical (or horizontal) axis. Instead of using the second derivative test, we view from the graph below (Figure 2) that the turning point we have calculated above occurs at a *maximum* value of hydraulic radius m (i.e., minimum wetted perimeter P). The graph clearly peaks at $\theta = \dfrac{\pi}{4} = 0.78539816^c$.

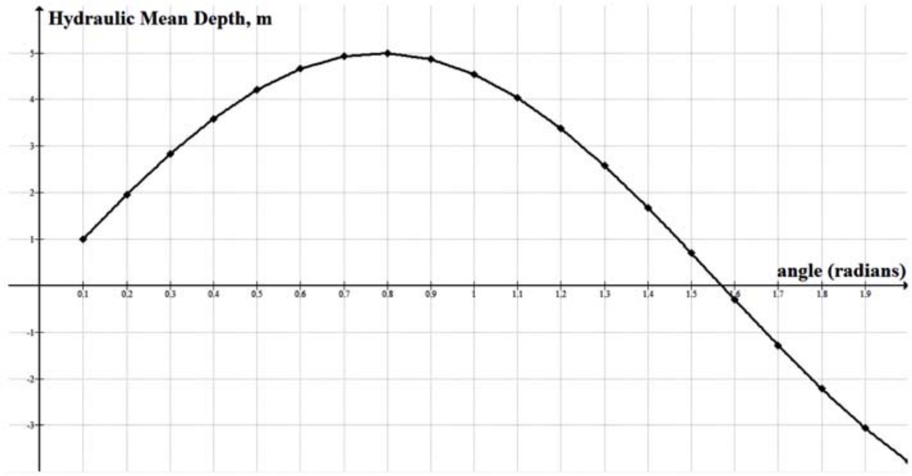

Figure 2. Economy of a triangular open channel at various angles, expressed in terms of hydraulic mean depth m (area $A = 100$).

2.1.2. Optimum channel width, height and area

In view of Figure 1 and equations (2.8) and (2.9), we derive the following relationships between the hydraulic mean radius and the most economical dimensions of the triangular channel:

$$m = \frac{\eta^2 \cot\theta}{2\eta \csc\theta} = \frac{\eta}{2\sqrt{2}} \Rightarrow \text{ optimum channel depth, } \eta = 2\sqrt{2}\,\text{m}. \qquad (2.14)$$

$$\text{Best channel width } = 2 \times \text{ optimum channel depth } = 4\sqrt{2}\,\text{m}. \qquad (2.15)$$

$$\text{Slanting side } = \sqrt{2} \times \text{ optimum channel depth } = 4\,\text{m}. \qquad (2.16)$$

2.2. Minimum velocity fluctuation design

Having established the relative dimensions of the most economical triangular channel, we now wish to construct the section *above* the free surface of the fluid in such a way that an overflow will not cause a change in velocity. See Figure 3:

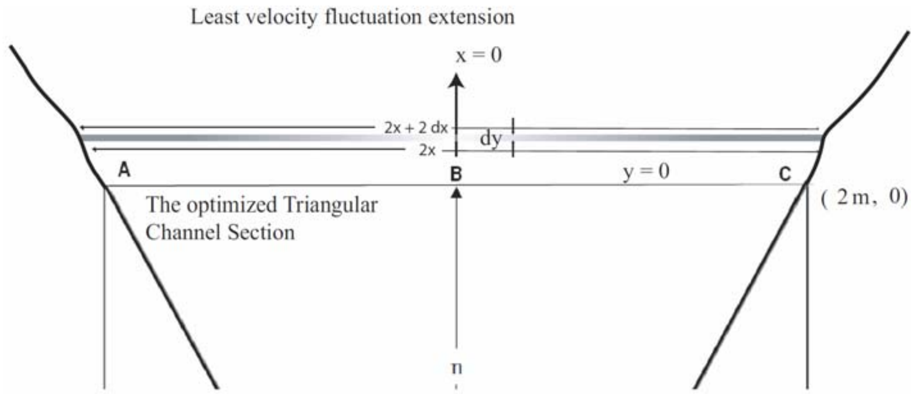

Figure 3. Least velocity fluctuation section of the channel.

Since the velocity is influenced only by the hydraulic mean depth $m = \frac{A}{P}$, we can suppress velocity fluctuations by keeping m constant, thus,

$$m = \frac{dA}{dP} = \frac{2xdy}{2\sqrt{dx^2 + dy^2}} \Rightarrow [dx^2 + dy^2]m^2 = x^2 dy^2 \tag{2.17}$$

which simplifies to

$$\left[\frac{dy}{dx}\right]^2 = \frac{m^2}{x^2 - m^2} \Rightarrow y = \int \frac{m^2}{\sqrt{x^2 - m^2}} dx = m \cosh^{-1}\left[\frac{x}{m}\right] + k. \tag{2.18}$$

In view of equations (2.14), (2.16) and (2.15), $\overline{BC}$ in Figure 3 has length $2m$. Since this constant velocity section starts *just above* the optimized triangle, we substitute the coordinates $(2m, 0)$ into equation (2.18) to find the value of k as $k = -m \cosh^{-1} 2$. Therefore, the channel walls of the minimum fluctuation section should bear the relationship

$$y = m \cosh^{-1}\left[\frac{x}{m}\right] - m \cosh^{-1} 2. \tag{2.19}$$

2.3. Comparison of efficiencies of regular trapezoidal channel sections with different slant angles θ

It has been shown above that the triangular cross-section is a special case of the trapezoidal one. It would be of interest to compare the efficiencies for other trapezoidal sections, with slant angles computed as $\frac{2\pi}{n}$, where n is the number of sides of the *regular polygon* of whose the channel section is a part. With $A = 100$ and $\eta = 50$ (note that these have base, unlike the triangle), the efficiencies for $3 \leq n \leq 10$ have been computed below, and the corresponding graph for $n \leq 50$ provided as well.

Table 1. Efficiency values for the first 8 polygons with $A = 100$ and $\eta = 50$

Polygon's sides n	Slope θ (rad)	Wetted perimeter P	Efficiency $\frac{1}{P}$
3	2.094395333	146.3375981	0.006833514
4	1.5707965	102.0000087	0.009803921
5	1.2566372	90.90024054	0.011001071
6	1.047197667	88.60254038	0.011286358
7	0.897598	90.03112933	0.011107269
8	0.78539825	93.42135265	0.010704191
9	0.698131778	97.9846981	0.010205675
10	0.6283186	103.3110595	0.009679506

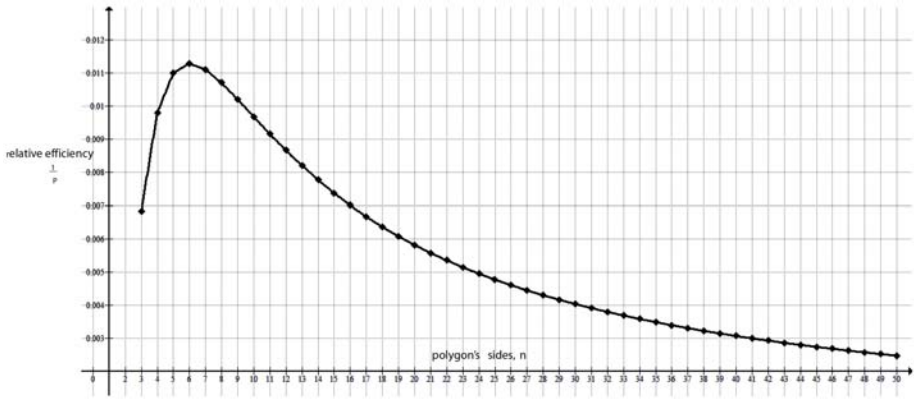

Figure 4. Relative efficiencies for *n*-sided trapezoidal channels.

This graph peaks at $n = 6$ which reveals that the regular hexagon is the most efficient *c*-sectional design. Interesting information also emerges, for instance, that a 7-sided (heptagonal) cross-section at 0.011107269 is more efficient than a 5-sided (pentagonal) one at 0.011001071.

3. Conclusion

For a triangular section, open channel flow with fluid depth η and hydraulic mean depth *m*, the optimum conditions for maximum discharge, and constant velocity during overflow are summarized below:

Table 2. Summary of optimal values for triangular base channel

Depth η	Slope θ (deg)	Slant side	Channel width
$2\sqrt{2}m$	45°	$4m$	$4\sqrt{2}m$

It was also found that the equation of the constant velocity section is

$$y = m\cosh^{-1}\left[\frac{x}{m}\right] - m\cosh^{-1} 2.$$

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