# Hybrid Crank-Nicolson-Du Fort and Frankel (CN-DF) Scheme for the Numerical Solution of the 2-D Coupled Burgers' System

**Article** *in* Applied Mathematical Sciences · February 2014



**3 authors:**

Kweyu Cleophas
Moi University


Benjamin Nyamai
Moi University


John Wahome
Laikipia University College

Applied Mathematical Sciences, Vol. 8, 2014, no. 48, 2353 - 2361
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2014.310596

# Hybrid Crank-Nicolson-Du Fort and Frankel (CN-DF) Scheme for the Numerical Solution of the 2-D Coupled Burgers' System

**Kweyu Cleophas[1], Nyamai Benjamin[2] and Wahome John[3]**

[1,2]Department of Mathematics and Computer Science
University of Eldoret, P.O. Box 1125, Eldoret, Kenya

[3]Department of Mathematics, Laikipia University
P.O. Box 441, Nyahururu, Kenya



**Abstract**

The two dimensional coupled Burgers' equations constitute an appropriate model for developing computational algorithms, for solving the incompressible Navier-Stokes equations. Moreover, they are generally used as transport equations because they model a number of fluid flow phenomena, for example, turbulent flow, shock wave formation and boundary layer formation. In this paper, we develop a hybrid Crank-Nicolson and Du Fort and Frankel (CN-DF) scheme. The hybrid CN-DF is developed by introducing the Du Fort and Frankel (DF) properties into the Crank-Nicolson scheme (CN). This is a three-level scheme and is also unconditionally stable. Numerical solutions from the hybrid scheme are obtained by the use of MATLAB software. By use of $\mathbf{L^1}$ error, it is determined that the hybrid scheme is fifth order accurate in space and produces better results in comparison to the pure Crank-Nicolson and the pure Du Fort and Frankel schemes.

**Keywords:** 2-D Burgers' equation, Hybrid CN-DF method, $\mathbf{L^1}$ error, fifth order accurate in space

# 1 Introduction

The Burgers' equation is an important non-linear parabolic partial differential equation (PDE) widely used to model several physical flow phenomena in fluid dynamics teaching and in engineering such as turbulence, boundary layer behaviour, shock wave formation, and mass transport, Pandey *et al.*, [10]. In general, this equation is suited to modelling fluid flows because it incorporates directly the interaction between the non-linear convection processes and the diffusive viscous processes, Fletcher[5]. Consequently, it is one of the principle model equations used to test the accuracy of new numerical methods or computational algorithms, Kanti and Lajja, [6]. The 2-D coupled non-linear Burgers' equations are a special form of incompressible Navier-Stokes equations without having the pressure term and the continuity equation, Vineet *et al.*,[14].

The Du Fort and Frankel scheme (DF) is a finite difference scheme that was first presented in 1953 as a numerical method for solving the heat equation with periodic boundary conditions, Du Fort and Frankel, [4]. This is a three level explicit scheme but which is unconditionally stable, however, it faces consistency problems for large values of $\Delta t$, Mitchell and Griffiths, [9]. The scheme has been used to solve both the one and two dimensional Burgers' equation, Pandey *et al.*, and Samir, [11, 12].

On the other hand, the Crank-Nicolson (CN) scheme is an implicit finite difference scheme that was developed by John Crank and Phyllis Nicolson in the mid 20th Century, Crank and Nicolson, [3]. It is a second order accurate method in both space and time and unconditionally stable and was used for solving the heat equation and similar partial differential equations, [13]. Consequently, it is a two level scheme.

The term hybrid means blended or "marrying" two or more numerical methods, Koross *et al.*, [7]. The hybrid Crank-Nicolson and Du Fort and Frankel (CN-DF) scheme was first developed in 2009 from operator splitting for solving the one dimensional Burgers' equation, Koross *et al.*, [7]. It was found that the DF method increases the efficacy of the CN method by increasing the number of grid points involved. Consequently, the CN-DF scheme is three level and unconditionally stable and produces more accurate numerical results than those of the parent pure schemes.

In this paper, the hybrid CN-DF scheme is developed by introducing or incorporating important properties from the DF scheme into the CN scheme to improve its efficacy. The terms that are affected in the Burgers' equation are the unsteady state and diffusion terms. Numerical solutions for the CN-DF scheme are compared with those of the exact solution and the pure CN scheme and the $\mathbf{L^1}$ error is used to determine the order of convergence and consistency of the CN-DF scheme. The Reynolds number is kept constant at 4000 and the time stepping is made very small at $\Delta t = 0.001$ to maintain consistency and accuracy.

# 2 Mathematical Formulation

The 2-D Burgers' model is given by;

$$u_t + uu_x + vu_y = \frac{1}{Re}(u_{xx} + u_{yy}) \tag{2.1}$$

$$v_t + uv_x + vv_y = \frac{1}{Re}(v_{xx} + v_{yy}) \tag{2.2}$$

subject to the initial conditions

$$u(x, y, 0) = \varphi_1(x, y), \quad v(x, y, 0) = \varphi_2(x, y) \; \big\} \, (x, y) \in \Omega \tag{2.3}$$

and Dirichlet boundary conditions

$$u(x, y, t) = \zeta(x, y, t), \quad v(x, y, t) = \xi(x, y, t) \; \big\} \, (x, y) \in \partial\Omega, \quad t > 0 \tag{2.4}$$

where $\Omega = \{(x, y) : a \leq x \leq b, a \leq y \leq b\}$ is the computational domain which in this study is taken to be a square domain, and $\partial\Omega$ is its boundary; $u(x, y, t)$ and $v(x, y, t)$ are the velocity components to be determined; $\varphi_1$, $\varphi_2$, $\zeta$, and $\xi$ are known functions; $u_t$ is the unsteady term; $uu_x$ is the non-linear convection term; $Re$ is the Reynolds number, and $\frac{1}{Re}(u_{xx} + u_{yy})$ is the diffusion term.
The discretization of the 2-D Burgers equations (2.1) and (2.2) is done by the CN-DF scheme. We begin by providing an insight into the discretization of the pure DF and pure CN schemes independently before embarking on the hybrid scheme.

## 2.1 The Pure Du Fort and Frankel (DF) Scheme

For the discretization of equations (2.1) and (2.2) by the DF scheme, we have;

$$\frac{1}{k}\mu\delta_t u(x, y, t^n) + u(x, y, t)\frac{1}{h}\mu\delta_x(u(x, y, t^n)) + v(x, y, t)\frac{1}{h}\mu\delta_y(u(x, y, t^n)) = \frac{1}{h^2\mathrm{Re}}[{\delta_x}^2 u(x, y, t^n + {\delta_y}^2 u(x, y, t^n)] \tag{2.5}$$

$$\frac{1}{k}\mu\delta_t v(x, y, t^n) + u(x, y, t)\frac{1}{h}\mu\delta_x(v(x, y, t^n)) + v(x, y, t)\frac{1}{h}\mu\delta_y(v(x, y, t^n)) = \frac{1}{h^2\mathrm{Re}}[{\delta_x}^2 v(x, y, t^n + {\delta_y}^2 v(x, y, t^n)] \tag{2.6}$$

The resultant $u^n_{i,j}$ and $v^n_{i,j}$ terms on the right hand side (RHS) are replaced by their average terms given by $\frac{u^{n+1}_{i,j}+u^{n-1}_{i,j}}{2}$ and $\frac{v^{n+1}_{i,j}+v^{n-1}_{i,j}}{2}$ respectively from which the following recurrence equations are obtained

$$\frac{u^{n+1}_{i,j}-u^{n-1}_{i,j}}{2k}+u^n_{i,j}(\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h})+v^n_{i,j}(\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2h})$$
$$=\frac{1}{Re}\{(\frac{u^n_{i+1,j}-u^{n+1}_{i,j}-u^{n-1}_{i,j}+u^n_{i-1,j}}{h^2})+(\frac{u^n_{i,j+1}-u^{n+1}_{i,j}-u^{n-1}_{i,j}+u^n_{i,j-1}}{h^2})\} \tag{2.7}$$

$$\frac{v^{n+1}_{i,j}-v^{n-1}_{i,j}}{2k}+u^n_{i,j}(\frac{v^n_{i+1,j}-v^n_{i-1,j}}{2h})+v^n_{i,j}(\frac{v^n_{i,j+1}-v^n_{i,j-1}}{2h})$$
$$=\frac{1}{Re}\{(\frac{v^n_{i+1,j}-v^{n+1}_{i,j}-v^{n-1}_{i,j}+v^n_{i-1,j}}{h^2})+(\frac{v^n_{i,j+1}-v^{n+1}_{i,j}-v^{n-1}_{i,j}+v^n_{i,j-1}}{h^2})\} \tag{2.8}$$

## 2.2 The Pure Crank-Nicolson (CN) Scheme

For the discretization of equations (2.1) and (2.2) by the CN scheme, we have;

$$\frac{1}{k}\Delta_t u(x,y,t^n)+u(x,y,t)\frac{1}{2h}\mu\delta_x[u(x,y,t^{n+1})+u(x,y,t^n)]$$
$$+v(x,y,t)\frac{1}{2h}\mu\delta_y[u(x,y,t^{n+1})+u(x,y,t^n)]=\frac{1}{2h^2\mathrm{Re}}[{\delta_x}^2(u(x,y,t^{n+1}+u(x,y,t^n))$$
$$+{\delta_y}^2(u(x,y,t^{n+1}+u(x,y,t^n))] \tag{2.9}$$

$$\frac{1}{k}\Delta_t v(x,y,t^n)+u(x,y,t)\frac{1}{2h}\mu\delta_x[v(x,y,t^{n+1})+v(x,y,t^n)]$$
$$+v(x,y,t)\frac{1}{2h}\mu\delta_y[v(x,y,t^{n+1})+v(x,y,t^n)]=\frac{1}{2h^2\mathrm{Re}}[{\delta_x}^2(v(x,y,t^{n+1})+v(x,y,t^n)$$
$$+{\delta_y}^2(v(x,y,t^{n+1})+v(x,y,t^n))] \tag{2.10}$$

from which the following recurrence equations are obtained

$$\frac{u^{n+1}_{i,j}-u^n_{i,j}}{k}+\frac{1}{2}[u^{n+1}_{i,j}(\frac{u^{n+1}_{i+1,j}-u^{n+1}_{i-1,j}}{2h})+u^n_{i,j}(\frac{u^n_{i+1,j}-u^n_{i-1,j}}{2h})]+\frac{1}{2}[v^{n+1}_{i,j}(\frac{u^{n+1}_{i,j+1}-u^{n+1}_{i,j-1}}{2h})$$
$$+v^n_{i,j}(\frac{u^n_{i,j+1}-u^n_{i,j-1}}{2h})]=\frac{1}{Re}[\frac{1}{2}\{(\frac{u^{n+1}_{i+1,j}-2u^{n+1}_{i,j}+u^{n+1}_{i-1,j}}{h^2})+(\frac{u^n_{i+1,j}-2u^n_{i,j}+u^n_{i-1,j}}{h^2})\}$$
$$+\frac{1}{2}\{(\frac{u^{n+1}_{i,j+1}-2u^{n+1}_{i,j}+u^{n+1}_{i,j-1}}{h^2})+(\frac{u^n_{i,j+1}-2u^n_{i,j}+u^n_{i,j-1}}{h^2})\}] \tag{2.11}$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{k}+\frac{1}{2}[u_{i,j}^{n+1}(\frac{v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}}{2h})+u_{i,j}^{n}(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2h})]+\frac{1}{2}[v_{i,j}^{n+1}(\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2h})$$
$$+v_{i,j}^{n}(\frac{v_{i,j+1}^{n}-v_{i,j-1}^{n}}{2h})]=\frac{1}{Re}[\frac{1}{2}\{(\frac{v_{i+1,j}^{n+1}-2v_{i,j}^{n+1}+v_{i-1,j}^{n+1}}{h^2})+(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{h^2})\}$$
$$+\frac{1}{2}\{(\frac{v_{i,j+1}^{n+1}-2v_{i,j}^{n+1}+v_{i,j-1}^{n+1}}{h^2})+(\frac{v_{i,j+1}^{n}-2v_{i,j}^{n}+v_{i,j-1}^{n}}{h^2})\}] \quad (2.12)$$

where $h=\Delta x=\Delta y$, $k=\Delta t$, and $h^2=\Delta x^2=\Delta y^2$.

### 2.3 The Hybrid Crank-Nicolson-Du Fort and Frankel (CN-DF) Scheme

To form the hybrid CN-DF scheme, we first replace the time discretization in the CN scheme by that of the DF scheme followed by a replacement of the terms $u_{i,j}^n$ and $v_{i,j}^n$ on the RHS of the CN scheme by their averages. This results in a three level implicit scheme given by;

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n-1}}{2k}+\frac{1}{2}[u_{i,j}^{n+1}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h})+u_{i,j}^{n}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h})]+\frac{1}{2}[v_{i,j}^{n+1}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2h})$$
$$+v_{i,j}^{n}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2h})]=\frac{1}{Re}[\frac{1}{2}\{(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2})+(\frac{u_{i+1,j}^{n}-u_{i,j}^{n+1}-u_{i,j}^{n-1}+u_{i-1,j}^{n}}{h^2})\}$$
$$+\frac{1}{2}\{(\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2})+(\frac{u_{i,j+1}^{n}-u_{i,j}^{n+1}-u_{i,j}^{n-1}+u_{i,j-1}^{n}}{h^2})\}] \quad (2.13)$$

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n-1}}{2k}+\frac{1}{2}[u_{i,j}^{n+1}(\frac{v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}}{2h})+u_{i,j}^{n}(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2h})]+\frac{1}{2}[v_{i,j}^{n+1}(\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2h})$$
$$+v_{i,j}^{n}(\frac{v_{i,j+1}^{n}-v_{i,j-1}^{n}}{2h})]=\frac{1}{Re}[\frac{1}{2}\{(\frac{v_{i+1,j}^{n+1}-2v_{i,j}^{n+1}+v_{i-1,j}^{n+1}}{h^2})+(\frac{v_{i+1,j}^{n}-v_{i,j}^{n+1}-v_{i,j}^{n-1}+v_{i-1,j}^{n}}{h^2})\}$$
$$+\frac{1}{2}\{(\frac{v_{i,j+1}^{n+1}-2v_{i,j}^{n+1}+v_{i,j-1}^{n+1}}{h^2})+(\frac{v_{i,j+1}^{n}-v_{i,j}^{n+1}-v_{i,j}^{n-1}+v_{i,j-1}^{n}}{h^2})\}] \quad (2.14)$$

The resultant non-linear algebraic systems of equations are linearized by Newton's method during the numerical implementation. The terms from the DF scheme are the ones which contribute to the increased grid points (three level) and therefore increased efficacy.

## 3 Numerical results by the hybrid CN-DF scheme

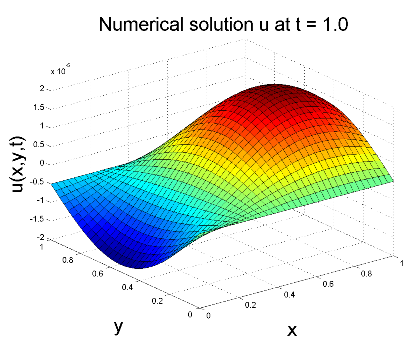
We employ the exact solutions and their initial and boundary conditions that were developed by Kweyu *et al.*, [8] to derive the numerical solutions using the

CN-DF scheme. The exact solutions are given by;

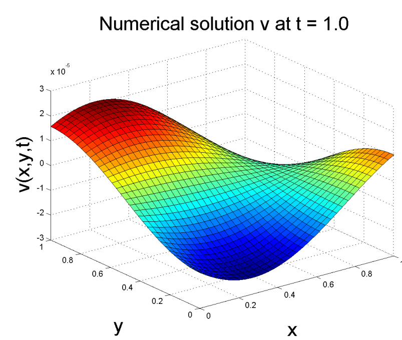
$$u(x,y,t) = \frac{-2y - 2\pi e^{\frac{-2\pi^2 t}{\mathrm{Re}}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))}{\mathrm{Re}(100 + xy + e^{\frac{-2\pi^2 t}{\mathrm{Re}}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))} \tag{3.1}$$

$$v(x,y,t) = \frac{-2x - 2\pi e^{\frac{-2\pi^2 t}{\mathrm{Re}}}((\cos(\pi x) + \sin(\pi x))\cos(\pi y))}{\mathrm{Re}(100 + xy + e^{\frac{-2\pi^2 t}{\mathrm{Re}}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))} \tag{3.2}$$

From which the initial and boundary conditions are derived.
The numerical solutions for $u$ and $v$ using the CN-DF scheme at Re = 4000 and 32×32 grid points are given graphically by;



(a)



(b)

Figure 1: Numerical solutions for $u$ and $v$ with dt = 0.001 and t = 1.0 seconds

The comparisons between the numerical solutions and the exact solutions are presented for u and v at random points of the solution matrices is given in the following table.

Table 1: Solutions for u at dt = 0.001, t = 1.0, and Re = 4000

| (x,y) | Exact Solution u | Hybrid CN-DF u | Pure C-N u |
|---|---|---|---|
| (0.1,0.1) | -3.586880e-06 | -3.586899e-06 | -3.586912e-06 |
| (0.5,0.1) | 4.314708e-06 | 4.314738e-06 | 4.314758e-06 |
| (0.3,0.3) | 1.282006e-06 | 1.282019e-06 | 1.282031e-06 |
| (0.7,0.3) | 1.610071e-05 | 1.610081e-05 | 1.610089e-05 |
| (0.1,0.5) | -1.237415e-05 | -1.237419e-05 | -1.237423e-05 |
| (0.5,0.5) | 1.296916e-05 | 1.296924e-05 | 1.296931e-05 |
| (0.5,0.9) | 3.276510e-07 | 3.276817e-07 | 3.277015e-07 |
| (0.9,0.9) | 1.576664e-06 | 1.576699e-06 | 1.576724e-06 |

Table 2: Solutions for v at dt = 0.001, t = 1.0, and Re = 4000

| (x,y) | Exact Solution v | Hybrid CN-DF v | Pure C-N v |
|---|---|---|---|
| (0.1,0.1) | -1.915564e-05 | -1.915574e-05 | -1.915582e-05 |
| (0.5,0.1) | -1.730376e-05 | -1.730384e-05 | -1.730391e-05 |
| (0.3,0.3) | -1.416108e-05 | -1.416114e-05 | -1.416120e-05 |
| (0.7,0.3) | -5.511169e-06 | -5.511183e-06 | -5.511191e-06 |
| (0.1,0.5) | -4.935646e-05 | -4.935647e-05 | -4.935647e-05 |
| (0.5,0.5) | -2.469256e-06 | -2.469255e-06 | -2.469255e-06 |
| (0.5,0.9) | 1.227266e-05 | 1.227274e-05 | 1.227280e-05 |
| (0.9,0.9) | -1.395881e-05 | -1.395886e-05 | -1.395890e-05 |

The above table shows that the CN-DF scheme provides accurate results closer to the exact solution as compared to those of the C-N scheme.
We now turn our attention to the $\mathbf{L^1}$ error analysis. It is used to determine the order of convergence or accuracy besides checking whether the scheme is consistent.

Table 3: Order of Convergence for solution $u$ and $v$ at Re = 4000, t = 1 sec, dt = 0.001

| No. of Cells | $\mathbf{L^1}$ error in $u$ | Order | No. of Cells | $\mathbf{L^1}$ error in $v$ | Order |
|---|---|---|---|---|---|
| (4,4) | 1.3464e-009 | - | (4,4) | 7.5869e-010 | - |
| (8,8) | 4.2381e-010 | 1.6676 | (8,8) | 3.2784e-010 | 1.2105 |
| (16,16) | 9.1106e-011 | 2.2178 | (16,16) | 7.8436e-010 | 2.0634 |
| (32,32) | 4.4870e-012 | 4.3437 | (32,32) | 3.5022e-012 | 4.4852 |
| (64,64) | 1.3904e-013 | 5.0122 | (64,64) | 1.0782e-013 | 5.0216 |

# 4 Conclusion

The hybrid CN-DF scheme is implicit, three level in time, and unconditionally stable. From tables 1 and 2, it is determined that the CN-DF scheme is more accurate than the CN scheme with regards to the exact solution. Consequently, it is fifth order convergent in space implying a higher accuracy than the second order convergent CN scheme. The decrease in the $\mathbf{L^1}$ error as the mesh is refined also verifies the consistency of the scheme. Therefore, the scheme is a great development in numerical analysis and can be applied to other partial differential equations where high accuracy is required.

# References

[1] A.R. Bahadir, A Fully Implicit Finite Difference Scheme for Two-Dimensional Burgers' Equations, *Applied Mathematics and Computation*, 137 (2003), 131-137.

[2] J.M. Burgers, On a Quasilinear Parabolic Equation Occuring in Aerodynamics, *Advances in Applied Mechanics*, 3 (1950), 201-230.

[3] J. Crank, P. Nicolson, A Practical Method for Numerical Evaluation of Partial Differential Equations of the Heat Conduction Type, *Proc. Camb. Phil. Soc.*, 1 (1947), 50-67.

[4] E.C. Du Fort and S.P. Frankel, Conditions in the Numerical Treatment of Parabolic Differential Equations, *Mathematical Tables and Other Aids to Computation*, 7 (1953), 135-152.

[5] C.A.J. Fletcher, Generating Exact Solutions of the Two-Dimensional Burgers' Equations, *International Journal for Numerical Methods in Fluids*, 3 (1983), 213-216.

[6] P. Kanti and V. Lajja, A Note on Crank-Nicolson Scheme for Burgers' Equation, *Applied Mathematics*, 2 (2011), 888-889.

[7] A. Koross, M.E.O. Oduor and O. Omollo, Hybrid Finite Difference Schemes from Operator Splitting for Solving Burgers' Equation, *Journal of Agriculture, Pure and Applied Science and Technology*, 1 (2009), 45-52.

[8] M.C. Kweyu, W.A. Manyonge, A. Koross and V. Ssemaganda, Numerical Solutions of the Burgers' System in Two Dimensions under Varied Initial and Boundary Conditions, *Applied Mathematical Sciences*, 6 (2012), 5603-5615.

[9] A.R. Mitchell and D.F. Griffiths, The Finite Difference Method in Partial Differential Equations, John Wiley and Sons Ltd, 1980, 337-360.

[10] K. Pandey, V. Lajja and K.V. Amit, On a Finite Difference Scheme for Burgers' Equation, *Applied Mathematics and Computation*, 215 (2009), 2208-2214.

[11] K. Pandey, V. Lajja and K.V. Amit, Du Fort-Frankel Finite Difference Scheme for Burgers' Equation, *Arab Journal of Mathematics*, 2 (2012), 91-101.

[12] F.R. Samir, On the Fourth-Order Accurate Compact ADI Scheme for Solving the Unsteady Nonlinear Coupled Burgers' Equations, *Journal of Nonlinear Mathematical Physics*, 6 (1998), 13-34.

[13] C. Tuncer, Convective Heat Transfer, Springer, (2002), ISBN 0-9668461-4-1.

[14] K.S. Vineet, T. Mohammad, B. Utkarsh and YVSS. Sanyasiraju, Crank-Nicolson Scheme for Numerical Solutions of Two-Dimensional Coupled Burgers' Equations, *International Journal of Scientific and Engineering Research*, 2 (2011), 2229-5518.

**Received: October 1, 2013**