COMPARATIVE ANALYSIS OF SPECTRAL THEORY OF DIFFERENTIAL AND DIFFERENCE OPERATORS ON HILBERT SPACES

BY

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Declaration

Declaration by the candidate

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Dedication

I dedicate this work to my family.

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Abstract

There has been effort to investigate the spectrum of difference operators to parallel that of differential operators. This has been done either through the subspace theory or direct definition of the domain of the operator. Even though much has been done to compare the spectral theory of differential and difference operators of order two, few or limited comparative analysis exists beyond order two operators. In particular, no comparative analysis has been done for order six operators on Hilbert spaces and that of the fourth order has not been exhausted especially when the odd order coefficients are unbounded. Of importance is to compare the results obtained in differential operators to those of their discrete counterparts if the two operators are of the same order under similar growth and decay conditions. The main aim of this study was to conduct a comparative analysis of spectral theory of higher order differential and difference operators on Hilbert spaces, when the odd order coefficients are unbounded. The specific objectives were; to evaluate and compare the deficiency indices of the second order differential and difference operators with unbounded odd order coefficients, discuss the spectrum of the fourth order differential and difference operators and finally apply asymptotic integration and summation to analyze the spectral properties of sixth order differential and difference operators on Hilbert spaces, with the third order coefficient unbounded. The comparative analysis was carried out by means of asymptotic integration and summation based on Levinson's and Levinson-Benzaid-Lutz theorems. For order two differential operator with unbounded odd order coefficients, the absolutely continuous spectrum was the whole of the real line with spectral multiplicity as one. On the other hand, the spectrum of their discrete counterparts only consisted of eigenvalues under similar growth conditions. Similarly, order four differential operator resulted into absolutely continuous spectrum with spectral multiplicity one whenever the third order coefficient is unbounded while the spectrum of fourth order difference operator under similar conditions is pure discrete. Finally, the absolutely continuous spectrum was found to be the whole real line in the case of order six differential operator with sixth order difference operator giving discrete spectrum when the third order coefficient is unbounded. Since spectral theory have wide applications in other fields like quantum mechanics, stability analysis of market prices as well as in epidemiology, the results obtained in this research are applicable in stability analysis of market prices because asymptotic integration and summation are perturbation processes. Due to complexity in computations and analysis of the roots of degree six polynomials, only three term sixth order operators were analyzed. In future, one can investigate the spectral properties of order six operators with all the coefficients taken as non-zero. This can be generalized to higher orders more than six.

Notations and Terminologies

LBL:	Levinson Benzaid Lutz
USI:	Uniformly Square Integrable
τ:	symmetric differential expression
$\mathcal{L}$:	symmetric difference expression
T:	minimal differential operator
T^*:	maximal differential operator
L^*:	maximal difference operator
L:	minimal difference operator
$O(.), o(.)$:	Landau Symbols (the 'big-O'and 'little-o')

Contents

Chapter 1

Introduction

This chapter presents a background of the study, definitions of terms used in this study, statement of the problem, study objectives and significance of the study.

1.1 Background of the Study

There has been a number of papers investigating the spectrum of difference operators to parallel that of differential operators. Attempts have been made to compare the spectral theory of differential and difference operators but only for second order and to some extent fourth order operators (Nyamwala, 2010; Behncke & Nyamwala, 2013). Thus for higher orders, that is, of orders more than four, this is lacking completely. We point out here that due to limiting techniques in constructing symmetric difference equations of odd orders, the analysis concentrated only on even order operators.

We are considering even order differential and difference operators particularly, second, fourth and sixth order generated by equations (1.1) and (1.2) respectively. Let

$$\begin{aligned} \tau y(x) &= w^{-1}(x)\{\sum_{k=0}^{3}(-1)^k(p_k(x)y^{(k)}(x))^{(k)} \\ &- i\sum_{j=1}^{3}(-1)^{(j)}(q_j(x)y^{(j)}(x))^{(j-1)} + (q_j(x)y^{(j-1)}(x))^{(j)}\} \end{aligned} \tag{1.1}$$

be a 6^{th} order symmetric differential equation defined on $\mathcal{L}^2([0,\infty))$ where $p_k(x), q_j(x)$, k=0,1,2,3 and j=1,2,3 are real-valued functions with $p_3(x), w(x) > 0$, $w(x)$ is a weighted function and $i = \sqrt{-1}$. Here, $y^{(k)}(x)$ is the k^{th} derivative of $y(x)$ with respect to x. In the case of 2^{nd} and 4^{th} orders, we will assume that $p_1(x) > 0$ and $p_2(x) > 0$ respectively. The other coefficients of higher order will be taken as zero.

Then $\tau y(x)$ in (1.1) generates a differential operator on $\mathcal{L}^2([0,\infty))$. Similarly, let

$$\begin{aligned} \mathcal{L}y(t) &= w^{-1}(t)\{\sum_{k=0}^{3}(-1)^k\Delta^k[p_k(t)\Delta^k y(t-k) \\ &- i\sum_{j=1}^{3}(-1)^{(j)}[\Delta^{j-1}(q_j(t)\Delta^j y(t-j)) + \Delta^j(q_j(t)\Delta^{j-1}y(t-j+1))]\} \end{aligned} \tag{1.2}$$

be a 6^{th} order symmetric difference equation defined on $\ell^2(\mathbb{N})$ where $w(t) > 0$ and $p_3(t) > 0$, $p_k(t), q_j(t)$, k=0,1,2,3, j=1,2,3 are real valued functions with Δ a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$. Then $\mathcal{L}y(t)$ generates a difference operator on $\ell^2(\mathbb{N})$.

The interest of this study was in obtaining the deficiency indices of minimal operators generated by (1.1) and (1.2) and the spectrum of self-adjoint extension of these minimal operators. The two, that is, deficiency indices and the spectrum of self-adjoint extensions, constitute the spectral theory of the operators generated by (1.1) and (1.2) respectively.

1.2 Definition of Terms

In this section, we define the basic concepts in spectral theory that are commonly used and are fundamental in comparative analysis of spectral theory of the differential and difference operators on Hilbert spaces.

Let T be an operator defined on the Hilbert space $\mathcal{H}$. The symbol $D(T)$ will be used to denote the domain of T.

Definition 1.2.1

Spectrum of T denoted by $\sigma(T)$ is defined as the set of all complex numbers λ, such that $(T-\lambda I)^{-1}$ does not exist. Mathematically, one writes $\sigma(T) = \{\lambda \in \mathbb{C}; (T-\lambda I)^{-1}$ does not exist$\}$. The spectrum has various components, namely; essential spectrum, residual spectrum, absolutely continuous spectrum, point spectrum and singular continuous spectrum.

The set of all complex numbers λ, such that $(T-\lambda I)^{-1}$ does not have a dense range in $\mathcal{H}$ is known as continuous spectrum of T and is denoted by $\sigma_c(T)$. In symbols, one writes $\sigma_c(T) = \{\lambda \in \mathbb{C}; (T-\lambda I)^{-1}$ does not have a dense range$\}$ (Kreyszig, 1989).

The set of all complex numbers λ, such that $(T-\lambda I)^{-1}$ does not exist since T is not injective is known as point spectrum of T and is denoted by $\sigma_p(T)$. Symbolically, one writes $\sigma_p(T) = \{\lambda \in \mathbb{C}; (T-\lambda I)^{-1}$ does not exist since T is not injective$\}$.

Residual spectrum is the set of all complex numbers λ, such that $(T-\lambda I)^{-1}$ does not exist since T is not bounded away from zero and is denoted by $\sigma_r(T)$. Mathematically, one writes $\sigma_r(T) = \{\lambda \in \mathbb{C}; (T-\lambda I)^{-1}$ does not exist since T is not bounded away from zero$\}$.

The set of all complex numbers λ, such that $(T-\lambda I)^{-1}$ does not exist since T is not a semi-fredholm operator (an operator whose range is closed and is finite dimensional) is known as essential spectrum of T and is denoted by $\sigma_{ess}(T)$. In symbols, one writes $\sigma_{ess}(T) = \{\lambda \in \mathbb{C}; (T-\lambda I)^{-1}$ does not exist since T is not a semi-fredholm operator$\}$.

Definition 1.2.2

An operator T is said to be densely defined if $D(T)$ is dense in $\mathcal{H}$, that is, $\overline{D(T)} = \mathcal{H}$, where, $\mathcal{H}$ is a Hilbert space.

Definition 1.2.3

An operator T defined on a Hilbert space $\mathcal{H}$ is said to be symmetric if T is densely defined and $T \subset T^*, D(T) \subset D(T^*)$, that is, $< Tu, v >=< u, T^*v >=< u, Tv >$ for all $u, v \in D(T)$.

Definition 1.2.4

The maximal operator T^* is defined on the largest possible domain in $\mathcal{L}^2((0,\infty), w)$ which is mapped onto $\mathcal{L}^2([0,\infty), w)$.

The domain of the maximal operator T^* generated by τ is given by;

$D(T^*) = \{y \in \mathcal{L}^2(0,\infty : w) : y^{[0]}, y^{[1]}, y^{[2]}, y^{[3]}, y^{[4]}, y^{[5]}$ are absolutely continuous in $(0,\ \infty)$, $\tau y \in \mathcal{L}^2((0,\infty : w)$, $T^*y = \tau y$ for all $y \in D(T^*)\}$.

Definition 1.2.5

An operator defined by restraining within bounds the domain of the maximal operator only to those functions y with compact support is known as pre-minimal operator. It is denoted by T_1 and its domain is defined by

$D(T_1) = \{y \in D(T^*); y \text{ has compact support in } (0, \infty)\}$.

$T_1 y = T^* y = \tau y$ for all $y \in D(T_1)$. For unbounded domains, T_1 is not necessarily closed but is densely defined. The closure of the preminimal operator $T_1, \overline{T}_1$, is the minimal operator spawned by (1.1) and (1.2) and is denoted by T.

Definition 1.2.6

In Eastham (1989), the deficiency index, def$T$, is defined as the pair

$$defT = (dimN_{T^*+i}, dimN_{T^*-i}).$$

N_{T^*+i} is the nullspace of $T^* + iI$ and N_{T^*-i} is the nullspace of $T^* - iI$. Thus N_{T^*-i} is the set of all elements such that $Ty = iy$. If one uses a nonreal complex spectral parameter z, then for $Imz > 0$, one has $dimN_{T^*-i} = dimN_{T^*-z}$ and $dimN_{T^*+i} = dimN_{T^*-\bar{z}}$ with $N_+ = dimN_{T^*-z}$ and $N_- = dimN_{T^*-\bar{z}}$. Although the definition of the deficiency indices depend on z, the dimension of the nullspaces are independent of z provided that z remains in either of the half-planes. For $Imz > 0$, N_+ and N_- may be finite or infinite. Thus $defT = (N_+, N_-)$.

Definition 1.2.7

An operator T has self-adjoint extensions if both its deficiency indices are equal, that is, $defT = (r, r)$ and r is not equal to zero. Otherwise, for $r = 0$, the operator is self-adjoint. Here, $3 \leq r \leq 6$ as per the results proved by Naimark (1967), is the range for order six.

Definition 1.2.8

The M-matrix generalises the m-function of the Weyl Titchmarsh and thus relates the asymptotics of the eigenfunction of higher order differential operators to the spectrum of their self-adjoint realisation. Given the Hamiltonian system of the form

$$Jy'(x) = [z\mathcal{A}(x) + B(x)]y(x), \tag{1.3}$$

where

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\mathcal{A} = diag(w, 0, ..., 0)$ with the assumption that $\mathcal{A}(x), B(x)$ are locally integrable in the underlying interval $[a, \infty)$ and $B(x) = B^*(x), \quad \mathcal{A}(x) > 0$, (in the positive definite sense), almost everywhere. The nonzero matrix elements of $\mathcal{A}(x)$ are $\mathcal{A}_{11} = w$ while

$$B = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix}$$

with the nonzero matrix elements of A, B and C given by

$$A_{j,j+1} = 1, \quad A_{n,n} = i\frac{q_n}{p_n}, \quad B_{n,n} = p_n^{-1}$$

$$C_{j,j} = p_{j-1}, \quad C_{j,j+1} = iq_j = -C_{j+1,j}.$$

Let $Y_\alpha(., z) = (U_\alpha(., z), V_\alpha(., z))$ be a fundamental matrix with initial values

$$Y_\alpha(a, z) = \begin{bmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{bmatrix}$$

where α_1, α_2 are n by n complex-valued matrices described with rank$(\alpha_1, \alpha_2) = n$ and

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n, \quad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0_n. \tag{1.4}$$

The boundary conditions at the regular endpoint a are given by

$$(\alpha_1, \alpha_2)y(a) = 0. \tag{1.5}$$

U_α, V_α are 2n by n complex-valued matrices whose every column solves $\tau u = zu$. Note that the boundary condition at a are satisfied by $V_\alpha(., z)$ and α_1, α_2 satisfy (1.4) and (1.5). Therefore, the columns of Y_α generates the 2n-dimensional vector space of solutions of (1.3).

In the limit point case, self-adjoint extensions are realised by fixing the boundary conditions at a. Now the boundary conditions to the right through $\alpha = (\alpha_1, \alpha_2)$ are fixed and the techniques of Hinton and Shaw (1981) applied for $Imz \neq 0$, the M-matrix, $M_\alpha(z) \in \mathbb{C}^{n\times n}$ is defined by;

$$\chi_\alpha(x,z) = Y_\alpha(x,z)\begin{bmatrix} I_n \\ M_\alpha(z) \end{bmatrix} \in \mathcal{L}^2[a,\infty).$$

$M_\alpha(z)$ is analytic for $Imz \neq 0$ and $ImM_\alpha(z)$ is positive definite in the upper half plane. The columns of $\chi_\alpha(x,z)$ form a basis for the square integrable solutions of (1.3).

Similarly, for the difference case, in the limit point case with $Imz > 0$, one has a matrix, $M \in \mathbb{C}^{n\times n}$ such that;

$$\chi_\alpha(x,z) = Y_\alpha(x,z)\begin{bmatrix} I_n \\ M(z) \end{bmatrix} = U_\alpha(t,z) + V_\alpha(t,z)M(z),$$

where $\chi_\alpha(t,z)$ satisfies the boundary condition, $(\alpha_1, \alpha_2)y(a) = 0$. As before $M(z)$ is determined from the solutions that stay absolutely square summable as $Imz \searrow 0$, it is unique, analytic in both half planes and satisfies $M^*(\bar{z}) = M(z)$. It has been shown by Shi (2006), that if L is limit point at $t = \infty$, then one can construct the M-matrix $M(z)$ for the Hamiltonian restricted to $[a,\infty)$ with Dirichlet boundary conditions. To do this, let

$$\begin{bmatrix} W_1(a,z) \\ W_2(a,z) \end{bmatrix}$$

be a system of n square summable solutions for $Imz > 0$. Then from the theory of Hinton and Shaw (1981) which was extended to discrete setting by Shi (2006), it follows that these solutions arise from $Y(t,z)\begin{bmatrix} I_n \\ M(z) \end{bmatrix}$ where $Y_\alpha(t,z)$ is the fundamental solution of the system satisfying the appropriate boundary conditions at a. If one compares both sets of solutions, it shows that there is an invertible n by n matrix C such that

$$\chi(a,z) = \begin{bmatrix} W_1(a,z) \\ W_2(a,z) \end{bmatrix} C = Y(t,z) \begin{bmatrix} I_n \\ M(z) \end{bmatrix}.$$

This in turn implies $M(z) = W_2(a,z)W_1^{-1}(a,z)$. Now let $F_\alpha(.,z)$ be n by 2n system of square summable solutions of the Hamiltonian system satisfying boundary conditions at a and infinity and $z, z' \notin \mathbb{R}$, then for $F_\alpha(.,z) - F_\alpha(.,z') \in D(H)$ and by results of Remling (1998), it follows that $\langle F_\alpha(.,z), F_\alpha(.,z)\rangle = (Imz)^{-1}ImM(z)$ if $z' = z$. Therefore, if $z = \mu + i\epsilon$ for some $\epsilon > 0$, then one has for $\mu_+ = lim_{\epsilon\to 0+}\mu + i\epsilon$

$$ImM(\mu_+) = lim_{\epsilon\to 0+}ImM(\mu + i\epsilon) = lim_{\epsilon\to 0+}\epsilon\langle F_\alpha(.,\mu_+ + i\epsilon), F_\alpha(.,\mu_+ + i\epsilon)\rangle.$$

1.3 Statement of the Problem

It is well known that Sturm-Liouville equations and their discrete counterparts, Jacobi matrices, can be analysed by closely related methods. Thus, many Schrödinger-type results have their discrete counterparts and often a result in the discrete or continuous sector leads to a result in the other area. A comparative analysis of the spectral theory for fourth order operators has not been exhausted especially when the odd order coefficients are unbounded while that of sixth order differential and difference operators on $\mathcal{L}^2[0,\infty)$ and $\ell^2(\mathbb{N})$ is lacking completely.

1.4 Objectives of the Study

In this section, we have given both the main and specific objectives of this research.

1.4.1 Main Objective

The main objective of this study was to conduct a comparative analysis of spectral theory of higher order differential and difference operators on Hilbert spaces, when the odd order coefficients are unbounded.

1.4.2 Specific Objectives

The specific objectives of this study were to:-

(i) Evaluate and compare the deficiency indices of second order differential and difference operators on Hilbert spaces with unbounded odd order coefficient.

(ii) Discuss the spectrum of fourth order differential and difference operators on Hilbert spaces when the third order coefficient is unbounded.

(iii) Apply the asymptotic integration and summation to investigate the spectral theory of sixth order differential and difference operators when the third order coefficient is unbounded.

The results obtained are explained in the following subsections: Objective (i) is achieved in sections 4.2 and 4.3, objective (ii) is achieved in sections 4.4 and 4.5 while those of the third objective are in sections 4.6 and 4.7.

1.5 Justification of the Study

In this study, we have conducted a clear and refined comparative analysis of the spectral theory of the second and fourth order differential and difference operators especially when the odd order coefficients are unbounded and gone ahead to do the analysis of the sixth order differential and difference operators which is lacking completely. This has been done for the completeness of the analysis.

1.6 Significance of the Study

The Differential operators have many applications in real life situation for example in quantum mechanics where it is used to represent momentum in the field of observable. Here, ψ represents a state with variables as q, that is, $\psi = \psi(q)$. Thus for the momentum we have the operator D defined by $D\psi(q) = h/2\Pi i = d\psi(q)/dq$ which represents 1-dimensional Schrödinger equation in modern physics (second order differential operator).

Symmetric and Hermitian matrices or operators are applicable in mathematical

modelling since any system of differential equations explaining this model results into first order system (Hamiltonian system) with symmetric matrices whose eigenvalues are real and have physical meaning.
The results obtained in this research are also applicable in stability analysis of market prices because asymptotic integration and summation are perturbation processes.

This thesis is divided into five chapters, namely; 1. Introduction, where the background of the study, definition of the terms used in the study, objectives and significance of the study have been outlined. 2.Literature Review, which examines and acknowledges the contribution of other scholars and researchers 3. Methodology, here, asymptotic integration and summation procedures are outlined. 4. Results and Discussions, which extends the situation studied in (Agure, Ambogo, & Nyamwala, 2013; Behncke & Nyamwala, 2012) to the sixth order case and 5. Conclusion, which is a summary of the main findings based on the research objectives and conclusions drawn out of the results.

Chapter 2

Literature Review

2.1 Introduction

This chapter examines and acknowledges the contributions of other researchers and scholars on spectral theory which has been done through review of books, journals and research work. A lot of research on spectrum of differential and difference operators on Hilbert spaces have been conducted (Hinton & Schneider, 1993; Nyamwala, 2010; Behncke, Hinton, & Remling, 2001; Remling, 1999; Nyamwala, 2015). The focus here, is on literature on the spectrum and deficiency indices of differential and difference operators of order two and four.

2.2 Spectral theory of difference and differential operators

A study on deficiency indices and spectrum of fourth order difference equations with unbounded coefficients was carried out by Agure et al. (2013). The study used subspace theory together with appropriate smoothness and decay conditions to calculate the deficiency indices of fourth order difference equations and absolutely continuous spectrum with unbounded coefficients. The results showed that if the coefficients $p_k,\ q_j,\ k = 0, 1, 2,\ j = 1, 2$, are allowed to be unbounded and satisfy appropriate smoothness and decay conditions, and the Hamiltonian satisfies the definiteness and regular conditions, the deficiency indices of the minimal subspace will be (n, n), where $2 \leq n \leq 4$ and the absolutely continuous spectrum of the selfadjoint extension subspace is the whole of $\mathbb{R}$ and has spectral multiplicity one.
An investigation on higher even order linear differential operators with unbounded coefficients had been conducted in (Behncke & Nyamwala, 2012) For these operators, the eigenvalues of the characteristic polynomials fall into distinct classes or clusters. In such a case, the spectral properties, deficiency indices and spectra, of

the underlying differential operators are superpositions of the contributions from the individual clusters. The results were based on a quantitative improvement of Levinson's Theorem. The investigation concludes that the method used can also be applicable to other classes of linear differential operators.

In (Nyamwala, 2015), it was shown that the absolutely continuous spectrum exists outside a certain bounded interval in a research on absolutely continuous spectrum of fourth order difference equation with bounded coefficients. In addition, the spectral multiplicity as well as the location of absolutely continuous spectrum of selfadjoint subspace extension under certain asymptotic conditions were computed.

In a study on spectrum and deficiency indices of four term differential operator where strengthening dichotomy condition and weakening decay conditions was applied, it has been proved by Nyamwala (2010) that a four term 2n-th order differential operator with unbounded coefficients on half line is a nonlimit-point operator. It is also proved that the deficiency index of this operator is determined by the behaviour of the coefficients themselves and that the absolutely continuous spectrum had multiplicity of two.

Suppose that a difference operator has almost constant coefficients, it has been proved by Behncke and Nyamwala (2011), that the operators whose coefficients are approximately constant in a general sense have an absolutely continuous spectrum which is equal to that of the corresponding constant coefficient operator or given by that of the limiting constant coefficient operator. For such operators, the absolutely continuous spectrum can be read off from the associated characteristic polynomial. The approach is based on an analysis of the associated difference equation with the help of uniform asymptotic summation techniques.

Levinson's theorem in asymptotic integration of linear differential systems is strengthened in a quantitative way by Behncke (2010a). The results showed that any decay in excess of absolute integration appears with a remainder.

The survey in the spectral theory of certain one-dimensional differential and finite difference operators : Jacobi matrices, Krein systems and Schrodinger operators was carried out by Killip and Simon (2003) and the connection for these results is the use of sum rules relating the coefficients and spectral data.

The asymptotic behaviour of large eigenvalues for a class of finite difference self-adjoint operators with compact resolvents in $\ell^2(\mathbb{N})$ was investigated by Anne Boutet De Monvel (2012). This was done by obtaining the simplest remainder estimates and then computing further terms of the asymptotics with smaller remainder under stronger conditions of smoothness imposed on the entries.

The relationship between the asymptotic behaviour of solutions of singular Sturm-Liouville equation and spectral properties of the corresponding self-adjoint operators is shown by Daphne (2005). The link between the number of points of the spectrum below an eigenvalue and the number of zeros in the associated eigenfunction was noted. The extension of the theory to the related differential and difference operators was also shown and the applications discussed in conjuction with other asympotic methods.

A study on spectral analysis of higher order differential operators was carried out by (Remling, 1998). The study interpreted the m-function in terms of Hilbert space notions and showed that the classical m-function could be recovered as a part of the more complicated one. Application of this led to results on spectral multiplicity and stability process of the spectrum.

A discussion on spectral properties of higher order ordinary differential operators was carried out by (Behncke et al., 2001). If the coefficients differed from constants by small perturbations, then the spectral properties were preserved. The results showed that the perturbed operators had the same spectral properties as the unperturbed one except that there may be additional point spectrum. Location and multiplicity of the spectrum was also determined.

A study on deficiency index problem in which, as part of the spectral theory of self-adjoint differential operators, the problem was to determine the number of linearly independent solutions of the associated differential equation on a Hilbert space was carried out in (Eastham, 1989). Applications of spectral theory were also carried out

and the location of eigenvalues embedded in the continuous spectrum determined.
In (Weidmann, 1980), classes of linear operators were studied. The spectral theory of self-adjoint operators (first for compact operators and then for the general case) as well as some important consequences and a detailed characterisation of the spectral properties was analysed. Von Neumann's extension theory for symmetric operators was developed and the results of perturbation theory for self-adjoint operators were found. Lastly, applications of partial differential operators, in particular to Schrodinger and Dirac operators were shown.

The absolutely continuous spectrum of the constant coefficient operator and its multiplicity was read off from the range of the characteristic polynomial on the unit circle by Nyamwala (2010). It is also proved that the absence of singular continuous spectrum under suitable smoothness assumptions for the coefficients is clearly a perturbation result. This perturbation is not in the operator sense but rather a perturbation of the M-function. These results may be extended to operators on $\mathcal{L}^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$ by using the decomposition method.

The spectral theory of higher order difference operators had been conducted by means of asymptotic summation, thereby extending many results of differential operators to discrete settings. The spectra of degenerate fourth-order operators was also investigated by Behncke and Nyamwala (2013) and the results then compared with those of corresponding differential operators. Even though there had been many similarities between both classes of operators, the spectral results may be quite distinct. The comparative analysis of spectral theory of the operators is not exhausted for order four operators and is missing completely for operators of order more than four.

This study has conducted a comparative analysis of spectral theory of order two, four and six for differential and difference operators. The deficiency indices and the spectral multiplicity have also been evaluated on the Hilbert space. The comparative analysis has been carried out by means of asymptotic integration and summation based on Levinson's and Levinson-Benzaid-Lutz theorems.

Chapter 3

Methodology

3.1 Introduction

In this chapter, we outline the methods that will be used to meet objectives of the study. It outlines the procedure used to evaluate the deficiency indices of the difference and the differential operator, prove the dichotomy condition and finally investigate the spectral properties of the sixth order differential and difference operators. The main procedure in solving symmetric differential equations of higher order has been asymptotic integration. The theorem states that, the solutions of a system

$$u'(x) = \{\wedge(x) + R(x)\}u(x), \qquad \wedge(x) = diag(\lambda_k(x))$$

looks like the solutions of the unperturbed system $u' = \wedge u$ if $R(x)$ is sufficiently small and $\wedge(x) = diag(\lambda_k(x))$ satisfies a dichotomy condition (Eastham, 1989), here, sufficiently small means absolutely integrable. The dichotomy condition amounts to; for every unequal pair $k, j,\quad a \leq t \leq x < \infty,\quad Re\{\lambda_k(x,z) - \lambda_j(x,z)\}$ has constant sign modulo $\mathcal{L}^1([a,\infty)$ for all $z \in \Omega$. Moreover, assuming that $\| R(x) \| \leq \rho(x)$ with $\rho(x) \in \mathcal{L}^1([a,\infty)$. Then

$$Y'(x,z) = [\wedge(x,z) + R(x,z)]Y(x,z)$$

has solutions $y_k(x,z),\ 1 \leq k \leq 2n$ with asymptotic form

$$Y_k(x,z) = (e_k + r_k(x,z)).\exp(\int_a^x \lambda_k(t,z)dt),$$

where e_k denotes the kth unit vector and $r_k(x,z)$ depends analytically on $z \in \Omega$ and tends to 0 z-uniformly as $x \longrightarrow \infty$.

For the difference operator, asymptotic summation which is based on the famous Levinson-Benzaid-Lutz's theorem has been used where the asymptotics of the eigenfunctions of the operators have been determined (Benzaid & Lutz, 1987). The dichotomy condition amounts to; for any pairs of indices k and j, such that $k \neq j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_k(t,z)| \geq \delta$ for all $t \geq a$. Then either $|\frac{\lambda_k(t,z)}{\lambda_j(t,z)}| \geq 1+\delta$ or $|\frac{\lambda_k(t,z)}{\lambda_j(t,z)}| \leq 1-\delta$ for a large t. Here, the form of the solution is given by;

$$Y(t,z) = [e_k + r_{kk}]\Pi_{t=a}^{t-1}(\wedge(l,z)),$$

where $r_{kk}(t,z) = o(1)$.

3.2 System formulation

The first objective was achieved by system formulation of order two difference and differential operators, converted them into first order by computing their quasidifferences and quasiderivatives respectively, then their characteristic polynomial. Since the characteristic polynomials of order two operators were quadratic expressions, the zeros were computed explicitly. Order four operators without odd order terms resulted into biquadratic characteristic polynomials whose roots were similarly computed explicitly. The deficiency indices were then read off from the asymptotics of the eigenfunctions. Asymptotic integration was used in solving symmetric differential equations of higher order while asymptotic summation was used for the symmetric difference equations.

For the second objective, we begun by obtaining the first objective before determining the spectrum from analysis of the solutions. The z-uniformly square integrable or summable eigenfunctions contributed to discrete spectrum. The eigenfunctions that lost their square integrability as $Imz \longrightarrow 0$ contributed to absolutely continuous spectrum with the multiplicity of the absolutely continuous spectrum equal to the number of such eigenfunctions. The behaviour of the correction term determined the range of the spectral parameter which was used to locate the absolutely continuous spectrum. The method of asymptotic integration excludes coefficients whose derivatives decay too slowly, hence no singular continuous spectrum. The last objective was a combination of the first and second objective.

The M-matrix $M(z)$ was constructed for the Hamiltonian restricted to $[a, \infty)$ with Dirichlet boundary conditions as outlined in Chapter One. This was done by letting

$$\begin{bmatrix} W_1(a,z) \\ W_2(a,z) \end{bmatrix}$$

be a system of $2n \times n$ square summable solutions for $Imz > 0$. Then from the theory of Hinton and Shaw (1981), it followed that these solutions arose from $Y(t,z)\begin{bmatrix} I_n \\ M(z) \end{bmatrix}$ where $Y_\alpha(t,z)$ was the fundamental solution of the system satisfying the appropriate boundary conditions at a. In this case, these are $Y_\alpha(a,z) = I_{2n}$. A comparison was made for both sets of solutions and was shown that there was an invertible n by n matrix C such that

$$\chi(a,z) = \begin{bmatrix} W_1(a,z) \\ W_2(a,z) \end{bmatrix} C = Y(t,z)\begin{bmatrix} I_n \\ M(z) \end{bmatrix}.$$

This in turn implied $M(z) = W_2(a,z)W_1^{-1}(a,z)$. Letting $F_\alpha(.,z)$ be n by 2n system of square summable solutions of the Hamiltonian system satisfying boundary conditions at a and infinity and $z, z' \notin \mathbb{R}$, then

$$F_\alpha(.,z) - F_\alpha(.,z') \in D(\mathcal{H})$$

and by results of Remling (1998), it followed that

$$\langle F_\alpha(.,z), F_\alpha(.,z)\rangle = (Imz)^{-1} ImM(z) \quad if \quad z' = z.$$

Therefore, if $z = \mu + i\epsilon$ for some $\epsilon > 0$, then one has for $\mu_+ = lim_{\epsilon\to 0+}\mu + i\epsilon$

$$ImM(\mu_+) = lim_{\epsilon\to 0+} ImM(\mu + i\epsilon) = lim_{\epsilon\to 0+}\epsilon\langle F_\alpha(.,\mu_+ + i\epsilon), F_\alpha(.,\mu_+ + i\epsilon)\rangle.$$

Chapter 4

Results and Discussions

4.1 Introduction

In this chapter, we present the main results and discussions of this research in terms of lemmas and theorems. The results are grouped according to the research objectives.

The main results and discussions are organized as follows:

Section 4.2 and 4.3: Evaluating and comparing the deficiency indices of second order differential and difference operators with unbounded odd order coefficient.

Section 4.4 and 4.5: Discussing the spectrum of fourth order differential and difference operators when the third order coefficient is unbounded.

Section 4.6 and 4.7: Application of asymptotic integration and summation to investigate the spectral theory of the sixth order differential and difference operators when the third order coefficient is unbounded.

4.2 Order Two Differential Operator

In this section, we consider (1.1) with $p_3 = q_3 = q_2 = p_2 = 0, \quad p_1, q_1, p_0 \neq 0$ so that we have order two differential equation generating a second order differential operator. Thus, we consider the symmetric differential equation of the form

$$\tau y(x) = -(p_1(x)y'(x))' + i[q_1(x)y'(x) + (q_1(x)y(x))'] + p_0(x)y(x). \tag{4.1}$$

The growth conditions are assumed to be;

$$|q_1(x)| \nearrow \infty, \qquad p_0, p_1 = o(q_1), \qquad \forall x \in [0, \infty). \tag{4.2}$$

Further, we assume that the coefficients obey the following regularity and decay conditions.

$$\frac{f'}{f} \in \mathcal{L}^2, \quad \frac{f''}{f}, \left(\frac{f'}{f}\right)^2 \in \mathcal{L}^1, \qquad f = p_0, p_1, q_1, \quad f = f(x). \tag{4.3}$$

4.2.1 System formulation

We study the spectral theory of differential operators generated by (4.1) on $\mathcal{L}^2(0,\infty)$ by means of asymptotic integration. Note that y', y'' will denote the first and second derivatives of y while $y^{[1]}, y^{[2]}$ will denote the quasiderivatives of y in line with Walker's definition, see (Walker, 1974). $\mathcal{L}^2(0,\infty)$ will denote the underlying Hilbert space of square integrable functions defined on $[0,\infty)$.

The coefficients p_1, p_0 and q_1 will satisfy conditions (4.2) and (4.3).

In (4.2), for $f = o(q_1)$, we mean that $|f(x)| \ll |q_1(x)|$ for all $x \in [0,\infty)$ and $f = O(g)$ means that there exists $k > 0$ such that $k^{-1}|f(x)| \approx |g(x)| \approx k|f(x)|$ for all $x \in [0,\infty)$.

Our starting point is the differential equation $\tau y(x) = zy(x)$, that is,

$$\tau y(x) = -(p_1(x)y'(x))' + i[q_1(x)y'(x) + (q_1(x)y(x))'] + p_0(x)y(x) = zy(x) \tag{4.4}$$

on $[0,\infty)$, where z is the spectral parameter. The coefficients p_1, p_0 and q_1 are assumed to be real-valued.

By application of quasiderivatives as defined in Walker (1974), we can convert (4.4) into its first order system. These are given by;

$$y^{[0]} = y, \quad y^{[0]\prime} = y' = i\frac{q_1}{p_1}y_1 + \frac{1}{p_1}y_2$$

$$y^{[1]} = p_1y' - iq_1y, \quad y^{[1]\prime} = (p_1y')' - (iq_1y)' = (p_0 - \frac{q_1^2}{p_1})y^{[0]} + \frac{iq_1}{p_1}y^{[1]}.$$

Here, $\quad y = y(x), \quad p_k = p_k(x)$ and $q_j = q_j(x)$.

One can define the maximal and minimal operator T^* and T respectively, corresponding to τ as done in Chapter One.

In order to apply asymptotic integration method, it is convenient to write (4.4) as a first order system. The solutions of (4.4) via asymptotic integration is based on the famous Levinson's theorem which has undergone various modifications either through strengthening the dichotomy or decay conditions. In our case, the generalized version which has a spectral parameter z, suffices and is stated below.

4.2.2 Asymptotic Integration

One of the most significant results in asymptotic integration theory which is crucial in solving the first order system of a higher order differential equation is the Levinson's Theorem. The theorem states that, the solutions of a system

$$U'(x) = \{\Lambda(x) + R(x)\}U(x), \qquad \wedge(x) = diag(\lambda_i(x)) \quad i = 1, 2, \ldots, 2n$$

looks like the solutions of the unperturbed system $u' = \wedge u$ if $R(x)$ is sufficiently small and $\Lambda(x) = diag(\lambda_i(x))$ satisfies a dichotomy condition. For more details, see the book of Eastham (1989). In spectral theory, the matrix elements of $\lambda_i(x)$ will generally depend also on the spectral parameter z. Thus, one writes $\lambda_i = \lambda_i(x, z)$ for this. In this case, it will be important to state Levinson's Theorem uniformly in z in order to control the z-dependence of the solution. The following z-uniform version will suffice and is stated for a 2nth order system. Its proof can be found in the paper of Behncke et al. (2001).

Theorem 4.2.1. *Let $\Lambda(x, z) = diag(\lambda_1(x, z), \ldots, \lambda_{2n}(x, z))$ and $R(x)$ be $2n \times 2n$ matrices which for all x, are analytic functions of $z \in \Omega \subset \mathbb{C}$. For any unequal pair of indices i and j, $i, j \in [1, \ldots, 2n]$, assume that $\Lambda = diag(\lambda_1(x, z), \ldots, \lambda_{2n}(x, z))$ satisfy the dichotomy condition uniformly in z, that is, for every unequal pair i, j, $a \leq t \leq x < \infty$, $Re\{\lambda_i(x, z) - \lambda_j(x, z)\}$ has constant sign modulo $\mathcal{L}^1([a, \infty)$ for all $z \in \Omega$. Moreover, assume that $\| R(x) \| \leq \rho(x)$ with $\rho(x) \in \mathcal{L}^1([a, \infty)$. Then*

$$Y'(x, z) = [\Lambda(x, z) + R(x, z)]Y(x, z) \tag{4.5}$$

has solutions $y_k(x,z),\ 1 \le k \le 2n$, *with asymptotic form*

$$Y_k(x,z) = (e_k + r_k(x,z)).\exp(\int_a^x \lambda_k(t,z)dt), \tag{4.6}$$

where e_k *denotes the kth unit vector and* $r_k(x,z)$ *depends analytically on* $z \in \Omega$ *and tends to 0 z-uniformly as* $x \longrightarrow \infty$.

Levinson's theorem thus requires that (4.4) be converted into first order system as well as in the form of (4.5). We need to compute the characteristic polynomial of the first order associated with (4.4), the eigenvalues, prove the dichotomy conditions and finally perform the diagonalisations. In order to apply asymptotic methods, it is convinient to write (4.4) as a first order system and we will assume that $f = f(x)$, where $f = p_0, p_1, q_1$ and $y = y(x)$.

$$Y' = AY,$$

$$Y = \begin{bmatrix} y^{[0]} \\ y^{[1]} \end{bmatrix} \textit{and} \quad A = \begin{bmatrix} \frac{iq_1}{p_1} & \frac{1}{p_1} \\ p_0 - \frac{q_1^2}{p_1} & \frac{iq_1}{p_1} \end{bmatrix}$$

Here, z has been absorbed into p_0, that is, p_0 can be read as $p_0 - z$.

We will now proceed and diagonalise the system. This requires the eigenvalues of A which are the roots of the characteristic polynomial $\mathcal{P}(A)$, that is, $\mathcal{P}(A) = det(A - \lambda I_2)$. This results into;

$$\mathcal{P}(\lambda, x, z) = \lambda^2 - \frac{2iq_1}{p_1}\lambda - \frac{p_0}{p_1}. \tag{4.7}$$

Multiplying althrough by p_1 and substituting λ by $-i\nu$ we obtain a Fourier polynomial denoted by $\mathcal{P}_F(\nu, x, z)$ and is given by,

$$p_1\mathcal{P}_F(\nu, x, z) = -p_1\nu^2 - 2q_1\nu - p_0.$$

Multiplying the resultant polynomial by -1 and equating it to zero we obtain,

$$-p_1\mathcal{P}_F(\nu, x, z) = p_1\nu^2 + 2q_1\nu + p_0 = 0. \tag{4.8}$$

This is a polynomial with real coefficients if z is real, reflecting the symmetry of τ. Remark 4.2.2 below explains how one avoids degeneracy cases, that is, double or multiple roots that may lead to complications in diagonalisations.

Remark 4.2.2. *Like in the paper of Behncke and Nyamwala (2012), it can be shown that there exist finitely many spectral values z for which the roots of $\mathcal{P}_F(\lambda, x, z)$ are not distinct. Let $w_1 < w_2 < \ldots < w_k$ denote all of the real spectral values z leading to multiple roots. Following (Behncke & Nyamwala, 2012), the analysis will be restricted to small complex neighborhoods of $z_0 \in (w_i, w_{i+1}), \quad i = 0, \ldots, k, \quad$ where, $w_0 = -\infty$ and $w_{k+1} = \infty$. For a given $z_0 \in (w_i, w_{i+1})$, now choose $a > 0$ and $\epsilon > 0$ so that $\mathcal{P}_F(\lambda, x, z) = 0$ has no multiple roots for any*

$$z \in K_\epsilon(z_0) = \{z||z - z_0| \le \epsilon, Imz \ge 0\} = K.$$

This is possible because the roots of $\mathcal{P}_F$ depends analytically on the coefficients. Throughout the proof of some results, it may be necessary to adjust ϵ repeatedly.

The zeros of equation (4.8) will be of the form,

$$\nu_{1/2} = \frac{-2q_1 \pm \sqrt{4q_1^2 - 4p_1p_0}}{2p_1}$$

$$= \frac{-q_1}{p_1} \pm \frac{q_1}{p_1}\left\{1 - \frac{p_0p_1}{2q_1^2}\right\}^{\frac{1}{2}}$$

$$\approx \frac{-q_1}{p_1} \pm \frac{q_1}{p_1} \mp \frac{p_0}{2q_1} + \ldots + O(q_1^{-2})$$

$$\nu_1 \approx -2\frac{q_1}{p_1} + \frac{p_0 - z}{2q_1} + O(q_1^{-2}) \quad \textit{and} \quad \nu_2 \approx -\frac{(p_0 - z)}{2q_1} + O(q_1^{-2}) \tag{4.9}$$

Thus the eigenvalues, $\lambda_k = \lambda_k(x, z), \quad k = 1, 2$ are analytic functions of x and z and can be approximated from ν_1 and ν_2 above for any $z = z_0 + i\eta$ where z_0 will be absorbed in p_0 and $Imz = \eta$ such that $0 < \eta < \epsilon$ as

$$\lambda_1 = \frac{2iq_1}{p_1} - \frac{(p_0 - z_0)i}{2q_1} - \frac{\eta}{2q_1}$$

$$\lambda_2 = \frac{(p_0 - z_0)i}{2q_1} + \frac{\eta}{2q_1}.$$

The proof of dichotomy condition is simplified by Lemma 4.2.3 below and it implies that the condition is proved only for real ν-roots since the ν-roots which are in the complex conjugate pairs with non-zero imaginary parts will lead to equal number of square and non-square integrable solutions irrespective of the uniform dichotomy conditions. We, therefore, state the following Lemma whose proof can be found in (Behncke, 2010a).

Lemma 4.2.3. *Consider the system $u' = (\Lambda + R)u$ and assume*

$$\lambda_i(x) = \lambda_{i0} + \lambda_{i1}(x) + \lambda_{i2}(x)$$

with $\lambda_{i1} = o(1)$ and $\lambda_{i2}(x)$ conditionally integrable, $i = 1, \ldots, 2n$. Sort the eigenvalues into classes $C_1, \ldots, C_k$ so that

(i) $\lambda_i \in C_1$ then $Re\lambda_{i0} = \alpha_1$, where α_1 is a constant.

(ii) $\lambda_i \in C_l, \lambda_j \in C_m, l \neq m$ then $|Re(\lambda_{i0} - \lambda_{j0})| \geq \delta > 0, l, m = l, ..., k$.

Now let $m_\pm = max_{\lambda_i \in C_l}(Re(\lambda_{i1}(x))_\pm$ and $|C_1|$ denote the number of elements in C_l. Then the system has $|C_1|$ independent solutions u associated to C_1 satisfying K_1 $exp\,(\alpha_1 x - \int_a^x m_l-(t)dt) \leq \| u(x) \| \leq K_2$ $exp\,(\alpha_1 x + \int_a^x m_l+(t)dt)$, where K_1 and K_2 are constants.

The transformation

$$exp\left(\int_0^x \Lambda_{i2}(t)dt\right),$$

eliminates the conditionally integrable terms $\lambda_{i2}(x)$ while preserving the $\mathcal{L}^1$ nature of the off-diagonal terms. Let $\alpha_1 \pm i\beta_1, \ldots, \alpha_k \pm i\beta_k, \gamma_{2k+1}, \ldots, \gamma_{2n}$ be the roots of the Fourier polynomial $\mathcal{P}(\lambda, x, z)$, where $\alpha_j, \beta_j, \gamma_l$ are functions of x and z, with

$\alpha_j(x, z), \quad \beta_j(x, z), \quad \gamma_l(x, z) \in \mathbb{R}$, then this lemma implies that the non-real eigenvalues lead to k square integrable solutions, which decay exponentially and a corresponding sets of exponentially increasing solutions in terms of magnitude. This holds regardless of the dichotomy conditions. For real eigenvalues, one has by the implicit function theorem

$$\gamma_l(x, z) = \gamma_l(x, z_0) + (\partial_\lambda \mathcal{P}(\gamma_l, x, z))^{-1}(z - z_0)$$

for small $|z - z_0|$. Thus, the dichotomy condition holds if

$$(\partial_\lambda \mathcal{P}(\gamma_l)) \neq (\partial_\lambda \mathcal{P}(\gamma_m)), \quad l \neq m,$$

because $(\partial_\lambda \mathcal{P}(\lambda))$ is real, $0 < \eta < \epsilon$. $\gamma_l = \gamma_l(x, z_0)$ will then contribute to the deficiency index if $(\partial_\lambda \mathcal{P}(\gamma_l)) < 0$, because the corresponding exponent is $-i\gamma_l(x, z_0 + i\eta) \approx -i\gamma_l(x, z_0) + \eta(\partial_\lambda \mathcal{P}(\gamma_l(x, z_0)))^{-1}$. The associated eigenfunctions, however, will lose their square integrability as $\eta \to 0_+$ if the coefficients are bounded. Since the signs of $(\partial_\lambda \mathcal{P}(\gamma_l))$ are evenly distributed, half of the γ's will lead to square integrable solutions for $\eta = Imz > 0$ and complementary γ's will lead to square integrable solutions on the lower half plane. This shows that, with an even number of eigenvalues say $2n$ for bounded coefficients, T is limit point and $defT = (n, n)$. Some of the γ-eigenfunctions may stay square integrable as $Imz \searrow 0$ if the coefficients are unbounded and T may be non-limit point. It suffices to check the dichotomy condition only for the real roots of $\mathcal{P}_F(x, \nu, z)$ in all cases.

Theorem 4.2.4. *Let T be the minimal differential operator generated by (4.1) and assume that condition (4.2) is satisfied, then for $z \in K$, the eigenvalues of the associated characteristic polynomial of τ satisfies the dichotomy condition. Here, $K = \{z||z - z_0| < \epsilon\}$.*

Proof. The proof will be offered in two different ways. Since the roots of $\mathcal{P}_F$ as given in (4.9) are real, we prove the dichotomy condition off the real axis. Let $z = z_0 + i\eta$

where $z_0, \eta \in \mathbb{R}, \eta > 0$ and relatively small, $\{0 < \eta < \epsilon\}$.

$$\nu_1 = -2\frac{q_1}{p_1} + \frac{p_0 - (z_0 + i\eta)}{2q_1} + O(q_1^{-2})$$

$$\nu_2 = -\frac{(p_0 - (z_0 + i\eta))}{2q_1} + O(q_1^{-2}).$$

This implies that

$$\lambda_1(x, z) = -i\left\{-2\frac{q_1}{p_1} + \frac{p_0 - (z_0 + i\eta)}{2q_1} + O(q_1^{-2})\right\}$$

$$= \frac{2iq_1}{p_1} - \frac{(p_0 - z_0)i}{2q_1} - \frac{\eta}{2q_1} + O(q_1^{-2})$$

and

$$Re\lambda_1(x, z) = \frac{-\eta}{2q_1} + O(q_1^{-2})$$

$$Im\lambda_1(x, z) = -\left\{\frac{-2q_1}{p_1} + \frac{p_0 - z_0}{2q_1} + O(q_1^{-2})\right\}.$$

Similarly, we have

$$\lambda_2(x, z) = -i\left\{-\frac{p_0 - (z_0 + i\eta)}{2q_1} + O(q_1^{-2})\right\}$$

and

$$Re\lambda_2(x, z) = \frac{\eta}{2q_1} + O(q_1^{-2})$$

$$Im\lambda_2(x, z) = \frac{p_0 - z_0}{2q_1} + O(q_1^{-2}).$$

Even if $q_1(x) < 0$ or $q_1(x) > 0$, then either $Re\lambda_1(x, z) > 0$ or $Re\lambda_1(x, z) < 0$ respectively. A similar analysis is true for $Re\lambda_2(x, z)$. This implies that in each case of the sign of $q_1(x)$, one eigensolution will be bounded while the other is unbounded. This is the required dichotomy condition. A similar proof can be obtained if we apply the results of Lemma 4.2.3 directly. The correction term is given by $\frac{\partial}{\partial_\nu}\mathcal{P}_F(\nu, x, z) \approx 2p_1\nu + 2q_1$, by application of implicit function theorem. Now substituting the values of ν_1 and ν_2, we obtain $\frac{\partial}{\partial_{\nu_1}}\mathcal{P}_F(\nu_1, x, z) \approx -2q_1 + O(q_1^{-2})$ and $\frac{\partial}{\partial_{\nu_2}}\mathcal{P}_F(\nu_2, x, z) \approx$

$2q_1 + O(q_1^{-2})$. Since $\frac{\partial}{\partial_{\nu_1}}\mathcal{P}_F(\nu_1, x, z) \neq \frac{\partial}{\partial_{\nu_2}}\mathcal{P}_F(\nu_2, x, z)$ and they are of different signs, this suffices for the z-uniform dichotomy condition between ν_1 and ν_2. □

4.2.3 Diagonalisation

In order to diagonalise the first order system, we require the corresponding eigenvectors. These are obtained by solving the equation $Av = \lambda v$, where A is a 2×2 matrix, v is the eigenvector, while λ is the eigenvalue. Normalising the first component of v, leads to the equation,

$$\begin{bmatrix} \frac{iq_1}{p_1} & \frac{1}{p_1} \\ p_0 - \frac{q_1^2}{p_1} & i\frac{q_1}{p_1} \end{bmatrix} \begin{bmatrix} 1 \\ \mu \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \mu \end{bmatrix}$$

$$\frac{iq_1}{p_1} + \frac{1}{p_1}\mu = \lambda_1$$

$$p_0 - \frac{q_1^2}{p_1} + i\frac{q_1}{p_1}\mu = \lambda_1 \mu.$$

Collecting like terms together we get

$$\mu = \frac{p_0 p_1 - q_1^2}{p_1 \lambda_1 - iq_1}.$$

The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \frac{p_0 p_1 - q_1^2}{p_1 \lambda_1 - iq_1} \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} 1 \\ \frac{p_0 p_1 - q_1^2}{p_1 \lambda_2 + iq_1} \end{bmatrix}.$$

Note: The eigenvectors will be approximated using the leading terms only for simplicity in computation. Thus we have,

$$\frac{p_0p_1 - q_1^2}{-iq_1} = \frac{-i(p_0p_1 - q_1^2)}{q_1}$$

$$= q_1^{-1}\{iq_1^2 - ip_0p_1\} \approx iq_1 + O(q_1^{-1})$$

$$v_1 \approx \begin{bmatrix} 1 \\ iq_1 \end{bmatrix}$$

Similarly, for v_2 we have,

$$v_2 \approx \begin{bmatrix} 1 \\ -iq_1 \end{bmatrix}.$$

So that

$$M = \begin{bmatrix} 1 & 1 \\ iq_1 & -iq_1 \end{bmatrix}.$$

Here, det $M(x,z) = O(q_1)$ and

$$M^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2iq_1} \\ \frac{1}{2} & -\frac{1}{2iq_1} \end{bmatrix}.$$

Using this matrix, $M(x,z)$ to diagonalise the system by making a transformation of the form $Y(x,z) = M(x,z)V(x,z)$, we have

$$V'(x,z) = [\Lambda(x,z) + R(x,z)]V(x,z)$$

$\Lambda(x,z) = diag(\lambda_1(x,z) + r_{11}(x,z), \quad \lambda_2(x,z) + r_{22}(x,z))$.

Here, $r_{11}(x,z) = O(q_1^{-1}), \quad r_{22}(x,z) = O(q_1^{-2})$ are correction terms added to the diagonals as a result of diagonalisation. The remainder matrix $R(x,z)$ has $R_{jj}(x,z) = 0, \quad j = 1,2$ while $R_{jl} = O(f^1.q_1^{-1}), \quad j,l = 1,2, \quad j \neq l$. These terms are both $\mathcal{L}^2$ and $\mathcal{L}^1$ terms.

A second diagonalisation can be carried out just like the first one. This is possible since the coefficients were assumed to be twice differentiable. This can be done by normalising the first component of $\Lambda_1(v)$ as shown in (Behncke, 2010b).

If T is in limit point at infinity, that is, $defT = (n, n)$, then the self-adjoint extensions H are defined by boundary conditions at a parametised by $n \times n$ matrices α_1 and α_2 with

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n, \quad \alpha_1\alpha_2^* = \alpha_2\alpha_1^*.$$

Then $Hy = T^*y$ for all $u \in D(H)$, where

$$D(H) = \{y \in D(T^*)|(\alpha_1, \alpha_2)y(0) = 0\}.$$

If T is non-limit-point, then additional boundary conditions at infinity are needed. These are given as,

$$lim_{x \longrightarrow \infty} v_k^* Jy(x) = 0.$$

The functions $v_1, \ldots, v_r$ are linearly independent modulo $D(T)$ at infinity and may be choosen as eigenfunctions of $T^*v_j = zv_j, z \in \mathbb{R}$. They also satisfy

$$lim_{x \longrightarrow \infty} v_k^*(x) Jv_j(x) = 0$$

for $j, k = 1, \ldots, n$ (Hinton & Schneider, 1993).

Theorem 4.2.5. *Let T be the minimal differential operator generated by (4.1) on $\mathcal{L}^2[0, \infty)$ and assume that conditions (4.2) and (4.3) are satisfied. Then*

(i) *If $|q_1|^{-1}$ is integrable, then $defT = (2, 2)$ and $\sigma(H)$ is discrete.*

(ii) *If $|q_1|^{-1}$ is not integrable, then $defT = (1, 1)$. Suppose $q_1 > 0$ then $\sigma_{ac}(H) \subset [\bar{p}_0, \infty)$ and if $q_1 < 0$ then $\sigma_{ac}(H) = \mathbb{R}$ with spectral multiplicity 1. Here $\bar{p}_0 = lim\ sup\ p_0(x)$.*

Proof. (i) The proof follows from the following steps; system formulation, diagonalisation, dichotomy condition and the M-matrix. One thus writes the differential equation (4.1) into first order by using quasiderivatives. One then applies two diagonalisations in order to bring the first order system into Levinson's form since the smooth part of the coefficients are twice differentiable. The deficiency indices can be read off from the asymptotics of the eigenvalue solution as $Imz \searrow 0$. The form of the solution is given by

$$y_{1/2}(x,z) = (e_1 + r_{1/2}(x,z)).exp\left(\int_a^x \frac{\mp\eta}{|2q_1(t)|}dt\right)dx.$$

Thus assume $|q_1|^{-1}$ is integrable, then both the solutions are square integrable in the upper and lower half planes and hence results into $defT = (2,2)$. All the solutions are z-uniformly square integrable and hence discrete spectrum.

(ii) If $|q_1|^{-1}$ is not integrable, then $y_1(x)$ is square integrable in the upper half plane if $q_1(x) > 0$ and fails to be square integrable in the lower half plane. The eigensolution $y_2(x)$ is square integrable in the lower half plane if $q_1(x) > 0$ but fails in the upper half plane. The situation is reversed if $q_1(x) < 0$. In each half plane with the appropriate sign of $q_1(x), \quad defT = (1,1)$. If $|q_1|^{-1}$ is not integrable, then the correction term is given by $\frac{\mp\eta}{2|q_1(x)|}$ for $y_{1/2}(x)$ solutions. Thus $y_1(x)$ is square integrable since $Re\lambda_1(x,z) = \frac{-\eta}{2|q_1(x)|}$, $\eta > 0$ but loses its square integrability as $\eta \longrightarrow 0^+$. Thus for $q_1(x) < 0$, it implies that $-\infty < z < \infty$, hence $\sigma_{ac}(H) = \mathbb{R}$ with spectral multiplicity 1. On the other hand, $y_2(x)$ is not integrable since $Re\lambda_2(x,z) = \frac{\eta}{2|q_1(x)|}$, $\eta > 0$. Thus for $q_1(x) > 0$, it implies that $\bar{p}_0 < z < \infty$, hence $\sigma_{ac}(H) \subset [\bar{p}_0, \infty)$.

$\square$

4.2.4 Power Coefficients

Example 4.2.6. *Let $p_1(x) = x^\alpha, q_1(x) = cx^\beta, p_0(x) = x^\gamma$ such that the following conditions are satisfied,*

$$\alpha, \gamma < 0 \quad and \quad \beta > 0 \quad so \quad that \quad q_1(x) \to \infty \quad as \quad x \to \infty. \tag{4.10}$$

Thus, we obtain the equation

$$\tau y(x) = -(x^{\alpha} y'(x))' + i[cx^{\beta} y'(x) + (cx^{\beta} y(x))'] + x^{\gamma} y(x) \tag{4.11}$$

Here, we solve the equation $\tau y(x) = zy(x)$ through asymptotic integration and in line with the results of Theorem 4.2.5. The appropriate Fourier polynomial is given by

$$P_F(\nu, x, z) = x^{\alpha} \nu^2 + 2cx^{\beta} \nu + x^{\gamma} - z \tag{4.12}$$

Equating (4.12) to zero and solving its roots gives,

$$\nu_{1/2}(x, z) = -cx^{\beta-\alpha} \pm x^{-\alpha} \{c^2 x^{2\beta} - x^{\alpha}(x^{\gamma} - z)\}^{\frac{1}{2}}$$

Approximating to $O(x^{-3\beta})$ we have,

$$\nu_1(x, z) = -\frac{1}{2c} x^{-\beta}(x^{\gamma} - z) + O(cx^{-3\beta})$$

$$\nu_2(x, z) = -2cx^{\beta-\alpha} + \frac{1}{2c} x^{-\beta}(x^{\gamma} - z) + O(cx^{-3\beta})$$

Since $\lambda = -i\nu$, then

$$Re\lambda_1(x, z) = \frac{1}{2cx} x^{-\beta} \eta + O(cx^{-3\beta})$$

$$Im\lambda_1(x, z) = \frac{1}{2c} x^{-\beta}(x^{\gamma} - z_0) + O(cx^{-3\beta})$$

Similarly, we have

$$Re\lambda_2(x, z) = -\frac{1}{2cx} x^{-\beta} \eta + O(cx^{-3\beta})$$

$$Im\lambda_2(x, z) = 2cx^{\beta-\alpha} - \frac{1}{2c} x^{-\beta}(x^{\gamma} - z_0) + O(cx^{-3\beta})$$

Given T, the minimal differential operator generated by (4.11) on $[0, \infty)$ and assume that conditions (4.10) are satisfied, with $\beta > 1$, then $x^{-\beta}$ is integrable, $defT = (2, 2)$

and $\sigma(H)$ is discrete. On the other hand, if $0 < \beta \leq 1$, it implies that $x^{-\beta}$ slowly decays, $defT = (1, 1)$ and the spectrum is absolutely continuous.

4.3 Order Two Difference Operator

In this section, we consider (1.2) when $p_3(t) = q_3(t) = q_2(t) = p_2(t) = 0$, $p_1(t), q_1(t), p_0(t) \neq 0$ so that we have second order difference equation of the form;

$$\mathcal{L}y(t) = -\Delta[(p_1(t)\Delta y(t-1)] + i[q_1(t)\Delta y(t-1) + \Delta(q_1(t)y(t))] + p_0(t)y(t). \quad (4.13)$$

We assume the following decay conditions,

$$\frac{\Delta^2 f}{f}, \left(\frac{\Delta f}{f}\right)^2 \in \ell^1, \qquad \frac{\Delta f}{f} \in \ell^2, \qquad f = p_0, p_1, q_1 \quad (4.14)$$

and growth conditions similar to (4.2). Generally, (4.14) are discrete counterparts of (4.3).

4.3.1 System formulation

Our starting point is the difference equation (4.13) defined on $\ell^2[0, \infty)$. We will solve the equation $\mathcal{L}y(t) = zy(t)$, where z is considered as the spectral parameter. Here, just like in Section (4.2), we are fixing the left regular point off the zero element so that the operator is well defined for all $t \geq a > 0$. The result can be extrapolated to $[0, \infty)$ because deficiency indices do not depend on left regular endpoint (Remling, 1999; Shi, 2006).

In order to define the discrete Hamiltonian system for (4.13), one introduces quasi-differences (Shi, 2006), for the equation $(\mathcal{L} - z)y(t) = 0$. These are given by;

$$u(t) = p_1(t)(\Delta y(t-1)) - iq_1(t)y(t)$$

$$x(t) = y(t-1).$$

Taking the quasidifference elements as vectors, we also have, $Y(t) = \{x(t), u(t)\}^{tr}$ and the partial shift operator $\mathcal{R}Y(t)$ defined by $\mathcal{R}Y(t) = \{x(t+1), u(t)\}^{tr}$ where tr

denotes the vector transpose. Therefore, after absorbing z into p_0, we have

$$\Delta u(t) = [p_0 - \frac{q_1^2}{p_1}]y(t) + \frac{iq_1}{p_1}u(t)$$

$$= [p_0 - \frac{q_1^2}{p_1}]x(t+1) + \frac{iq_1}{p_1}u(t)$$

since $x(t+1) = y(t)$.

$$\Delta x(t) = \frac{iq_1}{p_1}x(t+1) + \frac{1}{p_1}u(t) \tag{4.15}$$

which results to the first order of the form,

$$\Delta \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} \frac{iq_1}{p_1} & \frac{1}{p_1} \\ p_0 - \frac{q_1^2}{p_1} & \frac{iq_1}{p_1} \end{bmatrix} \begin{bmatrix} x(t+1) \\ u(t) \end{bmatrix} \tag{4.16}$$

The above is one of the many ways of writing (4.13) into its first order. The form that is easily convertible to Levinson-Benzaid-Lutz form is given by,

$$\Delta Y(t) = S(t,z)\mathcal{R}Y(t)$$

which is equivalent to,

$$\begin{bmatrix} x(t+1) \\ u(t+1) \end{bmatrix} = \begin{bmatrix} S(t,z) \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

Here, $S(t,z)$ is a 2×2 matrix whose entries can be computed as follows;
In (4.16) above, taking

$$\Delta x(t) = \frac{iq_1(t)}{p_1(t)}x(t+1) + \frac{1}{p_1(t)}u(t),$$

this implies that,

$$x(t+1) - x(t) = \frac{iq_1(t)}{p_1(t)}x(t+1) + \frac{1}{p_1(t)}u(t).$$

Hence,

$$x(t+1) = \frac{p_1(t)}{p_1(t) - iq_1(t)}x(t) + \frac{1}{p_1(t) - iq_1(t)}u(t).$$

On the other hand,

$$\Delta u(t) = [p_0(t) - \frac{q_1^2(t)}{p_1(t)}]x(t+1) + \frac{iq_1(t)}{p_1(t)}u(t) \tag{4.17}$$

and

$$\Delta u(t) = u(t+1) - u(t). \tag{4.18}$$

Rewriting $u(t+1)$ using (4.17) and (4.18) we obtain,

$$u(t+1) = (p_0 - \frac{q_1^2}{p_1})x(t+1) + (1 + \frac{iq_1}{p_1})u(t)$$

Hence, the first order is given by $Y(t+1) = S(t,z)Y(t)$ where

$$S(t,z) = \begin{bmatrix} \frac{p_1}{p_1 - iq_1} & \frac{1}{p_1 - iq_1} \\ p_0 - \frac{q_1^2}{p_1} & 1 + \frac{iq_1}{p_1} \end{bmatrix}$$

and the corresponding characteristic polynomial, $det\,(S(t,z) - \lambda I_2) = \mathcal{P}(t,\lambda,z)$ results into,

$$\mathcal{P}(t,\lambda,z) = \lambda^2 - \lambda\left\{1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1}\right\} - \frac{p_0 - q_1^2}{p_1 - iq_1} \tag{4.19}$$

The zeros of equation (4.19) will be approximated as follows:

$$\lambda_{1/2}(t,z) = \frac{1}{2}\left\{\left(1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1}\right) \pm \left\{\left(1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1}\right)^2 + \frac{4(p_0 - q_1^2)}{p_1 - iq_1}\right\}^{\frac{1}{2}}\right\}$$

$$\lambda_{1/2}(t,z) = \frac{1}{2}\left\{\left(1 + \frac{iq_1}{p_1} + \frac{p_1}{p_1 - iq_1}\right) \pm \frac{iq_1}{p_1} + O(q_1^{-2})\right\}$$

$$\lambda_1(t,z) \approx \frac{iq_1}{p_1} + \frac{1}{2} + O(q_1^{-1}), \quad \lambda_2(t,z) \approx \frac{1}{2} + \frac{ip_1}{2q_1} + O(q_1^{-2}) \tag{4.20}$$

This implies that as $t \longrightarrow \infty$, then $|\lambda_1(t,z)| \longrightarrow |\frac{q_1(t)}{p_1(t)}|$ and $|\lambda_2(t,z)| \longrightarrow \frac{1}{2}$ because $q_1(t)$ is dominant.

To apply asymptotic summation method, it is convenient to write (4.13) as a first order system. The solutions of (4.13) via asymptotic summation is based on the famous Levinson-Benzaid-Lutz's theorem, (Benzaid & Lutz, 1987) which has undergone various modifications either to strengthen the dichotomy condition or decay conditions. In our case, we will state a generalized version which has a spectral parameter z.

4.3.2 Asymptotic Summation

Levinson-Benzaid-Lutz Theorem states that, the solutions of a system

$$Y(t+1) = \{\Lambda(t) + R(t)\}Y(t), \tag{4.21}$$

where $\Lambda(t)$ is diagonal and invertible, looks like the solutions of the unperturbed system $Y(t+1) = \Lambda(t)Y(t)$ if $R(t)$ is sufficiently small and $\Lambda = diag(\lambda_i(t))$ satisfies a dichotomy condition. In Benzaid-Lutz results, small means absolutely summable, that is, for all $i = 1, \dots, 2n, \quad \lambda_i^{-1}R(t) \in \ell^1$. As in the continuous case, in the spectral theory of difference operators, the matrix elements $\lambda_i(t)$ will generally depend on the spectral parameter z. Thus, one writes $\lambda_i = \lambda_i(t,z)$ for this. In this case, it will be important to prove LBL-Theorem uniformly in z in order to control the z-dependence of the solution. The following z-uniform version will suffice for the application of asymptotic summation in our case.

Theorem 4.3.1. *Let $\Lambda(t,z) = diag(\lambda_1(t,z), \dots, \lambda_{2n}(t,z))$ for $t \geq a$ and where $z \in K$. Moreover, assume that the following conditions hold uniformly for all $z \in K$,*

(i) $\lambda_i(t,z) \neq 0$ for all $1 \leq i \leq 2n$

(ii) $R(t)$ be a $2n \times 2n$ matrix defined for all $t \geq a$, satisfying

$$\sum_{t=a}^{\infty} \left| \frac{1}{\lambda_i(t,z)} \right| \parallel R(t,z) \parallel < \infty,$$

for all $i = 1, 2, \dots, 2n$

(iii) $\Lambda(t,z)$ *satisfies the following uniform dichotomy condition. For any pairs of indices* i *and* j*, such that* $i \neq j$*, assume there exists* δ *with* $0 < \delta < 1$ *such that* $|\lambda_i(t,z)| \geq \delta$ *for all* $t \geq a$*. Then either* $|\frac{\lambda_i(t,z)}{\lambda_j(t,z)}| \geq 1$ *or* $|\frac{\lambda_i(t,z)}{\lambda_j(t,z)}| \leq 1$ *for a large* t*.*

Then the linear system has a fundamental matrix satisfying, as $t \longrightarrow \infty$*,*

$$Y(t,z) = [e_k + r_{kk}]\Pi_{l=a}^{t-1}(\Lambda(l,z)). \tag{4.22}$$

where $r_{kk}(t,z) = o(1)$*.*

Lemma 4.3.2 below greatly simplifies the proof of the dichotomy condition to only those eigenvalues with magnitude 1. The Lemma is the discrete version of Lemma 4.2.3. and its proof can be found in (Behncke, 2010).

Lemma 4.3.2. *Let*

$$U(t+1,z) = [\Lambda(t,z) + R(t,z)]U(t,z), \quad t \geq a, \tag{4.23}$$

$$\Lambda(t,z) = diag(\lambda_1(t,z), \ldots, \lambda_{2n}(t,z))$$

be asymptotically constant difference equation such that

$$\sum_{t=a}^{t-1} \| R(t,z) \| \, |\lambda_i^{-1}(t,z)| < \infty.$$

Assume the eigenvalues $\lambda_i(t,z)$ *for* $i = 1, \ldots, 2n$ *satisfy* $\lambda_i(t,z) = \lambda_{i,0} + \lambda_{i,1} + \lambda_{i,2}$*, with* $\lambda_{i,0}$ *constant,* $\lambda_{i,1}(t,z) \to 0$ *as* $t \to \infty, \lambda_{i,2}$ *is conditionally summable and* λ_{i0} *distinct. Let* $h(t) > 0$ *be a nonsummable, monotonic function in* $\mathbb{N}$ *and assume the eigenvalues can be sorted into classes* $C_1, \ldots, C_n$ *so that if* $\lambda_i(t,z), \lambda_j(t,z) \in C_k$*, then* $(\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|} - 1) = o(h(t))$*; if* $\lambda_i(t,z) \in C_k, \lambda_j(t,z) \in C_l, k \neq l$ *then* $(\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|}) \leq 1 - h(t)$ *or* $(\frac{|\lambda_i(t,z)|}{|\lambda_j(t,z)|}) \geq 1 + h(t)$*. For each* $\lambda(t,z)$ *now write* $|\lambda(t,z)| = 1 + \mu(t,z)$ *with* $\mu_+ = max(0,\mu)$ *and* $\mu_- = min(0,\mu)$ *and for each class k, define,*

$$\alpha_k(t,z) = max\mu(t,z)_+ \quad and \quad \alpha_k(t,z) = max\mu(t,z)_-.$$

The above conditions are assumed to hold uniformly for all $z \in K$. Then associated to each C_k there are $|C_k|(|C_k|$ is the number of elements in the kth class) solutions satisfying

$$K_1 \Pi_{s=a}^{t-1}(1 - b_k(s,z)) \leq \| u(t,z) \| \leq K_2 \Pi_{s=a}^{t-1}(1 + \alpha_k(s,z))$$

for all $t \geq a$.

The conditionally summable terms can be removed by a simple transformation $\Pi_a^{t-1}\Lambda_{i2}(s,z)$.
When λ is a root of a polynomial $\mathcal{P}(t,\lambda,z)$, then $\bar{\lambda}^{-1}$ is also a root. Thus let $\lambda_1, \bar{\lambda_1}^{-1}, \ldots, \lambda_n, \bar{\lambda_n}^{-1}$ be roots of the polynomial $\mathcal{P}(t,\lambda,z)$ which for $t \longrightarrow \infty$ converge to the appropriate limits. One can arrange these roots into two groups, that is, $\lambda_1, \lambda_1^{-1}, \ldots, \lambda_m, \lambda_m^{-1}$ and $\lambda_{2m+1}, \ldots, \lambda_{2n}$ such that $|\lambda_l| > 1, \quad |\lambda_l^{-1}| < 1, \quad l = 1, \ldots, m$ and $|\lambda_j| = 1, \quad j = 2m+1, \ldots, 2n$. This lemma implies that the first $2m \quad \lambda$-roots lead to m square summable solution and m non-square summable solution. This holds regardless of the uniform dichotomy condition. The other $2(n-m) \quad \lambda$-roots with magnitude 1 can be written with their first order correction terms as

$$\lambda_j(z) = \lambda_j(z_0) + (\partial_\lambda \mathcal{P}(t,\lambda_j,z))^{-1}(z - z_0)$$

for small $|z - z_0|$. Thus the dichotomy condition holds if

$$\partial_\lambda \mathcal{P}(t,\lambda_j,z) \neq \partial_\lambda \mathcal{P}(t,\lambda_i,z), i \neq j.$$

It thus suffices to check the uniform dichotomy condition only for the λ-roots of $\mathcal{P}(t,\lambda,z)$ with $|\lambda|$=1 .

Lemma 4.3.3. *Assume that (4.2) is satisfied, then the eigenvalues of the operator generated by (4.13) satisfy the z-uniform dichotomy condition.*

Proof. Since $|\lambda_1(t,z)| \approx |\frac{q_1(t)}{p_1(t)}| \nearrow \infty$ as $t \longrightarrow \infty$ while $|\lambda_2(t,z)| \approx \frac{1}{2}$ as $t \longrightarrow \infty$, by the results of Lemma 4.3.2, the z-uniform dichotomy condition is satisfied because

$|\lambda_1(t,z)| > 1$ and $|\lambda_2(t,z)| < 1$. □

4.3.3 Diagonalisation

In order to diagonalise the first order system, we require the corresponding eigenvectors. These are obtained by solving the equation $S(t,z)v = \lambda v$, where S is the 2×2 matrix, v is the eigenvector, while λ is the eigenvalue. This leads to the following expression if we normalise the first component of the vector,

$$\begin{bmatrix} \frac{p_1}{p_1 - iq_1} & \frac{1}{p_1 - iq_1} \\ \frac{p_0 p_1 - q_1^2}{p_1} & \frac{p_1 + iq_1}{p_1} \end{bmatrix} \begin{bmatrix} 1 \\ \mu \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ \mu \end{bmatrix}$$

$$\frac{p_1}{p_1 - iq_1} + \frac{1}{p_1 - iq_1}\mu = \lambda_1$$

$$\frac{p_0 p_1 - q_1^2}{p_1} + \frac{p_1 + iq_1}{p_1}\mu = \lambda_1 \mu.$$

Collecting like terms together we get

$$\mu = \frac{p_0 p_1 - q_1^2}{p_1 \lambda_1 - p_1 - iq_1}.$$

The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \frac{p_0 p_1 - q_1^2}{p_1 \lambda_1 - p_1 - iq_1} \end{bmatrix}$$

and

$$v_2 = \begin{bmatrix} 1 \\ \frac{p_0 p_1 - q_1^2}{p_1 \lambda_2 - p_1 - iq_1} \end{bmatrix}.$$

Just like in the differential case and for simplicity in computations, the eigenvectors will be approximated using the leading terms only. The above vectors can be simplified as;

$$v_1 = \begin{bmatrix} 1 \\ \frac{q_1^2}{p_1} \end{bmatrix}.$$

Similarly, the second component gives us,

$$v_2 = \begin{bmatrix} 1 \\ -iq_1 \end{bmatrix}$$

so that

$$M(t,z) = \begin{bmatrix} 1 & 1 \\ \frac{q_1^2}{p_1} & -iq_1 \end{bmatrix}$$

and

$$M^{-1}(t,z) = \begin{bmatrix} \frac{p_1}{p_1 - iq_1} & \frac{p_1}{iq_1(p_1 - iq_1)} \\ \frac{-iq_1}{p_1 - iq_1} & \frac{p_1}{-iq_1(p_1 - iq_1)} \end{bmatrix} \approx \begin{bmatrix} \frac{i}{q_1} & \frac{p_1}{q_1^2} \\ 1 & -\frac{p_1}{q_1^2} \end{bmatrix}.$$

Even though the diagonalising matrix is unbounded, its inverse is bounded. Here, the $(detM(t,z))^{-1} = O(p_1 q_1^{-2})$. The system is then transformed using

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = M(t,z)V(t,z).$$

After diagonalisation, we have a first order system of the form

$$V(t+1,z) = [\Lambda_1(t,z) + R_1(t,z)]V(t,z)$$

$\Lambda_1(t,z) = diag(\lambda_1(t,z) + \varrho_1(t,z), \quad \lambda_2(t,z) + \varrho_2(t,z))$. The $\varrho_k(t,z)$ terms, $k = 1,2$, are obtained as a result of diagonalisations and are basically bounded and summable. The remainder matrix after first diagonalisation, $R_1(t,z)$ has zeros in its main diagonal and the off-diagonal terms are given by $(R_1)_{jl}(t,z) = O(q_1^{-1}.\Delta f), \quad l,j = 1,2, \quad l \neq j$. These are ℓ^2 and ℓ^1 terms by assumptions. We can also construct a matrix $S(\tilde{t},z)$ consisting of $\Lambda_1(t,z)$ and ℓ^2-terms from $R_1(t,z)$, then a second diagonalisation is possible using the eigenvectors of $S(\tilde{t},z)$. For more details, see

(Behncke, 2010b).

Theorem 4.3.4. *Let L be the minimal difference operator generated by (4.13) on $\ell^2[0,\infty)$ and assume that conditions (4.2) and (4.14) are satisfied, then the* $def L = (1,1)$ *and the spectrum is discrete.*

Proof. The strategy of proof follows the outlay of Theorem 4.3.1. .The difference equation (4.13) is converted to the first order system by use of quasi-differences given on pages 30-32. Once the first order has been obtained, $det(S(t,z)-\lambda I)$ gives the characteristic polynomial whose zeros are the eigenvalues of $S(t,z)$ and likewise those of L.

By application of Levinson's-Benzaid-Lutz theorem (Theorem 4.3.1), we need to establish the z-uniform dichotomy condition which is immediate from Lemma 4.3.3. The system $Y(t+1,z) = S(t,z)Y(t,z)$ can be converted into LBL-form through diagonalisations. These have been outlined in section 4.3.2. Application of Theorem 4.3.1 now shows that the eigenvalue solutions are of the form;

$$y_k(t,z) = [e_k(t,z) + r_{kk}(t,z)]\Pi_{l=0}^{l=t-1}(\lambda_k(l,z)); \quad k = 1,2.$$

The asymptotics of $y_k(t,z)$ depends on $|\lambda_k(l,z)|$ as shown by Behncke and Nyamwala, (Behncke & Nyamwala, 2013) and also (Shi, 2006). Here, if $|\lambda_k(l,z)| > 1$, then $|y_k(t,z)|^2 \longrightarrow \infty$ as $t \longrightarrow \infty$ and for $|\lambda_k(l,z)| < 1$, then $|y_k(t,z)|^2 \longrightarrow 0$ as $t \longrightarrow \infty$. A decay or bounded solution implies that the solution is square summable and hence contribute to deficiency indices of L as shown by (Shi, 2006). Therefore, $\lambda_1(t,z)$ will lead to non-square summable solution and $\lambda_2(t,z)$ will lead to square summable solution irrespective of the spectral parameter z, $\quad def L = (1,1)$ and the spectrum of self-adjoint operator extension will consist of only eigenvalues. □

4.3.4 Power Coefficients

Example 4.3.5. *Let* $p_1(t) = t^\alpha, q_1(t) = ct^\beta, p_0(t) = t^\gamma$ *such that the following conditions are satisfied*

$$\alpha, \gamma < 0 \quad and \quad \beta > 0 \quad so \quad that \quad q_1(t) \to \infty \quad as \quad t \to \infty. \tag{4.24}$$

Thus we obtain the equation

$$\mathcal{L}y(t) = -\Delta[t^\alpha \Delta y(t-1)] + i[ct^\beta \Delta y(t-1) + \Delta(ct^\beta y(t)] + t^\gamma y(t). \tag{4.25}$$

Solving the equation $\mathcal{L}y(t) = zy(t)$ leads to the Fourier polynomial of the form

$$\mathcal{P}(t, \lambda, z) = \lambda^2 - \lambda \left\{ 1 + \frac{ict^\beta}{t^\alpha} + \frac{t^\alpha}{t^\alpha - ict^\beta} \right\} - \frac{t^\gamma - ct^{2\beta}}{t^\alpha - ict^\beta}. \tag{4.26}$$

Approximating the roots of (4.26) to $O(t^{-2\beta})$ leads to,

$$\lambda_1(t, z) \approx ict^{\beta-\alpha} + \frac{1}{2} + O(t^{-\beta}), \quad \lambda_2(t, z) \approx \frac{1}{2} + \frac{t^\alpha}{2(t^\alpha - ict^\beta)} + O(t^{-2\beta}). \tag{4.27}$$

The magnitude of $\lambda_1(t, z)$ is greater than 1 while the magnitude of $\lambda_2(t, z)$ is less than 1 as $t \to \infty$. Thus $defL = (1, 1)$ and since the two lead to one eigensolution which is z-uniformly square summable and the other z-uniformly non-square summable, the spectrum is discrete at most.

4.4 Order Four Differential Operator

In this section, we consider the fourth order version of (1.1), that is, $p_3(x) = q_3(x) = 0, \quad p_2(x) \neq 0$ so that we have order four differential operator.

$$\begin{aligned} \tau y(x) &= (p_2(x)y''(x))'' - (p_1(x)y'(x))' + p_0(x)y(x) \\ &- i[(q_2(x)y''(x))' + (q_2(x)y'(x))'' - q_1(x)y'(x) - (q_1(x)y(x))']. \end{aligned} \tag{4.28}$$

The case where $|p_1(x)| \longrightarrow \infty$ had been considered by Remling (1999). So in our case, we will assume the following,

$$|q_2(x)| \nearrow \infty, \qquad p_0, p_1, p_2, q_0, q_1 = o(q_2), \qquad \forall x \in [0, \infty). \tag{4.29}$$

Further, assume that the coefficients obey the following growth conditions

$$|p_0|^{\frac{2}{5}}|p_1| = o(q_2^{\frac{2}{3}}) \quad and \quad |p_0|^{\frac{1}{3}}|q_1| = o(q_2^{\frac{1}{3}}). \tag{4.30}$$

4.4.1 System formulation

We study the spectral theory of differential operators generated by (4.28) on $\mathcal{L}^2(0, \infty)$ by means of asymptotic integration.

Like in the case of order two, the coefficients are measurable, real-valued and satisfy conditions (4.29) and (4.30).

To begin, we consider the differential equation $\tau y = zy$, that is,

$$\begin{aligned} \tau y(x) &= (p_2(x)y''(x))'' - (p_1(x)y'(x))' + p_0(x)y(x) \\ &- i[(q_2(x)y''(x))' + (q_2(x)y'(x))'' - q_1(x)y'(x) - (q_1(x)y(x))'] = zy(x) \end{aligned} \tag{4.31}$$

on $(0, \infty)$ where z is considered as the spectral parameter.

4.4.2 Asymptotic Integration

In order to apply asymptotic integration methods (Eastham, 1989), it is more convenient to rewrite equation (4.28) as a first order system.

We introduce the quasiderivatives

$$y^{[0]} = y, \quad y^{[1]} = y', \quad y^{[2]} = p_2 y'' - iq_2 y',$$

$$y^{[3]} = -(p_2 y'')' + p_1 y' + iq_2 y'' + i(q_2 y')' - iq_1 y$$

$$(y^{[1]})' = \frac{iq_2}{p_2} y^{[1]} + \frac{1}{p_2} y^{[2]}$$

$$(y^{[2]})' = -iq_1 y^{[0]} + (p_1 - \frac{q_2^2}{p_2}) y^{[1]} + \frac{iq_2}{p_2} y^{[2]} - y^{[3]}$$

$$(y^{[3]})' = p_0 y^{[0]} + iq_1 y^{[1]}$$

to obtain;

$$u'(x) = C(x,z)u(x); \quad u = (y, y^{[1]}, y^{[3]}, y^{[2]},)^t \tag{4.32}$$

where

$$\mathcal{C} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & \frac{iq_2}{p_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix}, \quad C = \begin{bmatrix} p_0 & iq_1 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} \end{bmatrix}.$$

(4.32) explicitly becomes,

$$\begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{iq_2}{p_2} & 0 & \frac{1}{p_2} \\ p_0 & iq_1 & 0 & 0 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} & -1 & \frac{iq_2}{p_2} \end{bmatrix} \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[3]} \\ y^{[2]} \end{bmatrix},$$

where

$$\mathcal{C}(x,z) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{iq_2}{p_2} & 0 & \frac{1}{p_2} \\ p_0 & iq_1 & 0 & 0 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} & -1 & \frac{iq_2}{p_2} \end{bmatrix}.$$

Computing the characteristic polynomial of the matrix $\mathcal{C}$ by solving $det(C - \lambda I_4) = \mathcal{P}(\lambda, x, z)$, gives;

$$\mathcal{P}(\lambda, x, z) = \lambda^4 - \frac{2iq_2}{p_2}\lambda^3 - \frac{p_1}{p_2}\lambda^2 + \frac{2iq_1}{p_2}\lambda + \frac{p_0}{p_2}.$$

Multiplying all through by p_2 we obtain,

$$p_2 \mathcal{P}(\lambda, x, z) = p_2\lambda^4 - 2iq_2\lambda^3 - p_1\lambda^2 + 2iq_1\lambda + p_0.$$

The occurrence of the complex terms $-2iq_2$ and $2iq_1$ makes it desirable to use the Fourier polynomial $\mathcal{P}_F$ instead. One thus obtains this by replacing λ by $-i\nu$.

$$\mathcal{P}_F(\nu, x, z) = p_2\nu^4 + 2q_2\nu^3 + p_1\nu^2 + 2q_1\nu + p_0. \tag{4.33}$$

Because of the assumptions in (4.29) and (4.30), we can apply the techniques of Eastham, (Eastham, 1989), which have been used by Behncke and Nyamwala in their work, see (Behncke & Nyamwala, 2012) for more details. Therefore, the roots of $\mathcal{P}_F(\nu, x, z)$ can be approximated from the following polynomials

$$\mathcal{P}_{F_1} \approx p_2\nu + 2q_2 + p_1\nu^{-1} + 2q_1\nu^{-2} + p_0\nu^{-3}, \quad \mathcal{P}_{F_2} \approx 2q_2\nu^3 + p_0 + p_2\nu^4 + p_1\nu^2 + q_1\nu. \tag{4.34}$$

The magnitudes of these ν-roots are approximately $|\nu_1| \approx 2|\frac{q_2}{p_2}|$ and $|\nu_{2,3,4}| \approx |\frac{p_0}{2q_2}|^{\frac{1}{3}}$ as the following results suggest.

Theorem 4.4.1. *Assume (4.29) and (4.30) are satisfied. Then;*

(i) The roots of $\mathcal{P}_F(\nu, x, z)$ can be approximated from $\mathcal{P}_{F_1}$ and $\mathcal{P}_{F_2}$ in (4.34).

(ii) $|\nu_1| \approx 2|\frac{q_2}{p_2}|$ and $|\nu_{2,3,4}| \approx |\frac{p_0}{2q_2}|^{\frac{1}{3}}$.

(iii) If the $\tilde{\nu}$-root of $\mathcal{P}_{F_j}$ is real or complex with non-imaginary part, then the corresponding ν-root of $\mathcal{P}_F(\nu, x, z)$ is real or complex with non-zero imaginary part.

Proof. (i) It suffices to show that $p_1\nu^{-1}$, $2q_1\nu^{-2}$ and $p_0\nu^{-3}$ are $o(1)$ in $\mathcal{P}_{F_1}$. Note here that

$$|p_1\nu^{-1}| \approx |p_1||\frac{q_2}{p_2}|^{-1} = |p_1||\frac{p_2}{q_2}| = o(1)$$

since $|p_2| \ll |q_2|$ and $p_1 = o(q_2)$.

$$|2q_1\nu^{-2}| \approx 2|q_1||\frac{p_2}{q_2}|^2 = o(1)$$

again since $|p_2| \ll |q_2|$ and $|q_1|$ is bounded. Similarly,

$$|p_0\nu^{-3}| \approx |p_0||\frac{p_2}{q_2}|^3 = o(1)$$

because of (4.29).

In the case of $\mathcal{P}_{F_2}$, we show that $|p_2\nu^4|, \quad |p_1\nu^2|$ and $|p_0\nu|$ are $o(1)$.

$$|p_2\nu^4| \approx |p_2||\frac{p_0}{q_2}|^{\frac{4}{3}} \approx |p_2||\frac{p_0}{q_2}|^{\frac{1}{3}}|q_1||q_1|^{-1}|\frac{p_0}{q_2}|^3 = o(1)$$

because of (4.30).

(ii) This is immediate from (i) above since $\mathcal{P}_{F_1} \approx p_2\nu + 2q_2 + o(1)$ and $\mathcal{P}_{F_2} \approx 2q_2\nu^3 + o(1) + p_0$.

(iii) This is proved by application of Banach fixed point theorem. Assume that the root of $\mathcal{P}_{F_j}$ is $\tilde{w}_j$, with the corresponding root of $\mathcal{P}_F$ being w_j. Then for some constant $c_j > 0, \quad c_j < \frac{1}{8}$, we have $|w_j - \tilde{w}_j| \leq c_j$ through iteration and application of Banach fixed point theorem, $\{\tilde{w}_j\}_{j=1}^{\infty}$ converges to w_j uniquely. Here, $\frac{1}{8}$ has been picked for faster convergence of the iterates though other constants of absolute value less than 1 may be preferred but at the expense of compromising convergence rate. The ν-roots can be approximated as follows;

$$\nu_1 \approx \frac{-2q_2}{p_2}$$

$$\nu_{2,3,4} \approx \left(\frac{-(p_0 - z)}{2q_2}\right)^{\frac{1}{3}}$$

$$\approx \left|\frac{p_0 - z}{2q_2}\right|^{\frac{1}{3}} \left\{cos\left(\frac{\theta + 2\pi m}{3}\right) + isin\left(\frac{\theta + 2\pi m}{3}\right)\right\}$$

$m = 0, 1, 2$ and θ is the argand of $\frac{-(p_0-z)}{2q_2}$. By application of fundamental theorem of algebra, two of the $\nu_{2,3,4}$ roots will be complex conjugate pair of the other while one root will be real. Thus if $\tilde{w}_j$ has non-zero imaginary part, then w_j also has non-zero imaginary part.

□

Theorem 4.4.2. *Assume (4.29) and (4.30) are satisfied, then the eigenvalues of the operator generated by (4.28) satisfy the dichotomy condition.*

Proof. By application of implicit function theorem, the correction term is given by

$$\partial_\nu \mathcal{P}_F(x,\nu,z) \approx 4p_2\nu^3 + 6q_2\nu^2 + 2p_1\nu + 2q_1.$$

Now substituting the values of ν_1 and $\nu_{2,3,4}$, we obtain

$$|\partial_{\nu_1}\mathcal{P}_F| \approx |q_2(\frac{q_2}{p_2})^2| \approx |p_2|^{-2}|q_2|^3.$$

This implies that $|\partial_{\nu_1}\mathcal{P}_F|^{-1} \approx |p_2|^2|q_2|^{-3}$. Similarly,

$$|\partial_{\nu_{2,3,4}}\mathcal{P}_F| \approx |q_2(\frac{p_0}{q_2})^{\frac{2}{3}}| \approx |q_2|^{\frac{1}{3}}|p_0|^{-\frac{2}{3}}.$$

So that $|\partial_{\nu_{2,3,4}}\mathcal{P}_F|^{-1} \approx |q_2|^{-\frac{1}{3}}|p_0|^{\frac{2}{3}}$.
Since $|\partial_{\nu_1}\mathcal{P}_F|^{-1} \neq |\partial_{\nu_{2,3,4}}\mathcal{P}_F|^{-1}$ this suffices for the uniform dichotomy condition between ν_1 and $\nu_{2,3,4}$. Here, if $z = z_0 + i\eta, \quad \eta > 0$, then the eigenvalues $\lambda_k(x,z)$ will be expressed as $\lambda_k(x,z) = \lambda_{k0}(x,z) + i(\frac{\eta}{\partial_{\nu_k}\mathcal{P}_{\mathcal{F}}(x,z)})$ where $\lambda_{k0}(x,z)$ is the approximated root while $i(\frac{\eta}{\partial_{\nu_k}\mathcal{P}_{\mathcal{F}}(x,z)})$ is the correction term. Because of Lemma 4.2.3, this only affects the pure real ν-roots. Explicitly, we have $Re(\lambda_1(x,z) - \lambda_{2,3,4}(x,z)) \neq 0$. □

4.4.3 Diagonalisation

In order to diagonalise the first order system, we require the corresponding eigenvectors. These are obtained by solving the equation $Cv = \lambda v$, where C is a 4×4 matrix, v is the eigenvector, while λ is the eigenvalue. This leads to the expression

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{iq_2}{p_2} & 0 & \frac{1}{p_2} \\ p_0 & iq_1 & 0 & 0 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} & -1 & \frac{iq_2}{p_2} \end{bmatrix} \begin{bmatrix} 1 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \lambda_k \begin{bmatrix} 1 \\ \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix}$$

Collecting like terms together and solving for $\mu_i, i = 1, 2, 3$, we get

$$\mu_1 = \lambda_k, \quad \mu_2 = -iq_1 + p_1\lambda_k + 2iq_2\lambda_k^2 - p_2\lambda_k^3, \quad \mu_3 = p_2\lambda_k^2 - iq_2\lambda_k$$

The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ O(q_2) \\ O(q_2^{-1}) \\ O(q_2^2) \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ O(q_2^{-\frac{1}{3}}) \\ O(q_2^{\frac{1}{3}}) + O(q_1) \\ O(q_2^{\frac{2}{3}}) - O(q_2^{-\frac{2}{3}}) \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1 \\ O(q_2^{-\frac{1}{3}}) \\ O(q_2^{\frac{1}{3}}) - O(q_1) \\ O(q_2^{\frac{2}{3}}) + O(q_2^{-\frac{2}{3}}) \end{bmatrix}, \qquad v_4 = \begin{bmatrix} 1 \\ O(q_2^{-\frac{1}{3}}) \\ O(q_2^{\frac{1}{3}}) \\ O(q_2^{\frac{2}{3}}) \end{bmatrix}.$$

The eigenvectors can then be approximated from the above expressions. Hence, the diagonalising matrix is given by

$$M(x,z) = [v_1(x,z), v_2(x,z), v_3(x,z), v_4(x,z)].$$

Theorem 4.4.3. *Let T be the minimal differential operator generated by (4.31) on $\mathcal{L}^2[0,\infty)$ and assume that conditions (4.29) and (4.30) are satisfied. Then*

(i) Suppose that $|q_2|^{-\frac{1}{3}}$ is integrable, then $defT = (3,3)$ and $\sigma(H)$ is discrete.

(ii) Suppose that $|q_2|^{-3}$ is integrable but $|q_2|^{-\frac{1}{3}}$ is not integrable, then $defT$ will either be (3,2) or (2,3) depending on the sign of q_2 . Hence no self-adjoint extension.

(iii) Suppose that $|q_2|^{-\frac{1}{3}}$ is not integrable, then $defT = (2,2)$ and $\sigma_{ac}(H) = \mathbb{R}$

Proof. **(i)** Here, the eigensolution associated to $\lambda_1(x,z)$ will be square integrable both in the upper and lower half plane since $|q_2|^{-\frac{1}{3}}$ is integrable. On the other hand, two of the roots from the set $\lambda_{2,3,4}(x,z)$ will be in complex conjugate pair leading to one square integrable and one non-square integrable solution in the complex plane. The other real root from the set $\lambda_{2,3,4}(x,z)$, will also lead to a solution that is square integrable both in the upper and lower half plane. Thus we have $defT = (3,3)$ and hence the spectrum is pure discrete.

(ii) In this case, $\lambda_1(x,z)$ will be pure imaginary and since $|q_2|^{-3}$ is integrable, it implies that the associated eigensolution will be square integrable both in the upper and lower half planes. Similarly, two of the roots from the set $\lambda_{2,3,4}(x,z)$ will be in complex conjugate pair thus contributing to one square integrable and one non-square integrable solution in the complex plane. The remaining real root from the set $\lambda_{2,3,4}(x,z)$ will lead to a solution that is either square integrable in the upper complex plane but not in the lower complex plane and vice versa depending on the sign of $q_2(x)$. Thus we have either $defT = (3,2)$ or $(2,3)$ depending on the sign of $q_2(x)$. By Von Neumann theorem and since $defT \neq (r,r), \quad T$ has no self-adjoint extension.

(iii) If $|q_2|^{-\frac{1}{3}}$ is not integrable, then the eigensolution associated to $\lambda_1(x,z)$ will either be square integrable in the lower half plane but not in the upper half plane and conversely. On the other hand, two of the roots from the set $\lambda_{2,3,4}(x,z)$ will be in complex conjugate pair leading to one square integrable and one non-square integrable solution in the complex plane. The eigensolution associated to the remaining real root from the set $\lambda_{2,3,4}(x,z)$ will either be integrable in the upper complex plane but not in the lower complex plane and in reverse depending on the sign of $q_2(x)$. Thus we have $defT = (2,2)$ with discrete spectrum. By application of implicit function theorem and the sign of $q_2(x)$, there is no way the eigensolutions associated to $\lambda_1(x,z)$ and the real root from the set $\lambda_{2,3,4}(x,z)$ can both be square integrable in the same plane or non-square integrable in the same plane. This rules out the possibility of having $defT = (1,3) \quad or \quad (3,1)$. Suppose that $|q_2|^{-\frac{1}{3}}$ is not integrable, then the eigensolution associated to $\lambda_1(x,z)$ will lose its square integrability as $\eta \longrightarrow 0$

and hence will contribute to absolutely continuous spectrum of multiplicity one.

□

4.5 Order Four Difference Operator

In this section, we consider the fourth order version of (1.2), that is, $p_3(t) = q_3(t) = 0, \quad p_2(t) \neq 0$ so that we have order four symmetric difference equation

$$\begin{aligned} \mathcal{L}y(t) &= \Delta^2[p_2(t)\Delta^2 y(t-2)] - \Delta[p_1(t)\Delta y(t-1)] + p_0(t)y(t) \\ &- i\Delta\left(q_2(t)\Delta^2 y(t-2)\right) - i\Delta^2\left(q_2(t)\Delta y(t-1)\right) + i\Delta\left(q_1(t)y(t)\right) \\ &+ iq_1(t)\Delta y(t-1). \end{aligned} \tag{4.35}$$

The growth conditions in (4.29) and (4.30) will be assumed but with the independent variable taken as t. In order to apply asymptotic summation, the following smoothness and decay assumptions are necessary;

$$\frac{\Delta^2 f}{f}, \left(\frac{\Delta f}{f}\right)^2 \in \ell^1, \qquad \frac{\Delta f}{f} \in \ell^2, \qquad f = p_0, p_2, q_1, q_2. \tag{4.36}$$

4.5.1 System formulation

We study the spectral theory of the difference equation (4.35) defined on $\ell^2[0, \infty)$. In this case, we solve the equation $\mathcal{L}y = zy$. Here z is a spectral parameter.

4.5.2 Asymptotic Summation

In order to write the Hamiltonian system of (4.35) one introduces quasi-differences,

$$x_1(t) = y(t-1), \quad \Delta x_1(t) = \Delta y(t-1), \quad x_1(t+1) = y(t)$$

$$x_2(t) = \Delta y(t-2), \quad \Delta x_2(t) = \Delta^2 y(t-2), \quad x_2(t+1) = \Delta y(t-1).$$

This implies that, $\Delta x_1(t) = \Delta y(t-1) = x_2(t+1)$.

$$\Delta x_2(t) = \Delta^2 y(t-2) = \frac{1}{p_2}u_2(t) + \frac{iq_2}{p_2}x_2(t+1)$$

$$\begin{aligned} u_1(t) &= -\Delta\left(p_2(t)\Delta^2 y(t-2)\right) + p_1(t)\Delta y(t-1) \\ &+ i\{\Delta\left(q_2(t)\Delta y(t-1)\right) + q_2(t)\Delta^2 y(t-2)\} - iq_1(t)y(t) \end{aligned}$$

$$u_2(t) = p_2(t)\Delta^2 y(t-2) - i(q_2(t)\Delta y(t-1))$$

$$\Delta u_1(t) = (p_0 - z)(t)x_1(t+1) + iq_1(t)x_2(t+1)$$

$$\Delta u_2(t) = (p_1 - \frac{q_2^2}{p_2})x_2(t+1) - iq_1(t)x_1(t+1) + \frac{iq_2}{p_2}u_2(t) - u_1(t).$$

One can easily obtain the following relation

$$\Delta x_2(t) = \frac{1}{p_2}u_2(t) + \frac{iq_2}{p_2}\Delta y(t-1) = \frac{1}{p_2}u_2(t) + \frac{iq_2}{p_2}x_2(t+1).$$

In order to determine the number of solutions that are square summable, we use asymptotic summation that requires (4.35) to be converted into first order system. The first order system of (4.35) is therefore given by

$$\Delta Y(t,z) = G(t,z)RY(t,z)$$

where

$$Y(t,z) = [x_1(t), x_2(t), u_1(t), u_2(t)]^{tr}.$$

R is the partial shift operator such that

$$R(Y(t)) = [x_1(t+1), x_2(t+1), u_1(t), u_2(t)]^{tr},$$

here tr is the usual transpose of a vector or matrix.

$G(t,z)$ is given below

$$G(t,z) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{iq_2}{p_2} & 0 & \frac{1}{p_2} \\ p_0 & iq_1 & 0 & 0 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} & -1 & \frac{iq_2}{p_2} \end{bmatrix}.$$

We now write matrix $G(t,z)$ in block form as follows;

$$G(t,z) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & \frac{iq_2}{p_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{p_2} \end{bmatrix}, \quad C = \begin{bmatrix} p_0 - z & iq_1 \\ -iq_1 & p_1 - \frac{q_2^2}{p_2} \end{bmatrix}.$$

We can now apply the approach of Shi (Shi, 2006) so as to obtain the first order system in the form $Y(t+1,z) = S(t,z)Y(t,z)$ where

$$Y(t,z) = [x_1(t,z), x_2(t,z), u_1(t,z), u_2(t,z)]^{tr}.$$

Thus we obtain

$$Y(t+1,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix} Y(t,z)$$

where $E = (I_n - A)^{-1}$.

Explicitly this gives,

$$S(t,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}$$

$$S(t,z) = \begin{bmatrix} 1 & \frac{1}{1-\frac{iq_2}{p_2}} & 0 & \frac{1}{p_2}.\frac{1}{1-\frac{iq_2}{p_2}} \\ 0 & \frac{1}{1-\frac{iq_2}{p_2}} & 0 & \frac{1}{p_2}.\frac{1}{1-\frac{iq_2}{p_2}} \\ p_0 - z & \frac{p_0-z+iq_1}{1-\frac{iq_2}{p_2}} & 1 & \frac{1}{p_2}.\frac{p_0-z+iq_1}{1-\frac{iq_2}{p_2}} \\ -iq_1 & \frac{p_1-iq_1-q_2^2}{1-\frac{iq_2}{p_2}} & -1 & 1+\frac{iq_2}{p_2}+\frac{p_1-iq_1-\frac{q_2^2}{p_2}}{1-\frac{iq_2}{p_2}} \end{bmatrix}.$$

We thus compute the characteristic polynomial of matrix $S(t,z)$ using the formula $\mathcal{P}(t,\lambda,z) = det(S(t,z)-\lambda I_4)$. Multiplying the resultant polynomial by $p_2 - iq_2$ and dividing with λ^2 we obtain a polynomial of the form

$$\begin{aligned} \mathcal{F}(t,\lambda,z) &= \frac{p_2 - iq_2}{\lambda^2}\mathcal{P}(t,\lambda,z) = p_2(1-\lambda)^2(1-\lambda^{-1})2 + p_1(1-\lambda)(1-\lambda^{-1}) \\ &+ p_0 + q_2(1-\lambda)(1-\lambda^{-1})(i\lambda+(i\lambda)^{-1}) + q_1(i\lambda+(i\lambda)^{-1}). \end{aligned}$$

This leads to λ-roots such that λ^{-1} are also roots. If we suppose that $\gamma = (1-\lambda)(1-\lambda^{-1}) = 2-\lambda-\lambda^2$, then $|\lambda| = 1$ can only be obtained if and only if $\gamma \in [0,4]$. In order to eliminate the imaginary coefficients in the polynomial, we make unitary transformations by letting

$$\lambda = \frac{(is+1)}{(is-1)}$$

and then multiplying the resulting polynomial by $(s^2+1)^2$ we obtain

$$Q(t,s,z) = (s^2+1)^2\mathcal{F}\left(\frac{is+1}{is-1},t,z\right)$$

so that

$$Q(t,s,z) = (16p_2 + 4p_1 + p_0) + (16q_2 + 4q_1)s + (4p_1 + 2p_0)s^2 + 4q_1s^3 + p_0s^4 \quad (4.37)$$

Note that $p_0 = p_0 - z$.

We can now apply conditions (4.29) and (4.36) to obtain the s-roots and then make backward substitution to get λ-roots. The s-roots of this polynomial can then be approximated as,

$$s_1 \approx \frac{-(16p_2 + 4p_1 + p_0)}{16q_2 + 4q_1} + \quad lower \quad order \quad terms.$$

$$s_{2,3,4} = \left(\frac{-16q_2 + 4q_1}{p_0}\right)^{\frac{1}{3}} + \quad lower \quad order \quad terms$$

Here, lower order terms are those terms approximately $O(q_2^{-2})$.

To obtain the λ-roots, we make the following substitution

$$s = \frac{\lambda + 1}{i\lambda - i} = \frac{i\lambda^2 + 2i\lambda + i}{1 - \lambda^2}.$$

Theorem 4.5.1. *Assume (4.29) and (4.36) are satisfied, then the eigenvalues of the operator generated by (4.35) satisfy the dichotomy condition.*

Proof. Since $|\lambda_1(t,z)| \approx |\frac{16p_2+4p_1+p_0)}{16q_2+4q_1}|$ and $|\lambda_{2,3,4}(t,z)| \approx |\frac{-16q_2+4q_1}{p_0}|$ as $t \to \infty$, it follows that $|\lambda_1(t,z)| < 1$ and $|\lambda_{2,3,4}(t,z)| > 1$. Irrespective of the uniform dichotomy condition, $\lambda_1(t,z)$ will lead to z-uniformly summable solution while $\lambda_{2,3,4}(t,z)$ will lead to non-square summable solution. This is the required dichotomy condition. $\square$

Theorem 4.5.2. *Let L be the minimal difference operator generated by (4.35) on $\ell^2[0,\infty)$ and assume that condition (4.29) and (4.36) are satisfied. Then*

(i) If $|q_2|^{-1}$ is summable, then $def L = (3,3)$ and $\sigma(H)$ is discrete.

(ii) If $|q_2|^{-2}$ is not summable, then $def L = (2,2)$ with discrete spectrum.

Proof. **(i)** If $|q_2|^{-1}$ is summable, the eigenfunction associated to $\lambda_1(t,z)$ will be z-uniformly square summable both in the upper and lower half plane. Similarly, the eigensolution associated to the real root from the set $\lambda_{2,3,4}(t,z)$ will also be z-uniformly square summable both in the upper and lower complex plane. More so, two of the roots from the set $\lambda_{2,3,4}(t,z)$ are in complex conjugate pair leading to one square summable and one non-square summable solution in the complex plane. It follows that $def L = (3,3)$ and hence the spectrum is discrete at most.

(ii) If $|q_2|^{-2}$ is not summable, then the eigensolution associated to $\lambda_1(t,z)$ will either be z-square summable in the lower half plane but not in the upper half plane and vice versa. On the other hand, the complex conjugate pair from the set $\lambda_{2,3,4}(t,z)$ will contribute to one z-square summable and one non-square summable solution in the complex plane. The remaining real root from the set $\lambda_{2,3,4}(t,z)$ will lead to a solution that is either z-square summable in the upper complex plane but not in the lower complex plane and the other way round . Thus we have $defT = (2,2)$ with discrete spectrum.

$\square$

4.6 Order Six Differential Operator

In this section, we consider (1.1) with $p_3(x), p_0(x), q_2(x) \neq 0, \quad p_1(x) = p_2(x) = q_1(x) = q_3(x) = 0$ and $w(x) = 1$, then (1.1) generates a sixth order symmetric differential equation on $\mathcal{L}^2[0,\infty)$ of the form

$$\tau y(x) = -(p_3(x)y'''(x))''' - i[(q_2(x)y''(x))' + (q_2(x)y'(x))''] + p_0(x)y(x). \tag{4.38}$$

The growth conditions will be as follows;

$$|q_2(x)| \nearrow \infty, \quad p_0, p_3 = o(q_2), \quad p_1(x) = p_2(x) = q_1(x) = q_3(x) = 0 \quad \forall x \in [0,\infty) \tag{4.39}$$

$$|q_2|^{-\frac{4}{3}}|p_3|^{\frac{1}{3}}, \quad |p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}} = o(1) \tag{4.40}$$

$$\frac{f'}{f} \in \mathcal{L}^2, \quad \frac{f''}{f}, \left(\frac{f'}{f}\right)^2 \in \mathcal{L}^1, \qquad f = p_0, p_3, q_2. \tag{4.41}$$

4.6.1 System formulation

We study the spectral theory of differential operators generated by (4.38) using asymptotic integration. To begin, we consider the differential equation $\tau y(x) = zy(x)$, on $(0,\infty)$ where z is considered as the spectral parameter.

4.6.2 Asymptotic Integration

To apply asymptotic integration methods, it is more convenient to rewrite equation (4.38) as a first order system.

We introduce the quasiderivatives

$$y^{[0]} = y, \quad y^{[1]} = y'$$

$$y^{[2]} = y'', \quad y^{[3]} = p_3 y'''$$

$$y^{[4]} = -y^{(iv)} - iq_2 y'$$

$$y^{[5]} = -y^{(v)} + i\left\{(q_2 y')' + q_2 y''\right\}$$

$$u'(x) = F(x,z)u(x); \quad u = (y^{[0]}, y^{[1]}, y^{[2]}, y^{[5]}, y^{[4]}, y^{[3]})^{tr}. \tag{4.42}$$

This has been formulated in line with the formulations of Walker (Walker, 1974). Here,

$$F(x,z) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{p_3} \end{bmatrix}$$

$$C = \begin{bmatrix} p_0 - z & 0 & 0 \\ 0 & 0 & iq_2 \\ 0 & -iq_2 & 0 \end{bmatrix}.$$

(4.42) explicitly becomes,

$$\begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[5]} \\ y^{[4]} \\ y^{[3]} \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p_3} \\ p_0 - z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iq_2 & -1 & 0 & 0 \\ 0 & -iq_2 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} y^{[0]} \\ y^{[1]} \\ y^{[2]} \\ y^{[5]} \\ y^{[4]} \\ y^{[3]} \end{bmatrix}$$

so that the matrix $F(x, z)$ is given by

$$F(x,z) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p_3} \\ p_0 - z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iq_2 & -1 & 0 & 0 \\ 0 & -iq_2 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Solving $det(F - \lambda I_6) = \mathcal{P}(\lambda, x, z)$ we obtain,

$$\mathcal{P}(\lambda, x, z) = -p_3\lambda^6 - 2iq_2\lambda^3 + p_0 - z.$$

The occurrence of the complex term $-2iq_2$ makes it desirable to use the Fourier polynomial $\mathcal{P}_F$ instead. One thus obtains this by replacing λ by $-i\nu$

$$\mathcal{P}_F(\nu, x, z) = p_3\nu^6 + 2q_2\nu^3 + p_0 - z. \tag{4.43}$$

Therefore, the roots of $\mathcal{P}_F(\nu, x, z)$ can be approximated from the following polynomials

$$\mathcal{P}_{F_1} \approx p_3\nu^3 + 2q_2 + p_0\nu^{-3} \tag{4.44}$$

$$\mathcal{P}_{F_2} \approx p_0 + 2q_2\nu^3 + p_3\nu^6. \qquad (4.45)$$

The magnitudes of these ν-roots are approximately $|\nu_{1,2,3}| \approx |\frac{p_0}{q_2}|^{\frac{1}{3}}$ and $|\nu_{4,5,6}| \approx |\frac{q_2}{p_3}|^{\frac{1}{3}}$.

The roots of the polynomial in (4.43) can be approximated from (4.44) and (4.45) and their existence are immediate from Theorem 4.4.1.

Theorem 4.6.1. *Suppose (4.39), (4.40) and (4.41) are satisfied. Then* $|p_3\nu^6_{1,2,3}| = o(1)$ *and* $|p_0\nu^{-3}_{4,5,6}| = o(1)$ *where* $|\nu_{1,2,3}| \approx |\frac{p_0}{q_2}|^{\frac{1}{3}}$ *and* $|\nu_{4,5,6}| \approx |\frac{q_2}{p_3}|^{\frac{1}{3}}$. *Moreover, the* ν*-root of* $\mathcal{P}_F(\nu, x, z)$ *can be approximated from (4.43) and (4.44).*

Proof. It suffices to show that $p_0\nu^{-3}$ and $p_3\nu^6$ are $o(1)$ in $\mathcal{P}_{F_1}$. Here,

$$|p_0\nu^{-3}| \approx |p_0||\frac{q_2}{p_3}|^{-1} \approx |p_0||\frac{p_3}{q_2}| \approx |p_0||p_3||q_2|^{-1} = o(1).$$

$$|p_3\nu^6| \approx |p_3||\frac{p_0}{q_2}|^2 \approx |p_0||p_3||q_2|^{-2} = o(1)$$

since $p_0 = o(q_2)$. The rest of the proof is immediate from the results of Theorem 4.4.1. □

Lemma 4.6.2. *Assume that condition (4.39), (4.40) and (4.41) are satisfied, then the eigenvalues of the operator generated by (4.38) satisfies the* z*-uniform dichotomy condition.*

Proof. The correction term as a result of the spectral parameter z is approximately $(\partial_\nu \mathcal{P}(\nu, x, z))^{-1}$(Behncke & Nyamwala, 2012). Hence for the ν_1, ν_2 and ν_3, we have $|\partial_\nu \mathcal{P}_{\mathcal{F}}(\nu, x, z)|^{-1} \approx |p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ while for ν_4, ν_5 and ν_6 we have $|\partial_\nu \mathcal{P}_{\mathcal{F}}(\nu, x, z)|^{-1} \approx |p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$. Since $|\partial_{\nu_{1,2,3}} \mathcal{P}_F|^{-1} \neq |\partial_{\nu_{4,5,6}} \mathcal{P}_F|^{-1}$ this suffices for the uniform dichotomy condition between $\nu_{1,2,3}$ and $\nu_{4,5,6}$. To prove the dichotomy condition within the clusters, it suffices to analyze $Im(\lambda_k(x, z) - \lambda_j(x, z))$ for $k > j$. Since one of the $\lambda_{1,2,3}(x, z)$ roots is real, assuming it is $\lambda_1(x, z)$ then it suffices to check the dichotomy between $\lambda_2(x, z)$ verses $\lambda_3(x, z)$. But $\lambda = -i\nu$, then from the Implicit function theorem, $Im\lambda_2(x, z) \neq Im\lambda_3(x, z)$ and have different signs. Similarly, if $\lambda_6(x, z)$ is real, then $Im\lambda_4(x,z) \neq Im\lambda_5(x,z)$ and their signs are different, this suffices for the uniform dichotomy condition. $\square$

Theorem 4.6.3. *Let T be the minimal differential operator generated by (4.38) on $\mathcal{L}^2[0,\infty)$ and assume that condition (4.39), (4.40) and (4.41) are satisfied. Then*

(i) Suppose that $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ and $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ are integrable, then $defT = (4,4)$ and $\sigma(H)$=discrete.

(ii) Suppose that either $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ is integrable but $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ is not or $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ is not integrable while $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ is integrable, then either $defT = (3,4)$ or $(4,3)$ depending on the sign of $q_2(x), \forall x \subset [0,\infty)$. Then T has no selfadjoint extension.

(iii) If $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ and $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ are not integrable, then $defT = (3,3)$ and $\sigma_{ac}\mathcal{H} \subset [\bar{p}_0,\infty)$ if $q_2 > 0$ while $\sigma_{ac}(H) = \mathbb{R}$ if $q_2 < 0$ with spectral multiplicity 1.

Proof. **(i)** By Theorem 4.2.1 (Levinson's Theorem), the differential equation (4.38) is converted into first order system by use of quasiderivatives. We then compute the characteristic polynomial of the matrix in the first order system whose zeros can be obtained from the roots of $\mathcal{P}(\nu,x,z)$, then we show the z-uniform dichotomy condition by application of Lemma 4.2.3. Theorem 4.2.1 now shows that the solutions are of the form;

$$y_k(x,z) = (e_k + r_{kk}(x,z)).exp\left(\int_0^x \lambda_k(l,z)\right)dx \quad k = 1,2,3,4,5,6.$$

If $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ and $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ are integrable, then $\lambda_1(x,z)$ will be pure imaginary and since $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ is integrable, then the associated eigensolution will be square integrable both in the upper and lower half planes. Since $\lambda_{2,3}(x,z)$ are in complex conjugate pair, they will contribute to one square integrable and one non-square integrable solution in the complex plane. The eigensolution associated to $\lambda_4(x,z)$ will be square integrable both in the upper and lower halfplane. The remaining complex conjugate pair $\lambda_{5,6}(x,z)$, will also be contributing to one square integrable and one non-square integrable solutions to the complex plane. This leads to $defT = (4,4)$ and the spectrum is discrete at most.

(ii) If $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ is integrable but $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ is not or $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ is not integrable while $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ is integrable, then the eigensolution associated to $\lambda_1(x,z)$ will be square integrable both in the upper and lower half planes. Since $\lambda_{2,3}(x,z)$ are in complex conjugate pair, they will contribute to one square integrable and one non-square integrable solution in the complex plane. The eigensolution associated to $\lambda_4(x,z)$ will either be square integrable in the upper half plane but not in the lower half plane and the other way round. The remaining complex conjugate pair $\lambda_{5,6}(x,z)$, will also be contributing to one square integrable and one non-square integrable solution to the complex plane. This leads to $defT = (4,3)$ or $(3,4)$. Then T has no selfadjoint extension.

(iii) If $|p_3|^{\frac{1}{3}}|q_2|^{-\frac{4}{3}}$ and $|p_0|^{-\frac{1}{3}}|q_2|^{-\frac{2}{3}}$ are not integrable, then the eigensolution associated to $\lambda_1(x,z)$ will either be square integrable in the upper complex plane but not in the lower complex plane and conversely depending on the sign of $q_2(x)$. Since $\lambda_{2,3}(x,z)$ are in complex conjugate pair, they will contribute to one square integrable and one non-square integrable solution in the complex plane. Similarly, the eigensolution associated to $\lambda_4(x,z)$ will either be square integrable in the lower complex plane and not in the upper complex plane and vice versa. The other complex conjugate pair $\lambda_{5,6}(x,z)$, will be contributing to one square integrable and one non-square integrable solutions to the complex plane. Due to the sign of $q_2(x)$, $\lambda_{1,2,3}(x,z)$ can either contribute (1,2) or (2,1) to the deficiency indices and this will change if we consider the contribution to the deficiency indices of $\lambda_{4,5,6}(x,z)$ under similar conditions, that is, (2,1) or (1,2). This leads to $defT = (3,3)$ and hence discrete spectrum. If $|q_2|^{-\frac{2}{3}}$ and $|q_2|^{-\frac{4}{3}}$ are not integrable, then the eigenfunction associated with $\lambda_1(x,z)$ loses its square integrability as $\eta \longrightarrow 0^+$. Thus for $q_2(x) < 0$, it implies that

$-\infty < z < \infty$, hence $\sigma_{ac}(H) = \mathbb{R}$ with spectral multiplicity 1. On the other hand, for $q_2(x) > 0$, it implies that $\bar{p}_0 < z < \infty$, hence $\sigma_{ac}(H) \subset [\bar{p}_0, \infty)$.

$\square$

4.7 Order Six Difference Operator

In this section, we consider the sixth order version of (1.2), that is, $p_2(t) = q_1(t) = p_1(t) = 0, \quad p_3(t), q_2(t), p_0(t) \neq 0$ when n=3 so that we have order six symmetric difference equation.

$$\begin{aligned} Ly(t) &= -\Delta^3[p_3(t)\Delta^3 y(t-3)] - i\{\Delta\left(q_2(t)\Delta^2 y(t-2)\right) \\ &+ \Delta^2\left(q_2(t)\Delta y(t-1)\right)\} + p_0(t)y(t). \end{aligned} \tag{4.46}$$

The growth conditions in (4.39) will be assumed with the independent variable taken as t.

The following smoothness and decay assumptions are necessary;

$$\frac{\Delta^2 f}{f}, \left(\frac{\Delta f}{f}\right)^2 \in \ell^1, \qquad \frac{\Delta f}{f} \in \ell^2, \qquad f = p_0, q_2, p_3. \tag{4.47}$$

4.7.1 System formulation

We study the spectral theory of difference operators of the form (4.46) defined on $\ell^2[0,\infty)$. In this case, we solve the equation $Ly(t) = zy(t)$.

4.7.2 Asymptotic Summation and Results

In order to write the Hamiltonian system of (4.46) one introduces quasi-differences,

$$x_1(t) = y(t-1), \quad x_2(t) = \Delta y(t-2), \quad x_3(t) = \Delta^2 y(t-3)$$

$$\Delta x_1(t) = \Delta y(t-1) = x_2(t+1).$$

$$\begin{aligned} u_1(t) &= \Delta^2(p_3(t)\Delta^3 y(t-3)) + i\{\Delta(q_2(t)\Delta y(t-1)) \\ &+ q_2(t)\Delta^2 y(t-2)\} \end{aligned}$$

$$u_2(t) = -\Delta(p_3(t)\Delta^3 y(t-1)) - i(q_2(t)y(t-1)$$

$$u_3(t) = p_3(t)\Delta^3 y(t-3).$$

Now define the vector valued functions $x(t), u(t)$ and $Y(t)$ by

$$x(t) = (x_1(t), x_2(t), x_3(t)), \quad u(t) = (u_1(t), u_2(t), u_3(t)), \quad Y(t) = (x(t), u(t))^{tr}$$

and the partial shift $\mathcal{R}(Y(t))$ operator such that

$$\mathcal{R}(Y(t)) = [x_1(t+1), x_2(t+1), x_3(t+1), u_1(t), u_2(t), u_3(t)]^{tr},$$

where tr means the transpose of a vector or matrix.

Then (4.46) can be written in its discrete linear Hamiltonian form

$$\mathcal{J}\Delta Y(t) = [zW(t) + P(t)]RY(t)$$

where $t \in I, W(t)$ and $P(t)$ are 6×6 complex Hermitian matrices ,

$W(t) = diag\,(w(t), 0, 0, 0, 0, 0)\,, w(t) > 0$ is the weighted function and will be assumed to be $w(t) = 1$ and $\mathcal{J}$ is a canonical symplectic matrix, that is,

$$\mathcal{J} = \begin{bmatrix} 0 & -I_3 \\ I_3 & 0 \end{bmatrix}, \quad P(t) = \begin{bmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{bmatrix}.$$

The first order form of equation (4.46) leads to,

$$\Delta \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p_3} \\ p_0 - z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iq_2 & -1 & 0 & 0 \\ 0 & -iq_2 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \\ u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}.$$

So that

$$H(t,z) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{p_3} \\ p_0 - z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & iq_2 & -1 & 0 & 0 \\ 0 & -iq_2 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Writing matrix $H(t,z)$ in block form gives,

$$H(t,z) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{p_3} \end{bmatrix}, \quad C = \begin{bmatrix} p_0 - z & 0 & 0 \\ 0 & 0 & iq_2 \\ 0 & -iq_2 & 0 \end{bmatrix}.$$

By application of the approach of Shi, (Shi, 2006) we obtain the first order system in the form $Y(t+1,z) = S(t,z)Y(t,z)$ where,

$$Y(t,z) = (x_1(t,z), x_2(t,z), x_3(t,z), u_1(t,z), u_2(t,z), u_3(t,z))^{tr}.$$

Thus we obtain,

$$Y(t+1,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix} Y(t,z)$$

where

$$E = (I_n - A)^{-1} \quad and \quad A^* = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

so that

$$S(t,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}.$$

$$Y(t+1,z) = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & \frac{1}{p_3} \\ 0 & 1 & 1 & 0 & 0 & \frac{1}{p_3} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{p_3} \\ p_0 & p_0 & p_0 & 1 & 0 & \frac{p_0}{p_3} \\ 0 & 0 & iq_2 & -1 & 1 & \frac{iq_2}{p_3} \\ 0 & -iq_2 & -iq_2 & 0 & -1 & 1 - \frac{iq_2}{p_3} \end{bmatrix} Y(t,z)$$

and $p_0 = p_0 - z$.

One can now compute $det(S(t,z) - \lambda I_6) = \mathcal{P}(\lambda, t, z)$. Multiplying $\mathcal{P}(\lambda, t, z)$ by $p_3\lambda^{-3}$ we obtain a polynomial of the form

$$\mathcal{F}(\lambda, t, z) = p_3(1-\lambda)^3(1-\lambda^{-1})^3 + q_2(1-\lambda)(1-\lambda^{-1})(i\lambda + (i\lambda)^{-1}) + p_0 - z.$$

Substituting

$$\lambda = \frac{(is+1)}{(is-1)}$$

and then multiplying the resulting polynomial by $(s^2-1)^3$ we obtain

$$Q(s,z) = p_0 s^6 + 3p_0 s^4 + 16q_2 s^3 + 3p_0 s^2 + 16q_2 s + 64p_3 + p_0. \tag{4.48}$$

Note that $p_0 = p_0 - z$.

The s-roots of this polynomial can then be approximated as,

$$s_{1,2,3} \approx \left(\frac{-16q_2}{p_0}\right)^{\frac{1}{3}} + lower \quad order \quad terms$$

$$s_{4,5} \approx \frac{-3p_0 \pm (9p_0^2 - 1024q_2^2)^{\frac{1}{2}}}{32q_2}$$

$$\approx \pm i + lower \quad order \quad terms$$

$$s_6 \approx \frac{-4p_3}{q_2} - \frac{p_0}{16q_2} + lower \quad order \quad terms.$$

Here, lower order terms are terms of the size $O(q_2^{-2})$ for $s_{4,5,6}$ and $O(q_2^{-1})$ for $s_{1,2,3}$.

Lemma 4.7.1. *Assume that condition (4.39) and (4.47) are satisfied, then the eigenvalues of the operator generated by (4.46) satisfies the z-uniform dichotomy condition.*

Proof. In this case, the corresponding λ-roots are given by

$$\lambda_{1,2,3} = \frac{(is_{1,2,3} + 1)}{(is_{1,2,3} - 1)} \approx \frac{-\frac{16iq_2}{p_0} + 1}{-\frac{16iq_2}{p_0} - 1} \approx 1 \mp \frac{ip_0}{q_2} + \ldots.$$

$$\lambda_1 \approx 1, \quad \lambda_{2,3} \approx \mp \frac{ip_0}{q_2}.$$

Here, $\lambda_1(t,z)$ has magnitude approximately equal to one and so it remains to check the uniform dichotomy for $\lambda_2(t,z)$ verses $\lambda_3(t,z)$. If $|\lambda_2(t,z)| < 1$ and $|\lambda_3(t,z)| > 1$, this suffices for the z-uniform dichotomy condition between $\lambda_2(t,z)$ and $\lambda_3(t,z)$. Moreover, $s_4(t,z)$ and $s_5(t,z)$ are purely imaginary so that if $|\lambda_4(t,z)| > 1$ while $|\lambda_5(t,z)| < 1$ uniformly in t, then the dichotomy condition is satisfied. $|\lambda_6(t,z)| \approx 1$ as $t \longrightarrow \infty$ hence the desired dichotomy condition. $\square$

Theorem 4.7.2. *Let L be the minimal difference operator generated by (4.46) on $\ell^2[0,\infty)$ and assume that condition (4.39) and (4.47) are satisfied. Then*

(i) Suppose that $|q_2|^{-1}$ is summable, then $def\,L = (4,4)$ and $\sigma(H)$ is discrete.

(ii) *Suppose that $|q_2|^{-2}$ is not summable, then $defL = (3,3)$ and $\sigma(H)$ is discrete at most.*

Proof. **(i)** The difference equation (4.46) is first converted into first order by use of quasi-differences. Theorem 4.3.1. now shows that the eigenvalue solution are of the form;

$$y_k(t,z) = (e_k + r_{kk}(x,z)).\Pi_{l=0}^{l=t-1}\lambda_k(l,z) \quad k = 1,2,3,4,5,6.$$

If $|q_2|^{-1}$ is summable, then since one of the $s_{1,2,3}(t,z)$ roots is real, assuming it is $s_1(t,z)$, then the solution $y_1(t,z)$ of the associated root is square summable both in the upper and lower half planes. $s_{2,3}(t,z)$ are in complex conjugate pair and hence their solution contributes to one square summable both in the upper and lower half plane. Similarly, $s_{4,5}(t,z)$ are in complex conjugate pair and therefore their solutions contribute to one square summable both in the upper and lower half plane. The remaining solution $y_6(t,z)$ will also contribute to one square summable in the lower and upper half plane which leads to $defL = (4,4)$. These solutions will be uniformly square summable irrespective of the nature of z. Hence discrete spectrum.

(ii) If $|q_2|^{-2}$ is not summable, then $y_1(t,z)$ will either be square summable in the upper complex plane but not in the lower complex plane and vice versa depending on the sign of $q_2(t)$. Since $s_{2,3}(t,z)$ are in complex conjugate pair, their solutions, $y_{2,3}(t,z)$, will contribute to one square summable and one non-square summable solution in the complex plane. In the same way, $y_{4,5}(t,z)$ will be square summable both in the upper and lower complex plane since $s_{4,5}(t,z)$ are in complex conjugate pair. The solution, $y_6(t,z)$, associated to the root $s_6(t,z)$, will be contributing to either one square summable solution in the lower and fails in the upper complex plane and conversely depending on the sign of $q_2(t)$. This leads to $defL = (3,3)$ and hence discrete spectrum at most.

□

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

(i) For order two differential operator with unbounded odd order coefficients, the absolutely continuous spectrum was the whole of the real line with spectral multiplicity as one. On the other hand, the spectrum of their discrete counterparts only consisted of eigenvalues under the similar growth conditions. The deficiency indices for order two differential operator were $(2,2)$ whenever $|q_1(x)|^{-1}$ was integrable and $(1,1)$ when $|q_1(x)|^{-1}$ was not integrable. However, the deficiency indices were $(1,1)$ for the order two difference operator irrespective of the summability of $|q_1(t)|^{-1}$.

(ii) Order four differential operator with the third order coefficient unbounded resulted to absolutely continuous spectrum which is the whole of the real line with spectral multiplicity as one while the spectrum of order four difference operator was pure discrete under similar growth and decay conditions. For $|q_2(x)|^{-\frac{1}{3}}$ integrable and $|q_2(t)|^{-1}$ summable, the deficiency indices in both cases were $(3,3)$.

(iii) Under similar growth and decay conditions, order six differential operator with unbounded third order coefficients had absolutely continuous spectrum which was the whole of real line while the spectrum of their discrete counterparts was point spectrum at most. The deficiency index was $(4,4)$ in both cases when $|p_3(x)|^{\frac{1}{3}}|q_2(x)|^{-\frac{4}{3}}$ and $|p_0(x)|^{-\frac{1}{3}}|q_2(x)|^{-\frac{2}{3}}$ were integrable and when $|q_2(t)|^{-1}$ was summable.

5.2 Recommendations

In future, one can investigate the spectral properties of order six operators with all the coefficients taken as non- zero. This can be generalized to higher orders more than six. More so, when more than one coefficient is unbounded.

Different methods other than asymptotic integration and summation can be used, that is, scattering methods in order to handle singular continuous spectrum.

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Appendix : List of manuscript published/communicated

Nyamwala, F. O, Ambogo, D. O and Ngala, J. M (2018). Comparative Analysis of Spectral Theory of Second Order Difference and Differential Operators. *Kyungpook Mathematical Journal.*