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Optimizing a Trapezoidal Open-channel for Least Velocity Fluctuation during Overflow Using Mathematical Efficiency Criterion

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Abstract

In open channels, fluid velocity increases with depth of flow. Sewers are particularly susceptible to overwhelming storm water velocities during rains. When flow velocities exceed a certain threshold, damage of channel by scouring may result, or, conversely, siltation of suspended matter. Channel design must optimize dimensions and shapes which both minimize cost, maximizing discharge in normal seasons and regulate the discharge to minimize velocity fluctuations during overflow. Depending on the designer's objectives, channel design involves numerous parameters, including the characteristics of construction materials and earthwork. Traditional methods such as Langrage multipliers, Sequential Quadratic Programming (SQP), Differential Evolution Algorithm (DEA), genetic algorithms, ant-colony optimization, and lately, meta-heuristic algorithms are often used to minimize a cost function subject to channel cross-section. In this paper, using only the mathematical hydraulic efficiency criterion (other factors assumed optimum), a direct integro-differential

technique is applied to determine the optimum trapezoidal channel design that additionally minimizes velocity fluctuations during excessive discharge.

Keywords: Discharge, depth, top width, bottom width, area, wetted perimeter, hydraulic radius, Froude number, Manning equation, Chézy equation, channel slope, open channel, optimization, slope stability

1 Introduction

The main characteristic of open-channel flow (which includes flow in partially full conduit) is that one surface of the flow is exposed to the atmosphere. Flow in a closed, full pipe, however, differs in various aspects with the open-channel flow, mainly in that the cross-section of the flow for open channels is not determined entirely by the solid boundaries, but is free to change without restraint, depending on other parameters of the flow. The free surface is usually subjected to atmospheric pressure (which is constant) and therefore, the flow is caused by the component of the weight of the liquid. As expected therefore, uniform open channel flow is accompanied by drop in the piezometric pressure, $p + \rho g z$, but not a drop in the pressure at the free surface. See Wahome [7] for a synoptic comparison of the two flows.
The commonest examples of natural channel flow are rivers and streams, while irrigation canals, flumes and spillways are artificial open channels.
In practical channel hydrodynamics where conservation of resources is of prime importance, it is crucial to interrogate the issue of the most hydraulically optimum shape. This means the channel shape which allows maximum discharge for a fixed area, surface roughness and bed slope. All the common open channel formulas, including those of Chézy and Manning predict that for uniform flow with a given bed gradient, roughness and cross-sectional area, the mean velocity and discharge, Q - parameters which influence channel efficiency - all depend on the hydraulic mean depth, m, Massey, *et al*[8]. m is defined as the ratio of the flow area, A, to the wetted perimeter, P. The implication here is that the hydraulic efficacy of a channel will be predicated on the channel-shape. Maximizing m, which is equivalent to minizing the wetted perimeter, will therefore maximize the discharge. Cost of lining material will also be minimized.
Among the common channel designs, the semicircular shape has been found to possess the maximum hydraulic mean depth, Douglas,*et al*[9]. However, the consideration of other factors, such as *angle of repose* for loose granular banking material, cost of excavation and relative ease of construction make the semicircular shapes applicable only to small channels, leaving the trapezoidal shape more preferable in practice, Massey,*et al*[8]. Notably, rectangular and triangular channels are special cases of the trapezoidal channel. Hameed [4]

has studied rectangular canals with round bottoms and found them superior to trapezoidal channels.

Even when the channel cross-sectional is trapezoidal, it is still important to determine the most economical trapezium, that is, the one giving the maximum discharge for a given amount of excavation, based on dimensions' characteristics and side slopes.

In this paper, the dimensional characteristics of the most economical trapezoidal channel with minimum velocity fluctuations (to minimize scouring or siltation Stephenson [14], Chow [10]), are derived progressively. It is assumed that the free surface fluid flow is steady, inviscid, incompressible, irrotational and neglects surface tension and viscosity. It is further assumed that the mathematical hydraulic efficiency is the only consideration, and that the trapezoidal channel is symmetric. The condition for the turning points, and hence the optimal dimensions, are obtained by maximizing the hydraulic mean depth function using integro-differential calculus.

2 Mathematical Formulation

2.1 Most economical trapezoidal channel section

From the continuity equation (we state its convective form below), it is clear that discharge in a channel maximizes with velocity. (ρ = density, $\vec{V}$= velocity, t=time)

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \vec{V} = 0 \tag{2.1}$$

The Chézy formula for discharge in steady uniform flow, which is

$$\text{Discharge} = Q = AC\sqrt{(mi)} \tag{2.2}$$

with i being a gradient function, A the cross-sectional area, m the hydraulic mean depth, and C the Chézy constant shows that with slope and surface roughness held constant, flow velocity maximizes with the hydraulic mean depth, m. Other formulae, such as Manning's confirm this fact. The most economical (or efficient) section of a channel is defined as one *which gives the maximum discharge for a given amount of excavation,* Rajput [1].

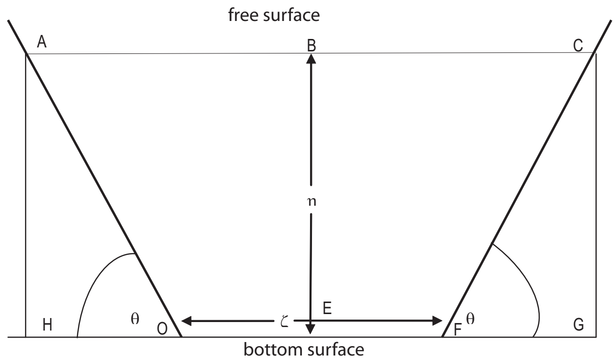

Figure 1: Physical setting of the open-channel problem

In the figure 1 above, $AOFC$ represents the symmetric trapezoidal cross section of the open channel. We use an integration approach to determine the area of the cross-section of the flow (this method is versatile for other non-regular shapes, when need arises). We take the origin to be at O, to have

$$\text{Area } A = \int_0^{\eta} \int_{y\cot(\pi-\theta)}^{y\cot\theta+\zeta} dxdy \tag{2.3}$$

with

$$y\cot(\pi-\theta) \leq x \leq y\cot\theta + \zeta \tag{2.4}$$

between the opposite banks of the channel. Since $\cot(\pi-\theta) = -\cot\theta$, the area evaluates to

$$A = \zeta\eta + \eta^2\cot\theta \tag{2.5}$$

The wetted perimeter in the channel is

$$P = \ \zeta + 2\eta\,\csc\theta \tag{2.6}$$

The hydraulic mean depth, m, which we need to maximize is

$$m = \frac{A}{P} = \frac{\zeta\eta + \eta^2\cot\theta}{\zeta + 2\csc\theta} = \frac{A}{\zeta + 2\eta\,\csc\theta} \tag{2.7}$$

From equations (2.6) and (2.5) we note that

$$\zeta = \frac{A}{\eta} - \eta\cot\theta \tag{2.8}$$

so that equation (2.7) becomes

$$m = \frac{A}{\frac{A}{\eta} - \eta\cot\theta + 2\eta\,\csc\theta} \tag{2.9}$$

2.1.1 Optimum hydraulic radius and area

We expect m to be maximum when the denominator of equation (2.9) is minimized (w.r.t. η). That is, when the *wetted perimeter is minimum*.
This happens when

$$\frac{\partial(\frac{A}{\eta} - \eta\cot\theta + 2\eta\,\mathrm{csc}\theta)}{\partial\eta} = -(\frac{A}{\eta^2}) - \cot\theta + 2\mathrm{csc}\theta = 0 \tag{2.10}$$

with the attached (and obviously true) condition that

$$\frac{\partial^2(\frac{A}{\eta} - \eta\cot\theta + 2\eta\,\mathrm{csc}\theta)}{\partial\eta^2} = \frac{2A}{\eta^3} > 0 \tag{2.11}$$

to guarantee that we have a minimum and not a maximum condition here. Equation (2.10) holds when

$$A_{optm} = (2\mathrm{csc}\theta - \cot\theta)\eta^2 \tag{2.12}$$

with A_{optm} representing the optimum area (i.e. area when the mean hydraulic depth is maximum). We now replace A in the equation (2.9), with the A_{optm} of equation (2.10) to have the maximum hydraulic mean depth, m_{max}, as

$$m_{max} = \frac{(2\mathrm{csc}\theta - \cot\theta)\eta^2}{\frac{(2\mathrm{csc}\theta-\cot\theta)\eta^2}{\eta} - \eta\cot\theta + 2\eta\,\mathrm{csc}\theta} = \frac{\eta}{2} \tag{2.13}$$

The last equation establishes that maximum efficiency for the symmetric trapezoidal channel obtains when m is half the level of free surface of the fluid from the channel bed.

2.1.2 Optimum polygonal shape and slope

We wish to establish the optimum angle, θ and hence the best shape for the trapezoidal channel. We find the value of θ which minimizes the denominator of equation (2.9). We seek,

$$\frac{\partial(\frac{A}{\eta} - \eta\cot\theta + 2\eta\,\mathrm{csc}\theta)}{\partial\theta} = 0 \tag{2.14}$$

That is

$$-\eta(-\csc^2\theta) + 2\eta(-\csc\theta\cot\theta) = 0 \tag{2.15}$$

therefore

$$\frac{1}{\sin\theta}(1 - 2\cos\theta) = 0 \Rightarrow \theta = \frac{\pi}{3} \tag{2.16}$$

Clearly, the derivative

$$\frac{\partial^2(\frac{A}{\eta} - \eta\cot\theta + 2\eta\csc\theta)}{\partial^2\theta} = \eta\left[2\csc^2\theta[\frac{1}{\sin\theta}(1-cos\theta)] + 2\csc\theta\cot^2\theta\right] \quad (2.17)$$

is positive-definite in the interval $0 < \theta < \pi$ which confirms that the angle $\frac{\pi}{3}$ of equation (2.16) occurs at a *maximum* point. This result confirms that *the most economical angle of slope for a trapezoidal channel is* $\frac{\pi}{3}$.
With $\theta = \frac{\pi}{3}$ in figure 1, we note that the slanting sides

$$OA = CF = \frac{2}{\sqrt{3}}\eta = \frac{4}{\sqrt{3}}m \quad (2.18)$$

by (2.13). Comparing equation (2.12) with (2.5) we have,

$$(2\csc\theta - \cos\theta)\eta^2 = \zeta\eta + \eta^2\cot\theta \Rightarrow 2\eta\csc\frac{\pi}{3} = \zeta\cot\frac{\pi}{3} \quad (2.19)$$

And therefore

$$\zeta = \frac{2}{\sqrt{3}}\eta = \frac{4}{\sqrt{3}}m \quad (2.20)$$

similar to equation (2.18). The results of equations (2.20) and (2.18) show that at the optimal channel dimensions,the base of the canal should be equal to the slanting side. This, together with the fact that $\theta = \frac{\pi}{3}$ proves that the most economical cross-sectional of a trapezoidal channel is a *half hexagon*. The various values for the first 8 polygons are tabulated in table below.

Table 1: Efficiency values for the first 8 polygons with $A = 100$ and $\eta = 50$

Polygon's sides n	Slope θ(rad)	Wetted Perimeter P	Efficiency $\frac{1}{P}$
3	2.094395333	146.3375981	0.006833514
4	1.5707965	102.0000087	0.009803921
5	1.2566372	90.90024054	0.011001071
6	1.047197667	88.60254038	0.011286358
7	0.897598	90.03112933	0.011107269
8	0.78539825	93.42135265	0.010704191
9	0.698131778	97.9846981	0.010205675
10	0.6283186	103.3110595	0.009679506

The graph below (figure 2) compares the relative efficiencies for polygonal channels of side n, $3 \leq n \leq 50$. We have computed canal efficiency in terms of the size of reciprocal $\frac{1}{P}$, P being the wetted perimeter(see equation (2.9). We have used the provisional values $A = 100$ and $\eta = 50$.
Besides peaking at $n = 6$ and therefore vindicating that the hexagon is the

most efficient c-sectional design, the graph reveals interesting information, for instance that a 7-sided (heptagonal) cross-section at 0.011107269 is more efficient than a 5-sided (pentagonal) one at 0.011001071.

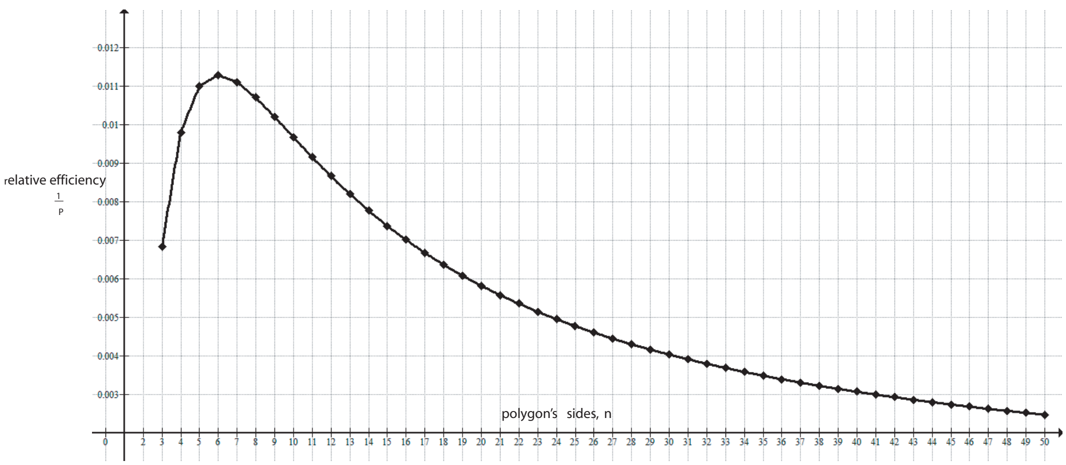

Figure 2: Relative efficiencies for n-sided trapezoidal channels

2.2 Minimum velocity fluctuation design

Having established the relative dimensions of the most economical trapezoidal channel, we now wish to construct the section *above* the free surface of the fluid in such a way that an overflow will not cause a change in velocity. See figure 3 below.

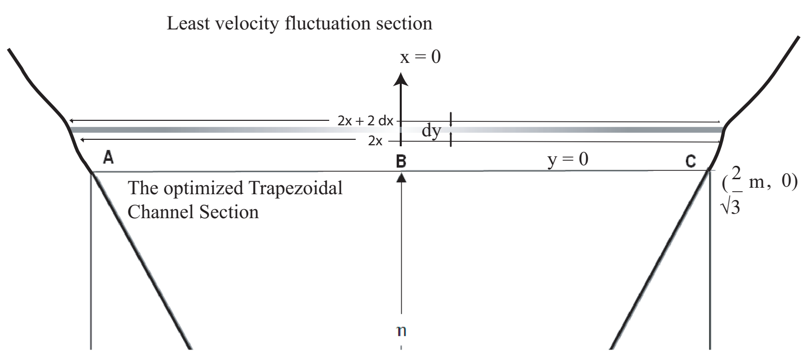

Figure 3: Least velocity fluctuation section of channel

Since the velocity is influenced only by the hydraulic mean depth $m = \frac{A}{P}$, we can suppress velocity fluctuations by keeping m constant. thus,

$$m = \frac{dA}{dP} = \frac{2xdy}{2\sqrt{dx^2 + dy^2}} \Rightarrow \left[dx^2 + dy^2\right]m^2 = x^2dy^2 \tag{2.21}$$

Which simplifies to

$$\Big[\frac{dy}{dx}\Big]^2 = \frac{m^2}{x^2 - m^2} \Rightarrow y = \int \frac{m^2}{\sqrt{x^2 - m^2}} dx = m \cosh^{-1}\Big[\frac{x}{m}\Big] + k \qquad (2.22)$$

In view of the equations (2.18), (2.13) and (2.20), $\overline{BC}$ in Figure 1 has length $\frac{2}{\sqrt{3}}m$. Since this constant velocity section starts *just above* the optimized trapezium, we substitute the coordinates $(\frac{2}{\sqrt{3}}m, 0)$ into the equation (2.22) to find the value of k as, $k = -m\cosh^{-1}(\frac{2}{\sqrt{3})}$. Therefore, the channel walls of the minimum fluctuation section should bear the relationship

$$y = m \cosh^{-1}\Big[\frac{x}{m}\Big] - m\cosh^{-1}\Big(\frac{2}{\sqrt{3}}\Big) \qquad (2.23)$$

3 Conclusion

For a trapezoidal section open-channel flow with base width ζ, fluid depth η and hydraulic mean depth m, the optimum conditions for maximum discharge, and constant velocity during overflow are summarized below.

Table 2: Summary of optimal values

Polygon	Slope θ(deg)	η	Slant Side	ζ
Hexagon	60^0	$2m$	$\frac{4}{\sqrt{3}}m$	$\frac{4}{\sqrt{3}}m$

It was also found that a heptagonal channel-base design would give better performance than a pentagonal one, and that the equation of the constant velocity section is $y = m\cosh^{-1}\big[\frac{x}{m}\big] - m\cosh^{-1}(\frac{2}{\sqrt{3}})$.

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