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Essential and continuous spectrum of symmetric difference equations

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Abstract

Essential and continuous spectrum of symmetric difference equations have been investigated. It has been shown that the deficiency indices and the existence of these components of the spectrum are determined by the growth conditions of the coefficients. In particular, the deficiency indices are superimposition of those clusters determined by the coefficient growth. Finally, we have proved the neccessary and sufficient conditions for the existence of essential spectrum of selfadjoint subspace extensions using subspace theory and asymtotic summation.

KEYWORDS

subspaces, deficiency indices, essential spectrum, continuous spectrum

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Primary: 39A20, Secondary: 47A55

1 | INTRODUCTION

In this paper, we consider $2n$th order J-symmetric difference equation of the form

$$\begin{aligned}\tau\hat{y}(t) = w^{-1}(t)\Bigg\{&\sum_{k=0}^{n}(-1)^k\Delta^k[p_k(t)\Delta^k\hat{y}(t-k)]\\ &- i\sum_{j=1}^{n}(-1)^j\left[\Delta^{j-1}(q_j(t)\Delta^j\hat{y}(t-j)) + (\Delta^j(q_j(t)\Delta^{j-1}\hat{y}(t-j+1))\right]\Bigg\}\\ = z\hat{y}(t)&\end{aligned} \tag{1.1}$$

defined on a weighted Hilbert space $\ell^2_w(\mathbb{N})$ with $w(t) > 0$ as the weight function, $t \in \mathbb{N}$. Here, we shall assume throughout unless otherwise stated that $p_n(t) \neq 0$, $p_k + q_k \neq 0$ for all $k = 0, 1, \dots, n$, the coefficients are twice differentiable and that $p_k(t), q_j(t)$, $k = 0, 1, 2, \dots, n$, $j = 1, 2, \dots, n$ are real valued functions. The second difference of these functions, that is, $\triangle^2 p_k(t)$ and $\triangle^2 q_j(t)$ exist and tend to zero as $t \to \infty$. We will be solving the equation $\tau\hat{y}(t) = z\hat{y}(t)$, where z is a spectral parameter. The difference equation (1.1) is said to be J-symmetric on a complex Hilbert space $\mathcal{H} = \ell^2_w(\mathbb{N})$ since there exists a linear conjugation operator J on $\mathcal{H}$ such that $\langle Jx, Jy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and the map L generated by (1.1) satisfies the relation $D(L) \subseteq D(JL^*J)$ where L^* is the Hilbert adjoint of L. J is a symplectic matrix that will be made clear in Section 2. In consistency with the notations in [29], $\triangle$ refers to a forward difference operator, that is, for any mapping f, $\triangle f(t) = f(t+1) - f(t)$. The notations that will be used in this study are largely standard and follow closely those of [29]. In most cases, the underlying interval will be taken as $I = [a, \infty)$ for a large regular end-point a, $a > 0$. For Hamiltonian systems, Hinton and Shaw's results [20] as well as Remlings's results [25] on self-adjoint extensions will be used to extend the spectral results to the integral interval $[0, \infty)$.

There are a number of papers that have looked into various components of the spectrum of Hamiltonian systems of differential operators, for example, the papers by Behncke [3], Sun and Shi [32]. Similarly, the author together with others have discussed the deficiency indices as well as the location of absolutely continous spectrum of both self-adjoint extenstion operators of differential operators as well as for self adjoint extenstion subspaces for difference equations. For more details see [1,8–12,24]. Therefore, the results of this paper can be considered as a continuation of the investigations carried out in the above mentioned papers and in particular, as an extension of some of the results in [11,12,24] to the discretised version. Here, we have applied asymptotic summation to compute the deficiency indices as well as the location of both essential and continuous spectrum. The computations are explicit and detailed with an example for illustrative purposes. The results have then been extended to $\ell^2_w(\mathbb{Z})$ using techniques in [28]. In this case, we have given the necessary and sufficient decay and smoothness conditions for the existence of the essential spectrum of selfadjoint realisations of minimal subspaces generated by (1.1).

A problem usually encountered in the difference systems is that the minimal operator generated by (1.1) may be neither densely defined nor single valued, its maximal operator may not be well defined and thus the selfadjoint extension operator for minimal operator cannot be discussed by application of von Neumann theory for densely defined Hermitian operators. This is unlike the continuous version where symmetric differential equations will generate densely defined minimal differential operators with single valued maximal operators. A linear operator in $\ell^2_w(\mathbb{N})$ is identified with a linear subspace of $\ell^2_w(\mathbb{N}) \times \ell^2_w(\mathbb{N})$ via its graph and a graph of non-densely defined or multivalued operator in $\ell^2_w(\mathbb{N})$ is also a linear subspace of $\ell^2_w(\mathbb{N}) \times \ell^2_w(\mathbb{N})$ [26,27,31].This, therefore, requires the theory of Hermitian subspaces where the von Neumann theory has been extended in order to discuss the selfadjoint extension of the minimal Hermitian subspaces and for more details see [26–28,31] and the references cited therein.

In addition to the problem mentioned above, computation of the zeros of polynomials of degree five or more can be quite involving and there is no closed form formula for doing this. In order to avoid such a problem, we have employed the techniques in [16, Section 3.3] which have also been used in [11,12] to approximate the roots of the associated characteristic polynomial which are partinent ingredients in the analysis of the deficiency indices of the minimal subspace generated by (1.1) and also the essential and continuous spectrum of the selfadjoint extension subspace of this minimal subspace. We have managed to circumvent the problem of non-densely defined minimal operators generated by (1.1) by using subspace theory to do our analysis.

The spectral multiplicity has been obtained by application of the theory of M-matrix as developed by Hinton and Schneider [18] which is a generalisation of the Weyl–Titchmarsh m-function and relates the asymptotics of the eigenfunctions of Hamiltonian systems to the spectrum of the selfadjoint realisations of the associated minimal subspaces [27,28,30]. Losely translated, the spectral multiplicity of the continuous spectrum is equal to the number of eigenfunctions that lose their square summability as $\operatorname{Im} z \to 0$. The M-matrix is the Borel transform of the spectral measure and the latter can be recovered from the M-matrix [25].

The paper is divided into three sections, namely; 1. Introduction, 2. Subspaces 3. Essential and continuous spectra.

2 | SUBSPACES

Discrete Hamiltonian systems originated from the discretisation of continuous Hamiltonian systems and from the discrete processes acting in accordance with the Hamiltonian principle such as discrete physical problems and discrete control problems. Thus like in the differential case, the coefficients will be assumed to be real-valued functions and will be allowed to be unbounded and satisfy the following conditions:

$$\frac{\triangle^2 f}{f} \in \ell^1, \quad \frac{\triangle f}{f} \in \ell^2, \quad \triangle f = o(1),$$
$$f = p_k, q_j, \quad k = 0, 1, 2, \dots, n, \quad j = 1, 2, \dots, n, \quad \forall t \in \mathbb{N}. \tag{2.1}$$

Besides, we will assume that

$$p_n(t), \quad w(t) > 0, \quad p_k(t) + q_k(t) \neq 0, \quad k = 0, 1, \dots, n. \tag{2.2}$$

For simplicity in computations, and actually throughout this paper, we will take $w(t) = 1$ for all $t \in \mathbb{N}$. In order to define the discrete Hamiltonian system for (1.1), one introduces quasi-difference, see [9,10,29],

$$x_k(t) = \Delta^{k-1} \hat{y}(t-k), \qquad k = 1, \dots, n,$$
$$u_n(t) = \hat{y}^{[n]} = p_n(t)\Delta^n \hat{y}(t-n) - i q_n(t)\Delta^{n-1}\hat{y}(t-(n-1)),$$

$$u_k(t) = \hat{y}^{[2n-k]} = \sum_{l=k}^{n}(-1)^{l-k}p_l(t)\Delta^l\hat{y}(t-l) - i\sum_{l=k+1}^{n}(-1)^{l-k}\left\{\Delta^{l-k}(q_l(t)\Delta^{l-1}\hat{y}(t-l+1))\right.$$
$$\left. + \Delta^{l-k-1}(q_l(t)\Delta^l\hat{y}(t-l))\right\} - iq_k(t)\Delta^{k-1}\hat{y}(t-k+1) \quad k = 1, \dots, n-1. \tag{2.3}$$

These formulae correspond very closely to the expressions for the quasiderivatives, which were first introduced by Walker [33]. Now define the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{tr}, \quad u(t) = (u_1(t), u_2(t), \dots, u_n(t))^{tr}, \quad y(t) = (x(t), u(t))^{tr}$$

and the partial shift operator $Ry(t)$ by $Ry(t) = (x(t+1), u(t))^{tr}$, where tr denotes the vector transpose. Then (1.1) can be written in its discrete linear Hamiltonian form, see [29],

$$J \bigtriangleup y(t) = [zW(t) + P(t)]Ry(t), \tag{2.4}$$

where $W(t)$ and $P(t)$ are $2n \times 2n$ complex Hermitian matrices, $W(t) = \operatorname{diag}(w(t), 0, \dots, 0)$, $x(t), u(t) \in \mathbb{C}^n$, J is a canonical symplectic matrix, that is,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \text{ and } P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix}.$$

(2.4) can then be rewritten as

$$\Delta\begin{pmatrix} x \\ u \end{pmatrix}(t) = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}\begin{pmatrix} x(t+1) \\ u(t) \end{pmatrix}. \tag{2.5}$$

The nonzero matrix elements of A, B and C are given by

$$A_{j,j+1} = 1, \quad A_{n,n} = i\frac{q_n}{p_n}, \quad B_{n,n} = p_n^{-1}, \quad C_{j,j} = p_{j-1}, \quad C_{j,j+1} = iq_j, \quad C_{j+1,j} = -iq_j.$$

Here p_0 and p_{n-1} should be read as $p_0 - zw$ and $p_{n-1} - \frac{q_n^2}{p_n}$, where z is the spectral parameter. In most of the remainder, the spectral term zW will be absorbed into C. (2.4) and (2.5) result into a first order system of (1.1) with a trasfer form $S(t, z)$ given by

$$\begin{pmatrix} x(t+1) \\ u(t+1) \end{pmatrix} = S(t, z)\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} E & EB \\ CE & I_n - A^* + CEB \end{pmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$$

which is appropriate in determination of the eigenfunctions of the Hamiltonian system (2.4) asymptotically. Here $E = (I_n - A)^{-1}$.

Now let $\ell^2_w(I)$ be a Hilbert space with weight function w and define this Hilbert space using the vector valued functions $x(t)$, $u(t)$ and $y(t)$ by

$$\ell^2_w(I) = \left\{ y : y = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C}^{2n} \text{ and } \sum_{t=0}^{\infty}(Ry^*)(t)W(t)(Ry)(t) < \infty \right\}.$$

Then the scalar product for the vector valued functions of the system is [29]

$$\sum_{t=0}^{\infty}\overline{y_1}(t+1)w(t)y(t+1) = \langle y_1, y\rangle_w, \quad y, y_1 \in \ell^2_w(I).$$

The system (2.4) will be assumed to satisfy some regularity conditions. There exists t_0 such that for all solutions $y(., z)$ of (2.4) and for all $z \in \mathbb{C}$

$$\sum_{s \geq t_0}(Ry(s, z))^* W(t) Ry(s, z) > 0, \qquad s \geq t_0, \tag{2.7}$$
$$I_n - A(t) \text{ is invertible}.$$

The assumption that $I_n - A(t)$ be invertible is to ensure the existence and uniqueness of the solutions of any initial value problem for (1.1).

It is well known that if definiteness conditions corresponding to

$$Jy'(t) = (P(t) + zW(t))y(t), \quad t \in [0, \infty),$$

is satisfied, then the minimal operator generated by this continuous system is densely defined and the maximal operator is well defined [2,21,22]. In this case the defect index of the minimal operator is equal to the number of linearly independent square integrable solutions [21,22]. But if this corresponding definiteness condition is not satisfied, the minimal operator may not be densely defined and the maximal operator may be multivalued [2,21,22]. However, in the case of the discrete systems, the minimal operator may fail to be densely defined even if the condition (2.7) is satisfied and this is an important difference between differential and difference equations. Due to these technical difficulties, the spectral properties for difference equations have not been studied that much compared to differential equations.

Since the minimal operator generated by (1.1) may be neither densely defined nor single valued, its maximal operator may not be well defined and thus the selfadjoint extension operator for minimal operator cannot be discussed by application of von Neumann theory for densely defined Hermitian operators. This, therefore, requires the theory of Hermitian subspaces where the von Neumann theory has been extended in order to discuss the selfadjoint extension of the minimal Hermitian subspaces and for more details see [26–28,31] and the references cited therein.

Let $\mathcal{M}$ be a linear subspace or a linear relation in $\ell^2_w(I) \times \ell^2_w(I)$, where the domain, range and kernel of $\mathcal{M}$ are defined by

$$D(\mathcal{M}) = \{y \in \ell^2_w(I) \ : \ (y, g) \in \mathcal{M} \ \text{ for some } \ g \in \ell^2_w(I)\},$$

$$R(\mathcal{M}) = \{g \in \ell^2_w(I) \ : \ (y, g) \in \mathcal{M} \ \text{ for some } \ y \in \ell^2_w(I)\},$$

$$K(\mathcal{M}) = \left\{y \in \ell^2_w(I) \ : \ (y, 0) \in \mathcal{M}\right\},$$

and finally define

$$\mathcal{M} - zI = \{(y, g - zy) \ : \ (y, g) \in \mathcal{M}\}$$

and

$$\mathcal{M}^* = \left\{(y, g) \in \ell^2_w(I) \times \ell^2_w(I) : \ \langle y, f\rangle = \langle g, x\rangle, \ \text{for all} \ \ (x, f) \in \mathcal{M}\right\}$$

so that $\mathcal{M} \subset \ell^2_w(I) \times \ell^2_w(I)$ is called a Hermitian subspace if $\mathcal{M} \subset \mathcal{M}^*$. Now define $\dim(R(\mathcal{M} - zI))^{\perp}$ as the defect index of $\mathcal{M}$ and z. But since $R(\mathcal{M} - zI)^{\perp} = K(\mathcal{M}^* - \bar{z}I)$, the defect indices of $\mathcal{M}$ and its closure with respect to the same z are equal. We will denote

$$\operatorname{def} \mathcal{M} = \dim_{\pm}(\mathcal{M}) = \dim_{\pm i}(\mathcal{M}) = (\mathcal{N}_+, \mathcal{N}_-)$$

as positive and negative defect indices of $\mathcal{M}$.

Now define two semi-scalar product spaces

$$\ell(I) = \left\{y : \ y = \{y(t)\}_{t=a}^{\infty} \subset \mathbb{C}^{2n}\right\}$$

and

$$\mathcal{L}^2_W(I) = \left\{y \in \ell(I) : \ \sum_{t \in I} R^*(y)(t)W(t)R(y)(t) < \infty\right\}$$

with the semi-scalar product

$$\langle y_1, y_2\rangle = \sum_{t \in I} R^*(y_2)(t)W(t)R(y_1)(t).$$

Then it follows that $\|y\| = (\langle y, y\rangle)^{\frac{1}{2}}$ for $y \in \mathcal{L}^2_W(I)$. Since $W(t)$ may be singular in I, $\|.\|$ is semi-norm. One thus defines a quotient space

$$L^2_W(I) = \mathcal{L}^2_W(I) / \left\{y \in \mathcal{L}^2_W(I) : \ \|y\| = 0\right\}.$$

It is true that $L^2_W(I)$ is a Hilbert space with an inner product $\langle .,. \rangle$. For a function y which is a solution of (1.1) and is summable, denote by $\tilde{y}$ the corresponding class in $L^2_W(I)$ and for any $\tilde{y} \in L^2_W(I)$ by $y \in \mathcal{L}^2_W(I)$ denote a representative of $\tilde{y}$. It is evident that $\langle \tilde{y_1}, \tilde{y_2} \rangle = \langle y_1, y_2 \rangle$ for any $\tilde{y_1}, \tilde{y_2} \in L^2_W(I)$. Now let π be a natural quotient map such that

$$\pi : \mathcal{L}^2_W(I) \to L^2_W(I), \quad y \to \tilde{y}.$$

Then π is surjective and not injective in general. One defines the natural difference operator corresponding to (1.1) by

$$\mathcal{L}(y)(t) = J\Delta y(t) - P(t)R(y)(t). \tag{2.8}$$

Further, we define

$$\mathcal{L}^2_{W0}(I) = \{ y \in \mathcal{L}^2_W(I) : \text{ there exists two integers } s, k \in I$$
$$\text{with } s \leq k \text{ such that } y(t) = 0 \text{ for } t \leq s \text{ and } t \geq k+1 \}$$

and

$$\tilde{H} = \left\{ (\tilde{y}, \tilde{g}) \in L^2_W(I) \times L^2_W(I) : \text{ there exists } y \in \tilde{y} \text{ such that } \mathcal{L}y(t) = W(t)R(g)(t), \ t \in I \right\},$$

$$H_{00} = \left\{ (\tilde{y}, \tilde{g}) \in \tilde{H} : \text{ there exists } y \in \tilde{y} \text{ such that } y \in \mathcal{L}^2_{W0}(I) \text{ and } \mathcal{L}y(t) = W(t)R(g)(t), \ t \in I \right\}.$$

Then $\tilde{H}$ and H_{00} are both linear subspaces in $L^2_W(I) \times L^2_W(I)$. $\tilde{H}$ and H_{00} are called maximal and preminimal subspaces corresponding to $\mathcal{L}$, the natural difference operator generated by (1.1), respectively. $H_0 = \overline{H_{00}}$ is the minimal subspace corresponding to $\mathcal{L}$ in $L^2_W(I) \times L^2_W(I)$.

It follows that $H_{00} \subset \tilde{H} \subset H^*_{00}$ and consequently H_{00} is a Hermitian subspace in $L^2_W(I) \times L^2_W(I)$. It has been proved in [27] that the adjoint of preminimal subspace is the maximal subspace and thus $H^*_{00} = H^*_0 = \tilde{H}$.

In order to discuss the selfadjoint extension subspace for H_0, one needs that the system (2.4) satisfies definiteness and some regularity conditions. These will be stated as follows:

A : There exists a finite subinterval $I_1 \subset I$ such that for any $z \in \mathbb{C}$ and for any non-trivial solution $y(t)$ of (1.1) the following always holds:

$$\sum_{t \in I_1} R(y)^*(t)W(t)R(y)(t) > 0.$$

B : Assume always that $I_n - A(t)$ is invertible for all $t \in [0, \infty)$.

We need to point out here that these assumptions have slight variations from those stated in (2.7). The definiteness condition A together with the regularity condition B now guarantees the existence of a unique solution for (2.4). If $z \in \mathbb{C} \backslash \mathbb{R}$, G. Ren and Y. Shi [27] have shown that the dimension of the defect space of H_0 and also of H_{00} are equal to the number of linearly independent square summable solutions of (1.1) or (2.4). Suppose that $n \leq p \leq 2n$, then H_0 has selfadjoint extension subspace in $L^2_W(I) \times L^2_W(I)$ denoted by H if and only if there exists two matrices $M_{p\times 2n}$ and $N_{p\times(2p-2n)}$ such that

$$\operatorname{rank}(M, N) = p, \qquad MJM^* - N\Phi^{tr}N^* = 0$$

and

$$H = \left\{ (\tilde{y}, \tilde{g}) \in \tilde{H} : \ M\tilde{y}(a) - N \begin{pmatrix} (\tilde{y}, y_1)(\infty) \\ (\tilde{y}, y_2)(\infty) \\ \vdots \\ (\tilde{y}, y_p)(\infty) \end{pmatrix} = 0 \right\}. \tag{2.9}$$

Here, $\Phi = \{y_1(a), \ldots, y_p(a)\}_{(2p-2n)\times(2p-2n)}$ is an invertible matrix of square summable solutions of $\tau\hat{y}(t) = z\hat{y}(t)$. For more details see [28, Theorems 5.7 & 5.8] and Section 5 of the same reference in general.

It has been shown in [27] that def H_0 is independent of the half-planes if z is nonreal. In that case, H_0 has a selfadjoint extension subspace H defined by (2.9). Moreover, if a closed Hermitian subspace has equal finite defect indices, then all its selfadjoint extension subspaces have the same essential spectrum [27]. For point spectrum of these subspaces, every isolated point of the spectrum of selfadjoint subspace is an eigenvalue of the subspace and therefore constitutes the point spectrum. Only those eigenfunctions that lose their square summability as Im $z \to 0$ contribute to absolutely continuous spectrum.

A relationship between the spectral results of selfadjoint subspace extension to that of the corresponding selfadjoint extension operator if the minimal and maximal operators generated by (1.1) were densely defined and single-valued respectively is given here below and for more details see [28] .

Theorem 2.1. *If H is a selfadjoint extension subspace in $L^2_W(I) \times L^2_W(I)$ and H_s is the selfadjoint operator defined on the subspace H, then*

$$\sigma_p(H) = \sigma_p(H_s), \quad \sigma_{ac}(H) = \sigma_{ac}(H_s), \quad \sigma_{ess}(H) = \sigma_{ess}(H_s).$$

3 | ESSENTIAL AND CONTINUOUS SPECTRA

3.1 | The eigenvalues

The aim of this section is the analysis of the essential and absolutely continuous spectrum of the subspace H, that is, $\sigma_{ess}(H)$ and $\sigma_{ac}(H)$ respectively as well as their spectral multiplicities. For this, we will determine the absolutely continuous spectrum via the M-function. This in turn requires the asymptotics of the solutions of (1.1). Since it has been shown in [25,32] that essential and absolutely continuous spectra together with their spectral multiplicities are independent of the boundary conditions and the left regular endpoints, we will pay little attention to the left regular endpoints. To obtain the asymptotics of the eigenfunctions of (1.1), rewrite (2.4) or (2.5) in the propagator form (2.6) and hence we determine the eigenvalues of the matrix $S(t,z)$. For this we compute the characteristic polynomial $\mathcal{P}(t,\lambda,z) = \det(S(t,z) - \lambda \cdot I_{2n})$ and then this is multiplied by $(p_n - iq_n)\lambda^{-n}$ to get

$$\begin{aligned}\mathcal{F}(\lambda,z,t) &= \frac{p_n - iq_n}{\lambda^n}\mathcal{P}(\lambda,t,z)\\ &= \sum_{k=0}^{n} p_k(1-\lambda)^k\left(1-\lambda^{-1}\right)^k + \sum_{j=1}^{n} q_j(1-\lambda)^{j-1}\left(1-\lambda^{-1}\right)^{j-1}\left(i\lambda + (i\lambda)^{-1}\right).\end{aligned}$$

Now a transformation of the form

$$\lambda = \frac{s+1}{s-1} \tag{3.2}$$

leads to $s = \frac{\lambda+1}{\lambda-1}$ and this kind of transformation maps the interior of the unit circle onto the left hand plane and the unit circle onto the imaginary axis. With

$$(1-\lambda)\left(1-\lambda^{-1}\right) = -4\left(s^2-1\right)^{-1} \quad \text{and} \quad i\left(\lambda - \lambda^{-1}\right) = 4is\left(s^2-1\right)^{-1}$$

one gets

$$(s^2-1)^n\mathcal{F}\left(\frac{s+1}{s-1}, z\right) = \sum_{k=0}^{n}(-1)^k p_k 2^{2k}\left(s^2-1\right)^{n-k} + \sum_{j=1}^{n}(-1)^{j-1}2^{2j}q_j\left(s^2-1\right)^{n-j} is - z\left(s^2-1\right)^n. \tag{3.3}$$

As in the previous studies, however, one should switch to the Fourier variant Q of this polynomial, by replacing s by is. Then

$$\begin{aligned}Q(s,z) &= \left(s^2+1\right)^n\mathcal{F}\left(\frac{is+1}{is-1}, z\right) \qquad (3.4)\\ &= \sum_{k=0}^{n} p_k 2^{2k}\left(s^2+1\right)^{n-k} + \sum_{j=1}^{n} 2^{2j}q_j\left(s^2+1\right)^{n-j} s - z\left(s^2+1\right)^n.\end{aligned}$$

Q is a polynomial with real coefficients. Now let $s^2 + 1$ be x and we convert the above polynomial to be a polynomial of x and z given by the expression

$$\Phi(x, z) = \sum_{k=0}^{n} 4^k p_k x^{n-k} + \sum_{k=1}^{n} 4^k q_k x^{n-k} - zx^n. \tag{3.5}$$

In order to solve for the x-roots of the above polynomial, besides (2.2), we make the following assumptions on the growth of the coefficients:

$$\left| \frac{p_k + q_k}{p_{k-1} + q_{k-1}} \right| \gg \left| \frac{p_{k+1} + q_{k+1}}{p_k + q_k} \right|$$

which implies generally that

$$| p_{k-1} + q_{k-1} || p_{k+1} + q_{k+1} |= o\left(| p_k + q_k |^2\right). \tag{3.6}$$

Thus using Eastham's approach [16] and generally applied in the spectral analysis of differential operators with unbounded coefficients by the author and others [1,11,12], the magnitude of the x-roots of $\Phi(x, z)$ can be estimated generally by $| x_{n-k} |\approx 4\left|\frac{p_{k+1}+q_{k+1}}{p_k+q_k}\right|$. This means that the x-roots are of the form

$$x_{n-k} \approx -4\frac{p_{k+1} + q_{k+1}}{p_k + q_k} + o(1). \tag{3.7}$$

An application of implicit function theorem shows that the correction term to the x-roots which is denoted by $o(1)$ in (3.7) is given by $\frac{\partial\Phi(x_k,z)}{\partial x}$ with the leading term computed at the pivotal coefficient, that is, $p_k + q_k$ since the application of (3.6) shows that the other terms of $\frac{\partial\Phi(x_k,z)}{\partial x}$ off the pivotal coefficient tend to zero as $t \to \infty$. Thus we have the correction term approximated by

$$4^k(p_k + q_k)\left(-4\frac{p_{k+1} + q_{k+1}}{p_k + q_k}\right)^{n-k-1}.$$

In approximating the x_{n-k}-roots of $\Phi(x, z)$ using the techniques of [1,11,12], one writes

$$\Phi(x, z) = \Phi_{(n-k)0}(x, z) + \Phi_{(n-k)1}(x, z)$$

where

$$\Phi_{(n-k)0}(x, z) = (p_k + q_k)x_{n-k} + 4(p_{k+1} + q_{k+1}),$$

and

$$\Phi_{(n-k)1}(x, z) = O\left(4^{k-1}(p_{k-1} + q_{k-1})\left(\frac{p_{k+1} + q_{k+1}}{p_k + q_k}\right)^2\right).$$

Theorem 3.1. *The zeros of $\Phi(x, z)$ are approximately equal to the zeros of $\Phi_{(n-k)0}(x, z)$ for each $k = 0, 1, \dots, n - 1$.*

Proof. *The strategy of the proof will follow closely the approaches in [12, Lemmas 3.1–3.3]. Thus by Lemma 3.1 of [12] if $P(\gamma)$ is a polynomial in γ such that the leading coefficient and the constant term have absolute value 1 and the other remaining coefficients are bounded, then there exists $K > 0$ such that $K^{-1} \leq| \gamma |\leq K$ for every root γ of $P(\gamma)$. This can be achieved for the roots of $\Phi_{(n-k)0}(x, z)$ by appropriate scalling of the coefficients. The roots of $\Phi_{(n-k)0}(x, z)$ are bounded. We need to show that $\Phi_{(n-k)1}(x, z) = o(1)$. The leading term in this case is given by $O\left(4^{k-1}(p_{k-1} + q_{k-1})\left(\frac{p_{k+1}+q_{k+1}}{p_k+q_k}\right)^2\right)$. This can be simplified as*

$$4^{k-1} | p_{k-1} + q_{k-1} | \left|\frac{p_{k+1} + q_{k+1}}{p_k + q_k}\right|^2 = 4^{k-1} | p_{k+1} + q_{k+1} | \left(| p_{k-1} + q_{k-1} || p_{k+1} + q_{k+1} || p_k + q_k |^{-2}\right).$$

Now application of (3.6) shows that the term within the brackets tends to zero as $t \to \infty$ and hence the desired result. Assume the zero of $\Phi_{(n-k)0}(x,z)$ is $x_{(n-k)0}$ and the corresponding root of $\Phi(x,z)$ within the same cluster is x_{n-k}, then by factorisation we have

$$\Phi_{(n-k)0}(x,z) = (x_{n-k} - x_{(n-k)0})\tilde{\Phi}_{(n-k)0}(x,z)$$

where $\tilde{\Phi}_{(n-k)0}(x,z)$ is a polynomial with lower degree than that of $\Phi_{(n-k)0}(x,z)$. Thus for some interval, one can show by application of Banach fixed point theorem that $x_{n-k} \approx x_{(n-k)0}$. For more details, see [12, Lemma 3.3]. □

The zeros of $\Phi(x,z)$ are approximately equal to the zeros of $\Phi_0(x,z)$ in such a away that if the x-root of $\Phi_{(n-k)0}(x,z)$ is real or complex with non-zero imaginary part, then the corresponding x-root of $\Phi(x,z)$ is also real or complex with non-zero imaginary part respectively. Thus one gets the x-roots of $\Phi(x,z)$ such that $| x_n | \gg | x_{n-1} | \gg \cdots \gg | x_2 | \gg | x_1 |$. Therefore, the s-roots of the polynomial $Q(s,z)$ can be obtained from the relation $s_{(n-k)\pm} = (x_{n-k} - 1)^{\frac{1}{2}}$.

In consideration of the resultant polynomial $\Phi(x,z)$ and its discriminant $\partial_x \Phi(x,z)$, one can show that there are only finitely many spectral values z for which $\Phi(x,z)$ has multiple roots, for more details, see [3]. Let $\omega_1 < \omega_2 < \cdots < \omega_m$ denote all the real spectral values z leading to multiple roots. Following [3], the analysis will be restricted to small complex neighbourhoods of $z_0 \in (\omega_j, \omega_{j+1})$, $j = 0, \dots, m$, where $\omega_0 = -\infty$ and $\omega_{m+1} = \infty$. For a given $z_0 \in (\omega_j, \omega_{j+1})$, one can choose $\epsilon > 0$ and $a > 0$ such that $\Phi(x,z) = 0$ has no multiple or double roots for any

$$z \in \mathcal{K}_\epsilon(z_0) = \{z| \; |z - z_0| \le \epsilon, \;\; \mathrm{Im}(z) > 0\} = \mathcal{K},$$

and $t \ge 0$. This is possible because for any $z \in \mathcal{K} \cap \mathbb{R}$ the roots x of $\Phi(x,z)$ depend analytically on the coefficients p_k, q_j and the spectral parameter z. Even though this analysis was done for the polynomial of differential operators, it is true for every polynomial that has a spectral parameter z in built on it.

Remark 3.2. *The above analysis leads to a pair of s-roots with equal magnitude. But this cannot stop one from making some other assumptions on the growth of the coefficients that lead to even number of s-roots with equal magnitude. Actually, the phrase "cluster of eigenvalues of the same magnitude" as used in [11] is applicable here, though by construction of appropriate polynomials, the cluster can only be of even number of eigenvalues. This conforms with the existing Hamiltonian theory of difference equations which so far has been developed only for even oder symmetric difference equations. For example, if in the above we assume that*

$$| p_{2(k-1)} + q_{2(k-1)} || p_{2(k+1)} + q_{2(k+1)} |= o\left(| p_{2k} + q_{2k} |^2\right),$$

$k = 1, 2, 3, \dots, n-1$, $q_0 = 0$ and $p_0 = p_0 - z$ and for the odd indexed coefficients, we assume that

$$(p_{2k-1} + q_{2k-1}) = O((p_{2(k-1)} + q_{2(k-1)})(p_{2k} + q_{2k}))^{\frac{1}{2}}, \quad k = 1, 2, \dots, n-1,$$

then this leads to two x-roots of $\Phi(x,z)$ of equal magnitude and therefore four s-roots of equal magnitude. This kind of clustering can be done depending on the growth of the coefficients and in each case a cluster of an even number of s-roots with equal magnitude will be achieved.

3.2 | Uniform dichotomy condition

Since the system (2.4) respectively (1.1) are solved by asymptotic summation which is based on the Levinson–Benzaid–Lutz theorem [8–10,24], the s-roots need satisfy the z-uniform dichotomy condition. As a result of that we need the following lemma which is Lemma 4.6 in [9]

Lemma 3.3. *Let*

$$u(t+1) = [\Lambda(t) + R(t)]u(t), \quad t \ge t_0, \quad \Lambda(t) = diag(\lambda_1(t), \dots, \lambda_n(t)) \tag{3.8}$$

be an asymptotically constant difference equation satisfying for $s = 1, \dots, n$

$$\lambda_s(t) = \lambda_{s0} + \lambda_{s1}, \quad with \;\; \lambda_{s0} \;\; constant, \tag{3.9}$$

$\lambda_{s1}(t) \to 0$ as $t \to \infty$, λ_{s0} distinct, $R(x) \in \ell^1$. Moreover assume that $\lambda = \mid \lambda_{s0} \mid \leq 1-\delta$, $s = 1, \dots, k$ and $\left|\frac{\lambda_{s0}}{\lambda_{j0}}\right| \geq 1+\delta$ or $\left|\frac{\lambda_{s0}}{\lambda_{j0}}\right| \leq 1-\delta$, $\delta > 0$ for $j = k+1, \dots, n$. Then (3.8) has k independent solutions

$$u_l(t) = O\left(\prod_{t_0}^{t-1}(\lambda(1+m(v)))\right), \quad 0 \leq m_l \to 0, \quad l = 1, \dots, k, \tag{3.10}$$

as $t \to \infty$, where m is defined as follows. For given $l \in \{1, \dots, k\}$ define $m_l(v)$ by $\mid \lambda_l(v) \mid = \lambda(1+m_l(v))$ and let $m(v) = \max m_l(v)_+$.

This lemma implies that the s-roots of the polynomial $Q(s,z)$ which are in the form $\alpha \pm i\beta$ with $\beta > 0$ will lead to square summable solutions for $s = \alpha + i\beta$ and non-square summable solutions for $s = \alpha - i\beta$ irrespective of the z-uniform dichotomy condition. This simplifies the proof of uniform dichotomy condition and hence it suffices to prove the dichotomy condition only for the real s-roots. We, therefore, have the following results.

Theorem 3.4. *Assume that (2.2) and (3.6) are satisfied, then the s-roots of the polynomial $Q(s,z)$ satisfy the z-uniform dichotomy condition.*

Proof. *By application of Lemma 3.3 it suffices to prove uniform dichotomy condition only for real s-roots. This will be considered in three cases:*

- **(i)** *Assume that $s_{j\pm}$ are real and are from the same cluster, then these roots are not equal since we have restricted our choice of z in $\mathcal{K}$. The s-roots have different signs. The uniform dichotomy condition is proved off the real axis. The correction terms to the s-roots is determined by $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-1}$ and leads to non-zero imaginary parts of different signs. Off the real axis, the moduli of the corresponding two λ-roots will be different with one of them having magnitude greater than one and the other of magnitude less than one. This is the required uniform dichotomy condition between s-real roots of the same cluster.*
- **(ii)** *Assume that s_k and s_j are real with $k \neq j$ and for simplicity let $k > j$. The correction terms of s_k and s_j are determined by $\left(\frac{\partial\Phi(x_k,z)}{\partial x}\right)^{-1}$ and $\left(\frac{\partial\Phi(x_j,z)}{\partial x}\right)^{-1}$ respectively. These correction terms are not equal, since*

  $$\left|\frac{\partial\Phi(x_k,z)}{\partial x}\right| \gg \left|\frac{\partial\Phi(x_j,z)}{\partial x}\right|.$$

  To see this, note that $\left|\frac{\partial\Phi(x_k,z)}{\partial x}\right| \approx \mid p_k + q_k \mid\mid x_k \mid^{n-k-1}$ and $\left|\frac{\partial\Phi(x_j,z)}{\partial x}\right| \approx \mid p_j + q_j \mid\mid x_j \mid^{n-j-1}$. It suffices to show that $\frac{|x_j|^{n-j-1}}{|x_j|^{n-j-1}}$ is $o(1)$ but this is true since this term is equal to $\mid \frac{x_j}{x_k} \mid^n \mid x_k \mid^{k+1} \mid x_j \mid^{-(j+1)}$ and $\mid x_k \mid \gg \mid x_j \mid$ by assumption that $k > j$ and (3.6). This implies that off the real axis, the two s_k and s_j will lead to λ-eigenvalues of different magnitudes which again implies that z-uniform dichotomy condition is satisfied.
- **(iii)** *If $p_0 \approx z$ and the s_n-roots are real, then the z-uniform dichotomy condition follows at once from [10, Section 4].* □

3.3 | Diagonalisations

As explained in Section 3.1 and also in [10, Section 3], we can chose $z \in \mathcal{K}_\epsilon(z_0)$ in such a way that $S(t,z)$ in (2.6) has distinct eigenvalues. Thus by applying results in [10], we can find a transforming matrix D, which diagonalises the matrix $S(t,z)$, $D^{-1}SD = \mathrm{diag}(\lambda_1, \dots, \lambda_{2n})$. Here the eigenvectors that form the columns of the matrix D in a more generalised case are obtained by replacing Δ by $\lambda - 1$ and $\hat{y}(t+k)$ by λ^k in (2.3). Therefore, a transformation of the form

$$v = D\begin{pmatrix} x \\ u \end{pmatrix}$$

leads to [10]

$$v(t+1) = (\Lambda(t) + R(t))v(t) \;\text{ with }\; \Lambda = \mathrm{diag}(\lambda_k(t)) \;\text{ and }\; \lambda_k(t) = (\lambda_k + R_{kk})(t). \tag{3.11}$$

Here $R_k(t)$, arises from the correction terms after the first diagonalisation. Recall that the coefficients were assumed to be twice differentiable and hence we need two diagonalisations to achieve the LBL-form. In particular, we have $R_{kk}(t) = 0$ and from (2.1) we have

$$\Delta^2 R_{kj},\ \ (\Delta R_{kj}(t))^2 \ \ \in \ell^1 \ \text{ and } \ R_{kj}(t) \to 0 \ \text{ as } \ t \to \infty.$$

Since the eigenvalues of Λ are distinct and since $R_{kj} = o(1)$, the matrix $\Lambda + R$ can be diagonalised again with a diagonalising transformation of the form $(1 + B)$ with $B_{kk} = 0$ and $B_{kj} = (\lambda_j - \lambda_k)^{-1}(R)_{kj}$ [4]. See [10] where it was also applied in the case of difference equations. Such a diagonalisation can be repeated and leads to a system in Levinson–Benzaid–Lutz form with $R \in \ell^1$. After the first diagonalisation, the elements of the remainder term are of the form $\frac{\Delta f_k}{f_k} x_{n-k}^{-\frac{1}{2}}$ while the second diagonalisation results into remainder terms of the form $\frac{\Delta^2 f_k}{f_k} x_{n-k}^{-1}$. Here, $f_k = p_k, q_k$. It should be noted that the $\lambda_k(t)$ may be taken as the roots of $\mathcal{P}(\lambda, z, t)$, because the final two diagonalisations create at most summable perturbations of the diagonal. One, therefore, requires decay conditions of the form

$$\frac{\Delta f_k}{f_k} x_1^{-\frac{1}{2}} \in \ell^2, \ \ \frac{\Delta^2 f_k}{f_k} x_1^{-1} \in \ell^1, \ \ f_k = p_k, q_k. \tag{3.12}$$

Thus by application of Levinson–Benzaid–Lutz Theorem [10], the eigenfunctions of τ for $z \in \mathcal{K}$ are of the form

$$v_k(t, z) = (\varrho_k(t, z) + r_k(t, z)) \prod_{l=a}^{t-1} \lambda_k(l, z) \quad \text{with} \quad r_k(t, z) = o(1). \tag{3.13}$$

Here $\varrho_k(t, z)$ is a suitable eigenvector of $S(t, z)$ for the eigenvalue $\lambda_k(t, z)$.

3.4 | Essential and continuous spectrum

In this subsection, we will assume that $\mathcal{B}_{n-k}$ refers to a cluster $n - k$ of s-roots of equal magnitude. Since the deficiency index of the minimal subspace is equal to the number of square summable solutions of (1.1) respectively (2.4), we will denote by $\operatorname{def} \mathcal{B}_{n-k}$ the contribution to the deficiency index by the solutions corresponding to the eigenvalues of cluster $n - k$ which are actually determined by the x_{n-k} root of the polynomial $\Phi(x, z)$.

Theorem 3.5. *Let H_0 be the minimal subspace generated by (1.1) and assume that (2.1), (2.2), (3.6) and (3.12) are satisfied with $w(t) = 1$ for all $t \in \mathbb{N}$. Moreover, assume that $\mathcal{B}_{n-k}$ are the clusters of s-roots of equal magnitude which are in pair. Then*

- **(i)** *$\operatorname{def} H_0 = \sum_{k=0}^{n-1} \operatorname{def} \mathcal{B}_{n-k}$,*
- **(ii)** *$\sigma_{ac}(H) = \mathbb{R}$ of spectral multiplicity less or equal to n.*

Proof. *The difference equation (1.1) is converted into its first order system using quasi-differences (2.3) [29]. Thus the appropriate minimal subspace generated by (1.1) respectively (2.3) can be determined just like in Section 2. In order to determine the deficiency indices of H_0, we apply asymptotic summation as outlined in Levinson–Benzaid–Lutz theorem. This implies that we compute the eigenvalues of the matrix $S(t, z)$, establish the z-uniform dichotomy condition, diagonalise the system and finally, the forms of the eigensolutions. These have been explained in the Sections 3.1–3.3 above and hence we do the analysis based on the x_{n-k} roots of the polynomial $\Phi(x, z)$, $k = 0, 1, \dots, n-1$, though one can calculate explicitly, the approximate values of the λ roots of the matrix $S(t, z)$. If x_{n-k} is negative, then the associated $s_{(n-k)\pm}$-roots are in complex conjugate pairs with non-zero imaginary parts. These will lead to $\lambda_{(n-k)\pm}$-roots such that if $\mid \lambda_{(n-k)+} \mid > 1$ then $\mid \lambda_{(n-k)-} \mid < 1$ therefore resulting into one non-square summable and one square summable solutions respectively irrespective of the z-uniforn dichotomy condition. To see this, we need to show that one of the eigensolutions associated with $s_{(n-k)\pm}$-roots is bounded while the other one is not. Therefore let $s_{(n-k)\pm} = \alpha_{n-k} \pm i\beta_{n-k}$ where $\alpha_{n-k}, \beta_{n-k} \in \mathbb{R}$, $\beta_{n-k} > 0$. Using the form of solutions in (3.13), we approximate the value of $\|v_k(t, z)\|^2$ as $t \to \infty$. In order to evaluate $\prod_{l=a}^{t-1} \mid \lambda_k(l, z) \mid^2$, take logarithms and use Euler summation formula to get*

$$\ln \prod_{l=a}^{t-1} \mid \lambda_k(l, z) \mid^2 \approx -2 \sum_{l=a}^{t-1} \beta_{n-k}(l, z).$$

Thus one has modulo some constant factor that

$$\|v_k(.,z)\|^2 \approx \int_a^\infty \varrho_k(t,z)\exp\left(-2\int_a^t \beta_{n-k}(l,z)dl\right)dt$$

where $\varrho_k(t,z)$ is a suitable eigenvector. This norm is bounded and converges to a unique limit as $t \to \infty$ for $\beta_{n-k} > 0$. The analysis for $-\beta_{n-k}$ is done in a similar way.

Since the number of square summable solutions contribute to the deficiency indices [23,28], an x_{n-k}-root of this nature contributes only $(1,1)$ to the deficiency indices. But if x_{n-k} is positive, then the associated $s_{(n-k)\pm}$-roots are real leading to $\lambda_{(n-k)\pm}$-roots with the property $|\lambda_{(n-k)+}| \approx 1$ and $|\lambda_{(n-k)-}| \approx 1$. In this case, the z-uniform dichotomy condition follows from Theorem 3.4 and is proved off the real axis. The square summability of the associated eigensolutions will depend on the correction terms of $x_{(n-k)}$ respectively $s_{(n-k)}$-roots which are of the form $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-1}$ for x_{n-k} and $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-\frac{1}{2}}$ for s_{n-k} both evaluated at x_{n-k}. Here, we will concentrate on the correction terms of s_{n-k}-roots only since it will give more information on whether the two eigensolutions of the x_{n-k}-root are square summable or only one of them. Thus in terms of magnitude, $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-\frac{1}{2}}$ is aproximately

$$|p_k+q_k|^{\frac{n-k-2}{2}} \cdot |p_{k+1}+q_{k+1}|^{\frac{-n+k+1}{2}}.$$

If $|p_k+q_k|^{\frac{n-k-2}{2}} \cdot |p_{k+1}+q_{k+1}|^{\frac{-n+k+1}{2}}$ is summable, then all the eigensolutions associated with $s_{(n-k)\pm}$-roots are square summable and hence the x_{n-k}-root contributes $(2,2)$ to the deficiency indices. But if this term is not summable, then the x_{n-k} contributes $(1,1)$ to the deficiency indices. The deficiency indices of the minimal subspace generated by (1.1) is the total sum of the contributions from the x_{n-k} roots $k = 0,1,2,\dots,n-1$.

If x_{n-k} is positive and $|p_k+q_k|^{\frac{n-k-2}{2}} \cdot |p_{k+1}+q_{k+1}|^{\frac{-n+k+1}{2}}$ is not summable, one of the eigensolutions will be losing its square summability as Im $z \to 0^+$ and such eigensolutions contribute to absolutely continuous spectrum. The number of eigensolutions with such behaviour from all the x_{n-k}-roots equals to the spectral multiplicity of the absolutely continuous spectrum and this can be determined from the rank of the M-matrix [25,29].

Here, it suffices to show that Im $M(z)$ exists non-trivially. Now let $F(.,z)$ be a n by $2n$ system of square summable eigenfunctions which satisfy the M,N- boundary conditions at a and ∞ and define the M-matrix as given in [25] by

$$\langle F(.,z), F(.,z')\rangle(\bar{z}-z') = M^*(z) - M(z')$$

whose extension to discrete setting is given in [29, Theorem 6.3]. Thus for $z = z_0 + i\eta$, $z_0 \in \mathbb{R}$, one obtains

$$Im\ M(z) = \lim_{\eta\to 0^+} Im\ M(z_0+i\eta) = \lim_{\eta\to 0^+} \eta\langle F(.,z_0+i\eta), F(.,z_0+i\eta)\rangle.$$

This is computed for those solutions that lose their square summability as $\eta \to 0^+$. In such a case, let such solution be from the $(n-j)$ cluster, then it implies that the $s_{(n-j)\pm}$-roots are real and hence the corresponding λ-roots have absolute value approximately equal to 1. The square summability is determined by the corresponding correction term given by

$$|p_j+q_j|^{\frac{n-j-2}{2}} |p_{j+1}+q_{j+1}|^{\frac{-n+j+1}{2}}.$$

Again taking Euler logarithmic summation formula as before we get

$$\|v_j(.,z)\|^2 \approx \int_a^\infty \varrho_j(t,z)\exp\left\{-\frac{1}{2}\eta\int_a^{t-1} |p_j+q_j|^{\frac{n-j-2}{2}} |p_{j+1}+q_{j+1}|^{\frac{-n+j+1}{2}}(l,z)dl\right\}dt$$

where $\varrho_j(t,z)$ is a suitable eigenvector. Hence $\lim_{\eta\to 0^+}\|v_j(.,\eta)\|^2$ exists nontrivially because of (3.6) and consequently Im $M(z)$ is non-trivial. The results of [25] can then be applied to extend these results to $a = 0$. Since each positive x_{n-k}-root can only have one such eigensolution, the spectral multiplicity of the self adjoint subspace extension cannot be more than n. The coefficients are allowed to be unbounded with various signs and therefore $\sigma_{ac}(H) = \mathbb{R}$ of spectral multiplicity less or equal to n. □

Remark 3.6. *The results of Theorem 3.5 can be extended to a cluster of any even number of s-roots since only half of those solutions associated with real s-roots will contribute to the absolutely continuous spectrum and therefore the spectral multiplicity.*

Generally, to decide that a real number z is an eigenvalue of H requires that the solution $\hat{y}$ in the equation $(\tau - z)\hat{y} = 0$ satisfies all the boundary conditions. This is not always easy. Therefore, one can easily make general statements about essential spectrum of H. Suppose that $\hat{y}$ is a solution of $(\tau - z)\hat{y} = 0$, $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$, $\eta > 0$ for small $\eta > 0$, such that $\hat{y}$ loses its square summability as $\eta \to 0^+$, then z is in the essential spectrum of H. This is the only possible way to determine whether $z \in \sigma_{ess}(H)$ since $\tau\hat{y}(t)$ (1.1) is a finite order difference equation and as such z cannot be an eigenvalue of infinite multiplicity. Note that all self-adjoint extension subspaces H have the same essential spectrum. We thus have the following result:

Theorem 3.7. *Assume all the conditions in Theorem 3.5 are satisfied. Moreover, assume a that some of the s-roots of (3.4) are real, then def* $H_0 = (\tilde{n}, \tilde{n})$, $\tilde{n} > n$ *if the correction term* $\mid p_k + q_k \mid^{\frac{n-k-2}{2}} \cdot \mid p_{k+1} + q_{k+1} \mid^{\frac{-n+k+1}{2}}$ *of their cluster is summable and* $\sigma_{ess}(H) \neq \emptyset$.

Proof. *Suppose that some s-roots of the polynomial* $Q(s, z)$ *in (3.4) of a particular cluster* $\mathcal{B}_{n-k}$ *are real, then these s-roots will contribute* $(2, 2)$ *to the deficiency indices if* $\mid p_k + q_k \mid^{\frac{n-k-2}{2}} \cdot \mid p_{k+1} + q_{k+1} \mid^{\frac{-n+k+1}{2}}$ *is summable. As a result, an application of Theorem 3.5 (i) leads to deficiency indices that are more than n both in the upper and lower half-planes. Let these deficiency indices be* $(\tilde{n}, \tilde{n})$ *where* $\tilde{n} > n$. *It follows that* H_0 *has self-adjoint extension subspaces with extra boundary conditions at infinity defined by (2.9). It remains to show that for any solution* $\hat{y}$ *that solves the equation* $(\tau - z)\hat{y} = 0$ *as* $\eta \to 0^+$ *is not an eigenvalue. Assume that* $z \notin \sigma(H)$, *then* z *is in the resolvent of* H *and also in the resolvent of the minimal subspace* H_0 *which is an n-dimensional restriction of the maximal subspace* $\tilde{H}$ *at 0. This leads to the relation*

$$(\tilde{n}, \tilde{n}) = def\, H_0 = \left(\dim \mathcal{K}\left(\left(\tilde{H}^* - z\right) y\right), \dim \mathcal{K}\left(\left(\tilde{H}^* - z\right) y\right)\right) = (n, n)$$

which is a contradiction. But since z *is not an eigenvalue of* H *because the solution* $\hat{y}$ *that solves* $(\tau - z)\hat{y} = 0$ *will not satisfy all the boundary conditions as* $\eta \to 0^+$, *it follows that* $z \in \sigma_{ess}(H)$. □

The results in Theorem 3.7 show that the solutions that lose their square summability as $\eta \to 0^+$ contributes also to the essential spectrum. The result can be extended by defining $\tau\hat{y}$ on $\ell^2(\mathbb{Z})$ using decomposition theorem techniques of two point interface [6,7]. This has also been extended to the discrete setting in [10]. If (1.1) is defined on $\ell^2(\mathbb{Z})$, then the situation is quite different. In this case the decomposition method studied by Behncke and Hinton [6,7] can be used. The methods of [6,7,28,29] show that the classical decomposition method can be applied not only in the computation of the deficiency indices but can also be extended to derive spectral results. In this particular case, we will follow the techniques used in [28] since they are closer to our workings.

Define (1.1) on $\ell^2(\mathbb{Z})$ and write $\mathbb{Z} = I_1 \cup I_2$ where $I_1 = (-\infty, 0]$ and $I_2 = [0, \infty)$. Each interval is a set of integral values. Now let $\hat{H}_0$ be the minimal subspace generated by (1.1) on $\ell^2(\mathbb{Z})$. The deficiency indices of $\hat{H}_0$ will be equal to the sum of the deficiency indices of the minimal subspaces generated by (1.1) when defined on $\ell^2(I_1)$ and $\ell^2(I_2)$. If H_{01} and H_{02} are minimal subspaces generated by (1.1) on $\ell^2(I_1)$ and $\ell^2(I_2)$ respectively, then

$$\text{def}\, \hat{H}_0 = \text{def}\, \hat{H}_{01} + \text{def}\, \hat{H}_{02}.$$

As a result, we obtain the following results.

Theorem 3.8. *Let* $\tau\hat{y}$ *in (1.1) be defined on* $\ell^2(\mathbb{Z})$. *Suppose that* $\hat{H}_0$ *is the minimal subspace generated and assume that all the conditions in Theorem 3.5 are satisfied. Then def* $\hat{H}_0 = (p, p)$, $2n < p < 4n$ *if there exists s-roots which are real from cluster* $\mathcal{B}_{n-k}$ *and* $\mid p_k + q_k \mid^{\frac{n-k-2}{2}} \cdot \mid p_{k+1} + q_{k+1} \mid^{\frac{-n+k+1}{2}}$ *is summable . The essential spectrum of its self-adjoint subspace extension is non-empty.*

Proof. *Let* $\hat{H}_0$ *be the minimal subspace generated by (1.1) on* $\ell^2(\mathbb{Z})$. *Thus def* $\hat{H}_0 = 2\sum_{k=0}^{n-1} def\, \mathcal{B}_{n-k}$ *as a result of contribution of def* H_{01} *and def* H_{02} *which are equal by application of the results of Theorem 3.5. Since there exists s-roots which are real from cluster* $\mathcal{B}_{n-k}$ *and because* $\mid p_k + q_k \mid^{\frac{n-k-2}{2}} \cdot \mid p_{k+1} + q_{k+1} \mid^{\frac{-n+k+1}{2}}$ *is summable, all the eigensolutions associated with these roots will be square summable so long as* $Im\, z > 0$. *The other clusters with no pure real roots will contribute to equal number of square and non-square summable solutions since roots with non-zero imaginary parts are always in complex conjugate pairs. It follows that* $\dim\ker \hat{H}_0 > 2n$ *and therefore def* $\hat{H}_0 = (p, p)$ *where* $2n < p < 4n$. *Thus by imposing boundary conditions both at* $-\infty$ *and also at* ∞, *there exist two square matrices* M *and* N *of order* $p - 2n$ *with the conditions:*

$$rank(M, N) = p - 2n, \qquad M\Phi_1^{tr} M^* - N\Phi_2^* N^* = 0$$

where Φ_1 and Φ_2 are invertible matrices of order $p-2n$ with square summable solutions satisfying boundary conditions at $-\infty$ and ∞ respectively. In such a case the selfadjoint subspace extension of $\hat{H}_0$ exists. Let us denote this space by $\hat{H}$. $\hat{H}$ is defined by

$$\hat{H} = \left\{ (\tilde{y}, \tilde{g}) \in \tilde{H} \ : \quad M \begin{pmatrix} (\tilde{y}, y_{11})(-\infty) \\ (\tilde{y}, y_{12})(-\infty) \\ \vdots \\ (\tilde{y}, y_{1p-2n})(-\infty) \end{pmatrix} - N \begin{pmatrix} (\tilde{y}, y_{21})(\infty) \\ (\tilde{y}, y_{22})(\infty) \\ \vdots \\ (\tilde{y}, y_{2p-2n})(\infty) \end{pmatrix} = 0 \right\}.$$

Here, $y_{kj} \in \Phi_k$, $k = 1, 2$ and $j = 1, \dots, p-2n$.The remaining part of the proof follow closely that of Theorem 3.5. Becuase (1.1) is defined on the whole of $\mathbb{Z}$ and moreover the coefficients are unbounded, the essential spectrum is the whole of $\mathbb{R}$ with spectral multiplicity equal to the number of eigensolutions that lose thier square summability as Im $z \to 0$. □

The following example reinforces the results of Theorem 3.5.

Example 3.9. *Consider the following eighth order symmetric difference equation with weight function $w(t) = 1$:*

$$\tau\hat{y}(t) = \sum_{k=0}^{4} (-1)^k \Delta^k \left[p_k(t)\Delta^k \hat{y}(t-k) \right] - i \sum_{j=i}^{4} (-1)^j \left[\Delta^{j-1}(q_j(t)\Delta^j \hat{y}(t-j)) + (\Delta^j(q_j(t)\Delta^{j-1}\hat{y}(t-j+1)) \right].$$

We solve the equation $\tau\hat{y}(t) = z\hat{y}(t)$. Then the associated $\Phi(x, z)$ polynomial is of the form

$$\Phi(x, z) = (p_0 - z)x^4 + 4(p_1 + q_1)x^3 + 16(p_2 + q_2)x^2 + 64(p_3 + q_3)x + 256(p_4 + q_4). \tag{3.14}$$

The analysis will be done for four clusters each of a pair of s-roots of equal magnitude. Now assume that

$$| p_0 || p_2 + q_2 |= o\left(| p_1 + q_1 |^2\right),$$

$$| p_1 + q_1 || p_3 + q_3 |= o\left(| p_2 + q_2 |^2\right)$$

and

$$| p_2 + q_2 || p_4 + q_4 |= o\left(| p_3 + q_3 |^2\right).$$

Then $| x_1 |\approx 4\left|\frac{p_4+q_4}{p_3+q_3}\right|$, $| x_2 |\approx 4\left|\frac{p_3+q_3}{p_2+q_2}\right|$, $| x_3 |\approx 4\left|\frac{p_2+q_2}{p_1+q_1}\right|$, and $| x_4 |\approx 4\left|\frac{p_1+q_1}{p_0}\right|$

(i) *Thus assume that $p_k + q_k$ all have the same sign, then all x_k, $k = 1, 2, 3, 4$, are negative and all the $s_{k\pm}$ are complex with non-zero imaginary parts. Since if a complex number is a root of a polynomial, then its complex conjugate is also a root of the same polynomial, there will be four square summable and four non-square summable eigensolutions irrespective of uniform dichotomy condition. def $H_0 = (4, 4)$ and the square summable eigensolutions are z-uniformly square summable. The spectrum of the self adjoint subspace extension consists only of the eigenvalues thus $\sigma(H)$ is pure discrete.*

(ii) *Suppose that the signs of the pair $p_k + q_k$ and $p_{k-1} + q_{k-1}$ for all $k = 1, 2, 3, 4$, $q_0 = 0$ and $p_0 = p_0 - z$ are different, then all x_k are positive and therefore all $s_{k\pm}$ roots are real. The z-uniform dichotomy condition is proved off the real axis and follows from Theorem 3.3. The correction terms of the x_k respectively $s_{k\pm}$ play an important role in determining the deficiency indices as well as the existence of the absolutely continuous spectrum. These terms are approximately $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-1}$ and $\left(\frac{\partial\Phi(x,z)}{\partial x}\right)^{-\frac{1}{2}}$ for x_k and $s_{k\pm}$ roots respectively. On approximation, these terms are given by $| p_3 + q_3 |^{-\frac{1}{2}}$ for the $s_{1\pm}$ and $s_{2\pm}$ roots, $| p_1 + q_1 |^{\frac{1}{2}} | p_2 + q_2 |^{-1}$ for $s_{3\pm}$ and $| p_0 || p_1 + q_1 |^{-\frac{3}{2}}$ for $s_{4\pm}$. The contribution of x_k roots to the deficiency index are as follows*

x-root	Correction Term	Cont. to def H_0
x_1 and x_2	$\mid p_3 + q_3 \mid^{-\frac{1}{2}}$ summable	$(2, 2)$
”	$\mid p_3 + q_3 \mid^{-\frac{1}{2}}$ non-summable	$(1, 1)$
x_3	$\mid p_1 + q_1 \mid^{\frac{1}{2}} \mid p_2 + q_2 \mid^{-1}$ summable	$(2, 2)$
”	$\mid p_1 + q_1 \mid^{\frac{1}{2}} \mid p_2 + q_2 \mid^{-1}$ non-summabe	$(1, 1)$
x_4	$\mid p_0 \mid\mid p_1 + q_1 \mid^{-\frac{3}{2}}$ summable	$(2, 2)$
”	$\mid p_0 \mid\mid p_1 + q_1 \mid^{-\frac{3}{2}}$ non-summabe	$(1, 1)$

Here Cont. to def H_0 means contribution to deficiency indices.
The above table gives various combinations of deficiency indices and spectral multiplicities. For example, if $\mid p_0 \mid\mid p_1 + q_1 \mid^{-\frac{3}{2}}$ and $\mid p_3 + q_3 \mid^{-\frac{1}{2}}$ are summable while $\mid p_1 + q_1 \mid^{\frac{1}{2}} \mid p_2 + q_2 \mid^{-1}$ is not summable, the def $H_0 = (7, 7)$ and the spectrum is discrete. But as Im $z \to 0$, three eigensolutions from x_1, x_2 and x_4 roots lose their square summability and contributes to absolutely continuous spectrum leading to def $H_0 = (4, 4)$ and $\sigma_{ac}(H) = \mathbb{R}$ of multiplicity three. Other combinations can easily be constructed from the above table.

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