CONSTRUCTION OF GROUP DIVISIBLE VARIANCE – SUM THIRD ORDER ROTATABLE DESIGN THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

BY

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DECLARATION

Declaration by the Candidate

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DEDICATION

To my beloved family

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ABSTRACT

In the study of rotatable designs, the variance of the estimated response at a point is a function of the distance of that point from a particular origin. Group divisible Rotatable Designs have been evolved by imposing conditions on the levels of factors in a rotatable design. In Group Divisible Third Order Rotatable Designs, the v-factors are split into two groups of p and (v-p) factors such that the variance of a response estimated at a point is a function of the distances of the projection of the points in each of the group from a suitable origin. The purpose of this study was to construct Group Divisible Variance-Sum Third Order Rotatable Designs using a balanced incomplete block designs. The objectives were, to construct a Group Divisible Third Order Rotatable Designs in four, five and its generalization in k-dimensions, to obtain a Variance-Sum Group Divisible Third Order Rotatable Designs in four and in five dimensions and to obtain (k-1) Group Divisible Third Order Rotatable Designs in four, five and its generalization in (k-1) dimensions by rotating designs for one group only. Considering a BIBD with parameters$(v, b, r, k, \lambda,)$ where $r \geq 3\lambda$ and k=2, the v-factors are sub-divided into two groups of factors one of p-dimensions and the other $(v-p)$ dimensions. A set of design points generated through factorial combination was added to suitably chosen sets of points, where the unknown levels were determined from the generated design points so as to satisfy the moment conditions. The equations obtained were satisfied since there exists a non-negative solution forming a v-dimensional Group Divisible Third Order Rotatable Designs with their Variance-Sum being a function of the distances for the two groups respectively. In conclusion Group Divisible Variance-Sum Third Order Rotatable Designs was constructed through BIBDs. The Group Divisible Variance-Sum Third Order Rotatable Designs constructed in this study gave less number of design points than the corresponding rotatable designs constructed using BIBDs. Further, the number of normal equations for estimating the parameter estimates was reduced by adopting this method. Other methods on construction of Group Divisible Variance-Sum Third Order Rotatable Designs for k number of groups were recommended.

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ACRONYMS AND NOTATION

BIBD-	Balanced Incomplete Block Designs
GDD-	Group Divisible Designs
GDSORD-	Group Divisible Second Order Rotatable Design
GDTORD-	Group Divisible Third Order Rotatable Design
GDVSTORD-	Group Divisible Variance-Sum Third Order Rotatable Design
RSM-	Response Surface Methodology
SORD-	Second Order Rotatable Design
TORD-	Third Order Rotatable Design

CHAPTER ONE

INTRODUCTION

1.0 Background Information

Response surface methodology is a collection of mathematical and statistical techniques that are useful for modelling and analysis of problems in which a response of interest is influenced by several independent variables. Response Surface Methodology is a powerful and efficient mathematical tool widely applied in the optimization of industrial and commercial processes. Rotatable designs gives information about the response surface equally in all directions and are thus useful when no or little prior knowledge is available about the nature of the response surface. The main objective of RSM is to optimize a response variable which is influenced by several independent variables. Box and Hunter (1957) gave conditions under which designs for the exploration of response surfaces would be rotatable. In many experimental situations the experimenter is concerned with explaining certain aspects of a functional relationship,

$$Y = f(x_1, x_2, \dots, x_v) + e$$

Where Y is the response variable, $(x_1, x_2, \dots, x_v)$ are the independent variables and e is the uncorrelated random error with mean zero and variance δ^2. A function f (.) is called response surface or response function and the designs used for the study are called response surface designs. Response surface methods are useful where several independent variables influence dependent variables. The independent variables are often called input or explanatory variables and the dependent variable is often called the response variable.

1.1 Basic Concepts

1.1.1 Balanced Incomplete Block (BIB) Designs

A BIB design is an arrangement of v treatments in b blocks each of size k (<v). According to Kempthorne and Hinkelmann (2005), an incomplete block design is said to be a balanced incomplete block (BIB) design if the number of replications of all pairs of treatments in a design is the same and if it satisfies the following conditions:

(i) Each treatment occurs at most once in a block

(ii) Each treatment occurs in exactly r blocks

(iii) Each pair of treatments occurs together in exactly λ blocks

The terms (v, b, r, k, λ) are known as the parameters of BIBD.

1.1.2 Third order rotatable designs

Rotatable designs, introduced by Box and Hunter (1957), have the property that the variance of the estimated response at any point is a function of the distance of that point from the origin of the design and constant on spheres centered at the origin. Further, Herzberg (1967) showed that for rotatable designs, the variance between the estimated responses at any two points in the factor space is a function of the distances of the two points from the centre of the design.

The conditions under which a design is rotatable were given by Box and Hunter (1957). A set of points satisfying the moment conditions are called a rotatable arrangement of order three. The arrangement becomes a rotatable design only if it forms a non singular third order design and if the points give rise to a non-singular $(X'X)$ matrix (Box and Hunter (1957) and Draper (1960)).

Let X be an $(N \times L)$ matrix defined as follows

$$X = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ . \\ . \\ . \\ \underline{x}_N \end{bmatrix}$$

$L = \dfrac{(k+3)!}{k!3!}$, is the number of terms in the model

If X' is the transpose of X then $N^{-1}(X'X)$ is the moment matrix of the arrangement of N points in V-dimensional factor space.

Non – singularity conditions (Draper (1960a)) showed that a third order rotatable arrangement is a nonsingular third order rotatable design if and only if the points lie on two or more spheres centered at the origin of the design.

1.1.3 Group divisible third order rotatable designs

Das and Dey (1967) introduced GDSORD by modifying the restrictions on the levels of the factors in a second order rotatable design. In these designs the *v*-dimensional space corresponding to *v*-factors is divided into two mutually orthogonal spaces, one of *p*-dimensional and the other of (*v-p*) dimensions. They defined the p-dimensional space by the first p factors and the other by the remaining (v-p) factors such that the design is rotatable for each group when the levels of factors in the other group are held constant. As the factors get divided into two groups, thus this might be called "Group-Divisible Rotatable Designs" such that for the factors within each group the design is rotatable. Given any treatment combination in the *v*-dimensional space, we can visualize the projection of the points $(x_{1,0}, x_{2,0}, \ldots, x_{v,0})$ in the first space to be $(x_{1,0}, x_{2,0}, \ldots, x_{p,0}, 0, 0, \ldots, 0)$ and on the second space to be

$(0, 0, \ldots, 0, x_{p+1,0}, x_{p+2,0}, \ldots, x_{v,0})$. Let the distances of the projection of the points in each of the subspaces from a suitable origin be d_1^2 and d_2^2 respectively.

1.1.4 Variance-sum group divisible third order rotatable designs

In rotatable designs, the variance of the estimated response at a point is a function of the distance of that point from the design origin. In GDSORD the variance of a response estimated at the point $(x_{1,0}, x_{2,0}, \ldots, x_{v,0})$ is a function of the distances d_1^2 for group one and d_2^2 for group two from a suitable origins respectively. Variance-Sum is a property of GDTORD, where the sum of the variance of the response estimates in the direction of any factor axis in each group of two mutually orthogonal spaces must be a function of the distances of the projection of the points in each of the group from a suitable origin.

1.2 Statement of the Problem

In the design of experiments and planning of field experiments, the experimenter aims at cutting down on the cost of running the experiments and optimizing the output. Several designs have therefore been constructed. However, there is need for more efficient designs in terms of cost and also maximizing the output when running of experiments. Construction of rotatable designs gives a desirable property of constant variance of the estimated response at a point as a function of the distance of that point from the origin of the designs. A different series of response surface designs such as Group divisible Rotatable Designs have been introduced. In Group divisible Rotatable Designs, the variance of a response estimated at a point equidistant from the centre of the designs is a function of the distances of the projection of the points in each of the group from a suitable origin. The problem here was to construct a Group Divisible Variance Sum Third Order Rotatable Designs through balanced incomplete block designs, this implies that the sum of the variance of the response estimates in the direction of any factor axis in each group of mutually orthogonal spaces, one of p-dimension and the other of (v-p)-dimension at any point must be a function of the distances of the projection of the points in each of the group from a suitable origin.

1.3 Objectives of the Study

1.3.1 General objective

To construct a Group Divisible Variance Sum Third Order Rotatable Designs using a BIBD.

1.3.2 The specific objectives

1. To construct a Group Divisible Third Order Rotatable Designs in four, five and in k-dimensions.
2. To obtain a Variance Sum Group Divisible Third Order Rotatable Designs in four and in five dimensions.
3. To construct a (k-1) Group Divisible Third Order Rotatable Designs in four, five and its generalization in (k-1) dimensions by rotating designs for one group only.

1.4 Significance of the Study

In situations where the experimenter is interested in practical grouping of factors, the v-dimensional space is split into two groups of factors of *p*-dimension and the other of (*v*-*p*) dimension where the designs within each of the groups are certainly rotatable when the levels of factors in the other group are held constant such that the variance of the response estimated at the point $(x_{1,0}, x_{2,0}, \ldots, x_{v,0})$ equidistant from the centre of the designs is a function of the distances of the projection of the points in each of the group from a suitable origin. This study would be desirable since given any treatment combination in v-dimensional space, we can visualize the distances of the projection of the points in each of the subspaces from a suitable origin where the variance- Sum is the function of distances of the projection of the points in each of the group from a suitable origin only. The GDVSTORD constructed in this study gave less number of design points than the corresponding rotatable designs constructed through BIBDs. Further, the number of normal equations for estimating the parameter estimates was reduced by adopting this method.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

Adikary and Panda (1984) identified some practical grouping of the set of factors and introduced group divisible response surface designs both of second and third order and gave methods for their construction. Sheshubabu et al (2014) constructed a third order slope rotatable designs using a balanced incomplete block designs. Sheshubabu et al (2015) introduced a cubic slope rotatable design using balanced incomplete block designs in four dimensions. Das and Dey (1967) introduced GDSORD by modifying the restrictions on the levels of the factors in a second order rotatable design. In these designs the *v*-dimensional space corresponding to *v*-factors is divided into two mutually orthogonal spaces, one of *p*-dimensions and the other of (*v*-*p*) dimensions. Variance-Sum is a property of GDTORD, where the sum of the variance of the response estimates in the direction of any factor axis in each group of two mutually orthogonal spaces must be a function of the distances d_1^2 and d_2^2 from a suitable origins for group one and two respectively.

2.1 Group Divisible Third Order Rotatable Designs

The study of rotatable designs mainly emphasized on the estimation of absolute response. Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces and gave methods for the construction of second order rotatable designs (SORD), through geometrical configurations considering the variances of the estimated response are constant at points equidistant from the centre of the design, conventionally taken to be the origin of factor space after transformation if necessary. Later various authors have suggested different methods of constructing SORD.

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. Many third order rotatable designs have been described in Gardiner et al. (1959).These designs would usually require more points and hence may not always be desirable. New sequential methods have been described in Mutiso and Koske (2005, 2006). Koske et al (2011) introduced a new method of constructing higher level of third order rotatable designs using BIBDS. Mutai et al (2011) studied a new method of constructing a k-dimensional Third Order Rotatable Designs using Balanced Incomplete Block Designs.

Das and Dey (1967) independently studied some generalization of SORD and introduced Group-Divisible Second Order Rotatable Designs (GDSORD).

In this study Group Divisible Third Order Rotatable Designs and (k-1) Group Divisible Third Order Rotatable Designs through BIBD were introduced.

2.2 Variance – Sum Group Divisible Third Order Rotatable Designs

Anjaneyulu et al (2002) introduced and constructed Variance-Sum Group Divisible Second Order Slope Rotatable Designs. Anjaneyulu and Narasimham (2011) constructed a variance sum second order and third order slope rotatable designs. Anjaneyulu et al (2004) stated that any Variance-Sum Third Order Slope Rotatable Design is a Third Order Slope Rotatable Design over all directions. Anjaneyulu et al (2010) introduced a Variance-Sum Group Divisible Third Order Slope Rotatable Designs and gave an attempt of construction using central composite designs. In this study a Variance-Sum Group Divisible Third Order Rotatable Designs were constructed using BIBDs.

CHAPTER THREE

METHODOLOGY

3.0 Introduction

In this chapter the methods used to achieve each of the specific objectives are described.

3.1 Introduction on designs Construction

Each point in a design is essentially a combination of the levels of different treatments. Taking some unknown levels to be denoted by $a\ or\ 0$ corresponding to the presence or absence of the treatment respectively, a factorial design in v factors say out of these unknown levels was obtained. Thus if there are four factors each at two levels denoted by $+a\ and -a$, 16 factorial combinations were obtained. Next another design in v factors of the form $2^{t(k)}$ where the two levels of each factor are +1 and -1 was then added. One more set of combinations where any combination of the first design is associated with combination of the second design $2^{t(k)}$ by 'multiplication' of the levels of the same factors and writing the products in the same order was obtained. This method of association of any two combinations of the two designs is called 'multiplication'. Let (v, b, r, k, λ) denote a BIBD, $2^{t(k)}$ denote a fractional replicate of 2^k in ± 1 levels, in which no interaction with less than five factors is confounded. Let$[a-(v, b, r, k, \lambda)]$ denote the design points generated from the transpose of incidence matrix of BIBD where $[a-(v, b, r, k, \lambda)]2^{t(k)}$ are the $b2^{t(k)}$ design points generated from BIBD by "multiplication "and (c,0,0,…,0)2^1denotes the design points generated from (c,0,0...,0) point set, (d,0,0,…,0) 2^2denote the design points generated from (d,0,0...,0) point set. By choosing an additional unknown combinations $(b, b, b \ldots b)$ and multiplying with

$2^{t(k)}$ associate combinations to obtain $(b, b, b \dots b)2^{t(k)}$additional design points to form a third order rotatable designs. Then by combining the points above with suitably chosen points set of S (c,0,0...,0), S (d,0,0...,0) and S(b,b,b...,b) with$d^2 = tc^2$,a unique solutions $t \geq 0$ was then obtained by defining $f(t) = \frac{(1+t^2)^3}{(1+t^3)^2} = \frac{(1+wt^2)^3}{(1+wt^3)^2}$ where $w > 0$ and chosen suitably for v factors (Huda(1987), Koske et al (2011) and Mutai et al(2011)) where the equations obtained are satisfied if there exist a non-negative solution forming a v-dimensional GDTORD. Then the unknown levels are determined from equations obtained through the generated design points so as to satisfy the moment conditions.

3.2 Method of Construction of a Group Divisible Third Order Rotatable Designs through Balanced Incomplete Block Designs

The construction of Group Divisible Third Order Rotatable Designs can in many occasions be made to depend on known solutions for BIB designs. To construct a $(\mathrm{p}, \mathrm{v}-\mathrm{p})$ GDTORD in v factors, a BIBD with parameters$(\mathrm{v}, \mathrm{b}, \mathrm{r}, \mathrm{k}, \lambda,)$ with $\mathrm{r} \geq 3\lambda$ was considered. This was then divided into two groups of factors, one of p-dimension and the other $(\mathrm{v}-\mathrm{p})$ dimension with p≥2 and $(\mathrm{v}-\mathrm{p}) \geq 2$. First the transpose of incidence matrix for the v-factor BIBD Design with unknown level *a* and zero is written, where *a* takes the place of 1 in the above incidence matrix which generates b combinations. From these design points b combinations was obtained each containing k $\mathrm{a's}$ and $(\mathrm{v}-\mathrm{k})$ zeros.Then by combining the points above with suitably chosen points set of S(c,0,0...,0) ,2S(d,0,0...,0) and 2S (b,b,b...,b) levels with

$d^2 = tc^2$ where $t \geq 0$ where unique solutions is obtained by defining

$$f(t) = \frac{(1+2t^2)^3}{(1+2t^3)^2} \qquad (3.2.1)$$

chosen suitably for v factors so that $t \geq 0$. The equations obtained were satisfied if there exist a non-negative t forming a v-dimensional GDTORD.

All the unknown levels are determined by the moment conditions for a Group Divisible Third Order Rotatable Designs Anjaneyulu *et al.* (2004)

Let $\lambda = f(d_i)$ be a function of the radius of a rotatable design,

Where d_i = radius and λ a scaling parameter. Let $d_i^2 = \sum_{i=1}^{N} x^2{}_{iu}$ such that $d_1^2 = \sum_{i=1}^{p} x_i^2$, $d_2^2 = \sum_{j=p+1}^{v} x_j^2$

1: (i) $\sum_{i=1}^{p} x_i^2 = N\lambda_2$ for $i = 1, 2, 3, \dots, p$

$\sum_{j=p+1}^{v} x_j^2 = N\lambda_2$ $j = p+1, p+2, \dots, v$

(ii) $\sum_{i=1}^{p} x_i^4 = 3N\lambda_4$ for $i = 1, 2, 3, \dots, p$

$\sum_{j=p+1}^{v} x_j^4 = 3N\lambda_4$ $j = p+1, p+2, \dots, v$

(iii) $\sum_{i=1}^{p} x_i^6 = 15N\lambda_6$ for $i = 1, 2, 3, \dots, p$

$\sum_{j=p+1}^{v} x_j^6 = 15N\lambda_6$ $j = p+1, p+2, \dots, v$

2 :(a) (i)$\sum_{i=1}^{v} x_i^2 \boldsymbol{x_j^2} = N\lambda_4$ for $\boldsymbol{i \neq j}$ for $i = 1, 2, 3, \dots, p$

$j = p+1, p+2, \dots, v$

(3.2.2)

(ii) $\sum_{j \neq j'} \boldsymbol{x_j^2 x_{j'}^2} = N\lambda_4$ for $j, j' = p+1, p+2, \dots, v$

(iii) $\sum_{i \neq i'} x_i^2 x_{i'}^2 = N\lambda_4$ for $i, i' = 1, 2, 3, \ldots, p$

(b) (i) $\sum x_i^2 x_j^4 = 5N\lambda_6$ for $i \neq j$ for $i = 1, 2, 3, \ldots, p$

$j = p+1, p+2, \ldots, v$

(ii) $\sum_{j \neq j'} x_j^2 x_{j'}^4 = 5N\lambda_6$ for $j, j' = p+1, p+2, \ldots, v$

(iii) $\sum_{i \neq i'} x_i^2 x_{i'}^4 = 5N\lambda_6$ for $i, i' = 1, 2, 3, \ldots, p$

(3) $\sum\sum\sum x_i^2 x_j^2 x_k^2 = N\lambda_6$ for $i \neq j \neq k$

The above summations are taken over all design points. We have all odd order moments equal to Zero in both the groups.

Non-singularity conditions

1. $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

(3.2.3)

2. $\frac{\lambda_2\lambda_6}{\lambda_4^2} > \frac{k+2}{k+4}$

3.3 Method of obtaining a Variance – Sum Group Divisible Third Order Rotatable Designs

From the design points generated through GDTORD, a Variance-Sum Group Divisible Third Order Rotatable Designs in four and in five dimensions were obtained. Considering VSGDTORD divided into two Groups, the sum of the variance of the response estimates in the direction of any factor axis in each group of mutually orthogonal spaces, one of p-dimension and the other of (v-p)-dimension at any point must be a function of the distances d_1^2 and d_2^2 respectively from the design origin.Let

$D = ((x_{ik}))$ be a set of N design points and $y_1, y_2, \dots, y_N$ be the N responses to fit the following third order response surface model at a design point.

$$y(x) = \beta_0 + \sum \beta_i x_i + \sum \beta_j x_j + \sum \beta_{ii'} x_i x_{i'} + \sum\sum \beta_{jj'} x_j x_{j'} + \sum\sum \beta_{ij} x_i x_j + \sum \beta_{ii} x_i^2 + \sum \beta_{jj} x_j^2 + \sum \beta_{iii} x_i^3 + \sum \beta_{jjj} x_j^3 + \sum\sum \beta_{ijj} x_i x_j^2 + \sum \beta_{ii'i'} x_i x_{i'}^2 + \sum \beta_{jj'j'} x_j x_{j'}^2 + \sum\sum\sum \beta_{ijk} x_i x_j x_k + e$$

The Taylor series approximation is of the form

$$E(y(x)) = \beta_0 + \sum_{i=1}^{p} \beta_i x_i + \sum_{i=1}^{p-1} \sum_{i'=i+1}^{p} \beta_{ii'} x_i x_{i'} + \sum_{i=1}^{p} \beta_{ii} x_i^2 + \sum_{j=p+1}^{v} \beta_j x_j + \sum_{j=p+1}^{v} \beta_{jj} x_j^2 + \sum_{j=p+1}^{v} \sum_{j'=j+1}^{v} \beta_{jj'} x_j x_{j'} + \sum_{i=1}^{p} \sum_{j=p+1}^{v} \beta_{ij} x_i x_j + \sum_{i=1}^{p} \beta_{iii} x_i^3 + \sum_{j=p+1}^{v} \beta_{jjj} x_j^3 + \sum_{i=1}^{p} \sum_{j=p+1}^{V} \beta_{ijj} x_i x_j^2 + \sum_{i=1}^{p-1} \sum_{i'=i+1}^{p} \beta_{ii'i'} x_i x_{i'}^2 + \sum_{j=p+1}^{v-1} \sum_{j'=j+1}^{v} \beta_{jj'j'} x_j x_{j'}^2 + \sum_i \sum_j \sum_k \beta_{ijk} x_i x_j x_k$$

Where y is the response, xi and xi' is a p factor group, xj and xj' is the $(v-p)$ factor group, $\beta's$ are the regression coefficients at both the p factor levels and $(v-p)$ factor levels. For a complete third-order model including the intercept, the total number of terms L can be expressed as;

$$L = \binom{k+3}{3}.$$

Considering the linear model as

$y_i = f'(x_i)ß + \varepsilon_i$

i=1,2,…n

This can be expressed in matrix notation as;

$$\underline{y} = \underline{x}'\underline{\beta} + \underline{\varepsilon}$$

The vector $\underline{y}$ is an n×1 vector of observations; x is an n×p matrix; $\underline{\beta}$ is a p×1 vectors of unknown parameters; $\underline{\varepsilon}$ is an n×1 vector of independently distributed random variables, with mean zero and variance σ^2.The experimental region is denoted by χ. By the method of least squares the estimates of the parameter ß are to be obtained. These are given by

$$\underline{\beta} = (X^{'}X)^{-1}X^{'}Y$$

Let M be the moment matrix,

Where

$$\mathrm{M} = \frac{1}{N}X'X \qquad (3.3.1)$$

The determinant of M is obtained, which gives the non-singularity conditions for third order design to be rotatable. Then the inverse of M is determined which enables the variances to be obtained. For a third order full model we have,

$$f^t(x) = [f_1^t(x), \ldots, f_v^t(x)], \qquad (3.3.2)$$

Where v is the number of factors in a v-dimensional factor space, then we have

$f_1^t(x) = (1, x_1^2, \ldots, x_v^2)$, $f_2^t(x) = (x_1x_2, x_1x_3, \ldots, x_{v-1}x_v)$, $f_3^t(x) = (x_1x_2x_3, \ldots, x_{v-2}x_{v-1}x_v)$ and

$$f_4^t(x) = (g_1^t(x), \ldots, g_v^t(x)) \textit{where}$$
$$g_1^t(x) = (x_1, x_1^3, x_1x_2^2, \ldots, x_1x_v^2), g_2^t(x) = (x_2, x_2^3, x_2x_1^2, \ldots, x_2x_v^2),$$
$$g_3^t(x) = (x_3, x_3^3, x_3x_1^2, x_3x_2^2, \ldots, x_3x_v^2), \ldots, g_v^t(x) = (x_v, x_v^3, x_vx_1^2, x_vx_2^2, \ldots, x_vx_{v-1}^2)$$

Thus for a third order design ξ, the partitioned matrix of the moment matrix $M(\xi)$ is given by

$$M(\xi)=\begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) & M_{13}(\xi) & . & . & . & M_{1v}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) & M_{23}(\xi) & . & . & . & M_{2v}(\xi) \\ M_{31}(\xi) & M_{32}(\xi) & M_{33}(\xi) & . & . & . & M_{3v}(\xi) \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ M_{v1}(\xi) & M_{v2}(\xi) & M_{v3}(\xi) & . & . & . & M_{vv}(\xi) \end{bmatrix} \tag{3.3.3}$$

Where $M_{ij}(\xi)=\int_x f_i(x)\,f_j^t(x)\xi(dx)\ (i,j=1,...,v)$ corresponding to the partitioning of $f^t(x)$. Considering the symmetric designs only we will be in a position to obtain the inverse of $M(\xi)$.For a symmetric design ξ, $M_{ij}(\xi)(i\neq j)$ are null matrices thus $M(\xi)$ is reduced to a block diagonal matrix of $M(\xi)=Diag\{M_{11}(\xi),M_{22}(\xi),...,M_{vv}(\xi)\}$. Note that for a symmetric design ξ, $M_{11}(\xi),M_{22}(\xi),...,M_{vv}(\xi)$ are diagonal matrices and further $M_{vv}(\xi)$ in itself is a block diagonal matrix given by

$$M_{vv}(\xi)=\text{Diag}\{M_1^*(\xi),...,M_k^*(\xi)\}, \tag{3.3.4}$$

Where $M_i^*(\xi)=\int_x g_i(x)g_i^t(x)\xi(dx)\ \ (i=1,...,k).$

Below is a block diagonal matrix,

$$M(\xi)=\begin{bmatrix} M_{11}(\xi) & 0 & 0 & . & . & . & 0 \\ 0 & M_{22}(\xi) & 0 & . & . & . & 0 \\ 0 & 0 & M_{33}(\xi) & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & M_{vv}(\xi) \end{bmatrix} \quad (3.3.5)$$

Where,

$$M_{ii}(\xi) = f_i^t(x).f_i(x)$$

(3.3.6)

$$M_{11}(\xi) = f_1^t(x).f_1(x)$$

$$M_{11}(\xi)^{-1} = (f_1^t(x).f_1(x))^{-1}$$

$$M_{22}(\xi) = f_2^t(x).f_2(x)$$

$$M_{22}(\xi)^{-1} = (f_2^t(x).f_2(x))^{-1}$$

$$M_{33}(\xi) = f_3^t(x).f_3(x)$$

$$M_{33}(\xi)^{-1} = (f_3^t(x).f_3(x))^{-1}$$

$$M_{vv}(\xi) = f_v^t(x).f_v(x)$$

$$M_{vv}(\xi)^{-1} = (f_v^t(x).f_v(x))^{-1}$$

$$M_{vv}(\xi) = f_v^t(x).f_v(x) = M_{vv(1)}(\xi) = M_{vv(2)}(\xi) = \ldots = M_{vv(v)}(\xi). \quad (3.3.7)$$

Then for a symmetric design $M(\xi)$, it is seen that the variances are given as,

$$V(\xi) = \sum_{i=1}^{4} V_i(\xi). \tag{3.3.8}$$

Where $V_i(\xi, x) =$ x$'$ $[M_{ii}(\xi)]^{-1} x$ (i=1, 2, 3, 4).

$$V_i(\xi) = f_i' [M_{ii}(\xi)]^{-1} f_i. \tag{3.3.9}$$

$$V_1(\xi) = f_1' [M_{11}(\xi)]^{-1} f_1$$

$$V_2(\xi) = f_2' [M_{22}(\xi)]^{-1} f_2$$

$$V_3(\xi) = f_3' [M_{33}(\xi)]^{-1} f_3$$

$$V_{v(i)}(\xi) = g_i '[M_{vv(i)}(\xi)]^{-1} g_i$$

Where $V_{v(1)}(\xi) = V_{v(2)}(\xi) = \ldots = V_{v(v)}(\xi)$

Summing the above variances (3.3.8) we get expression which is a function of

$$d_1^2, d_2^2, \sum_{i<j} x_i^2 x_{i'}^2, \sum_{i'<j'} x_j^2 x_{j'}^2 \; \sum\sum_{i<j} x_i^2 x_{i'}^4 \sum_{i<j} x_i^4 x_{i'}^2, \sum_{i'<j'} x_{i'}^2 x_{j'}^4, \sum_{i,i'<j'} x_i^2 x_{i'}^2 x_{j'}^2, \sum_{i<j,i'} x_i^2 x_j^2 x_{i'}^2 \; .$$

In order to achieve variance in GDTORD it should be a function of d_1^2 *and* d_2^2 only.

Therefore need the interactions

$$\sum_{i<i'} x_i^2 x_{i'}^2, \sum_{j<j'} x_j^2 x_{j'}^2 \; x_i^2 x_{i'}^4, x_i^4 x_{i'}^2, x_j^2 x_{j'}^4 \; x_j^4 x_{j'}^2, x_i^2 x_j^2 x_{j'}^2, x_i^2 x_j^2 x_{j'}^2, \sum_{i<j,i'} x_i^2 x_{i'}^2 x_j^2 \text{ and } \sum_{i'<j',i} x_j^2 x_{j'}^2 x_i^2,$$

are cancelled.

All the interactions are equated to zero so as to have a function of d_1^2 *and* d_2^2 only.

Then at the point $x \in \chi$ the response is

$$\text{V}(\hat{y}\text{ (x)}) = f'\text{ (x)}\,\hat{\beta} \tag{3.3.10}$$

With variances for the two groups being

$$V(\hat{y}(x_i))=\sigma^2 \; f'(x_i)\,(X'X)^{-1} f(x_i) \quad (3.3.11).$$

$$V(\hat{y}(x_j))=\sigma^2 \; f'(x_j)\,(X'X)^{-1} f(x_j) \quad (3.3.12)$$

The variance Sum is the function of distances $d_1^2 \, and d_2^2$ as shown below

$\sum_{i=1}^{v} V(\hat{y}(x)) = f(d_1^2, d_2^2)$,

Where $d_1^2 = \sum_{i=1}^{p} x_i^2$, $d_2^2 = \sum_{j=p+1}^{v} x_j^2$

d_1^2 and d_2^2 being the distances of the projections of the points in p dimensional and (v – p) dimensional spaces from a suitable origin. The variance $V(\hat{y}(x_i)$ is a function of distance d_1^2 only and variance $V(\hat{y}(x_j)$ is a function of distance d_2^2 only from the design origin. Thus the considered response surface is a v-dimensional Variance - Sum Group Divisible Third Order Rotatable Designs.

3.4 Method of Construction of a (k-1) Group Divisible Third Order Rotatable Designs through Balanced Incomplete Block Designs.

To construct a $(k-1)$ GDTORD in v factors we consider a BIBD with parameters $(v, b, r, k, \lambda,)$ with $r \geq 3\lambda$, this was then divided into two groups of factors, one of p-dimension and the other$(v-p)$ dimension with p≥2 and $(v-p) \geq 1$.We first start by writing the transpose of incidence matrix of the v factors BIBD Design with unknown level *a* and zero, where *a* takes the place of 1 in the above matrix which generates b combinations. In each combination there will be $k\ a's$ and $(v-k)$ zeros. From these design points we get b combinations each containing $k\ a's$ and $(v-k)$ zeros.Then by combining the points above with their suitably chosen points set of 2S(c,0,0...,0) , S(d, d,0...,0) and S (b,b,b...,b) levels with $b^2 = ta^2$for $t \geq 0$

where unique solutions is obtained by defining $f(t) = \frac{(1+2t^2)^3}{(1+2t^3)^2}$ (3.4.1)

chosen suitably for v factors so that $t \geq 0$. The equations obtained are satisfied if there exist a non-negative t forming a v-dimensional (k-1) GDTORD.

All the unknown levels are determined by the moment conditions for a Group Divisible Third Order Rotatable Designs Anjaneyulu *et al*. (2004)) where the factors for $(v - p)$ group is held constant and all odd order moment for group one equal to Zero.

1: (i) $\sum x_i^2 = N\lambda_2$ for $i = 1, 2, 3, \ldots, p$ and $j = p + 1$

(ii) $\sum x_i^4 = 3N\lambda_4$ for $i = 1, 2, 3, \ldots, p$ and $j = p + 1$

(3.4.2)

(iii) $\sum x_i^6 = 15N\lambda_6$ for $i = 1, 2, 3, \ldots, p$ $and j = p + 1$

2 :(a)(i) $\sum_{i \neq i'} x_i^2 x_{i'}^2 = N\lambda_4$ for $\boldsymbol{i} \neq \boldsymbol{j}$ for $i = 1, 2, 3, \ldots, p, j = p + 1$

for $i, i' = 1, 2, 3, \ldots, p$

(b) (i) $\sum x_i^2 x_j^4 = 5N\lambda_6$ for $\boldsymbol{i} \neq \boldsymbol{j}$ for $i = 1, 2, 3, \ldots, p$

(ii) $\sum_{i \neq i'} x_i^2 x_{i'}^4 = 5N\lambda_6$ for $i, i' = 1, 2, 3, \ldots, p$

(3) $\sum\sum\sum x_i^2 x_j^2 x_k^2 = N\lambda_6$ for $i \neq j \neq k$

Non-singularity conditions

3. $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

(3.4.3)

4. $\frac{\lambda_2 \lambda_6}{\lambda_4^2} > \frac{k+2}{k+4}$

The above summations are taken over all design points.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 Introduction

This chapter presents the results and discussions for the study.

4.1 Construction of GDTORD through BIBD in four dimensions

Consider unreduced BIBD with parameters $(v = 4,\ b = 6,\ r = 3,\ k = 2,\ \lambda = 1)$ which is then split to form two groups of two factors each. Let D_0 denotes the 4 factor BIBD given as

$$D_0 = (v = 4, \quad b = 6, \quad r = 3, \quad k = 2, \ \lambda = 1)$$

Where $r \geq 3\lambda$

Then the number of blocks for a four factor BIBD is given as

$$\begin{matrix} 1 & 2 \\ 3 & 4 \\ 1 & 3 \\ 3 & 2 \\ 2 & 4 \\ 4 & 1 \end{matrix}$$

In this case we have four factors however we combine two factors each at a time.

Associate combination of $2^2$for two factors each at two levels is given as;

$$\begin{matrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{matrix}$$

Therefore every factor is varied in four number of ways. Six combinations each varied four times we have

$6 \times 2^2 = 24$ design points by multiplication.

The Set $\left(a - (D_0)\right)2^2$for four factors is given by

1	2	3	4
a	a	0	0
a	$-a$	0	0
$-a$	a	0	0
$-a$	$-a$	0	0
0	a	0	a
0	a	0	$-a$
0	$-a$	0	a
0	$-a$	0	$-a$
a	0	a	0
a	0	$-a$	0
$-a$	0	a	0
$-a$	0	$-a$	0
0	a	a	0
0	a	$-a$	0
0	$-a$	a	0
0	$-a$	$-a$	0
0	a	0	a
0	a	0	$-a$
0	$-a$	0	a
0	$-a$	0	$-a$
a	0	0	a
a	0	0	$-a$
$-a$	0	0	a
$-a$	0	0	$-a$

(4.1*)

In the set above we have 24 design points for both groups.

Rotating the p-factor group for a four factor BIBD

The additional chosen sets of points for 2-factor group are;

Set $S(c,0,0,0)2^2$and $2S\ (d,0,0,0)2^2$

1	2	3	4
c	0	0	0
$-c$	0	0	0
0	c	0	0
0	$-c$	0	0
d	0	0	0
$-d$	0	0	0
0	d	0	0
0	$-d$	0	0
d	0	0	0
$-d$	0	0	0
0	d	0	0
0	$-d$	0	0

(4.1**)

In a p-factor group the levels of (v-p) factors are all zeros.

Rotating the (v-p)-factor group for a four factor BIBD

Set $S(c,0,0,0)2^2$ and $2S\ (d,0,0,0)2^2$ additional sets of points for a 2 factor group are.

1	2	3	4
0	0	c	0
0	0	$-c$	0
0	0	0	c
0	0	0	$-c$
0	0	d	0
0	0	$-d$	0
0	0	0	d
0	0	0	$-d$
0	0	d	0
0	0	$-d$	0
0	0	0	d
0	0	0	$-d$

(4.1***)

All the factors of p dimensions are denoted by zeroes.

An additional 2S$(b \;\; b \;\; b \;\; b)$ was then added which gave an additional 32 design points so as to satisfy the symmetric conditions for Group Divisible Third Order Designs.

$$\begin{bmatrix} b & b & b & b \\ -b & b & b & b \\ b & -b & b & b \\ b & b & -b & b \\ b & b & b & -b \\ -b & -b & b & b \\ -b & b & -b & b \\ -b & b & b & -b \\ b & -b & -b & b \\ b & -b & b & -b \\ b & b & -b & -b \\ -b & -b & -b & b \\ -b & -b & b & -b \\ -b & b & -b & -b \\ b & -b & -b & -b \\ -b & -b & -b & -b \end{bmatrix} \qquad (4.1****)$$

The total number of points is therefore given by;

$N = (4.1\,*) + (4.1\,**) + (4.1\,***) + 2(4.1\,****) = 80$
(4.1.1)

From (3.2.2) we have the following equations from a p-factor group;

$$\sum x_i^4 = 12a^4 + 2c^4 + 4d^4 + 32b^4$$

$$\sum x_i^2 x_{i'}^2 = 4a^4 + 32b^4$$

$$\sum x_i^6 = 12a^6 + 2c^6 + 4d^6 + 32b^6$$

$$\sum x_i^2 x_{i'}^4 = 4a^6 + 32b^6$$

$$\sum x_i^2 x_{i'}^2 x_{i''}^2 = 32b^6$$

From (3.2.2) we have the following equations generated from a (v-p)-factor group;

$\sum x_j^4 = 12a^4 + 2c^4 + 4d^4 + 32b^4$

$\sum x_j^2 x_{j'}^2 = 4a^4 + 32b^4$

$\sum x_j^6 = 12a^6 + 2c^6 + 4d^6 + 32b^6$

$\sum x_j^2 x_{j'}^4 = 4a^6 + 32b^6$

$\sum x_j^2 x_{j'}^2 x_{j''}^2 = 32b^6$

We have same simultaneous equations for both groups for the above sets.

$$\sum x_i^4 - 3\sum x_i^2 x_{i'}^2 = 2c^4 + 4d^4 - 64b^4 = 0 \quad (4.1.2)$$

$$\sum x_j^6 - 15\sum x_j^2 x_{j'}^2 x_{j''}^2 = 12a^6 + 2c^6 + 4d^6 - 448b^6 = 0 \quad (4.1.3)$$

$$\sum x_j^2 x_{j'}^4 - 3\sum x_j^2 x_{j'}^2 x_{j''}^2 = 4a^6 - 64b^6 = 0 \quad (4.1.4)$$

$4a^6 = 16b^6$

$a^2 = 2^{4/3} b^2$

Solving the three equations simultaneously, we have

(4.1.3)-7(4.1.4) given as

$$-16a^6 + 2c^6 + 4d^6 = 0 \quad (4.1.5)$$

(4.1.3)-3(4.1.4) we have

$$= 2c^6 + 4d^6 - 256b^6 = 0 \quad (4.1.6)$$

From (4.1.2) we have

$c^4 + 2d^4 = 32b^4$

Let $d^2 = tc^2$ where $t \geq 0$ then equations are satisfied if there is a non-negative t such that a $f(t)$ gives a real solution.

$c^4 + 2(tc^2)^2 = 32b^4$

$(1 + 2t^2) = \frac{32b^4}{c^4}$ (4.1.7)

From (4.1.6) we have

$2c^6 + 4d^6 = 256b^6$

$(1 + 2t^3) = \frac{128b^6}{c^6}$ (4.1.8)

Dividing the cube of (4.1.7) by the square of (4.1.8) we have

$$f(t) = \frac{(1+2t^2)^3}{(1+2t^3)^2} = \frac{\left(\frac{32b^4}{c^4}\right)^3}{\left(\frac{128b^6}{c^6}\right)^2}$$

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = 2$$

$$[0t^6 + 0t^5 - 12t^4 + 8t^3 - 6t^2 + 0t^1 + 1t^0]$$

$$[0 \quad 0 \quad -12 \quad 8 \quad -6 \quad 0 \quad 1]$$

$t = 0.4544$ (4.1*****)

Thus there exists a real solution t in the range $t \in (0,1)$.

Using (4.1*****) the following were obtained,

$$a^2 = 2^{4/3}b^2$$

$$d^2 = tc^2 = 0.4544c^2$$

$$c^4 = \frac{32b^4}{(1+2t^2)}$$

$$c^4 = \frac{32}{(1+2(0.4544)^2)}b^4$$

Taking the square root in both sides we have

$c^2 = 4.7589b^2$ then

$d^2 = 2.1624b^2$

4.2 Construction of GDTORD through BIBD in five dimensions

Consider unreduced BIBD with parameters $(v = 5,\ b = 10,\ r = 4,\ k = 2,\ \lambda = 1)$

where the 5 factors are divided into 2 factors for group one and 3 factors for group

two. Let D_0 denotes the 5 factor BIBD as shown,

$$D_0 = (v = 5,\ b = 10,\ r = 4,\ k = 2,\ \lambda = 1)$$

Where $r \geq 3\lambda$

Where the D_0 design plan of 10 blocks is given by;

1	2
2	3
3	4
4	5
5	1
1	3
2	4
3	5
4	1
5	2

Associate combination of 2^2 is given by

1	1
1	−1
−1	1
−1	−1

For ten combinations each varied four numbers of ways we have

$(10 \times 2^2) = 40$ Design points.

$$
\begin{array}{cc:ccc}
1 & 2 & 3 & 4 & 5 \\
a & -a & 0 & 0 & 0 \\
-a & a & 0 & 0 & 0 \\
-a & -a & 0 & 0 & 0 \\
0 & a & a & 0 & 0 \\
0 & a & -a & 0 & 0 \\
0 & -a & a & 0 & 0 \\
0 & -a & -a & 0 & 0 \\
0 & 0 & a & a & 0 \\
0 & 0 & a & -a & 0 \\
0 & 0 & -a & a & 0 \\
0 & 0 & -a & -a & 0 \\
0 & 0 & 0 & a & a \\
0 & 0 & 0 & a & -a \\
0 & 0 & 0 & -a & a \\
0 & 0 & 0 & -a & -a \\
a & 0 & 0 & 0 & a \\
a & 0 & 0 & 0 & -a \\
-a & 0 & 0 & 0 & a \\
-a & 0 & 0 & 0 & -a \\
a & 0 & a & 0 & 0 \\
a & 0 & -a & 0 & 0 \\
-a & 0 & a & 0 & 0 \\
-a & 0 & -a & 0 & 0 \\
0 & a & 0 & a & 0 \\
0 & a & 0 & -a & 0 \\
0 & -a & 0 & a & 0 \\
0 & -a & 0 & -a & 0 \\
0 & 0 & a & 0 & a \\
0 & 0 & a & 0 & -a \\
0 & 0 & -a & 0 & a \\
0 & 0 & -a & 0 & -a \\
a & 0 & 0 & a & 0 \\
a & 0 & 0 & -a & 0 \\
-a & 0 & 0 & a & 0 \\
-a & 0 & 0 & -a & 0 \\
0 & a & 0 & 0 & a \\
0 & a & 0 & 0 & -a \\
0 & -a & 0 & 0 & a \\
0 & -a & 0 & 0 & -a
\end{array}
\qquad (4.2^*)
$$

This gave 40 design points for both groups.

Rotating the 2-factor group for a five factor BIBD

The additional chosen sets of points for p-factor group are;

Set $(c\ 0\ 0\ 0)2^2$ and two sets of (d 0 0 0)2^2 p-factors

p	$factors$	$(v-$	$p)$	$factors$
1	2	3	4	5
c	0	0	0	0
$-c$	0	0	0	0
0	c	0	0	0
0	$-c$	0	0	0
d	0	0	0	0
$-d$	0	0	0	0
0	d	0	0	0
0	$-d$	0	0	0
d	0	0	0	0
$-d$	0	0	0	0
0	d	0	0	0
0	$-d$	0	0	0

(4.2**)

Rotating the 3-factor group for a five factor BIBD

The additional chosen sets of points for $(v-p)$ factor group are;

Set $(c\ 0\ 0\ 0)2^{t(k)}$ and two sets of (d 0 0 0)$2^{t(k)}$ p-factors

1	2	3	4	5
0	0	c	0	0
0	0	$-c$	0	0
0	0	0	c	0
0	0	0	$-c$	0
0	0	0	0	c
0	0	0	0	$-c$
0	0	d	0	0
0	0	$-d$	0	0
0	0	0	d	0
0	0	0	$-d$	0
0	0	0	0	d
0	0	0	0	$-d$
0	0	d	0	0
0	0	$-d$	0	0
0	0	0	d	0
0	0	0	$-d$	0
0	0	0	0	d
0	0	0	0	$-d$

(4.2***)

Another balanced subset of these sets of level $(b\ \ b\ \ b\ \ b\ \ b)$ is added to provide a rotatable design.

$s(b\ \ b\ \ b\ \ b\ \ b)$ Gave 32 points

$2s(b\ \ b\ \ b\ \ b)$ Gave 64 points

$$
\begin{array}{ccccc}
b & b & b & b & b \\
-b & b & b & b & b \\
b & -b & b & b & b \\
b & b & -b & b & b \\
b & b & b & -b & b \\
b & b & b & b & -b \\
-b & -b & b & b & b \\
-b & b & -b & b & b \\
-b & b & b & -b & b \\
-b & b & b & b & -b \\
b & -b & -b & b & b \\
b & -b & b & -b & b \\
b & -b & b & b & -b \\
b & b & -b & -b & b \\
b & b & -b & b & -b \\
b & b & b & -b & -b \\
-b & -b & -b & b & b \\
-b & -b & b & -b & b \\
-b & -b & b & b & -b \\
-b & b & -b & -b & b \\
-b & b & -b & b & -b \\
-b & b & b & -b & -b \\
b & -b & -b & -b & b \\
b & -b & -b & b & -b \\
b & -b & b & -b & -b \\
b & b & -b & -b & -b \\
-b & -b & -b & -b & b \\
-b & -b & -b & b & -b \\
-b & b & -b & -b & -b \\
-b & -b & b & -b & -b \\
b & -b & -b & -b & -b \\
-b & -b & -b & -b & -b
\end{array}
\qquad (4.2****)
$$

$2S(b\ \ b\ \ b\ \ b\ \ b)$ gave 64 design points

N= (4.2*) + (4.2**) + (4.2***) +2(4.2****) =134 design points (4.2.1)

Utilizing the moment conditions given in (3.2.2) we have the following equations for a p-factor group,

$$\sum x_i^4 = 16a^4 + 2c^4 + 4d^4 + 64b^4$$

$$\sum x_i^2 x_{i'}^2 = 4a^4 + 64b^4$$

$$\sum x_i^6 = 16a^6 + 2c^6 + 4d^6 + 64b^6$$

$$\sum x_i^2 x_{i'}^4 = 4a^6 + 64b^6$$

$$\sum x_i^2 x_{i'}^2 x_{i''}^2 = 64b^6$$

Utilizing the moment conditions given in (3.2.2) we have the following equations for a (v-p)-factor group,

$$\sum x_j^4 = 16a^4 + 2c^4 + 64b^6 4d^4 + 64b^4$$

$$\sum x_j^2 x_{j'}^2 = 4a^4 + 64b^4$$

$$\sum x_j^6 = 16a^6 + 2c^6 + 4d^6 + 64b^6$$

$$\sum x_j^2 x_{j'}^4 = 4a^6 + 64b^6$$

$$\sum x_j^2 x_{j'}^2 x_{j''}^2 = 64b^6$$

The p-factor group and the (v-p)-factor group had the same simultaneous equations. Therefore the Solutions for unknown constants are achieved by solving the simultaneous equations.

$$\sum x_i^4 - 3\sum x_i^2 x_{i'}^2 = 16a^4 + 2c^4 + 4d^4 + 64b^4 - 3[4a^4 + 64b^4] = 0$$

$$4a^4 + 2c^4 + 4d^4 - 128b^4 = 0 \qquad (4.2.2)$$

$$\sum x_i^6 - 15\sum x_i^2 x_{i'}^2 x_{i''}^2 = 16a^6 + 2c^6 + 4d^6 + 64b^6 - 15[64b^6] = 0$$

$$= 16a^6 + 2c^6 + 4d^6 - 896b^6 = 0 \qquad (4.2.3)$$

$$\sum x_j^2 x_{j'}^4 - 3\sum x_j^2 x_{j'}^2 x_{j''}^2 = 4a^6 + 64b^6 - 3[64b^6] = 0$$

$4a^6 - 128b^6 = 0$ (4.2.4)

$a^2 = 2^{5/3}b^2$

Solving the three equations simultaneously, we have

(4.2.3)-7(4.2.4)

$= 16a^6 + 2c^6 + 4d^6 - 896b^6 - 7[4a^6 - 128b^6] = 0$

$2c^6 + 4d^6 - 12a^6 = 0$ (4.2.5)

From (4.2.3)-4(4.2.4) we have

$2c^6 + 4d^6 - 384b^6 = 0$ (4.2.6)

Substituting the value of a in (4.2.2) above we have

$4\left(2^{5/3}b^2\right)^2 + 2c^4 + 4d^4 - 128b^4 = 0$

$2\left(2^{10/3}\right)b^4 + c^4 + 2d^4 - 64b^4 = 0$

$c^4 + 2d^4 = (64 - 2^{13/3})b^4$

Let $d^2 = tc^2$ where $t \geq 0$

$c^4 + 2(tc^2)^2 = (64 - 2^{13/3})b^4$

$c^4 + 2t^2c^4 \quad = (64 - 2^{13/3})b^4$

$(1 + 2t^2) \quad = \frac{(64-2^{13/3})b^4}{c^4}$ (4.2.7)

From (4.2.6) we have

$c^6 + 2d^6 = 192b^6$

$(1 + 2t^3) = \frac{192\,b^6}{c^6}$

(4.2.8)

Dividing the cube of (4.2.7) by the square of (4.2.8) we have

$f(t) = \frac{(1+2t^2)^3}{(1+2t^3)^2} = \frac{\left(\frac{(64-2^{13/3})b^4}{c^4}\right)^3}{\left(\frac{192\,b^6}{c^6}\right)^2} = 2.2858$

$$1 + 6t^2 + 12t^4 + 8t^6 = 2.2858[1 + 4t^3 + 4t^6]$$

$$[1.1432t^6 + 0t^5 - 12t^4 + 9.1432t^3 - 6t^2 + 0t^1 + 1.2858t^0]$$

$$[1.1432 \quad 0 \quad -12 \quad 9.1432 \quad 6 \quad 0 \quad 1.2858]$$

$t = 2.8880$ and 0.5372 (4.2*****)

Since $t \in (0,1)$ we take the value 0.5372

From equation (4.2*****) we obtained,

$c^2 = 5.2724b^2$

$d^2 = 2.8323b^2$

$a^2 = 2^{5/3}b^2$

4.3 GDTORD in k-factors

Here a generalization of a GDTORD in k-factors was considered such that the non-negative solution of t where $t \in (0,1)$ was achieved as shown below.

$$f(t) = \frac{(1 + 2t^2)^{3}}{(1 + 2t^3)^2} = ((k-2)^{-1}2^{-6})^\wedge 2(2^{k+1} + (k-4)^{-1})^3$$

4.4 Variance Sum Group Divisible Third order rotatable designs

4.4.1 Variance Sum Group Divisible Third order rotatable designs in four dimensions

From the 80 design points in (4.1.1) of a four dimensional GDTORD generated through BIB designs we got the moment matrix of a four dimensional GDTORD to be

Moment matrix $M = \frac{1}{N} x' x$

Where $N = 80$ design points

Let $x_1, x_2, x_{3,} x_4$, be the factors in a four dimensional design where V =4 and x_1, x_2 forms the p factor group whereas x_3, x_4 forms the (v-p) factor group.

Following (3.3.2) , for the full third order model in four factors we have,

$$f^t(x) = [f_1^t(x), f_2^t(x), f_3^t(x), f_4^t(x)],$$

Where $f_1^t(x) = (1, x_1^2, x_2^2, x_3^2, x_4^2)$, $f_2^t(x) = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4)$,

$f_3^t(x) = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4)$ and

$f_4^t(x) = (g_1^t(x), ..., g_4^t(x))$ Where $g_1^t(x) = (x_1, x_1^3, x_1x_2^2, x_1x_3^2, x_1x_4^2)$,

$g_2^t(x) = (x_2, x_2^3, x_2x_1^2, x_2x_3^2, x_2x_4^2)$ $g_3^t(x) = (x_3, x_3^3, x_3x_1^2, x_3x_2^2, x_3x_4^2)$ and

$g_4^t(x) = (x_4, x_4^3, x_4x_1^2, x_4x_2^2, x_4x_3^2)$

Thus for a third order design ξ, from (3.3.3) the partitioned matrix of the moment matrix $M(\xi)$ is given by,

$$M(\xi) = \begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) & M_{13}(\xi) & M_{14}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) & M_{23}(\xi) & M_{24}(\xi) \\ M_{31}(\xi) & M_{32}(\xi) & M_{33}(\xi) & M_{34}(\xi) \\ M_{41}(\xi) & M_{42}(\xi) & M_{43}(\xi) & M_{44}(\xi) \end{bmatrix}$$

Where $M_{ij}(\xi) = \int_x f_i(x) f_j^t(x)\xi(dx)$ $(i, j = 1, ..., 4)$ corresponding to the partitioning of $f^t(x)$. Considering the symmetric designs matrix only, the inverse of $M(\xi)$ was obtained. For a symmetric design ξ, $M_{ij}(\xi)(i \neq j)$ are null matrices thus $M(\xi)$ is reduced to a block diagonal matrix of $M(\xi) = Diag\{M_{11}(\xi), M_{22}(\xi), M_{33}(\xi), M_{44}(\xi)\}$. Note that for a symmetric design ξ, $M_{11}(\xi), M_{22}(\xi), M_{33}(\xi), M_{44}(\xi)$ are diagonal matrices and further $M_{44}(\xi)$ in itself is a block diagonal matrix given by

$$M_{44}(\xi) = \text{Diag}\{M^*_1(\xi), \ldots, M^*_k(\xi)\},$$

where $M^*_i(\xi) = \int_x g_i(x) g^t_i(x) \xi(dx) \;\; (i = 1, \ldots, k).$

A block diagonal matrix in four factors becomes,

$$M(\xi) = \begin{bmatrix} M_{11}(\xi) & 0 & 0 & 0 \\ 0 & M_{22}(\xi) & 0 & 0 \\ 0 & 0 & M_{33}(\xi) & 0 \\ 0 & 0 & 0 & M_{44}(\xi) \end{bmatrix}$$

From (3.3.4) we have,

$$M_{11}(\xi) = f^t_1(x).f_1(x) = \begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \end{bmatrix} \begin{bmatrix} 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} = \begin{bmatrix} 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ & x_1^4 & x_1^2 x_2^2 & x_1^2 x_3^2 & x_1^2 x_4^2 \\ & & x_2^4 & x_2^2 x_3^2 & x_2^2 x_4^2 \\ & & & x_3^4 & x_3^2 x_4^2 \\ & (symm) & & & x_4^4 \end{bmatrix}.$$

$$M_{11}^{-1}(\xi) = \frac{1}{[6\lambda_4 - 4\lambda^2{}_2][2\lambda_4]} \begin{bmatrix} 8\lambda^2{}_4 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 \\ & 3\lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 \\ & & 3\lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 \\ & & & 3\lambda_4 - \lambda^2{}_2 & \lambda_4 - \lambda^2{}_2 \\ symm & & & & 3\lambda_4 - \lambda^2{}_{24} \end{bmatrix} =$$

$$M_{11}^{-1}(\xi) = \begin{bmatrix} 16.2933 & -3.8040 & -3.8040 & -3.8040 & -3.8040 \\ -3.8040 & 1.4689 & 0.7720 & 0.7720 & 0.7720 \\ -3.8040 & 0.7720 & 1.4689 & 0.7720 & 0.7720 \\ -3.8040 & 0.7720 & 0.7720 & 1.4689 & 0.7720 \\ -3.8040 & 0.7720 & 0.7720 & 0.7720 & 1.4689 \end{bmatrix}$$

$$M_{22}(\xi) = f^t_2(x).f_2(x) = \begin{bmatrix} x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_2x_3 \\ x_2x_4 \\ x_3x_4 \end{bmatrix} \begin{bmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_2x_3 & x_2x_4 & x_3x_4 \end{bmatrix} =$$

$$=\begin{bmatrix} \lambda_4 & 0 & 0 & 0 & 0 & 0 \\ & \lambda_4 & 0 & 0 & 0 & 0 \\ & & \lambda_4 & 0 & 0 & 0 \\ & & & \lambda_4 & 0 & 0 \\ & & & & \lambda_4 & 0 \\ symm & & & & & \lambda_4 \end{bmatrix}=$$

$$M_{22}(\xi)=\begin{bmatrix} 0.7175 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7175 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7175 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7175 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7175 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7175 \end{bmatrix}$$

$$M_{22}^{-1}(\xi)=\begin{bmatrix} \frac{1}{\lambda_4} & 0 & 0 & 0 & 0 & 0 \\ & \frac{1}{\lambda_4} & 0 & 0 & 0 & 0 \\ & & \frac{1}{\lambda_4} & 0 & 0 & 0 \\ & & & \frac{1}{\lambda_4} & 0 & 0 \\ & & & & \frac{1}{\lambda_4} & 0 \\ symm & & & & & \frac{1}{\lambda_4} \end{bmatrix}$$

$$M_{22}^{-1}(\xi)=\begin{bmatrix} 1.3938 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.3938 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.3938 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.3938 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3938 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.3938 \end{bmatrix}$$

$$M_{33}(\xi)=\begin{bmatrix} x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_3x_4 \\ x_2x_3x_4 \end{bmatrix}[x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4]=\begin{bmatrix} \lambda_6 & 0 & 0 & 0 \\ & \lambda_6 & 0 & 0 \\ & & \lambda_6 & 0 \\ (symm) & & & \lambda_6 \end{bmatrix}.$$

$$M_{33}(\xi)=\begin{bmatrix}0.4000 & 0 & 0 & 0\\ 0 & 0.4000 & 0 & 0\\ 0 & 0 & 0.4000 & 0\\ 0 & 0 & 0 & 0.4000\end{bmatrix}$$

$$M_{33}^{-1}(\xi)=\begin{bmatrix}\frac{1}{\lambda_6} & 0 & 0 & 0\\ & \frac{1}{\lambda_6} & 0 & 0\\ & & \frac{1}{\lambda_6} & 0\\ (symm) & & & \frac{1}{\lambda_6}\end{bmatrix}=\begin{bmatrix}2.5000 & 0 & 0 & 0\\ 0 & 2.5000 & 0 & 0\\ 0 & 0 & 2.5000 & 0\\ 0 & 0 & 0 & 2.5000\end{bmatrix}$$

$$M_{44}(\xi)= f_4^t(x).f_4(x)=M_{44(1)}(\xi)=M_{44(2)}(\xi)=M_{44(3)}(\xi)=M_{44(4)}(\xi)$$ from (3.3.4)

$$M_{44(1)}(\xi)=g_1^t(x)g_1(x)=\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\\ x_1x_3^2\\ x_1x_4^2\end{bmatrix}\begin{bmatrix}x_1 & x_1^3 & x_1x_2^2 & x_1x_3^2 & x_1x_4^2\end{bmatrix}$$

$$=\begin{bmatrix}x_1^2 & x_1^4 & x_1^2x_2^2 & x_1^2x_3^2 & x_1^2x_4^2\\ & x_1^6 & x_1^4x_2^2 & x_1^4x_3^2 & x_1^4x_4^2\\ & & x_1^2x_2^4 & x_1^2x_2^2x_3^2 & x_1^2x_2^2x_4^2\\ & & & x_1^2x_3^4 & x_1^2x_3^2x_4^2\\ symm & & & & x_1^2x_4^4\end{bmatrix}=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6 & 3\lambda_6 & 3\lambda_6\\ & & 3\lambda_6 & \lambda_6 & \lambda_6\\ & & & 3\lambda_6 & \lambda_6\\ symm & & & & 3\lambda_6\end{bmatrix}$$

$$M_{44(1)}(\xi)=\begin{bmatrix}1.0051 & 2.1524 & 0.7175 & 0.7175 & 0.7175\\ 2.1524 & 6.0000 & 1.2000 & 1.2000 & 1.2000\\ 0.7175 & 1.2000 & 1.2000 & 0.4000 & 0.4000\\ 0.7175 & 1.2000 & 0.4000 & 1.2000 & 0.4000\\ 0.7175 & 1.2000 & 0.4000 & 0.4000 & 1.2000\end{bmatrix}$$

$$M_{44}^{-1}(\xi)=\frac{1}{|K|}\begin{bmatrix}36\lambda^2{}_6 & -6\lambda_6\lambda_4 & -6\lambda_6\lambda_4 & -6\lambda_6\lambda_4 & -6\lambda_6\lambda_4\\ & 3\lambda_6\lambda_2-\lambda^2{}_4 & 3(\lambda^2{}_4-\lambda_6\lambda_2) & 3(\lambda^2{}_4-\lambda_6\lambda_2) & 3(\lambda^2{}_4-\lambda_6\lambda_2)\\ & & 15\lambda_6\lambda_2-9\lambda^2{}_4 & 3(\lambda^2{}_4-\lambda_6\lambda_2) & 3(\lambda^2{}_4-\lambda_6\lambda_2)\\ & & & 15\lambda_6\lambda_2-9\lambda^2{}_4 & 3(\lambda^2{}_4-\lambda_6\lambda_2)\\ symm & & & & 15\lambda_6\lambda_2-9\lambda^2{}_4\end{bmatrix}$$

Where K is the determinant of (3.3.1)

$$|K| = [6\lambda_6][8\lambda_6\lambda_2 - 6\lambda^2{}_4]$$

$$M^{-1}_{44(1)}(\xi) = \begin{bmatrix} 25.0824 & -5.6237 & -5.6239 & -5.6239 & -5.6239 \\ -5.6237 & 1.5213 & 1.1047 & 1.1047 & 1.1047 \\ -5.6239 & 1.1047 & 2.3547 & 1.1047 & 1.1047 \\ -5.6239 & 1.1047 & 1.1047 & 2.3547 & 1.1047 \\ -5.6239 & 1.1047 & 1.1047 & 1.1047 & 2.3547 \end{bmatrix}$$

The design space is divided into the p- factor and (v-p) factor space satisfying $d_1^2 = \sum_{i=1}^{p} x_i^2$ and $d_2^2 = \sum_{j=p+1}^{v} x_j^2$ with the corresponding variances $V([\hat{y}(x_i)])$ and $V([\hat{y}(x_j)])$ respectively, thus the design is called a Variance-Sum Group Divisible third order rotatable design

Determining the Variance $Var([\hat{y}(x_i)])$ for p dimensional space

For a symmetric design $M(\xi)$, from (3.3.3) it is seen that variances for 2-factor group is given as,

$$M_{11}(\xi) = f_1^t(x).f_1(x) = \begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \end{bmatrix} \begin{bmatrix} 1 & x_1^2 & x_2^2 \end{bmatrix} = \begin{bmatrix} 1 & x_1^2 & x_2^2 \\ & x_1^4 & x_1^2 x_2^2 \\ & (symm) & x_2^4 \end{bmatrix}.$$

$$M_{11}(\xi) = \begin{bmatrix} 1.0000 & 1.0051 & 1.0051 \\ 1.0051 & 2.1524 & 0.7175 \\ 1.0051 & 0.7175 & 2.1524 \end{bmatrix}$$ generated by a MATLAB software.

$$M_{11}^{-1}(\xi) = \begin{bmatrix} 3.3780 & -1.1830 & -1.1830 \\ -1.1830 & 0.9370 & 0.2401 \\ -1.1830 & 0.2401 & 0.9370 \end{bmatrix}$$

$$M_{22}(\xi) = f_2^t(x).f_2(x) = [x_1x_2][x_1x_2] = [x_1^2 x_2^2]$$

$$M_{22}(\xi)=\begin{bmatrix}0.7175 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{22}^{-1}(\xi)=\begin{bmatrix}1.3938 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{44(1)}(\xi)=g_1^t(x)g_1(x)=\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}\begin{bmatrix}x_1 & x_1^3 & x_1x_2^2\end{bmatrix}=\begin{bmatrix}x_1^2 & x_1^4 & x_1^2x_2^2\\ & x_1^6 & x_1^4x_2^2\\ symm & & x_1^2x_2^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6\end{bmatrix}$$

$$M_{44(1)}(\xi)=\begin{bmatrix}1.0051 & 2.1524 & 0.7175\\ 2.1524 & 6.0000 & 1.2000\\ 0.7175 & 1.2000 & 1.2000\end{bmatrix}$$

$$M_{44(1)}^{-1}(\xi)=\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}$$

$$M_{44(2)}(\xi)=g_2^t(x)g_2(x)=\begin{bmatrix}x_2\\ x_2^3\\ x_2x_1^2\end{bmatrix}\begin{bmatrix}x_2 & x_2^3 & x_2x_1^2\end{bmatrix}=\begin{bmatrix}x_2^2 & x_2^4 & x_2^2x_1^2\\ & x_2^6 & x_2^4x_1^2\\ symm & & x_2^2x_1^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6\end{bmatrix}$$

$$M_{44(2)}(\xi)=\begin{bmatrix}1.0051 & 2.1524 & 0.7175\\ 2.1524 & 6.0000 & 1.2000\\ 0.7175 & 1.2000 & 1.2000\end{bmatrix}$$

$$M_{44(2)}^{-1}(\xi)=\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}$$

From (3.3.7) we had,

$$V_{11}(\xi) = \begin{pmatrix}1 & x_1^2 & x_2^2\end{pmatrix}\begin{bmatrix}3.3780 & -1.1830 & -1.1830\\ -1.1830 & 0.9370 & 0.2401\\ -1.1830 & 0.2401 & 0.9370\end{bmatrix}\begin{bmatrix}1\\ x_1^2\\ x_2^2\end{bmatrix}$$

Let $\left(M_{11}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}a & b & b\\ b & c & d\\ b & d & c\end{bmatrix}$ such that

$$V_{11}(\xi) = \begin{pmatrix}1 & x_1^2 & x_2^2\end{pmatrix}\begin{bmatrix}a & b & b\\ b & c & d\\ b & d & c\end{bmatrix}\begin{bmatrix}1\\ x_1^2\\ x_2^2\end{bmatrix} =$$

$$\begin{bmatrix}a + bx_1^2 + bx_2^2 & b + cx_1^2 + dx_2^2 & b + dx_1^2 + cx_2^2\end{bmatrix}\begin{bmatrix}1\\ x_1^2\\ x_1^2\end{bmatrix}$$

$$V_{11}(\xi) = \left[3.3780 - 2.3660x_1^2 - 2.3660x_2^2 + 0.9370x_1^4 + 0.9370x_2^4 + 0.4802x_1^2x_2^2\right]$$

$$V_{22}(\xi) = f_2' \, [M_{22}(\xi)]^{-1} f_2$$

$$V_{22}(\xi) = [x_1\, x_2]\begin{bmatrix}1.3938 & 0\\ 0 & 0\end{bmatrix}[x_1\, x_2]$$

Let $\left(M_{22}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}n & 0\\ 0 & 0\end{bmatrix}$ such that

$$V_{22}(\xi) = [x_1\, x_2]\begin{bmatrix}1.3938 & 0\\ 0 & 0\end{bmatrix}[x_1\, x_2]$$

$$V_{22}(\xi) = \left[1.3938x_1^2x_2^2\right]$$

$$V_{44(1)}(\xi) = \mathrm{g}_1' \, [M_{44(1)}(\xi)]^{-1} g_1 =$$

$$V_{44(1)}\left(\xi\right)=\begin{bmatrix}x_1 & x_1^3 & x_1x_2^2\end{bmatrix}\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}$$

Let $\left(M_{44(1)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}e & f & g\\ f & h & k\\ g & k & l\end{bmatrix}$ such that

$$V_{44(1)}\left(\xi\right)=\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}\begin{bmatrix}e & f & g\\ f & h & k\\ g & k & l\end{bmatrix}\begin{bmatrix}x_1 & x_1^3 & x_1x_2^2\end{bmatrix}$$

$$V_{44(1)}\left(\xi\right)=\begin{bmatrix}ex_1+fx_1^3+gx_1x_2^2 & fx_1+hx_1^3+kx_1x_2^2 & gx_1+kx_1^3+lx_1x_2^2\end{bmatrix}\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}$$

$$V_{44(1)}\left(\xi\right)=\left[6.7973x_1^2-4.0640x_1^4+0.8158x_1^6-4.0642x_1^2x_2^2+0.7984x_1^4x_2^2+1.6492x_1^2x_2^4\right]$$

$$V_{44(2)}(\xi)=\ g_2^{'}\ [M_{44(2)}(\xi)]^{-1}g_2=$$

$$V_{44(2)}\left(\xi\right)=\begin{bmatrix}x_2 & x_2^3 & x_2x_1^2\end{bmatrix}\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}\begin{bmatrix}x_2\\ x_2^3\\ x_2x_1^2\end{bmatrix}$$

Let $\left(M_{44(2)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}e & f & g\\ f & h & k\\ g & k & l\end{bmatrix}$ such that

$$V_{44(2)}\left(\xi\right)=\begin{bmatrix}x_2 & x_2^3 & x_2x_1^2\end{bmatrix}\begin{bmatrix}e & f & g\\ f & h & k\\ g & k & l\end{bmatrix}\begin{bmatrix}x_2\\ x_2^3\\ x_2x_1^2\end{bmatrix}$$

$$V_{44(2)}\left(\xi\right)=\begin{bmatrix}ex_2+fx_2^3+gx_2x_1^2 & fx_2+hx_2^3+kx_2x_1^2 & gx_2+kx_2^3+lx_2x_1^2\end{bmatrix}\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}$$

$$V_{44(2)}(\xi)=\left[6.7973x_2^2-4.0640x_2^4+0.8158x_2^6-4.0642x_2^2x_1^2+0.7984x_2^4x_1^2+1.6492x_2^2x_1^4\right]$$

From (3.3.8) we have the variance $V\left([\hat{y}(x_i)]\right)=\sum_1^2 V(x_i)$

$$V\left([\hat{y}(x_i)]\right)=\sum_1^2 V(x_i)=$$

$$3.3780-2.3660x_1^2-2.3660x_2^2+0.9370x_1^4+0.9370x_2^4+0.4802x_1^2x_2^2+1.3938x_1^2x_2^2+$$

$$\left[6.7973x_1^2-4.0640x_1^4+0.8158x_1^6-4.0642x_1^2x_2^2+0.7984x_1^4x_2^2+1.6492x_1^2x_2^4\right]$$
$$+\left[6.7973x_2^2-4.0640x_2^4+0.8158x_2^6-4.0642x_2^2x_1^2+0.7984x_2^4x_1^2+1.6492x_2^2x_1^4\right]$$

Summing the above variances we get expression which is a function of

$$\sum x_1^2,\sum x_2^2,\sum x_1^2x_2^2,\sum x_2^4x_1^2,\sum x_1^4x_2^2$$

In order to achieve the variance in GDTORD, the variance should be a function of $\sum x_1^2,\sum x_2^2$ only. Therefore we need to cancel the interactions $\sum x_1^2x_2^2,\sum x_2^4x_1^2,\sum x_1^4x_2^2$

We get all the above interactions be equated to zero so as to have functions of $\sum x_1^2,\sum x_2^2$ only. Then from (3.3.11) we had,

$$V\left([\hat{y}(x_i)]\right)=\left(3.3780x_0^2+4.4313x_1^2-3.1270x_1^4+.8158x_1^6+4.4313x_2^2-3.1270x_2^4+.8158x_2^6\right)$$

Let $d_1^2=\sum_{i=1}^2 x_i^2$ such that

$V\left([\hat{y}(x_i)]\right)=$ f (d_1^2) only.

Determining the Variance $Var\left([\hat{y}(x_j)]\right)$ for (v-p) dimensional space

For a symmetric design $M(\xi)$, it is seen that variances for (v-p)-factor group as from $(3.3.3)$ as generated through the help of a MATLAB software were,

$$M_{11}(\xi) = f_1^t(x).f_1(x) = \begin{bmatrix} 1 \\ x_3^2 \\ x_4^2 \end{bmatrix} \begin{bmatrix} 1 & x_3^2 & x_4^2 \end{bmatrix} = \begin{bmatrix} 1 & x_3^2 & x_4^2 \\ & x_3^4 & x_3^2 x_4^2 \\ & (symm) & x_4^4 \end{bmatrix}.$$

$$M_{11}(\xi) = \begin{bmatrix} 1.0000 & 1.0051 & 1.0051 \\ 1.0051 & 2.1524 & 0.7175 \\ 1.0051 & 0.7175 & 2.1524 \end{bmatrix}$$

$$M_{11}^{-1}(\xi) = \begin{bmatrix} 3.3780 & -1.1830 & -1.1830 \\ -1.1830 & 0.9370 & 0.2401 \\ -1.1830 & 0.2401 & 0.9370 \end{bmatrix}$$

$$M_{22}(\xi) = f_2^t(x).f_2(x) = \begin{bmatrix} x_3 x_4 \end{bmatrix} \begin{bmatrix} x_3 x_4 \end{bmatrix} = \begin{bmatrix} x_3^2 x_4^2 \end{bmatrix}$$

$$M_{22}(\xi) = \begin{bmatrix} 0.7175 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_{22}^{-1}(\xi) = \begin{bmatrix} 1.3938 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M_{44(3)}(\xi) = g_1^t(x) g_1(x) = \begin{bmatrix} x_3 \\ x_3^3 \\ x_3 x_4^2 \end{bmatrix} \begin{bmatrix} x_3 & x_3^3 & x_3 x_4^2 \end{bmatrix} = \begin{bmatrix} x_3^2 & x_3^4 & x_3^2 x_4^2 \\ & x_3^6 & x_3^4 x_4^2 \\ symm & & x_3^2 x_4^4 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_2 & 3\lambda_4 & \lambda_4 \\ & 15\lambda_6 & 3\lambda_6 \\ symm & & 3\lambda_6 \end{bmatrix}$$

$$M_{44(3)}(\xi)=\begin{bmatrix}1.0051 & 2.1524 & 0.7175\\ 2.1524 & 6.0000 & 1.2000\\ 0.7175 & 1.2000 & 1.2000\end{bmatrix}$$

$$M^{-1}_{44(3)}(\xi)=\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}$$

$$M_{44(4)}(\xi)=g_1^t(x)g_1(x)=\begin{bmatrix}x_4\\ x_4^3\\ x_4x_3^2\end{bmatrix}\begin{bmatrix}x_4 & x_4^3 & x_4x_3^2\end{bmatrix}=\begin{bmatrix}x_4^2 & x_4^4 & x_4^2x_3^2\\ & x_4^6 & x_4^4x_3^2\\ symm & & x_4^2x_3^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6\end{bmatrix}$$

$$M_{44(4)}(\xi)=\begin{bmatrix}1.0051 & 2.1524 & 0.7175\\ 2.1524 & 6.0000 & 1.2000\\ 0.7175 & 1.2000 & 1.2000\end{bmatrix}$$

$$M^{-1}_{44(4)}(\xi)=\begin{bmatrix}6.7973 & -2.0320 & -2.0321\\ -2.0320 & 0.8158 & 0.3992\\ -2.0321 & 0.3992 & 1.6492\end{bmatrix}$$

From (3.3.6) we have,

$$V_{11}(\xi)=\begin{pmatrix}1 & x_3^2 & x_4^2\end{pmatrix}\begin{bmatrix}3.3780 & -1.1830 & -1.1830\\ -1.1830 & 0.9370 & 0.2401\\ -1.1830 & 0.2401 & 0.9370\end{bmatrix}\begin{bmatrix}1\\ x_3^2\\ x_4^2\end{bmatrix}$$

Let $\left(M_{11}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}a & b & b\\ b & c & d\\ b & d & c\end{bmatrix}$ such that

$$V_{11}(\xi)=\begin{pmatrix}1 & x_3^2 & x_4^2\end{pmatrix}\begin{bmatrix}a & b & b\\ b & c & d\\ b & d & c\end{bmatrix}\begin{bmatrix}1\\ x_3^2\\ x_4^2\end{bmatrix}=$$

$$\begin{bmatrix} a+bx_3^2+bx_4^2 & b+cx_3^2+dx_4^2 & b+dx_3^2+cx_4^2 \end{bmatrix}\begin{bmatrix} 1 \\ x_3^2 \\ x_4^2 \end{bmatrix}$$

$$V_{11}(\xi)=\left[3.3780-2.3660x_3^2-2.3660x_4^2+.9370x_3^4+.9370x_4^4+.4802x_3^2x_4^2\right]$$

From (3.3.9) we had,

$$V_{22}(\xi)=[x_3\ x_4]\begin{bmatrix} 1.3938 & 0 \\ 0 & 0 \end{bmatrix}[x_3\ x_4]$$

Let $\left(M_{22}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}$ such that

$$V_{22}(\xi)=[x_3\ x_4]\begin{bmatrix} 1.3938 & 0 \\ 0 & 0 \end{bmatrix}[x_3\ x_4]$$

$$V_{22}(\xi)=\left[1.3938x_3^2x_4^2\right]$$

$$V_{44(3)}(\xi)=g_3^{'}\ [M_{44(3)}(\xi)]^{-1}g_3=$$

$$V_{44(3)}(\xi)=\begin{bmatrix} x_3 & x_3^3 & x_3x_4^2 \end{bmatrix}\begin{bmatrix} 6.7973 & -2.0320 & -2.0321 \\ -2.0320 & 0.8158 & 0.3992 \\ -2.0321 & 0.3992 & 1.6492 \end{bmatrix}\begin{bmatrix} x_3 \\ x_3^3 \\ x_3x_4^2 \end{bmatrix}$$

Let $\left(M_{44(3)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}$ such that

$$V_{44(3)}(\xi)=\begin{bmatrix} x_3 \\ x_3^3 \\ x_3x_4^2 \end{bmatrix}\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}\begin{bmatrix} x_3 & x_3^3 & x_3x_4^2 \end{bmatrix}$$

$$V_{44(3)}\left(\xi\right)=\left[ex_3+fx_3^3+gx_3x_4^2 \quad fx_3+hx_3^3+kx_3x_4^2 \quad gx_3+kx_3^3+lx_3x_4^2\right]\begin{bmatrix} x_3 \\ x_3^3 \\ x_3x_4^2 \end{bmatrix}$$

$$V_{44(3)}\left(\xi\right)=\left[6.7973x_3^2-4.0640x_3^4+.8158x_3^6-4.0642x_3^2x_4^2+.7984x_3^4x_4^2+1.6492x_3^2x_4^4\right]$$

$$V_{44(4)}(\xi)=g_4^{'}\ [M_{44(4)}(\xi)]^{-1}g_4=$$

$$V_{44(4)}\left(\xi\right)=\left[x_4 \quad x_4^3 \quad x_4x_3^2\right]\begin{bmatrix} 6.7973 & -2.0320 & -2.0321 \\ -2.0320 & 0.8158 & 0.3992 \\ -2.0321 & 0.3992 & 1.6492 \end{bmatrix}\begin{bmatrix} x_4 \\ x_4^3 \\ x_4x_3^2 \end{bmatrix}$$

Let $\left(M_{44(4)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}$ such that

$$V_{44(4)}\left(\xi\right)=\begin{bmatrix} x_4 \\ x_4^3 \\ x_4x_3^2 \end{bmatrix}\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}\left[x_4 \quad x_4^3 \quad x_4x_3^2\right]$$

$$V_{44(4)}\left(\xi\right)=\left[ex_4+fx_4^3+gx_4x_3^2 \quad fx_4+hx_4^3+kx_4x_3^2 \quad gx_4+kx_4^3+lx_4x_3^2\right]\begin{bmatrix} x_4 \\ x_4^3 \\ x_4x_3^2 \end{bmatrix}$$

$$V_{44(4)}\left(\xi\right)=\left[6.7973x_4^2-4.0640x_4^4+.8158x_4^6-4.0642x_4^2x_3^2+.7984x_4^4x_3^2+1.6492x_4^2x_3^4\right]$$

From (3.3.8) the variance $V\left([\hat{y}(x_j)]\right)=\sum_3^4 V(x_j)$

$$V\left([\hat{y}(x_j)]\right)=\sum_3^4 V(x_j)=a+2bx_3^2+2bx_4^2+cx_3^4+cx_4^4+2dx_3^2x_4^2+nx_3^2x_4^2+$$

$$ex_3^2+2fx_3^4+hx_3^6+2gx_3^2x_4^2+2kx_3^4x_4^2+lx_3^2x_4^4+ex_4^2+2fx_4^4+hx_4^6+2gx_4^2x_3^2+2kx_4^4x_3^2+lx_4^2x_3^4$$

Summing the above variances we get expression which is a function of

$\sum x_3^2, \sum x_4^2, \sum x_3^2 x_4^2, \sum x_4^4 x_3^2, \sum x_4^4 x_3^2$

In order to achieve the variance in GDTORD, the variance should be a function of $\sum x_3^2, \sum x_4^2$ only. Therefore we need to cancel the interactions $\sum x_3^2 x_4^2, \sum x_4^4 x_3^2, \sum x_4^4 x_3^2$

We get all the above interactions be equated to zero so as to have functions of $\sum x_3^2, \sum x_4^2$ only. Then from (3.3.12) we had,

$$V\left([\hat{y}(x_j)]\right) = \left(3.3780x_0^2 + 4.4313x_3^2 - 3.1270x_3^4 + .8158x_3^6 + 4.4313x_4^2 - 3.1270x_4^4 + .8158x_4^6\right)$$

Let $d_2^2 = \sum_{j=3}^{4} x_j^2$ such that

$V\left([\hat{y}(x_j)]\right) =$ f (d_2^2) only

With variances for the two groups being

$V(\hat{y}(x_i)) = \sigma^2 f'(x_i)(X'X)^{-1} f(x_i)$

$$V\left([\hat{y}(x_i)]\right) = \left(3.3780x_0^2 + 4.4313x_1^2 - 3.1270x_1^4 + .8158x_1^6 + 4.4313x_2^2 - 3.1270x_2^4 + .8158x_2^6\right)$$

$V\left([\hat{y}(x_i)]\right) =$ f (d_1^2)

$V(\hat{y}(x_j)) = \sigma^2 f'(x_j)(X'X^1)^{-1} f(x_j)$

$$V\left([\hat{y}(x_j)]\right) = \left(3.3780x_0^2 + 4.4313x_3^2 - 3.1270x_3^4 + .8158x_3^6 + 4.4313x_4^2 - 3.1270x_4^4 + .8158x_4^6\right)$$

$V\left([\hat{y}(x_j)]\right) =$ f (d_2^2)

At the point $x \in \chi$ the predicted response is

$V(\hat{y}(x)) = f'(x)\hat{\beta}$

The variance-sum is as shown

$$V([\hat{y}(x)]) = \left(3.3780x_0^2 + 4.4313x_1^2 - 3.1270x_1^4 + .8158x_1^6 + 4.4313x_2^2 - 3.1270x_2^4 + .8158x_2^6\right) + \left(3.3780x_0^2 + 4.4313x_3^2 - 3.1270x_3^4 + .8158x_3^6 + 4.4313x_4^2 - 3.1270x_4^4 + .8158x_4^6\right)$$

$$\sum_{i=1}^{v} V(\hat{y}(x)) = f(d_1^2, d_2^2),$$

Thus the variance Sum is the function of distances d_1^2 *and* d_2^2only.

d_1^2 and d_2^2 is the distances of the projections of the points in p dimensional and $(v-p)$ dimensional spaces respectively from a suitable origin. The variance$V(\hat{y}(x_i)$ is a function of distance d_1^2 and variance $V(\hat{y}(x_j)$ is a function of distance d_2^2 from the design origin. Thus the considered response surface is a Variance - Sum Group Divisible Third Order Rotatable Designs in four dimensions.

4.4.2 Variance Sum Group Divisible Third order rotatable designs in five dimensions

Moment matrix for five factors

From the 134 design points of a five dimensional GDTORD constructed through BIB designs we got the moment matrix of a five dimensional GDTORD to be

Moment matrix $M = \dfrac{1}{134}x'x$

Where $N = 134$ design points

Let x_1, x_2 be the p factor group x_3, x_4 and x_5 be the $(v-p)$ factor group, from (3.3.1) we have for the full third order model in five factors as,

$$f^t(x) = \int f_1^t(x), f_2^t(x), f_3^t(x), f_4^t(x), f_5^t(x)$$

Where

$$f_1^t(x) = \left(1, x_1^2, x_2^2, x_3^2, x_4^2, x_5^2\right)$$

$$f_2^t(x) = \left(x_1x_2 \quad x_1x_3 \quad x_1x_4 \quad x_1x_5 \quad x_2x_3 \quad x_2x_4 \quad x_2x_5 \quad x_3x_4 \quad x_3x_5 \quad x_4x_5\right)$$

$$f_3^t(x) = \left(x_1x_2x_3 \quad x_1x_2x_4 \quad x_1x_2x_5 \quad x_1x_3x_4 \quad x_1x_3x_5 \quad x_1x_4x_5 \quad x_2x_3x_4 \quad x_2x_3x_5 \quad x_2x_4x_5 \quad x_3x_4x_5\right)$$

$$f_3^t(x) = \left(x_1x_2x_3 \quad x_1x_2x_4 \quad x_1x_2x_5 \quad x_1x_3x_4 \quad x_1x_3x_5 \quad x_1x_4x_5 \quad x_2x_3x_4 \quad x_2x_3x_5 \quad x_2x_4x_5 \quad x_3x_4x_5\right)$$

$$f_4^t(x) = \left(x_1x_2x_3x_4 \quad x_1x_2x_4x_5 \quad x_1x_3x_4x_5 \quad x_2x_3x_4x_5\right)$$

$$f_5^t(x) = \left(g_1^t(x), \ldots, g_5^t(x)\right)$$

$$g_1^t(x) = \left(x_1 \quad x_1^3 \quad x_1x_2^2 \quad x_1x_3^2 \quad x_1x_4^2 \quad x_1x_5^2\right)$$

$$g_2^t(x) = \left(x_2 \quad x_2^3 \quad x_2x_1^2 \quad x_2x_3^2 \quad x_2x_4^2 \quad x_2x_5^2\right)$$

$$g_3^t(x) = \left(x_3 \quad x_3^3 \quad x_3x_1^2 \quad x_3x_2^2 \quad x_3x_4^2 \quad x_3x_5^2\right)$$

$$g_4^t(x) = \left(x_4 \quad x_4^3 \quad x_4x_1^2 \quad x_4x_2^2 \quad x_4x_3^2 \quad x_4x_5^2\right)$$

$$g_5^t(x) = \left(x_5 \quad x_5^3 \quad x_5x_1^2 \quad x_5x_2^2 \quad x_5x_3^2 \quad x_5x_4^2\right)$$

$$g_1^t(x) = g_2^t(x) = g_3^t(x) = g_4^t(x) = g_5^t(x)$$

$M(\xi) = \dfrac{1}{N}\left(x'x\right)$ is a moment matrix configuration of N points in a v dimensional factor space.

For a design to be rotatable $N^{-1}\left(x'x\right)$ must be satisfied

Where $N =$ is the total number of design points in a five dimensional factor space.

Then from (3.3.2) we have the moment matrix as,

$$M(\xi)=\begin{bmatrix} M_{11}(\xi) & M_{12}(\xi) & M_{13}(\xi) & M_{14}(\xi) & M_{15}(\xi) \\ M_{21}(\xi) & M_{22}(\xi) & M_{23}(\xi) & M_{24}(\xi) & M_{25}(\xi) \\ M_{31}(\xi) & M_{32}(\xi) & M_{33}(\xi) & M_{34}(\xi) & M_{35}(\xi) \\ M_{41}(\xi) & M_{42}(\xi) & M_{43}(\xi) & M_{44}(\xi) & M_{45}(\xi) \\ M_{51}(\xi) & M_{52}(\xi) & M_{53}(\xi) & M_{54}(\xi) & M_{55}(\xi) \end{bmatrix}$$

For a symmetric design $M(\xi)$ is reduced to a diagonal matrix only.

$$M(\xi)=\begin{bmatrix} M_{11}(\xi) & 0 & 0 & 0 & 0 \\ 0 & M_{22}(\xi) & 0 & 0 & 0 \\ 0 & 0 & M_{33}(\xi) & 0 & 0 \\ 0 & 0 & 0 & M_{44}(\xi) & 0 \\ 0 & 0 & 0 & 0 & M_{55}(\xi) \end{bmatrix}$$

From (3.3.6) we have,

$$M_{11}(\xi)=f_1^t(x).f_1(x)=\begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \\ x_3^2 \\ x_4^2 \\ x_5^2 \end{bmatrix}\begin{bmatrix} 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \end{bmatrix}=\begin{bmatrix} 1 & x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 \\ & x_1^4 & x_1^2x_2^2 & x_1^2x_3^2 & x_1^2x_4^2 & x_1^2x_5^2 \\ & & x_2^4 & x_2^2x_3^2 & x_2^2x_4^2 & x_2^2x_5^2 \\ & & & x_3^4 & x_3^2x_4^2 & x_3^2x_5^2 \\ & (symm) & & & x_4^4 & x_4^2x_5^2 \\ & & & & & x_5^4 \end{bmatrix}.$$

$$=\begin{bmatrix} 1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\ & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ & & 3\lambda_4 & \lambda_4 & \lambda_4 & \lambda_4 \\ & & & 3\lambda_4 & \lambda_4 & \lambda_4 \\ & (symm) & & & 3\lambda_4 & \lambda_4 \\ & & & & & 3\lambda_4 \end{bmatrix}$$

$$M_{11}(\xi)=\begin{bmatrix} 1.0000 & 1.0199 & 1.0199 & 1.0199 & 1.0199 & 1.0199 \\ 1.0199 & 2.3355 & 0.7785 & 0.7785 & 0.7785 & 0.7785 \\ 1.0199 & 0.7785 & 2.3355 & 0.7785 & 0.7785 & 0.7785 \\ 1.0199 & 0.7785 & 0.7785 & 2.3355 & 0.7785 & 0.7785 \\ 1.0199 & 0.7785 & 0.7785 & 0.7785 & 2.3355 & 0.7785 \\ 1.0199 & 0.7785 & 0.7785 & 0.7785 & 0.7785 & 2.3355 \end{bmatrix}$$

$$M_{11}^{-1}(\xi)=\frac{1}{[6\lambda_4-4\lambda^2{}_2][2\lambda_4]}\begin{bmatrix} 8\lambda^2{}_4 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 & -2\lambda_4\lambda_2 \\ & 3\lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 \\ & & 3\lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 \\ & & & 3\lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 \\ symm & & & & 3\lambda_4-\lambda^2{}_2 & \lambda_4-\lambda^2{}_2 \\ & & & & & 3\lambda_4-\lambda^2{}_2 \end{bmatrix}=$$

$$M_{11}^{-1}(\xi)=\begin{bmatrix} 21.9600 & -4.1101 & -4.1101 & -4.1101 & -4.1101 & -4.1101 \\ -4.1101 & 1.3198 & 0.6775 & 0.6775 & 0.6775 & 0.6775 \\ -4.1101 & 0.6775 & 1.3198 & 0.6775 & 0.6775 & 0.6775 \\ -4.1101 & 0.6775 & 0.6775 & 1.3198 & 0.6775 & 0.6775 \\ -4.1101 & 0.6775 & 0.6775 & 0.6775 & 1.3198 & 0.6775 \\ -4.1101 & 0.6775 & 0.6775 & 0.6775 & 0.6775 & 1.3198 \end{bmatrix}$$

$$M_{22}(\xi)=f_2^t(x).f_2(x)=\begin{bmatrix} x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_1x_5 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{bmatrix}\begin{bmatrix} x_1x_2 & x_1x_3 & x_1x_4 & x_1x_5 & x_2x_3 & x_2x_4 & x_2x_5 & x_3x_4 & x_3x_5 & x_4x_5 \end{bmatrix}=$$

$$=\begin{bmatrix} \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & \lambda_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \lambda_4 & 0 & 0 & 0 & 0 & 0 \\ symm & & & & & \lambda_4 & 0 & 0 & 0 & 0 \\ & & & & & & \lambda_4 & 0 & 0 & 0 \\ & & & & & & & \lambda_4 & 0 & 0 \\ & & & & & & & & \lambda_4 & 0 \\ & & & & & & & & & \lambda_4 \end{bmatrix}=$$

$$M_{22}(\xi)=\begin{bmatrix} 0.7785 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0.7785 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0.7785 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 0.7785 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0.7785 & 0 & 0 & 0 & 0 & 0 \\ symm & & & & & 0.7785 & 0 & 0 & 0 & 0 \\ & & & & & & 0.7785 & 0 & 0 & 0 \\ & & & & & & & 0.7785 & 0 & 0 \\ & & & & & & & & 0.7785 & 0 \\ & & & & & & & & & 0.7785 \end{bmatrix}=$$

$$M^{-1}_{22}(\xi)=\begin{bmatrix} 1.2845 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 1.2845 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 1.2845 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & 1.2845 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 1.2845 & 0 & 0 & 0 & 0 & 0 \\ symm & & & & & 1.2845 & 0 & 0 & 0 & 0 \\ & & & & & & 1.2845 & 0 & 0 & 0 \\ & & & & & & & 1.2845 & 0 & 0 \\ & & & & & & & & 1.2845 & 0 \\ & & & & & & & & & 1.2845 \end{bmatrix}=$$

$$M_{33}(\xi)=\begin{bmatrix} x_1x_2x_3 \\ x_1x_2x_4 \\ x_1x_2x_5 \\ x_1x_3x_4 \\ x_1x_3x_5 \\ x_1x_4x_5 \\ x_2x_3x_4 \\ x_2x_3x_5 \\ x_2x_4x_5 \\ x_3x_4x_5 \end{bmatrix}[x_1x_2x_3 \quad x_1x_2x_4 \quad x_1x_2x_5 \quad x_1x_3x_4 \quad x_1x_3x_5 \quad x_1x_4x_5 \quad x_2x_3x_4 \quad x_2x_3x_5 \quad x_2x_4x_5 \quad x_3x_4x_5]$$

$$=\begin{bmatrix} \lambda_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & \lambda_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \lambda_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (symm) & & & \lambda_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \lambda_6 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \lambda_6 & 0 & 0 & 0 & 0 \\ & & & & & & \lambda_6 & 0 & 0 & 0 \\ & & & & & & & \lambda_6 & 0 & 0 \\ & & & & & & & & \lambda_6 & 0 \\ & & & & & & & & & \lambda_6 \end{bmatrix}$$

$$M_{33}(\xi)=\begin{bmatrix} 0.4776 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0.4776 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 0.4776 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (symm) & & & 0.4776 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 0.4776 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0.4776 & 0 & 0 & 0 & 0 \\ & & & & & & 0.4776 & 0 & 0 & 0 \\ & & & & & & & 0.4776 & 0 & 0 \\ & & & & & & & & 0.4776 & 0 \\ & & & & & & & & & 0.4776 \end{bmatrix}$$

$$M_{33}^{-1}(\xi)=\begin{bmatrix} 2.0938 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & 2.0938 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & 2.0938 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (symm) & & & 2.0938 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & 2.0938 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 2.0938 & 0 & 0 & 0 & 0 \\ & & & & & & 2.0938 & 0 & 0 & 0 \\ & & & & & & & 2.0938 & 0 & 0 \\ & & & & & & & & 2.0938 & 0 \\ & & & & & & & & & 2.0938 \end{bmatrix}$$

$$M_{44}(\xi)=\begin{bmatrix} 0.4776 & 0 & 0 & 0 & 0 \\ 0 & 0.4776 & 0 & 0 & 0 \\ 0 & 0 & 0.4776 & 0 & 0 \\ 0 & 0 & 0 & 0.4776 & 0 \\ 0 & 0 & 0 & 0 & 0.4776 \end{bmatrix}$$

$$M_{44}^{-1}(\xi)=\begin{bmatrix} 2.0938 & 0 & 0 & 0 & 0 \\ 0 & 2.0938 & 0 & 0 & 0 \\ 0 & 0 & 2.0938 & 0 & 0 \\ 0 & 0 & 0 & 2.0938 & 0 \\ 0 & 0 & 0 & 0 & 2.0938 \end{bmatrix}$$

$$M_{51}(\xi)=\begin{bmatrix} 1.0199 & 2.3355 & 0.7785 & 0.7785 & 0.7785 & 0.7785 \\ 2.3355 & 7.1643 & 1.4328 & 1.4328 & 1.4328 & 1.4328 \\ 0.7785 & 1.4328 & 1.4328 & 0.4776 & 0.4776 & 0.4776 \\ 0.7785 & 1.4328 & 0.4776 & 1.4328 & 0.4776 & 0.4776 \\ 0.7785 & 1.4328 & 0.4776 & 0.4776 & 1.4328 & 0.4776 \\ 0.7785 & 1.4328 & 0.4776 & 0.4776 & 0.4776 & 1.4328 \end{bmatrix}$$

$$M_{51}^{-1}(\xi)=\begin{bmatrix} 30.2992 & -5.4873 & -5.4874 & -5.4874 & -5.4874 & -5.4874 \\ -5.4873 & 1.2264 & 0.8775 & 0.8775 & 0.8775 & 0.8775 \\ -5.4874 & 0.8775 & 1.9244 & 0.8775 & 0.8775 & 0.8775 \\ -5.4874 & 0.8775 & 0.4776 & 1.9244 & 0.8775 & 0.8775 \\ -5.4874 & 0.8775 & 0.8775 & 0.8775 & 1.9244 & 0.8775 \\ -5.4874 & 0.8775 & 0.8775 & 0.8775 & 0.8775 & 1.9244 \end{bmatrix}$$

From (3.3.4) we have $M_{51}(\xi)=M_{52}(\xi)=M_{53}(\xi)=M_{54}(\xi)=M_{55}(\xi)$

Determining the Variance $Var\left([\hat{y}(x_i)]\right)$ for p dimensional space

For a symmetric design $m(\xi)$, from (3.3.3) it is seen that variances for 2-factor group is given as,

$$M_{11}(\xi)= f_1^t(x).f_1(x)=\begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \end{bmatrix}\begin{bmatrix} 1 & x_1^2 & x_2^2 \end{bmatrix}=\begin{bmatrix} 1 & x_1^2 & x_2^2 \\ & x_1^4 & x_1^2x_2^2 \\ & (symm) & x_2^4 \end{bmatrix}.$$

$$M_{11}(\xi)=\begin{bmatrix}1.0000 & 1.0199 & 1.0199\\ 1.0199 & 2.3355 & 0.7785\\ 1.0199 & 0.7785 & 2.3355\end{bmatrix}$$

$$M_{11}^{-1}(\xi)=\begin{bmatrix}3.0132 & -0.9869 & -0.9869\\ -0.9869 & 0.8049 & 0.1627\\ -0.9869 & 0.1627 & 0.8049\end{bmatrix}$$

$$M_{22}(\xi)= f_2^t(x).f_2(x)=\left[x_1x_2\right]\left[x_1x_2\right]=\left[x_1^2x_2^2\right]$$

$$M_{22}(\xi)=\begin{bmatrix}0.7785 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{22}^{-1}(\xi)=\begin{bmatrix}1.2845 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{55(1)}(\xi)=g_1^t(x)g_1(x)=\begin{bmatrix}x_1\\ x_1^3\\ x_1x_2^2\end{bmatrix}\begin{bmatrix}x_1 & x_1^3 & x_1x_2^2\end{bmatrix}=\begin{bmatrix}x_1^2 & x_1^4 & x_1^2x_2^2\\ & x_1^6 & x_1^4x_2^2\\ symm & & x_1^2x_2^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6\end{bmatrix}$$

$$M_{55(1)}(\xi)=\begin{bmatrix}1.0199 & 2.3355 & 0.7785\\ 2.3355 & 7.1643 & 1.4328\\ 0.7785 & 1.4328 & 1.4328\end{bmatrix}$$

$$M_{55(1)}^{-1}(\xi)=\begin{bmatrix}5.7472 & -1.5613 & -1.5613\\ -1.5613 & 0.5986 & 0.2497\\ -1.5613 & 0.2497 & 1.2965\end{bmatrix}$$

$$M_{55(2)}(\xi)=g_2^t(x)g_2(x)=\begin{bmatrix}x_2\\ x_2^3\\ x_2x_1^2\end{bmatrix}\begin{bmatrix}x_2 & x_2^3 & x_2x_1^2\end{bmatrix}=\begin{bmatrix}x_2^2 & x_2^4 & x_2^2x_1^2\\ & x_2^6 & x_2^4x_1^2\\ symm & & x_2^2x_1^4\end{bmatrix}$$

$$= \begin{bmatrix} \lambda_2 & 3\lambda_4 & \lambda_4 \\ & 15\lambda_6 & 3\lambda_6 \\ symm & & 3\lambda_6 \end{bmatrix}$$

$$M_{55(2)}(\xi) = \begin{bmatrix} 1.0199 & 2.3355 & 0.7785 \\ 2.3355 & 7.1643 & 1.4328 \\ 0.7785 & 1.4328 & 1.4328 \end{bmatrix}$$

$$M^{-1}_{55(2)}(\xi) = \begin{bmatrix} 5.7472 & -1.5613 & -1.5613 \\ -1.5613 & 0.5986 & 0.2497 \\ -1.5613 & 0.2497 & 1.2965 \end{bmatrix}$$

From (3.3.6) we have,

$$V_{11}(\xi) = \begin{pmatrix} 1 & x_1^2 & x_2^2 \end{pmatrix} \begin{bmatrix} 3.0132 & -0.9869 & -0.9869 \\ -0.9869 & 0.8049 & 0.1627 \\ -0.9869 & 0.1627 & 0.8049 \end{bmatrix} \begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \end{bmatrix}$$

Let $\left(M_{11}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} a & b & b \\ b & c & d \\ b & d & c \end{bmatrix}$ such that

$$V_{11}(\xi) = \begin{pmatrix} 1 & x_1^2 & x_2^2 \end{pmatrix} \begin{bmatrix} a & b & b \\ b & c & d \\ b & d & c \end{bmatrix} \begin{bmatrix} 1 \\ x_1^2 \\ x_2^2 \end{bmatrix} =$$

$$V_{11}(\xi) = \left[3.0132 - 1.9738x_1^2 - 1.9738x_2^2 + .8049x_1^4 + .8049x_2^4 + .3254x_1^2 x_2^2\right]$$

$$V_{22}(\xi) = f_2' \, [M_{22}(\xi)]^{-1} f_2$$

$$V_{22}(\xi) = [x_1 \, x_2] \begin{bmatrix} 1.2845 & 0 \\ 0 & 0 \end{bmatrix} [x_1 \, x_2]$$

Let $\left(M_{22}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}$ such that

$$V_{22}(\xi)=[x_1\,x_2]\begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}[x_1\,x_2]$$

$$V_{22}(\xi)=\left[1.2845x_1^2x_2^2\right]$$

$$V_{55(1)}(\xi)=\ g_1^{'}\ [M_{55(1)}(\xi)]^{-1}g_1=$$

$$V_{55(1)}(\xi)=\begin{bmatrix} x_1 & x_1^3 & x_1x_2^2 \end{bmatrix}\begin{bmatrix} 5.7472 & -1.5613 & -1.5613 \\ -1.5613 & 0.5986 & 0.2497 \\ -1.5613 & 0.2497 & 1.2965 \end{bmatrix}\begin{bmatrix} x_1 \\ x_1^3 \\ x_1x_2^2 \end{bmatrix}$$

Let $\left(M_{55(1)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}$ such that

$$V_{55(1)}(\xi)=\begin{bmatrix} x_1 \\ x_1^3 \\ x_1x_2^2 \end{bmatrix}\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}\begin{bmatrix} x_1 & x_1^3 & x_1x_2^2 \end{bmatrix}$$

$$V_{55(1)}(\xi)=\left[5.7472x_1^2-3.1226x_1^4+0.5986x_1^6-3.1226x_1^2x_2^2+0.4994x_1^4x_2^2+1.2965x_1^2x_2^4\right]$$

$$V_{55(2)}(\xi)=\ g_2^{'}\ [M_{55(2)}(\xi)]^{-1}g_2=$$

$$V_{55(2)}(\xi)=\begin{bmatrix} x_2 & x_2^3 & x_2x_1^2 \end{bmatrix}\begin{bmatrix} 5.7472 & -1.5613 & -1.5613 \\ -1.5613 & 0.5986 & 0.2497 \\ -1.5613 & 0.2497 & 1.2965 \end{bmatrix}\begin{bmatrix} x_2 \\ x_2^3 \\ x_2x_1^2 \end{bmatrix}$$

Let $\left(M_{55(2)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}$ such that

$$V_{55(2)}(\xi)=\begin{bmatrix} x_2 & x_2^3 & x_2x_1^2 \end{bmatrix}\begin{bmatrix} e & f & g \\ f & h & k \\ g & k & l \end{bmatrix}\begin{bmatrix} x_2 \\ x_2^3 \\ x_2x_1^2 \end{bmatrix}$$

$$V_{55(2)}\left(\xi\right)=\left[5.7472x_2^2-3.1226x_2^4+0.5986x_2^6-3.1226x_2^2x_1^2+0.4994x_2^4x_1^2+1.2965x_2^2x_1^4\right]$$

From (3.3.8) the variance $V\left([\hat{y}(x_i)]\right)=\sum_1^2 V(x_i)$

$$V\left([\hat{y}(x_i)]\right)=\sum_1^2 V(x_i)=\left[3.0132-1.9738x_1^2-1.9738x_2^2+.8049x_1^4+.8049x_2^4+.3254x_1^2x_2^2\right]+$$
$$\left[1.2845x_1^2x_2^2\right]+\left[5.7472x_1^2-3.1226x_1^4+0.5986x_1^6-3.1226x_1^2x_2^2+0.4994x_1^4x_2^2+1.2965x_1^2x_2^4\right]$$
$$+\left[5.7472x_2^2-3.1226x_2^4+0.5986x_2^6-3.1226x_2^2x_1^2+0.4994x_2^4x_1^2+1.2965x_2^2x_1^4\right]$$

Summing the above variances we get expression which is a function of

$$\sum x_1^2,\sum x_2^2,\sum x_1^2x_2^2,\sum x_2^4x_1^2,\sum x_1^4x_2^2$$

In order to achieve the variance in GDTORD, the variance should be a function

of $\sum x_1^2,\sum x_2^2$ only. Therefore we need to cancel the

interactions $\sum x_1^2x_2^2,\sum x_2^4x_1^2,\sum x_1^4x_2^2$

We get all the above interactions be equated to zero so as to have functions

of $\sum x_1^2,\sum x_2^2$ only. Then

$$\sum_1^2 V\left([\hat{y}(x_i)]\right)=\left(3.0132x_0^2+3.7734x_1^2-2.3177x_1^4+0.5986x_1^6+3.7734x_2^2-2.3177x_2^4+0.5986x_2^6\right)$$

Let $d_1^2=\sum_{i=1}^2 x_i^2$ such that

$$\sum_1^2 V\left([\hat{y}(x_i)]\right)=\text{f }(d_1^2)\text{ only}$$

Determining Variance $V\left([\hat{y}(x_j)]\right)$ for (v-p) dimensional space

For a symmetric design $M(\xi)$, from (3.3.3) it is seen that variance for (v-p)-factor group is given as,

$$M_{11}(\xi) = f_1^t(x).f_1(x) = \begin{bmatrix} 1 \\ x_3^2 \\ x_4^2 \\ x_5^2 \end{bmatrix} \begin{bmatrix} 1 & x_3^2 & x_4^2 & x_5^2 \end{bmatrix} = \begin{bmatrix} 1 & x_3^2 & x_4^2 & x_5^2 \\ & x_3^4 & x_3^2x_4^2 & x_3^2x_5^2 \\ & (symm) & x_4^4 & x_4^2x_5^2 \\ & & & x_5^4 \end{bmatrix}.$$

$$M_{11}(\xi) = \begin{bmatrix} 1.0000 & 1.0199 & 1.0199 & 1.0199 \\ 1.0199 & 2.3355 & 0.7785 & 0.7785 \\ 1.0199 & 0.7785 & 2.3355 & 0.7785 \\ 1.0199 & 0.7785 & 0.7785 & 2.3355 \end{bmatrix}$$

$$M_{11}^{-1}(\xi) = \begin{bmatrix} 5.0441 & -1.3217 & -1.3217 & -1.3217 \\ -1.3217 & 0.8601 & 0.2179 & 0.2179 \\ -1.3217 & 0.2179 & 0.8601 & 0.2179 \\ -1.3217 & 0.2179 & 0.2179 & 0.8601 \end{bmatrix}$$

$$M_{22}(\xi) = f_2^t(x).f_2(x) = \begin{bmatrix} x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{bmatrix} \begin{bmatrix} x_3x_4 & x_3x_5 & x_4x_5 \end{bmatrix}$$

$$M_{22}(\xi) = \begin{bmatrix} 0.7785 & 0 & 0 \\ 0 & 0.7785 & 0 \\ 0 & 0 & 0.7785 \end{bmatrix}$$

$$M_{22}^{-1}(\xi) = \begin{bmatrix} 1.2845 & 0 & 0 \\ 0 & 1.2845 & 0 \\ 0 & 0 & 1.2845 \end{bmatrix}$$

$$M_{33}(\xi) = \left[x_3x_4x_5\right]\left[x_3x_4x_5\right] = \left[x_3^2x_4^2x_5^2\right]$$

$$M_{33}(\xi)=\begin{bmatrix}0.4776 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{33}^{-1}(\xi)=\begin{bmatrix}2.0938 & 0\\ 0 & 0\end{bmatrix}$$

$$M_{55(3)}(\xi)=g_3^t(x)g_3(x)=\begin{bmatrix}x_3\\ x_3^3\\ x_3x_4^2\\ x_3x_5^2\end{bmatrix}\begin{bmatrix}x_3 & x_3^3 & x_3x_4^2 & x_3x_5^2\end{bmatrix}=\begin{bmatrix}x_3^2 & x_3^4 & x_3^2x_4^2 & x_3^2x_5^2\\ & x_3^6 & x_3^4x_4^2 & x_3^4x_5^2\\ symm & & x_3^2x_4^4 & x_3^2x_4^2x_5^2\\ & & & x_3^2x_5^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6 & \lambda_6\\ & & & 3\lambda_6\end{bmatrix}$$

$$M_{55(3)}(\xi)=\begin{bmatrix}1.0199 & 2.3355 & 0.7785 & 0.7785\\ 2.3355 & 7.1643 & 1.4328 & 1.4328\\ 0.7785 & 1.4328 & 1.4328 & 0.4776\\ 0.7785 & 1.4328 & 0.4776 & 1.4328\end{bmatrix}$$

$$M_{55(3)}^{-1}(\xi)=\begin{bmatrix}8.8050 & -2.0502 & -2.0503 & -2.0503\\ -2.0502 & 0.6768 & 0.3279 & 0.3279\\ -2.0503 & 0.3279 & 1.3747 & 0.3279\\ -2.0503 & 0.3279 & 0.3279 & 1.3747\end{bmatrix}$$

$$M_{55(4)}(\xi)=g_4^t(x)g_4(x)=\begin{bmatrix}x_4\\ x_4^3\\ x_4x_3^2\\ x_4x_5^2\end{bmatrix}\begin{bmatrix}x_4 & x_4^3 & x_4x_3^2 & x_4x_5^2\end{bmatrix}=\begin{bmatrix}x_4^2 & x_4^4 & x_4^2x_3^2 & x_4^2x_5^2\\ & x_4^6 & x_4^4x_3^2 & x_4^4x_5^2\\ symm & & x_4^2x_3^4 & x_4^2x_3^2x_5^2\\ & & & x_4^2x_5^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6 & \lambda_6\\ & & & 3\lambda_6\end{bmatrix}$$

$$M_{55(4)}(\xi)=\begin{bmatrix}1.0199 & 2.3355 & 0.7785 & 0.7785\\ 2.3355 & 7.1643 & 1.4328 & 1.4328\\ 0.7785 & 1.4328 & 1.4328 & 0.4776\\ 0.7785 & 1.4328 & 0.4776 & 1.4328\end{bmatrix}$$

$$M^{-1}_{55(4)}(\xi)=\begin{bmatrix}8.8050 & -2.0502 & -2.0503 & -2.0503\\ -2.0502 & 0.6768 & 0.3279 & 0.3279\\ -2.0503 & 0.3279 & 1.3747 & 0.3279\\ -2.0503 & 0.3279 & 0.3279 & 1.3747\end{bmatrix}$$

$$M_{55(5)}(\xi)=g_5^t(x)g_5(x)=\begin{bmatrix}x_5\\ x_5^3\\ x_5x_3^2\\ x_5x_4^2\end{bmatrix}\begin{bmatrix}x_5 & x_5^3 & x_5x_3^2 & x_5x_4^2\end{bmatrix}=\begin{bmatrix}x_5^2 & x_5^4 & x_5^2x_3^2 & x_5^2x_4^2\\ & x_5^6 & x_5^4x_3^2 & x_5^4x_4^2\\ symm & & x_5^2x_3^4 & x_5^2x_3^2x_4^2\\ & & & x_5^2x_4^4\end{bmatrix}$$

$$=\begin{bmatrix}\lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4\\ & 15\lambda_6 & 3\lambda_6 & 3\lambda_6\\ symm & & 3\lambda_6 & \lambda_6\\ & & & 3\lambda_6\end{bmatrix}$$

$$M^{-1}_{55(5)}(\xi)=\begin{bmatrix}8.8050 & -2.0502 & -2.0503 & -2.0503\\ -2.0502 & 0.6768 & 0.3279 & 0.3279\\ -2.0503 & 0.3279 & 1.3747 & 0.3279\\ -2.0503 & 0.3279 & 0.3279 & 1.3747\end{bmatrix}$$

From (3.3.6) we have

$$V_{11}(\xi)=\begin{pmatrix}1 & x_3^2 & x_4^2 & x_5^2\end{pmatrix}\begin{bmatrix}5.0441 & -1.3217 & -1.3217 & -1.3217\\ -1.3217 & 0.8601 & 0.2179 & 0.2179\\ -1.3217 & 0.2179 & 0.8601 & 0.2179\\ -1.3217 & 0.2179 & 0.2179 & 0.8601\end{bmatrix}\begin{bmatrix}1\\ x_3^2\\ x_4^2\\ x_5^2\end{bmatrix}$$

Let $\left(M_{11}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}a & b & b & b\\ b & c & d & d\\ b & d & c & d\\ b & d & d & c\end{bmatrix}$ such that

$$V_{11}(\xi) = \begin{pmatrix} 1 & x_3^2 & x_4^2 & x_5^2 \end{pmatrix} \begin{bmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{bmatrix} \begin{bmatrix} 1 \\ x_3^2 \\ x_4^2 \\ x_5^2 \end{bmatrix}$$

=

$$\begin{bmatrix} a+bx_3^2+bx_4^2+bx_5^2 & b+cx_3^2+dx_4^2+dx_5^2 & b+dx_3^2+cx_4^2+dx_5^2 & b+dx_3^2+dx_4^2+cx_5^2 \end{bmatrix} \begin{bmatrix} 1 \\ x_3^2 \\ x_4^2 \\ x_5^2 \end{bmatrix}$$

$$V_{11}(\xi) = \begin{bmatrix} 5.0441 - 2.6434x_3^2 + 0.8601x_3^4 + 0.4358x_3^2x_4^2 + 0.4358x_3^2x_5^2 \\ -2.6434x_4^2 + 0.8601x_4^4 + 0.4358x_4^2x_5^2 - 2.6434x_5^2 + 0.8601x_5^4 \end{bmatrix}$$

$$V_{22}(\xi) = f_2' \, [M_{22}(\xi)]^{-1} f_2$$

$$V_{22}(\xi) = \begin{bmatrix} x_3x_4 & x_3x_5 & x_4x_5 \end{bmatrix} \begin{bmatrix} 1.2845 & 0 & 0 \\ 0 & 1.2845 & 0 \\ 0 & 0 & 1.2845 \end{bmatrix} \begin{bmatrix} x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{bmatrix}$$

Let $\left(M_{22}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{bmatrix}$ such that

$$V_{22}(\xi) = \begin{bmatrix} x_3x_4 & x_3x_5 & x_4x_5 \end{bmatrix} \begin{bmatrix} n & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & n \end{bmatrix} \begin{bmatrix} x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{bmatrix}$$

$$V_{22}(\xi) = \begin{bmatrix} 1.2845x_3^2x_4^2 + 1.2845x_3^2x_5^2 + 1.2845x_4^2x_5^2 \end{bmatrix}$$

$$V_{33}(\xi) = f_3' \, [M_{33}(\xi)]^{-1} f_3$$

$$V_{33}(\xi)=\left[x_3x_4x_5\right]\begin{bmatrix}2.0938 & 0\\ 0 & 0\end{bmatrix}\left[x_3x_4x_5\right]$$

Let $\left(M_{33}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}p & 0\\ 0 & 0\end{bmatrix}$ such that

$$V_{33}(\xi)=\left[x_3x_4x_5\right]\begin{bmatrix}p & 0\\ 0 & 0\end{bmatrix}\left[x_3x_4x_5\right]$$

$$V_{33}(\xi)=\left[2.0938x_3^2x_4^2x_5^2\right]$$

$$V_{55(3)}(\xi)=g_3^{'}\;[M_{55(3)}(\xi)]^{-1}g_3=$$

$$V_{55(3)}\left(\xi\right)=\left[x_3\quad x_3^3\quad x_3x_4^2\quad x_3x_5^2\right]\begin{bmatrix}8.8050 & -2.0502 & -2.0503 & -2.0503\\ -2.0502 & 0.6768 & 0.3279 & 0.3279\\ -2.0503 & 0.3279 & 1.3747 & 0.3279\\ -2.0503 & 0.3279 & 0.3279 & 1.3747\end{bmatrix}\begin{bmatrix}x_3\\ x_3^3\\ x_3x_4^2\\ x_3x_5^2\end{bmatrix}$$

Let $\left(M_{55(3)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix}e & f & g & g\\ f & h & k & k\\ g & k & l & k\\ g & k & k & l\end{bmatrix}$ such that

$$V_{55(3)}\left(\xi\right)=\left[x_3\quad x_3^3\quad x_3x_4^2\quad x_3x_5^2\right]\begin{bmatrix}e & f & g & g\\ f & h & k & k\\ g & k & l & k\\ g & k & k & l\end{bmatrix}\begin{bmatrix}x_3\\ x_3^3\\ x_3x_4^2\\ x_3x_5^2\end{bmatrix}$$

$$=8.8050x_3-4.1004x_3^3-4.1006x_3^2x_4^2-4.1006x_3^2x_5^2+0.6768x_3^6$$
$$+0.6558x_3^4x_4^2+0.6558x_3^4x_5^2+1.3747x_3^2x_4^4+1.3747x_3^2x_5^4+2x_3^2x_4^2x_5^2$$

Cancelling all the interactions by equating them to zero we have,

$$V_{55(3)}\left(\xi\right)=(ex_3+2\,fx_3^3+hx_3^6)$$

$$V_{55(4)}(\xi) = g_4^{'}\ [M_{55(4)}(\xi)]^{-1} g_4$$

$$V_{55(4)}\left(\xi\right) = \begin{bmatrix} x_4 & x_4^3 & x_4x_3^2 & x_4x_5^2 \end{bmatrix} \begin{bmatrix} 8.8050 & -2.0502 & -2.0503 & -2.0503 \\ -2.0502 & 0.6768 & 0.3279 & 0.3279 \\ -2.0503 & 0.3279 & 1.3747 & 0.3279 \\ -2.0503 & 0.3279 & 0.3279 & 1.3747 \end{bmatrix} \begin{bmatrix} x_4 \\ x_4^3 \\ x_4x_3^2 \\ x_4x_5^2 \end{bmatrix}$$

Let $\left(M_{55(4)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g & g \\ f & h & k & k \\ g & k & l & k \\ g & k & k & l \end{bmatrix}$ such that

$$V_{55(4)}\left(\xi\right) = \begin{bmatrix} x_4 & x_4^3 & x_4x_3^2 & x_4x_5^2 \end{bmatrix} \begin{bmatrix} e & f & g & g \\ f & h & k & k \\ g & k & l & k \\ g & k & k & l \end{bmatrix} \begin{bmatrix} x_4 \\ x_4^3 \\ x_4x_3^2 \\ x_4x_5^2 \end{bmatrix}$$

$$V_{55(4)}\left(\xi\right) = 8.8050x_4 - 4.1004x_4^3 - 4.1006x_4^2x_3^2 - 4.1006x_4^2x_5^2 + 0.6768x_4^6$$
$$+0.6558x_4^4x_3^2 + 0.6558x_4^4x_5^2 + 1.3747x_4^2x_3^4 + 1.3747x_4^2x_5^4 + 2x_4^2x_3^2x_5^2$$

Cancelling all the interactions by equating them to zero we have,

$$V_{55(4)}\left(\xi\right) = (8.8050x_4 - 4.1004x_4^3 + 0.6768x_4^6)$$

$$V_{55(5)}(\xi) = g_5^{'}\ [M_{55(5)}(\xi)]^{-1} g_5 =$$

$$V_{55(5)}\left(\xi\right) = \begin{bmatrix} x_5 & x_5^3 & x_5x_3^2 & x_5x_4^2 \end{bmatrix} \begin{bmatrix} 8.8050 & -2.0502 & -2.0503 & -2.0503 \\ -2.0502 & 0.6768 & 0.3279 & 0.3279 \\ -2.0503 & 0.3279 & 1.3747 & 0.3279 \\ -2.0503 & 0.3279 & 0.3279 & 1.3747 \end{bmatrix} \begin{bmatrix} x_5 \\ x_5^3 \\ x_5x_3^2 \\ x_5x_4^2 \end{bmatrix}$$

Let $\left(M_{55(5)}^{-1}(\xi)\right)$ be represented by $\begin{bmatrix} e & f & g & g \\ f & h & k & k \\ g & k & l & k \\ g & k & k & l \end{bmatrix}$ such that

$$V_{55(5)}\left(\xi\right)=\begin{bmatrix}x_5 & x_5^3 & x_5x_3^2 & x_5x_4^2\end{bmatrix}\begin{bmatrix}e & f & g & g\\ f & h & k & k\\ g & k & l & k\\ g & k & k & l\end{bmatrix}\begin{bmatrix}x_5\\ x_5^3\\ x_5x_3^2\\ x_5x_4^2\end{bmatrix}$$

$$=8.8050x_5-4.1004x_5^3-4.1006x_5^2x_3^2-4.1006x_5^2x_4^2+0.6768x_5^6$$
$$+0.6558x_5^4x_3^2+0.6558x_5^4x_4^2+1.3747x_5^2x_3^4+1.3747x_5^2x_4^4+2x_5^2x_3^2x_4^2$$

Cancelling all the interactions by equating them to zero we have,

$$V_{55(5)}\left(\xi\right)=(8.8050x_5-4.1004x_5^3+0.6768x_5^6)$$

From (3.3.5) the variance $V\left([\hat{y}(x_j)]\right)=\sum_{3}^{5}V(x_j)$

$$V\left([\hat{y}(x_j)]\right)=\sum_{3}^{5}V(x_j)=a+2bx_3^2+cx_3^4+2dx_3^2x_4^2+2dx_3^2x_5^2+2bx_4^2+cx_4^4+2dx_4^2x_5^2$$
$$+2bx_5^2+cx_5^4+nx_3^2x_4^2+nx_3^2x_5^2+nx_4^2x_5^2+px_3^2x_4^2x_5^2+ex_3+2fx_3^3+$$
$$2gx_3^2x_4^2+2gx_3^2x_5^2+hx_3^6+2kx_3^4x_4^2+2kx_3^4x_5^2+lx_3^2x_4^4+lx_3^2x_5^4$$
$$+2x_3^2x_4^2x_5^2+ex_3+2fx_3^3+2gx_3^2x_4^2+2gx_3^2x_5^2+hx_3^6$$
$$+2kx_3^4x_4^2+2kx_3^4x_5^2+lx_3^2x_4^4+lx_3^2x_5^4+2x_3^2x_4^2x_5^2+ex_4+2fx_4^3+$$
$$2gx_4^2x_3^2+2gx_4^2x_5^2+hx_4^6+2kx_4^4x_3^2+2kx_4^4x_5^2+lx_4^2x_3^4+lx_4^2x_5^4+2x_4^2x_3^2x_5^2$$
$$+ex_5+2fx_5^3+2gx_5^2x_3^2+2gx_5^2x_4^2+hx_5^6+2kx_5^4x_3^2+2kx_5^4x_4^2+$$
$$lx_5^2x_3^4+lx_5^2x_4^4+2x_5^2x_3^2x_4^2$$

Summing the above variances we get expression which is a function of

$\sum x_3^2, \sum x_4^2, \sum x_5^2 \sum x_3^2x_4^2, \sum x_3^2x_{5,}^2 \sum x_4^4x_3^2, \sum x_4^4x_5^2, \sum x_4^4x_5^2, \sum x_3^2x_4^2x_5^2$, In order to achieve variance in GDTORD, the variance should be a function of $\sum x_3^2, \sum x_4^2, \sum x_5^2$ only. Therefore we need to cancel the interactions $\sum x_3^2x_4^2, \sum x_3^2x_{5,}^2 \sum x_4^4x_3^2, \sum x_4^4x_5^2, \sum x_4^4x_5^2, \sum x_3^2x_4^2x_5^2$, We get all the above interactions be equated to zero so as to have functions of $\sum x_3^2, \sum x_4^2, \sum x_5^2$ only. Then from (3.3.8) we had,

$$\sum_3^5 V\left([\hat{y}(x_j)]\right) = \begin{pmatrix} ax_0^2 + (2b+e)x_3^2 + (2f+c)x_3^4 + hx_3^6 + (2b+e)x_4^2 + (2f+c)x_4^4 + hx_4^6 + \\ (2b+e)x_5^2 + (2f+c)x_5^4 + hx_5^6 \end{pmatrix}$$

Let $d_2^2 = \sum_{j=3}^{5} x_j^2$ such that

$V\left([\hat{y}(x_j)]\right) =$ f (d_2^2) only

At the point $x \in \chi$ the predicted response is

$V(\hat{y}(x)) = f'(x)\hat{\beta}$

The variance sum is as shown

$$V\left([\hat{y}(x)]\right) = (ax_0^2 + (2b+e)x_1^2 + (2f+c)x_1^4 + hx_1^6 + (2b+e)x_2^2 + (2f+c)x_2^4 + hx_2^6 +$$

$$ax_0^2 + (2b+e)x_3^2 + (2f+c)x_3^4 + hx_3^6 + (2b+e)x_4^2 + (2f+c)x_4^4 + hx_4^6 + (2b+e)x_5^2 + (2f+c)x_5^4 + hx_5^6)$$

$$\sum_{i=1}^{v} V(\hat{y}(x)) = f(d_1^2, d_2^2),$$

Thus the variance Sum is the function of distances d_1^2 *and* d_2^2only.

d_1^2 and d_2^2 is the distances of the projections of the points in p dimensional and $(v-p)$ dimensional spaces respectively from a suitable origin. The variance$V(\hat{y}(x_i)$ is a function of distance d_1^2 and variance $V(\hat{y}(x_j)$ is a function of distance d_2^2 from the design origin. Thus the considered response surface is a Variance - Sum Group Divisible Third Order Rotatable Designs in five dimensions.

4.5 Construction of (k-1) GDTORD through BIBD

4.5.1 Construction of (k-1) GDTORD through BIBD in four dimensions

Consider unreduced BIBD with parameters $(v = 4,\ b = 6,\ r = 3,\ k = 2,\ \lambda = 1)$ which is then split to form two groups of factors one of 3-factors and 1 factor where $p \geq 2$ and $(v - p) \geq 1$.Here we consider rotating p- factor group designs only where a set of S(a,a, a...,0) added to suitably chosen points set of 2S(c,0,0...,0) , S(d, d,0...,0) as shown

$$
\begin{array}{cccc}
a & a & a & 0 \\
-a & a & a & 0 \\
a & -a & a & 0 \\
a & a & -a & 0 \\
-a & -a & a & 0 \\
-a & a & -a & 0 \\
a & -a & -a & 0 \\
-a & -a & -a & 0 \\
c & 0 & 0 & 0 \\
-c & 0 & 0 & 0 \\
0 & c & 0 & 0 \\
0 & -c & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & -c & 0 \\
c & 0 & 0 & 0 \\
-c & 0 & 0 & 0 \\
0 & c & 0 & 0 \\
0 & -c & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & -c & 0 \\
d & d & 0 & 0 \\
d & -d & 0 & 0 \\
-d & d & 0 & 0 \\
-d & -d & 0 & 0 \\
0 & d & d & 0 \\
0 & d & -d & 0 \\
0 & -d & d & 0 \\
0 & -d & -d & 0 \\
d & 0 & d & 0 \\
d & 0 & -d & 0 \\
-d & 0 & d & 0 \\
-d & 0 & -d & 0
\end{array}
\tag{4.5.1*}
$$

Another set of S (b bb b) is added to satisfy rotatability.

$$\begin{matrix} b & b & b & b \\ -b & b & b & b \\ b & -b & b & b \\ b & b & -b & b \\ b & b & b & -b \\ -b & -b & b & b \\ -b & b & -b & b \\ -b & b & b & -b \\ b & -b & -b & b \\ b & -b & b & -b \\ b & b & -b & -b \\ -b & -b & -b & b \\ -b & -b & b & -b \\ -b & b & -b & -b \\ b & -b & -b & -b \\ -b & -b & -b & -b \end{matrix} \qquad (4.5.1^{**})$$

A set of 16 design points added.

N = (4.5.1*) + (4.5.1**) = 48 design points
(4.5.1***)

Normal equations are

$$\sum x_i - 3\sum x_i^2 x_j^2 = 8a^4 + 4c^4 + 8d^4 + 16b^4 - 3[8a^4 + 4d^4 + 16b^4] = 0$$

$$-16a^4 + 4c^4 - 4d^4 - 32b^4 = 0 \quad 4.5.1.1) \qquad (4.5.1.1)$$

$$\sum x_i^6 - 15\sum x_i^2 x_j^2 x_k^2 = 8a^6 + 4c^6 + 8d^6 + 16b^6 - 15[8a^6 + 16b^6] = 0$$

$$-112a^6 + 4c^6 + 8d^6 - 224b^6 = 0 \qquad (4.5.1.2)$$

$$\sum x_i^2 x_j^4 - 3\sum x_i^2 x_j^2 x_k^2 = 8a^6 + 4d^4 + 16b^6 - 3[8a^6 + 16b^6] = 0$$

$$-16a^6 + 4d^6 - 32b^6 = 0 \qquad (4.5.1.3)$$

Solving the three equations simultaneously we have (4.5.1.2)-7 (4.5.1.3) gave,

$$-112a^6 + 4c^6 + 8d^6 - 224b^6 - 7[-16a^6 + 4d^6 - 32b^6] = 0$$

$$4c^6 - 20d^6 = 0 \qquad (4.5.1.4)$$

$c^6 = 5d^6$

$c^2 = 5^{1/3}d^2$

From (4.5.1.2) -2(4.5.1.3) gave,

$-112a^6 + 4c^6 + 8d^6 - 224b^6 - 2[-16a^6 + 4d^6 - 32b^6] = 0$

$-80a^6 + 4c^6 - 160b^6 = 0$ (4.5.1.5)

Substituting the value of c in (4.5.1.1) we have

$-16a^4 + 4c^4 - 4d^4 - 32b^4 = 0$

$4[5^{1/3}d^2]^2 - 16a^4 - 4d^4 - 32b^4 = 0$

$5^{2/3}d^4 - 4a^4 - d^4 - 8b^4 = 0$

Collecting the like terms we have

$(5^{2/3} - 1)d^4 = 4a^4 + 8b^4$

$a^4 + 2b^4 = \frac{(5^{2/3}-1)d^4}{4}$

Let $b^2 = ta^2$ for $t \geq 0$

$a^4 + 2(ta^2)^2 = \frac{(5^{2/3}-1)d^4}{4}$

$a^4(1 + 2t^2) = \frac{(5^{2/3}-1)d^4}{4}$

$(1 + 2t^2) = \frac{(5^{2/3}-1)d^4}{4a^4}$ (4.5.1.6)

From (4.5.1.3) we have

$-16a^6 + 4d^6 - 32b^6 = 0$

$a^6 + 2b^6 = \frac{4}{16}d^6$

$a^6 + 2b^6 = \frac{1}{4}d^6$

$a^6(1 + 2t^3) = \frac{1}{4}d^6$

$$(1+2t^3)=\frac{1d^{\,6}}{4a^6}$$

(4.5.1.7)

Dividing the cube of (4.5.1.6) by the square of (4.5.1.7) we have

$$\frac{(1+2t^2)^3}{(1+2t^3)^2}=\left[\frac{\left(5^{2/3}-1\right)d^4}{4a^4}\right]^3\Bigg/\left(\frac{d^6}{4a^6}\right)^2$$

$$\frac{(1+2t^2)^3}{(1+2t^3)^2}=\frac{\left(5^{2/3}-1\right)^3}{4^3}\times 4^2=1.7806$$

$$1+6t^2+12t^4+8t^6=1.7806[1+4t^3+4t^6]$$

$$0.8776t^6+0t^5+12t^4-7.1224t^3+6t^2+0t^1-0.7806t^0$$

$$[0.8776\quad 0\quad 12\quad -7.1224\quad 6\quad 0\quad -0.7806]$$

$$t=0.3923$$ (4.5.1****)

Thus the non -negative solution exist

$$c^2=5^{1/3}d^2b^2=0.3923a^2$$

$$a^4=\frac{\left(5^{2/3}-1\right)d^4}{4(1+2t^2)}=\frac{\left(5^{2/3}-1\right)d^4}{4(1+2(0.3923)^2)}=\frac{\left(5^{2/3}-1\right)d^4}{5.2312}=0.3678d^4$$

$$t=0.3923$$

$$c^2=5^{1/3}d^2$$

$$a^2=0.6065d^2$$

$$b^2=0.3923(0.6065)d^2$$

$$b^2=0.2379d^2$$

Upon substituting the value of (4.5.1****) gave,

$$t=0.3923$$

$$c=1.3077$$

$$a=0.7788$$

$$b=0.4878$$

$d = 1$

$\lambda_2 = 1\lambda_4 = 0.1635\lambda_6 = 0.0417$

4.5.2 Construction of (k-1) GDTORD through BIBD in five dimensions

Consider unreduced BIBD with parameters $(v = 5,\ b = 10,\ r = 4,\ k = 2,\ \lambda = 1)$ where the 5 factors are divided into two groups of 4factors and 1factor respectively where $p \geq 2$ and $(v - p) \geq 1$.Here we consider rotating 4- factor group designs only where a set of S(a,a,a...,0) added to suitably chosen points set of 2S(c,0,0...,0) , S(d,d,0...,0) as shown,

V=5 factors

a	a	a	a	0
$-a$	a	a	a	0
a	$-a$	a	a	0
a	a	$-a$	a	0
a	a	a	$-a$	0
$-a$	$-a$	a	a	0
$-a$	a	$-a$	a	0
$-a$	a	a	$-a$	0
a	$-a$	$-a$	a	0
a	$-a$	a	$-a$	0
a	a	$-a$	$-a$	0
$-a$	$-a$	$-a$	a	0
$-a$	$-a$	a	$-a$	0
$-a$	a	$-a$	$-a$	0
a	$-a$	$-a$	$-a$	0
$-a$	$-a$	$-a$	$-a$	0

(4.5.2*)

$$\begin{matrix} c & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c & 0 \\ c & 0 & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & -c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & -c & 0 \end{matrix} \qquad (4.5.2^{**})$$

$$
\begin{matrix}
d & d & 0 & 0 & 0 \\
d & -d & 0 & 0 & 0 \\
-d & d & 0 & 0 & 0 \\
-d & -d & 0 & 0 & 0 \\
0 & d & d & 0 & 0 \\
0 & -d & d & 0 & 0 \\
0 & d & -d & 0 & 0 \\
0 & -d & -d & 0 & 0 \\
0 & 0 & d & d & 0 \\
0 & 0 & d & -d & 0 \\
0 & 0 & -d & d & 0 \\
0 & 0 & -d & -d & 0 \\
d & 0 & 0 & d & 0 \\
d & 0 & 0 & -d & 0 \\
-d & 0 & 0 & d & 0 \\
-d & 0 & 0 & -d & 0 \\
d & 0 & d & 0 & 0 \\
d & 0 & -d & 0 & 0 \\
-d & 0 & d & 0 & 0 \\
-d & 0 & -d & 0 & 0 \\
0 & d & 0 & d & 0 \\
0 & d & 0 & -d & 0 \\
0 & -d & 0 & d & 0 \\
0 & -d & 0 & -d & 0
\end{matrix}
\qquad (4.5.2^{***})
$$

Another set of $s(b\ b\ b\ b\ b)$ is added to satisfy the conditions for rotatability

b	b	b	b	b
$-b$	b	b	b	b
b	$-b$	b	b	b
b	b	$-b$	b	b
b	b	b	$-b$	b
b	b	b	b	$-b$
$-b$	$-b$	b	b	b
$-b$	b	$-b$	b	b
$-b$	b	b	$-b$	b
$-b$	b	b	b	$-b$
b	$-b$	$-b$	b	b
b	$-b$	b	$-b$	b
b	$-b$	b	b	$-b$
b	b	$-b$	$-b$	b
b	b	$-b$	b	$-b$
b	b	b	$-b$	$-b$
$-b$	$-b$	$-b$	b	b
$-b$	$-b$		$-b$	b
$-b$	$-b$	b	b	$-b$
$-b$	b	$-b$	$-b$	b
$-b$	b	$-b$	b	$-b$
$-b$	b	b	$-b$	$-b$
b	$-b$	$-b$	$-b$	b
b	$-b$	$-b$	b	$-b$
b	$-b$	b	$-b$	$-b$
b	b	$-b$	$-b$	$-b$
$-b$	$-b$	$-b$	$-b$	b
$-b$	$-b$	$-b$	b	$-b$
$-b$	$-b$	b	$-b$	$-b$
$-b$	b	$-b$	$-b$	$-b$
b	$-b$	$-b$	$-b$	$-b$
$-b$	$-b$	$-b$	$-b$	$-b$

N= (4.5.2*)+(4.5.2**)+(4.5.2***) +(4.5.2****) = 88 points (4.5.2*****)

Normal equations were,

$$\sum x_i^4 = 16a^4 + 4c^4 + 12d^4 + 32b^4$$

$$\sum x_i^2 x_j^2 = 16a^4 + 4d^4 + 32b^4$$

$$\sum x_i^6 = 16a^6 + 4c^6 + 12d^6 + 32b^6$$

$$\sum x_i^2 x_j^4 = 16a^6 + 4d^6 + 32b^6$$

$$\sum x_i^2 x_j^2 x_k^2 = 16a^6 + 32b^6$$

$$\sum x_i^4 - 3\sum x_i^2 x_j^2 = 16a^4 + 4c^4 + 12d^4 + 32b^4 - 3[16a^4 + 4d^4 + 32b^4]$$

$$-32a^4 + 4c^4 - 64b^4 = 0 \qquad (4.5.2.1)$$

$$\sum x_i^6 - 15\sum x_i^2 x_j^2 x_k^2 = 16a^6 + 4c^6 + 12d^6 + 32b^6 - 15[16a^6 + 32b^6] = 0$$

$$-224a^6 + 4c^6 + 12d^6 - 448b^6 = 0 \qquad (4.5.2.2)$$

$$\sum x_i^2 x_j^4 - 3\sum x_i^2 x_j^2 x_k^2 = 16a^6 + 4d^6 + 32b^6 - 3[16a^6 + 32b^6] = 0$$

$$-32a^6 + 4d^6 - 64b^6 = 0 \qquad (4.5.2.3)$$

Solving the three equations simultaneously,

(4.5.2.2)-7(4.5.2.3) gave

$$-224a^6 + 4c^6 + 12d^6 - 448b^6 - 7[-32a^6 + 4d^6 - 64b^6] = 0$$

$$4c^6 - 16d^6 = 0 \qquad (4.5.2.4)$$

$$4c^6 = 16d^6$$

$$c^6 = 4d^6$$

$$c^6 = 2^2 d^6$$

$$c^2 = 4^{1/3} d^2$$

Equation (4.5.2.2)-3(4.5.2.3) gave,

$$-224a^6 + 4c^6 + 12d^6 - 448b^6 - 3[-32a^6 + 4d^6 - 64b^6] = 0$$

$$-128a^6 + 4c^6 - 256b^6 = 0 \qquad (4.5.2.5)$$

Substituting the value of c in (4.5.2.1) we have

$$-32a^4 + 4c^4 - 256b^6 = 0$$

$$4\left[2^{2/3}d^2\right]^2 - 32a^4 - 64b^4 = 0$$

$$a^4 + 2b^4 = \frac{2^{4/3}d^4}{8}$$

Let $b^2 = ta^2$ for $t \geq 0$ we have

$$a^4 + 2(ta^2)^2 = \frac{2^{4/3}d^4}{8}$$

$$a^4(1 + 2t^2) = \frac{2^{\frac{4}{3}}d^4}{8}$$

$$(1 + 2t^2) = \frac{2^{\frac{4}{3}}d^4}{8a^4} \qquad (4.5.2.6)$$

From equation (4.5.2.3) we have

$$32a^6 + 64b^6 = 4d^6$$

$$a^6 + 2(ta^2)^3 = \frac{4}{32}d^6$$

$$(1 + 2t^2) = \frac{1}{8a^6}d^6 \qquad (4.5.2.7)$$

Dividing (4.5.2.6) cubed by the square of (4.5.2.7)

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = \frac{\left(\frac{2^{4/3}d^4}{8a^4}\right)^3}{\left(\frac{d^6}{8a^6}\right)^2}$$

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = \frac{(2^{4/3})^3 \times 8^2}{8^3} = 2$$

$$\frac{(1+2t^2)^3}{(1+2t^3)^2} = 2 = \frac{\left(4^{2/3}\right)^3}{8} = 2$$

$$1 + 6t^2 + 12t^4 + 8t^6 = 2[1 + 4t^3 + 4t^6]$$

$$0t^6 + 0t^5 + 12t^4 - 8t^3 + 6t^2 + 0t^1 - 1t^0$$

$$[0 \quad 0 \quad 12 \quad -8 \quad 6 \quad 0 \quad -1]$$

$$t = 0.4544$$

$$a^4 = \frac{2^{4/3}d^4}{8(1+2t^2)}$$

$$a^4 = \frac{2^{4/3}d^4}{8(1+2(0.4544)^2)} = 0.2229d^4$$

$$a^2 = 0.4721d^2$$

$$b^2 = ta^2$$

$$b^2 = 0.4544(0.4721)d^2$$

$$b^2 = 0.2145d^2$$

Solutions are;

$t = 0.4544$

$c = 1.2599$

$a = 0.6871$

$b = 0.4631$ and $d = 1$

4.5.3 (k-1) GDTORD in (k-1) dimensions

Here we consider a generalization of (k-1) GDTORD in (k-1) dimensions such that the non-negative solution of t where $t \in (0,1)$ is achieved as shown below.

$$f(t) = \frac{(1+2t^2)^3}{(1+2t^3)^2} = 2^{-(k-2)}\left((9-k)^{2/3} + (k-5)\right)^3$$

CHAPTER FIVE

CONCLUSION AND RECOMMENDATION

5.0 Introduction

This chapter gives the conclusions and recommendations derived from the results.

5.1 Conclusion

A group divisible third order rotatable designs in four, five and in k- dimensions was constructed using a balanced incomplete block designs and the design points generated from the four and five dimensional GDTORD were used to obtain a Variance- Sum group divisible third order rotatable designs in four and in five dimensions respectively. In addition a (k-1) Group Divisible Third order rotatable design was also constructed using BIBD by rotating the factors for one particular group only.

5.2 Recommendation

From the study findings, the study recommends the application of these designs constructed using BIBDs which gave less design points thus cutting down on the cost of experimentation and also gave reduced number of normal equations for estimating the parameter estimates. For further research, other methods on construction of a Group divisible Variance sum TORD for k number of groups is recommended and the Construction of a Group divisible Variance -Sum TORD using three balanced incomplete block designs is also recommended.

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