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Research Article

Counting formulas and bijections of nondecreasing 2-noncrossing trees

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Abstract

In this paper, we introduce nondecreasing 2-noncrossing trees and enumerate them according to their number of vertices, root degree, and number of forests. We also introduce nondecreasing 2-noncrossing increasing trees and count them by considering their number of vertices, label of the root, label of the leftmost child of the root, root degree, and forests. We observe that the formulas enumerating the newly introduced trees are generalizations of little and large Schröder numbers. Furthermore, we establish bijections between the sets of nondecreasing 2-noncrossing trees, locally oriented noncrossing trees, labelled complete ternary trees, and 3-Schröder paths.

Keywords: bijection; complete ternary tree; forest; locally oriented noncrossing tree; nondecreasing 2-noncrossing tree; root degree; 3-Schröder path.

2020 Mathematics Subject Classification: 05A15, 05A18, 05A19, 05C30.

1. Introduction

In the past quarter-century, noncrossing trees have received considerable attention as counting objects and have been generalised in the following two ways: assigning labels to the vertices [16,20] and considering block graphs [9]. *Noncrossing trees* were defined in [5] as trees whose vertices lie on the circumference of a circle and whose edges do not intersect inside the circle. Consider a noncrossing tree T and a vertex v of T. The *degree* of v is the number of vertices that are adjacent to v. The *outdegree* of v is the number of edges that are oriented away from v if we orient all edges away from the root (vertex 1). Note that if v is the root vertex, then its degree is equal to its outdegree. A vertex of outdegree 0 is a *leaf* and the *outdegree sequence* is the arrangement of outdegrees of all vertices of the tree. A collection of trees is a *forest*. An edge (a, b) is an *ascent* (respectively, *descent*) if $a < b$ (respectively, $a > b$) if we consider paths from the root.

One approach that is prominent in the enumeration of noncrossing trees is by first obtaining the (l, r)-*representation* of these trees introduced by Panholzer and Prodinger in [17], which we now review. Given a noncrossing tree, we can represent it as a plane tree such that if (i, j) is an ascent (respectively, a descent) in a path from the root then j is marked r (respectively, l). Figure 1.1 gives a noncrossing tree and its (l, r)-representation.

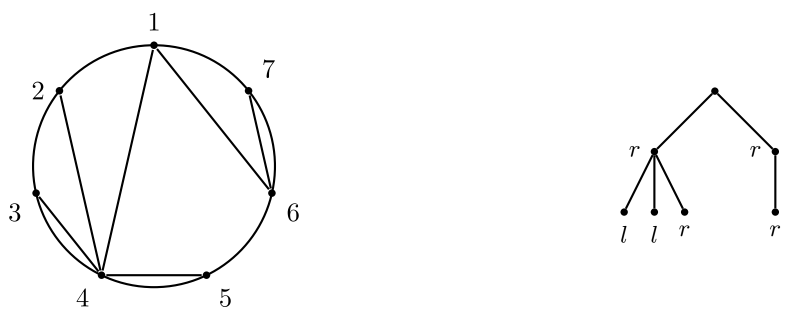

Figure 1.1: A noncrossing tree on 7 vertices with its (l, r)-representation.

It was shown by Noy [7] that the number of noncrossing trees with n edges is enumerated by generalized Catalan number $\binom{3n}{n}/(2n+1)$. Sequence $A001764$ of [18] lists a number of other combinatorial structures counted by the same formula. Noncrossing trees have also been enumerated by root degree [7], leaves, degree sequence, forests [5], and descents

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[6] among other parameters. Also, the *butterfly decomposition*, introduced by Flajolet and Noy in [5], plays an important role in the enumeration of noncrossing trees. If a vertex v of a noncrossing tree has outdegree d (that is, the vertex v has d children in the (l, r)-representation of the tree), then it can be decomposed into d butterflies all rooted at v as illustrated in Figure 1.2.

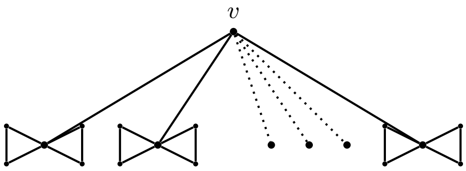

Figure 1.2: Decomposition of noncrossing trees into butterflies.

Formally, a *butterfly* is defined as an ordered pair of noncrossing trees sharing a common root vertex, where such a pair appears as wings of the butterfly. This method has been used in various enumerations of noncrossing trees and their generalizations as in [12–14]. Let $N(x)$ and $B(x)$ be the generating functions for noncrossing trees and butterflies respectively, where x marks a vertex. Since a noncrossing tree comprises of a root vertex and a sequence of butterflies, we have

$$N(x) = \frac{x}{1 - B(x)}. \tag{1}$$

Also, a butterfly is formed by bringing together two noncrossing trees, i.e., gluing two noncrossing trees together, so that every tree forms a wing of a butterfly. Thus,

$$B(x) = \frac{N(x)^2}{x}. \tag{2}$$

Note that $N(x)^2$ is divided by x to avoid over counting of vertices. Using (2) in (1), we obtain

$$N(x) = \frac{x}{1 - \frac{N(x)^2}{x}}, \tag{3}$$

as the generating function for noncrossing trees with x marking a vertex. Equation (3) is easily solved using the following form of the Lagrange inversion formula by letting $N(x) = \sqrt{x}\, M(x)$.

Theorem 1.1 (Lagrange inversion formula, [19])**.** *Let $M(z)$ be a generating function that satisfies the functional equation $M(z) = z\phi(M(z))$, where $\phi(0) \neq 0$. Then, $n[z^n]M(z)^k = k[m^{n-k}]\phi(m)^n$.*

In 2009, Yan and Liu [20] generalized noncrossing trees by introducing 2-*noncrossing trees*. These are noncrossing trees whose vertices are labelled 1 or 2 such that there are no ascents whose endpoints are both labelled 2 if we consider paths from the root. The said authors obtained the number of these trees with a given number of vertices and label of the root. Their formulas were later refined by Okoth in [10] by considering 2-noncrossing trees with a given number of vertices labelled 2. The results of Yan and Liu were generalized by Pang and Lv [16], in 2010, by introducing and enumerating k-*noncrossing trees*, which are noncrossing trees whose vertices are labelled from the set $\{1, 2, \ldots, k\}$ such that if (i, j) is an ascent in the path from the root then the sum of the labels of endpoints of the ascent is no more than $k + 1$. Pang and Lv [16] found a counting formula for these trees and their result was refined by Okoth [12] by taking root degree into consideration if the label of the root is k. In the same paper, Okoth also found a formula for the number of forests of k-noncrossing trees with the roots labelled k given the number of components. The number of k-noncrossing trees in which the number of occurrences of a certain label is considered was obtained by Okoth and Wagner in [15] and bijections with other combinatorial structures were obtained in [8]. The (l, r)-representation on noncrossing trees can be extended to k-noncrossing trees. We use the subscript of l and r to denote the label of the vertex in the k-noncrossing trees. This is illustrated in Figure 1.3.

We now define our main combinatorial structures under study.

Definition 1.1. *A* nondecreasing 2-noncrossing tree *is a 2-noncrossing tree whose (l, r)-representation is a plane tree in which for every internal vertex, all its children labelled l_1, l_2, r_1 and r_2 appear in this order from left to right.*

Figure 1.4 provides a nondecreasing 2-noncrossing tree on 7 vertices with the root labelled 1.

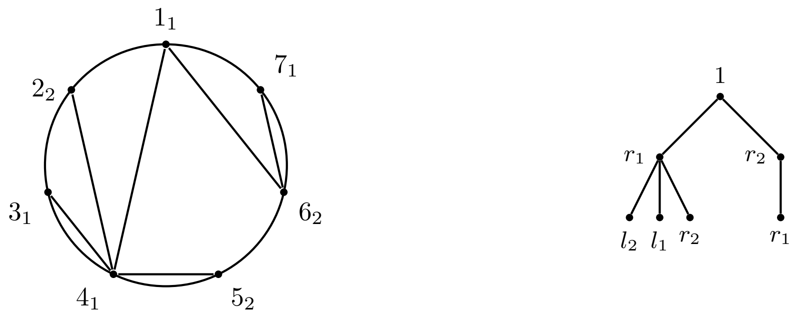

Figure 1.3: A 2-noncrossing tree on 7 vertices with the root labelled 1 together with its (l, r)-representation.

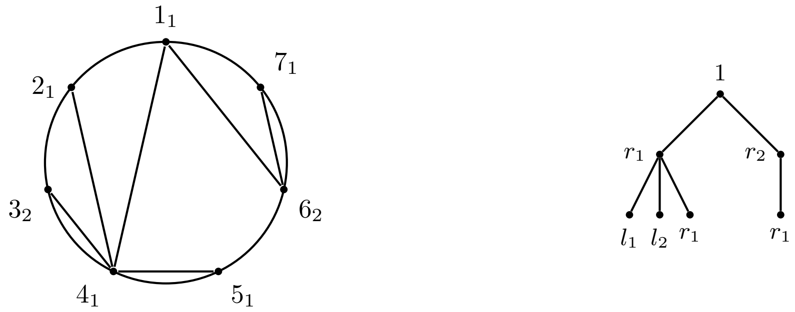

Figure 1.4: A nondecreasing 2-noncrossing tree on 7 vertices with the root labelled 1 together with its (l, r)-representation.

Now, we define another new class of combinatorial structures.

Definition 1.2. *A* nondecreasing 2-noncrossing increasing tree *is a nondecreasing 2-noncrossing tree whose edges are all ascents if paths from the root are considered in the (l, r)-representation of the 2-noncrossing tree.*

In the (l, r)-representation of the nondecreasing 2-noncrossing increasing trees, there are no vertices labelled l_1 or l_2. *Locally oriented noncrossing trees* were introduced by Okoth and Wagner in [14]. These are noncrossing trees whose edges are oriented from a vertex of a lower label towards a vertex of a higher label. If all edges incident to a given vertex v are directed towards (respectively, away from) v, then such a vertex is said to be a *sink* (respectively, *source*). In the same paper, it was shown that the number of locally oriented noncrossing trees on n vertices with k sources and ℓ sinks is

$$\frac{1}{n-1}\binom{n-1}{k-1}\binom{n-1}{\ell-1}\binom{n-1}{k+\ell-1}. \tag{4}$$

If we set $k + \ell = i + 2$ in (4), we have

$$\frac{1}{n-1}\binom{n-1}{k-1}\binom{n-1}{i-k+1}\binom{n-1}{i+1}. \tag{5}$$

Finally, summing over all k (making use of the Vandermonde convolution) in (5), we get

$$\frac{1}{n-1}\binom{2n-1}{i}\binom{n-1}{i+1}.$$

Note that $i + 1$ is the number of non-root sources and sinks. So, if these vertices, i.e. non-root sources and sinks, come in two colours, then the total number of these trees is

$$\frac{1}{n-1}\sum_{i=0}^{n-1}\binom{2n-1}{i}\binom{n-1}{i+1}2^{i+1}.$$

There is a relationship between noncrossing trees and d-ary trees. A *d-ary tree* is a plane tree in which every internal vertex has at most d children. They are known to be enumerated by generalized Catalan numbers. If every internal vertex of a d-ary tree has exactly d children then we get a *complete d-ary tree*. Further, by setting $d = 3$, we obtain a complete ternary tree. For every internal vertex, an edge that is incident to this vertex and appears on the far left, middle, and far right are called *left-edge, middle-edge*, and *right-edge*, respectively. A bijection between the set of locally oriented noncrossing trees on n vertices with k sources and ℓ sinks and the set of ternary trees on $n - 1$ vertices such that there are

$k-1$ left-edges and $\ell-1$ middle-edges was constructed by Okoth and Wagner in [14]. Their bijection was a modification of the bijection between noncrossing trees with $n-1$ edges and ternary trees on $n-1$ vertices that was obtained independently by Simion and Postnikov as stated by Stanley in [19, Solution 5.46]. There are other known bijections; for example, see [4,17].

Noncrossing trees and complete ternary trees are also known to be in bijection with lattice paths (consisting of unit vertical and unit horizontal steps) which lie weakly below the line $y = 2x$. The following lattice paths were introduced by Yang and Jiang in [21].

Definition 1.3 (see [21]). *A* 3-Schröder path *of length* n *is a lattice path lying in the first quadrant from* $(0,0)$ *to* $(3n,0)$ *consisting of* $(1,2)$, $(2,1)$ *and* $(1,-1)$ *steps. If the path starts from the* x*-axis with* $(1,2)$ *then we get a* small 3-Schröder path.

The set of 3-Schröder paths and small 3-Schröder paths are enumerated by sequences $A027307$ and $A034015$, respectively, of the celebrated encyclopedia [18].

This paper is organized as follows. In Section 2, we enumerate nondecreasing 2-noncrossing trees by the number of vertices and label of the root. The study is extended to enumerate these trees by root degree in Section 3 and number of forests in Section 4. In Section 5, we enumerate nondecreasing 2-noncrossing increasing trees by the number of vertices and label of the root, label of the leftmost child of the root, root degree, and number of forests of these trees. Section 6 is concerned with bijections. A bijection between the set of nondecreasing 2-noncrossing trees and the sets of locally oriented noncrossing trees with a given number of sources and sinks is given in Subsection 6.1. Subsection 6.2 gives a bijection concerning complete ternary trees and Subsection 6.3 provides a bijection about 3-Schröder paths. The paper is concluded in Section 7, which contains also proposals for extending this study.

2. Enumeration by number of vertices and vertices of a given type

Let $C(x,u)$ and $D(x,u)$ be the bivariate generating functions for nondecreasing 2-noncrossing trees with the roots labelled 2 and 1, respectively, where x marks a vertex and u marks a vertex labelled 2. Since a butterfly rooted at a vertex labelled 2 can be considered to have one wing rooted at a vertex labelled 2 and the other wing rooted at a vertex labelled 1, we have

$$C(x,u) = xu \cdot \frac{1}{1-\frac{D^2}{x}}, \tag{6}$$

and

$$D(x,u) = x \cdot \frac{1}{\left(1-\frac{D^2}{x}\right)} \cdot \frac{1}{\left(1-\frac{CD}{x}\right)}. \tag{7}$$

Let $C = xu(1+y)$. Then, from (6), we get

$$D^2 = \frac{xy}{1+y}. \tag{8}$$

Using $C = xu(1+y)$ and (8) in (7), we obtain

$$D^2u(1+y) - D + x(1+y) = 0. \tag{9}$$

Also, using (8) in (9), we have

$$D = x(1+(1+u)y). \tag{10}$$

Now, using (10) in (8), we arrive at the following equation:

$$y = x(1+y)(1+(1+u)y)^2. \tag{11}$$

Equation (11) is in the right form to apply the Lagrange inversion formula. Hence, we have

$$\begin{aligned}[x^nu^k]C &= [x^nu^k]uxy = [x^{n-1}u^{k-1}]y = \frac{1}{n-1}[t^{n-2}u^{k-1}]((1+(1+u)t)^2(1+t))^{n-1}\\ &= \frac{1}{n-1}[t^{n-2}u^{k-1}]\sum_{i\geq 0}\binom{2n-2}{i}((1+u)t)^i\sum_{j\geq 0}\binom{n-1}{j}t^j = \frac{1}{n-1}[t^{n-2}u^{k-1}]\sum_{i\geq 0}\binom{2n-2}{i}(1+u)^i\sum_{j\geq 0}\binom{n-1}{j}t^{i+j}\\ &= \frac{1}{n-1}[t^{n-2}u^{k-1}]\sum_{i\geq 0}\binom{2n-2}{i}\sum_{a\geq 0}\binom{i}{a}\sum_{j\geq 0}\binom{n-1}{j}t^{i+j}u^a = \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{i}{k-1}\binom{n-1}{n-i-2}.\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
[x^n u^k]D &= [x^n u^k](1+u)xy = [x^{n-1}u^k](1+u)y \\
&= \frac{1}{n-1}[t^{n-2}u^k](1+u)((1+(1+u)t)^2(1+t))^{n-1} \\
&= \frac{1}{n-1}[t^{n-2}u^k](1+u)\sum_{i\geq 0}\binom{2n-2}{i}((1+u)t)^i \sum_{j\geq 0}\binom{n-1}{j}t^j \\
&= \frac{1}{n-1}[t^{n-2}u^k]\sum_{i\geq 0}\binom{2n-2}{i}(1+u)^{i+1}\sum_{j\geq 0}\binom{n-1}{j}t^{i+j} \\
&= \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{i+1}{k}\binom{n-1}{n-i-2}. \qquad (13)
\end{aligned}$$

Adding (12) and (13), we obtain

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{k+i+1}{i+1}\binom{2n-2}{i}\binom{i+1}{k}\binom{n-1}{i+1}.$$

We formally state these results in the following theorem:

Theorem 2.1. *The number of nondecreasing* 2*-noncrossing trees on* n *vertices with the root labelled* 2 *such that there are* k *vertices labelled 2 is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{i}{k-1}\binom{n-1}{n-i-2}$$

and with the root labelled 1 *such that there are* k *vertices labelled 2 is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{i+1}{k}\binom{n-1}{n-i-2}.$$

In total, there are

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{k+i+1}{i+1}\binom{2n-2}{i}\binom{i+1}{k}\binom{n-1}{i+1}$$

nondecreasing 2*-noncrossing trees on* n *vertices such that there are* k *vertices labelled 2.*

Corollary 2.1. *The number of nondecreasing* 2*-noncrossing trees on* n *vertices with the root labelled* 2 *is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{n-1}{i+1}2^i \qquad (14)$$

and the number of nondecreasing 2*-noncrossing trees on* n *vertices with the root labelled* 1 *is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{n-1}{i+1}2^{i+1}. \qquad (15)$$

In total, there are

$$\frac{3}{n-1}\sum_{i=0}^{n-2}\binom{2n-2}{i}\binom{n-1}{i+1}2^i$$

nondecreasing 2*-noncrossing trees on* n *vertices.*

Proof. We sum over all k in Theorem 2.1 or set $u = 1$ in (11) and extract the coefficient of x^n in C and D, respectively. Finally, the respective equations are added together. □

Setting $k = 1$ in (12) or $k = 0$ in (13) and summing over all i, we rediscover the formula for the number of noncrossing trees on n vertices, i.e.,

$$\frac{1}{n-1}\binom{3n-3}{n-2}.$$

3. Enumeration by root degree

Let $D(x) = \sum_{n=0}^{\infty} D_n x^n$ and $C(x) = \sum_{n=0}^{\infty} C_n x^n$ be the generating functions for nondecreasing 2-noncrossing trees with the roots labelled 1 and 2, respectively. Let $d(n,r)$ be the number of nondecreasing 2-noncrossing trees on n vertices such that vertex 1 is labelled 2 and is of degree r. We prove the following lemma, with the proof mimicked from the previous work of Noy [7].

Lemma 3.1. *The numbers $d(n,r)$ can be written in terms of the numbers D_n as follows:*

$$d(n,r) = \sum_{\substack{j_1+j_2+\cdots+j_{2r}=n+r-1\\ j_1,j_2,\ldots,j_{2r}\geq 1}} D_{j_1} D_{j_2} \cdots D_{j_{2r}}.$$

Proof. Let N be a nondecreasing 2-noncrossing tree on n vertices with the root labelled 2 such that the root has degree r. The root is adjacent to r vertices $v_1 < v_2 < \cdots < v_r$ all labelled 1. For every $j = 1, 2, \ldots, r-1$, the subgraph induced by N on the vertex set $\{v_j, \ldots, v_{j+1}\}$ is the disjoint union of two nondecreasing 2-noncrossing trees with the roots labelled 1. This procedure gives a total of $2r-2$ trees. Also, the subgraphs induced on $\{2, \ldots, v_1\}$ and $\{v_r, \ldots, n\}$ are both nondecreasing 2-noncrossing trees with the roots labelled 1. We thus have $2r$ nondecreasing 2-noncrossing trees with the roots labelled 1. Let these trees have $j_1, j_2, \ldots, j_{2r}$ vertices, respectively. So, $j_1 + j_2 + \cdots + j_{2r} = n + r - 1$. In addition, a family of $2r$ nondecreasing 2-noncrossing trees with the roots labelled 1 on the corresponding vertex set determines a unique nondecreasing 2-noncrossing tree N with the root labelled 2. This completes the proof. □

The next result can be found in [2].

Theorem 3.1 (Lagrange-Bürmann)**.** *Let $q(x)$ be a power series with complex coefficients and β be a complex number, satisfying the following equation:*

$$q(x) = \beta + x \cdot g(q(x)).$$

Then

$$[x^n] f(q(x)) = f(\beta) + \frac{1}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}s^{n-1}} \left(\frac{\mathrm{d}}{\mathrm{d}s}(f(s)) \cdot (g(s))^n \right)_{s=\beta}.$$

We now prove the following result:

Theorem 3.2. *The number of nondecreasing 2-noncrossing trees with the root labelled 2 on n vertices such that the root has degree r and the total number of vertices labelled 2 equals k, is given by*

$$\frac{2r}{n-r-1}\binom{2n-2r-2}{k-1}\sum_{i\geq 0}\binom{k-1}{i}\binom{3n-r-i-4}{n-r-2}(-1)^i. \tag{16}$$

Proof. The generating function $C(x,u)$ of nondecreasing 2-noncrossing trees with the root labelled 2 is given by $C(x,u) = xu(1+y)$, where x marks a vertex and u marks a vertex labelled 2, respectively, and $y = x(1+y)(1+(1+u)y)^2$. Let $F = 1+y$ so that $C = xuF$, $y = F-1$ and $F = 1 + xF(F + u(F-1))^2$. The convolution of Lemma 3.1 implies that

$$N_r = \frac{1}{x^{r-1}} N^{2r} = x^{r+1} F^{2r},$$

where the coefficients in the series N_r give the number of nondecreasing 2-noncrossing trees with a root labelled 2 and of degree r. Since

$$F = 1 + xF(F + u(F-1))^2,$$

we apply Lagrange-Bürmann Inversion Formula (Theorem 3.1) to the series N_r with $\beta = 1, g(s) = s(s+u(s-1))^2$ and $f(s) = s^{2r}$ to get

$$\begin{aligned}
[x^n u^{k-1}] F^{2r} &= \frac{2r}{n!} \frac{d^{n-1}}{ds^{n-1}} \left\{ s^{n+2r-1} (s + u(s-1))^{2n} \right\}_{s=1} \\
&= \frac{2r}{n!} \frac{d^{n-1}}{ds^{n-1}} \left\{ \sum_{i\geq 0} \binom{2n}{i} (s-1)^i u^i s^{3n+2r-i-1} \right\}_{s=1} \\
&= \frac{2r}{n!} \binom{2n}{k-1} \frac{d^{n-1}}{ds^{n-1}} \left\{ (s-1)^{k-1} s^{3n+2r-k} \right\}_{s=1} \\
&= \frac{2r}{n!} \binom{2n}{k-1} \sum_{i\geq 0} \binom{k-1}{i} (-1)^{k-i-1} \frac{d^{n-1}}{ds^{n-1}} \left\{ s^{3n+2r+i-k} \right\}_{s=1}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}[x^n u^{k-1}]F^{2r} &= \frac{2r}{n!}\binom{2n}{k-1}\sum_{i\geq 0}\binom{k-1}{i}(-1)^{k-i-1}(3n+2r+i-k)(3n+2r+i-k-1)\cdots(2n+2r+i-k+2)\\ &= \frac{2r}{n!}\binom{2n}{k-1}\sum_{i\geq 0}\binom{k-1}{i}(-1)^{k-i-1}\frac{(3n+2r+i-k)!}{(2n+2r+i-k+1)!}\\ &= \frac{2r}{n}\binom{2n}{k-1}\sum_{i\geq 0}\binom{k-1}{i}(-1)^{k-i-1}\binom{3n+2r+i-k}{n-1}.\end{aligned}$$

Therefore,

$$[x^n u^k]N_r = [x^{n-r-1}u^{k-1}]F^{2r} = \frac{2r}{n-r-1}\binom{2n-2r-2}{k-1}\sum_{i\geq 0}\binom{k-1}{i}(-1)^{k-i-1}\binom{3n-r+i-k-3}{n-r-2}.$$

This completes the proof. □

By setting $k = 1$ in (16), we rediscover the next result.

Corollary 3.1. *There are*

$$\frac{2r}{n-r-1}\binom{3n-r-4}{n-r-2}$$

noncrossing trees on n vertices with the root of degree r.

Theorem 3.3. *The number of nondecreasing 2-noncrossing trees on n vertices whose root has label 2 and degree r is given by*

$$\frac{2r}{n-r-1}\sum_{i=0}^{2n-2r-2}\binom{2n-2r-2}{i}\binom{n+r+i-2}{2r+i}(-2)^i. \tag{17}$$

Proof. The generating function $C(x)$ of nondecreasing 2-noncrossing trees with the root labelled 2 is given by $C(x) = x(1+y)$, where x marks a vertex. We have $y = x(1+y)(1+2y)^2$. Let $F = 1+y$ so that $C = xF$, $y = F-1$ and $F = 1 + xF(2F-1)^2$. The convolution of Lemma 3.1 implies that

$$N_r = \frac{1}{x^{r-1}}N^{2r} = x^{r+1}F^{2r},$$

where the coefficients in the series N_r give the number of nondecreasing 2-noncrossing trees with a root labelled 2 and of degree r. Since

$$F = 1 + xF(2F-1)^2,$$

we apply Lagrange-Bürmann Inversion Formula to the series N_r with $\beta = 1, g(s) = s(2s-1)^2$ and $f(s) = s^{2r}$ to get

$$\begin{aligned}[x^n]F^{2r} &= \frac{2r}{n!}\frac{d^{n-1}}{ds^{n-1}}\left\{s^{n+2r-1}(2s-1)^{2n}\right\}_{s=1}\\ &= \frac{2r}{n!}\sum_{i\geq 0}\binom{2n}{i}(-1)^i 2^{2n-i}\frac{d^{n-1}}{ds^{n-1}}\left\{s^{3n+2r-i-1}\right\}_{s=1}\\ &= \frac{2r}{n!}\sum_{i\geq 0}\binom{2n}{i}(-1)^i 2^{2n-i}(3n+2r-i-1)(3n+2r-i-2)\cdots(2n+2r-i+1)\\ &= \frac{2r}{n!}\sum_{i\geq 0}\binom{2n}{i}\frac{(3n+2r-i-1)!}{(2n+2r-i)!}(-1)^i 2^{2n-i}\\ &= \frac{2r}{n}\sum_{i\geq 0}\binom{2n}{i}\binom{3n+2r-i-1}{n-1}(-1)^i 2^{2n-i}.\end{aligned}$$

Therefore,

$$[x^n]N_r = [x^{n-r-1}]F^{2r} = \frac{2r}{n-r-1}\sum_{i=0}^{2n-2r-2}\binom{2n-2r-2}{i}\binom{3n-r-i-4}{n-r-2}(-1)^i 2^{2n-2r-i-2},$$

which completes the proof. □

We remark that (17) can also be obtained by summing over all k in (16).

4. Forests of nondecreasing 2-noncrossing trees

In Theorem 4.1, we consider forests of nondecreasing 2-noncrossing trees with the following two properties:

- every component is rooted at a vertex with the smallest label,
- the components are nondecreasing 2-noncrossing trees with the root labelled 2, and the components do not intersect each other.

The case in which the roots of the components are all labelled 1 is considered in Theorem 4.2.

To prove our result, we use generating functions and adopt the proof of Okoth [12].

Theorem 4.1. *There are*

$$\frac{1}{n-k}\binom{n}{k-1}\sum_{i=0}^{n-k-1}\binom{n-1}{i+k}\binom{2n-2k}{i}2^i \tag{18}$$

forests of nondecreasing 2-noncrossing trees on n vertices with k components such that the roots of the components are labelled 2.

Proof. Let $C(x)$ be the generating function for the components, i.e. nondecreasing 2-noncrossing trees satisfying the given conditions, where x marks a vertex. We decompose the forests according to components containing vertex 1 as shown in Figure 4.1. If this component is on p vertices then there are p spaces that are to be filled by forests $T_1, T_2, \dots, T_p$. Some of these components may be empty.

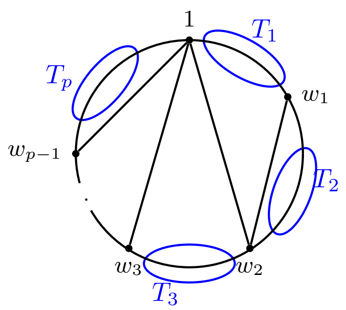

Figure 4.1: Decomposition of forests of nondecreasing 2-noncrossing trees according to vertex 1.

If $C(x) = \sum_{n\geq 1} c_n x^n$, then the generating function $T(x,z)$ for forests satisfies

$$T(x,z) = 1 + z\sum_{m\geq 1} c_m x^m T(x,z)^m = 1 + zC(xT(x,z)),$$

where z marks the number of components. Let $C(x) = x(1+y(x))$, then by (11) with $u=1$, we have $y(x) = x(1+y)(1+2y)^2$. Now, let $T(x,z) = 1+zs$. Then, $1+zs = 1+zC(x(1+zs))$ or $s = C(x(1+zs))$.

We define a as $a(t) = \frac{t}{C^{-1}(t)}$. Since $x(1+zs) = C^{-1}(s)$, we have $x(1+zs) = \frac{s}{a(s)}$. This implies that $s = x(1+zs)a(s)$. We now can apply the Lagrange inversion formula to obtain

$$[x^n z^k]T(x,z) = [x^n z^{k-1}]s = \frac{1}{n}[t^{n-1}z^{k-1}]((1+zt)a(t))^n = \frac{1}{n}\binom{n}{k-1}[t^{n-k}]a(t)^n.$$

To complete the proof, we only need to obtain $[t^{n-k}]a(t)^n$. By definition, we have $t = C\left(\frac{t}{a(t)}\right)$. We write $C(x) = x(1+y)$, to have

$$t = \frac{t}{a(t)}\left(1+y\left(\frac{t}{a(t)}\right)\right) \quad \text{or} \quad a(t) = 1 + y\left(\frac{t}{a(t)}\right)$$

where $y(x)$ satisfies $y(x) = x(1+y)(1+2y)^2$. It follows that

$$a(t)-1 = y\left(\frac{t}{a(t)}\right) = \frac{t}{a(t)}\left(1+y\left(\frac{t}{a(t)}\right)\right)\left(1+2y\left(\frac{t}{a(t)}\right)\right)^2 = \frac{t}{a(t)}\cdot a(t)(2a(t)-1)^2.$$

Therefore, $a(t) = 1 + t(2a(t)-1)^2$. Let $b(t) = a(t)-1$, then

$$b(t) = t(2(a(t)-1)-1)^2 = t\left(2b(t)+1\right)^2.$$

We use the Lagrange inversion formula to obtain the following equation:

$$[t^{n-r}]a(t)^n = [t^{n-r}](1+b(t))^n = \sum_{i\geq 0}\binom{n}{i}[t^{n-k}]b(t)^i = \sum_{i\geq 0}\binom{n}{i}\frac{i}{n-k}[t^{n-r-i}](2t+1)^{2n-2k}$$

$$= \frac{n}{n-k}\sum_{i\geq 0}\binom{n-1}{i-1}[t^{n-k-i}]\sum_{j\geq 0}\binom{2n-2k}{j}2^j t^j = \frac{n}{n-k}\sum_{i\geq 0}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-k-i}.$$

So, the number of forests of nondecreasing 2-noncrossing trees with n vertices and k components, such that the components have roots labelled 2, is

$$[x^n z^k]T(x,z) = \frac{1}{n}\binom{n}{k-1}[t^{n-k}]a(t)^n = \frac{1}{n}\binom{n}{k-1}\frac{n}{n-k}\sum_{i\geq 0}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-k-i}$$

$$= \frac{1}{n-k}\binom{n}{k-1}\sum_{i=1}^{n-k}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-k-i}.$$

This completes the proof. □

By setting $k = 1$ in (18), we obtain the formula for the number of nondecreasing 2-noncrossing trees with n vertices such that the root of every tree is labelled 2. The formula was already obtained in (14).

Theorem 4.2. *There are*

$$\frac{1}{n-k}\binom{n}{k-1}\sum_{i=0}^{n-k-1}\binom{n-1}{i+k}\binom{2n-2k}{i}2^{k+i} \tag{19}$$

forests of nondecreasing 2*-noncrossing trees on* n *vertices with* k *components such that the roots of the components are labelled 1.*

Proof. Let B be the generating function for the components, i.e. nondecreasing 2-noncrossing trees such that every component is rooted at the vertex with the smallest label, the roots of the components are labelled 1, and the components do not intersect. We decompose the forests according to components containing vertex 1 as shown in Figure 4.1. If this component is on p vertices, then there are p spaces that are to be filled by forests $T_1, T_2, \ldots, T_p$. Some of these components may be empty. Let $D(x) = \sum_{n\geq 1} d_n x^n$ be the generating function for nondecreasing 2-noncrossing trees with the root labelled 1. The generating function $T(x,z)$ for forests satisfies the following equation:

$$T(x,z) = 1 + z\sum_{m\geq 1} c_m x^m T(x,z)^m = 1 + zD(xT(x,z)),$$

where z marks the number of components. Let $D(x) = x(1+2y(x))$, then by (11) with $u = 1$, we have $y(x) = x(1+y)(1+2y)^2$. Now, let $T(x,z) = 1 + 2zs$. Then, $1 + 2zs = 1 + zD(x(1+2zs))$ or $2s = D(x(1+2zs))$.

We define a as $a(t) = \frac{t}{D^{-1}(t)}$. Since $x(1+2zs) = D^{-1}(s)$, then $x(1+2zs) = \frac{s}{a(s)}$. This implies that $s = x(1+2zs)a(s)$. Now, we can apply the Lagrange inversion formula to obtain the following equation:

$$[x^n z^k]T(x,z) = [x^n z^{k-1}]2s = 2\cdot\frac{1}{n}[t^{n-1}z^{k-1}]((1+2zt)a(t))^n = \frac{2}{n}\binom{n}{k-1}2^{k-1}[t^{n-k}]a(t)^n.$$

By the proof of Theorem 4.1, we have

$$[t^{n-k}]a(t)^n = \frac{n}{n-k}\sum_{i\geq 0}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-k-i}.$$

Therefore, the number of forests of nondecreasing 2-noncrossing trees with n vertices and k components whose roots are labelled 1 is

$$[x^n z^k]T(x,z) = \frac{2^k}{n}\binom{n}{k-1}[t^{n-k}]a(t)^n = \frac{2^k}{n}\binom{n}{k-1}\frac{n}{n-k}\sum_{i\geq 0}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-k-i}$$

$$= \frac{1}{n-k}\binom{n}{k-1}\sum_{i=1}^{n-k}\binom{n-1}{i-1}\binom{2n-2k}{n-k-i}2^{n-i},$$

which completes the proof. □

By setting $k = 1$ in (19), we obtain (15), which is the number of nondecreasing 2-noncrossing trees on n vertices with the root labelled 1.

5. Nondecreasing 2-noncrossing increasing trees

In this section, we enumerate nondecreasing 2-noncrossing increasing trees according to their number of vertices, root degree, label of the leftmost child of the root, and number of forests. We start with the number of vertices.

5.1. Number of vertices

Let $I(x,u)$ and $J(x,u)$ be the bivariate generating functions for nondecreasing 2-noncrossing increasing trees with the roots labelled 2 and 1, respectively, where x and u are marking vertices and vertices labelled 2, respectively. Since the left wings butterflies are missing in nondecreasing 2-noncrossing increasing trees, we have

$$I(x,u) = xu \cdot \frac{1}{1-J}, \tag{20}$$

where u marks a vertex labelled 2, and

$$J(x,u) = x \cdot \frac{1}{1-J} \cdot \frac{1}{1-I}. \tag{21}$$

Let $I = xu(1+y)$. Then, from (20), we get

$$J = \frac{y}{1+y} = y(1+y)^{-1}. \tag{22}$$

Using $I = xu(1+y)$ and (22) in (21), we obtain

$$y = x(1+y)(1+(1+u)y). \tag{23}$$

We note that (23) is in the right form to apply the Lagrange inversion formula. We now extract the coefficient of $x^n u^k$ in the generating functions $I(x,u)$ as follows:

$$\begin{aligned}
[x^n u^k]I &= [x^n u^k]uxy = [x^{n-1}u^{k-1}]y \\
&= \frac{1}{n-1}[t^{n-2}u^{k-1}]((1+(1+u)t)(1+t))^{n-1} \\
&= \frac{1}{n-1}[t^{n-2}u^{k-1}]\sum_{i\geq 0}\binom{n-1}{i}((1+u)t)^i \sum_{j\geq 0}\binom{n-1}{j}t^j \\
&= \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{i}{k-1}\binom{n-1}{n-i-2}.
\end{aligned} \tag{24}$$

Similarly, we have

$$\begin{aligned}
[x^n]J &= [x^n u^k]y(1+y)^{-1} = [x^n u^k]y\sum_{a\geq 0}\binom{-1}{a}y^a = \sum_{a\geq 0}\binom{-1}{a}[x^n u^k]y^{a+1} \\
&= \sum_{a\geq 0}\binom{-1}{a}\frac{a+1}{n}[t^{n-a-1}u^k]((1+(1+u)t)(1+t))^n \\
&= \sum_{a\geq 0}\binom{-1}{a}\frac{a+1}{n}[t^{n-a-1}u^k]\sum_{i\geq 0}\binom{n}{i}((1+u)t)^i\sum_{j\geq 0}\binom{n}{j}t^j \\
&= \sum_{a\geq 0}\binom{-1}{a}\frac{a+1}{n}[t^{n-a-1}]\sum_{i\geq 0}\binom{n}{i}\binom{i}{k}\sum_{j\geq 0}\binom{n}{j}t^{i+j} \\
&= \frac{1}{n}\sum_{i\geq 0}\sum_{a\geq 0}\left[\binom{-1}{a}-\binom{-2}{a-1}\right]\binom{n}{i}\binom{i}{k}\binom{n}{n-i-a-1}.
\end{aligned}$$

By the Vandermonde convolution, we have

$$\begin{aligned}
[x^n]J &= \frac{1}{n}\sum_{i\geq 0}\left[\binom{n-1}{n-i-1}-\binom{n-2}{n-i-2}\right]\binom{n}{i}\binom{i}{k} = \frac{1}{n}\sum_{i\geq 0}\frac{i}{n-1}\binom{n}{i}\binom{i}{k}\binom{n-1}{n-i-1} \\
&= \frac{1}{n-1}\sum_{i=1}^{n-1}\binom{n-1}{i-1}\binom{i}{k}\binom{n-1}{n-i-1} = \frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{i+1}{k}\binom{n-1}{n-i-2}.
\end{aligned} \tag{25}$$

Adding (24) and (25), we get

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{k+i+1}{k}\binom{n-1}{i}\binom{i}{k-1}\binom{n-1}{i+1}.$$

We thus have the following theorem:

Theorem 5.1. *The number of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with the roots labelled* 2*, such that there are* k *vertices labelled* 2*, is given by*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{i}{k-1}\binom{n-1}{i+1}, \tag{26}$$

and if the roots are labelled 1 *such that there are* k *vertices with label* 2*, then the mentioned number is given as follows:*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{i+1}{k}\binom{n-1}{i+1}. \tag{27}$$

Also, total number of nondecreasing 2*-noncrossing increasing trees on* n *vertices such that there are* k *vertices labelled* 2 *is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{k+i+1}{k}\binom{n-1}{i}\binom{i}{k-1}\binom{n-1}{i+1}.$$

By summing over all k in (26) and (27), we obtain the next result.

Corollary 5.1. *The number of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with the roots labelled* 2 *is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{n-1}{i+1}2^i, \tag{28}$$

and if the roots are labelled 1*, then the mentioned number is*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{n-1}{i+1}2^{i+1}. \tag{29}$$

In total, there are

$$\frac{3}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i}\binom{n-1}{i+1}2^i$$

nondecreasing 2*-noncrossing increasing trees on* n *vertices.*

By setting $k=1$ in (26) or $k=0$ in (27), we deduce that there are

$$\frac{1}{n-1}\binom{2n-2}{n}$$

noncrossing increasing trees on n vertices; hence, this is another manifestation of Catalan numbers as proved in [1].

5.2. Label of leftmost child of the root

The leftmost child of the root of a nondecreasing 2-noncrossing increasing tree whose root is labelled 2, in its (l,r)-representation, is always labelled 1. Thus, it is not interesting to study such trees. We therefore enumerate trees in which the root is labelled 1.

Theorem 5.2. *The number of nondecreasing* 2*-noncrossing trees on* n *vertices with the root labelled 1 such that the leftmost child of the root is labelled* 1 *is given by*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{(2i-n+2)(i+1)}{n-2}\binom{n-1}{i+1}\binom{n-1}{i}2^i. \tag{30}$$

Proof. Since the generating function for the number of nondecreasing 2-noncrossing increasing trees with the root labelled 1 is $J(x) = y(1+y)^{-1}$, where $y = x(1+y)(1+2y)$ and x marks a vertex, the required formula is

$$
\begin{aligned}
[x^n]J(x)^2 = [x^n]y^2(1+y)^{-2} &= \sum_{a\geq 0}\binom{-2}{a}[x^n]y^{a+2} \\
&= \sum_{a\geq 0}\binom{-2}{a}\frac{a+2}{n}[t^{n-a-2}](1+t)^n(1+2t)^n \\
&= \frac{1}{n}\sum_{a\geq 0}\left[\binom{-2}{a} - 2\binom{-3}{a-1}\right][t^{n-a-2}]\sum_{j\geq 0}\binom{n}{j}\sum_{i\geq 0}\binom{n}{i}2^i t^{i+j} \\
&= \frac{1}{n}\sum_{a\geq 0}\left[\binom{-2}{a} - 2\binom{-3}{a-1}\right]\sum_{i\geq 0}\binom{n}{n-a-i-2}\binom{n}{i}2^i \\
&= \frac{1}{n}\sum_{i\geq 0}\left[\binom{n-2}{i} - 2\binom{n-3}{i}\right]\binom{n}{i}2^i \\
&= \frac{1}{n-2}\sum_{i=0}^{n-2}\frac{2i-n+2}{n-i}\binom{n-2}{i}\binom{n-1}{i}2^i.
\end{aligned}
$$

The second-last equality follows from the Vandermonde convolution. □

Corollary 5.2. *The number of nondecreasing* 2*-noncrossing trees on* n *vertices with the root labelled 1, such that the leftmost child of the root is labelled* 2*, is given by*

$$\frac{1}{n-1}\sum_{i=0}^{n-2}\frac{(n-2i-2)(i+1)+2n-4}{n-2}\binom{n-1}{i+1}\binom{n-1}{i}2^i. \tag{31}$$

Proof. We obtain (31) by subtracting (30) from (29). □

We remark here that (31) gives the number of nondecreasing 2-noncrossing increasing trees in which the root is labelled 1 and all the vertices attached to the root are labelled 2.

5.3. Root degree

Theorem 5.3. *There are*

$$\frac{1}{n-q-1}\sum_{i=0}^{n-p-q-1}\frac{q(n-q-1)+i(p-q)}{n-p-1}\binom{n-p-1}{q+i}\binom{n-q-1}{i}2^i \tag{32}$$

nondecreasing 2*-noncrossing increasing trees on* n *vertices with the root labelled 1 such that there are* p *and* q *vertices adjacent to the root that are labelled 1 and 2, respectively.*

Proof. We extract the coefficient of x^n in the generating function $xJ(x)^pI(x)^q$.

$$
\begin{aligned}
[x^n]xJ(x)^pI(x)^q = [x^{n-1}]\left(\frac{y}{1+y}\right)^p(x(1+y))^q = [x^{n-q-1}]y^p(1+y)^{q-p} &= \sum_{a\geq 0}\binom{q-p}{a}[x^{n-q-1}]y^{p+a} \\
= \sum_{a\geq 0}\binom{q-p}{a}\frac{p+a}{n-q-1}[t^{n-q-p-a-1}](1+t)^{n-q-1}(1+2t)^{n-q-1} \\
= \frac{1}{n-q-1}\sum_{a\geq 0}\left[p\binom{q-p}{a} + (q-p)\binom{q-p-1}{a-1}\right][t^{n-p-q-a-1}]\sum_{j\geq 0}\binom{n-q-1}{j}\sum_{i\geq 0}\binom{n-q-1}{i}2^i t^{i+j}.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
[x^n]xJ(x)^pI(x)^q &= \frac{1}{n-q-1}\sum_{a\geq 0}\left[p\binom{q-p}{a}+(q-p)\binom{q-p-1}{a-1}\right]\sum_{i\geq 0}\binom{n-q-1}{n-p-q-a-i-1}\binom{n-q-1}{i}2^i\\
&= \frac{1}{n-q-1}\sum_{i\geq 0}\left[p\binom{n-p-1}{q+i}+(q-p)\binom{n-p-2}{q+i}\right]\binom{n-q-1}{i}2^i\\
&= \frac{1}{n-q-1}\sum_{i=0}^{n-p-q-1}\frac{q(n-q-1)+i(p-q)}{n-p-1}\binom{n-p-1}{q+i}\binom{n-q-1}{i}2^i.
\end{aligned}
$$

The second-last equality follows from the Vandermonde convolution. □

We remark that the label of the root of the tree can be changed to 2 if all the vertices adjacent to the root are labelled 1. So, by setting $q=0$ in (32), we obtain the following result:

Corollary 5.3. *The number of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with the root labelled 2, such that the root has degree* p*, is given as follows:*

$$
\frac{p}{n-1}\sum_{i=0}^{n-p-2}\binom{n-p-2}{i}\binom{n-1}{i+1}2^{i+1}. \tag{33}
$$

If we substitute $p=1$ in (33), we get the total number of planted nondecreasing 2-noncrossing increasing trees on n vertices with the root labelled 2. This formula is given by

$$
\frac{1}{n-1}\sum_{i=0}^{n-3}\binom{n-3}{i}\binom{n-1}{i+1}2^{i+1}.
$$

We rediscover the following result, already proved at the beginning of this section:

Corollary 5.4. *The number of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with the root labelled 2 is*

$$
\frac{1}{n-1}\sum_{i=0}^{n-1}\binom{n-1}{i+1}\binom{n-1}{i}2^{i}.
$$

Proof. We sum over all p in (33). □

By setting $p=0$ in (32), we obtain the next result.

Corollary 5.5. *The number of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with the root labelled 1 and of degree* q*, when all the vertices adjacent to the root are labelled 2, is*

$$
\frac{q}{n-1}\sum_{i=0}^{n-q-1}\binom{n-1}{q+i}\binom{n-q-2}{i}2^{i}. \tag{34}
$$

By taking $q=1$ in (34), we get the number of planted trees of nondecreasing 2-noncrossing increasing trees on n vertices, such that the root is labelled 1, as follows:

$$
\frac{1}{n-1}\sum_{i=0}^{n-2}\binom{n-1}{i+1}\binom{n-3}{i}2^{i}.
$$

5.4. Forests

Theorem 5.4. *There are*

$$
\frac{1}{n-q}\binom{p+q}{p}\sum_{i=0}^{n-p-q}\frac{q(n-q)+i(p-q)}{n-p}\binom{n-p}{q+i}\binom{n-q}{i}2^{i} \tag{35}
$$

forests of nondecreasing 2*-noncrossing increasing trees on* n *vertices with the root labelled 1 such that there are* p *and* q *components whose roots are labelled by 1 and 2, respectively.*

Proof. We extract the coefficient of x^n in the generating function $J(x)^p I(x)^q$.

$$[x^n]J(x)^p I(x)^q = [x^n]\left(\frac{y}{1+y}\right)^p (x(1+y))^q = [x^{n-q}]y^p(1+y)^{q-p} = \sum_{a\geq 0}\binom{q-p}{a}[x^{n-q}]y^{p+a}$$
$$= \sum_{a\geq 0}\binom{q-p}{a}\frac{p+a}{n-q}[t^{n-q-p-a}](1+t)^{n-q}(1+2t)^{n-q}$$
$$= \frac{1}{n-q}\sum_{a\geq 0}\left[p\binom{q-p}{a} + (q-p)\binom{q-p-1}{a-1}\right][t^{n-p-q-a}]\sum_{j\geq 0}\binom{n-q}{j}\sum_{i\geq 0}\binom{n-q}{i}2^i t^{i+j}$$
$$= \frac{1}{n-q}\sum_{a\geq 0}\left[p\binom{q-p}{a} + (q-p)\binom{q-p-1}{a-1}\right]\sum_{i\geq 0}\binom{n-q}{n-p-q-a-i}\binom{n-q}{i}2^i.$$

Summing over all values of a, we get

$$[x^n]J(x)^p I(x)^q = \frac{1}{n-q}\sum_{i\geq 0}\left[p\binom{n-p}{q+i} + (q-p)\binom{n-p-1}{q+i}\right]\binom{n-q}{i}2^i$$
$$= \frac{1}{n-q}\sum_{i=0}^{n-p-q}\frac{q(n-q)+i(p-q)}{n-p}\binom{n-p}{q+i}\binom{n-q}{i}2^i.$$

Since there are $\binom{p+q}{p}$ different arrangements for the components, we have the desired result by the product rule of counting. □

By setting $q = 0$ in (35), we obtain the next result.

Corollary 5.6. *The number of forests of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with* p *components, whose roots are all labelled 1, is given as follows:*

$$\frac{p}{n}\sum_{i=0}^{n-p-1}\binom{n-p-1}{i}\binom{n}{i+1}2^{i+1}. \tag{36}$$

If we substitute $p = 1$ in (36), we get the total number of nondecreasing 2-noncrossing increasing trees on n vertices with the root labelled 1:

$$\frac{1}{n}\sum_{i=0}^{n-2}\binom{n-2}{i}\binom{n}{i+1}2^{i+1}.$$

By setting $p = 0$ in (35), we obtain the following corollary:

Corollary 5.7. *The number of forests of nondecreasing* 2*-noncrossing increasing trees on* n *vertices with* q *components, whose roots are labelled 2, is given as follows:*

$$\frac{q}{n}\sum_{i=0}^{n-q}\binom{n}{q+i}\binom{n-q-1}{i}2^i. \tag{37}$$

By taking $q = 1$ in (37), we get the number of nondecreasing 2-noncrossing increasing trees on n vertices, such that the root is labelled 2, as given below:

$$\frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i+1}\binom{n-2}{i}2^i.$$

6. Bijections of nondecreasing 2-noncrossing trees

6.1. Locally oriented noncrossing trees

Theorem 6.1. *There is a bijection between the set of nondecreasing* 2*-noncrossing trees on* n *vertices with the root labelled* 1 *and the set of locally oriented noncrossing trees on* n *vertices with non-root sources and sinks coming in 2 colours.*

Proof. Let T be a nondecreasing 2-noncrossing tree on n vertices with the root labelled 1. We obtain its (l, r)-representation. It is worth noting that there are no (l_2, r_2) and (r_2, r_2) edges. We orient all the edges of T from a vertex of a lower label towards a vertex of a higher label. Note that the edges are labelled away from the vertices labelled l at a lower level and also towards r if the other endpoint is labelled l. If both endpoints are labelled r, then the edges are oriented away from the root. We proceed as follows:

(i) We traverse the tree in preorder.

(a) If a vertex v, which is neither a source nor a sink, labelled l_2 is encountered, then detach the subtrees rooted at the children of v labelled r_1 and attach them to the parent of v in order from left to right so that the detached children of v appear on the immediate right of v. Relabel these detached children of v as l_1 and orient the edges towards the parent of v. Also, relabel the remaining children of v labelled l_1 (respectively, l_2) as r_1 (respectively, r_2) and orient the edges away from v. Note that the vertex v becomes a source labelled l_2 by this procedure.

(b) If a vertex v, which is neither a source nor a sink, labelled r_2 is encountered, then detach the subtrees rooted at the children of v labelled r_1 and attach them to the parent of v in order from left to right so that the detached children of v appear on the immediate right of v. By this procedure, the vertex v becomes a sink labelled r_2.

(ii) By step (i), we find that vertices, which are neither sources nor sinks, are labelled r_1 or l_1. Drop the label 1 so that they are labelled r or l. Finally, the vertices labelled r_1 and r_2 are sinks (respectively, labelled l_1 and l_2 are sources), where the subscripts 1 and 2 are the colours.

The obtained tree is a locally oriented noncrossing tree in which non-root sources and sinks come in two colours.

For the reverse procedure, we obtain the (l, r)-representation of the locally oriented noncrossing tree in which non-root sources and sinks come in two colours 1 and 2; that is, r_1 and r_2 are sinks, and l_1 and l_2 are sources. We label the root as 1 and all vertices which are neither sources nor sinks as l_1 and r_1 if they were initially labelled l and r, respectively. We traverse the locally oriented noncrossing tree, in preorder and using the following:

(I) If a non-root source v, labelled l_2, is encountered, detach the subtrees rooted at its siblings, labelled l_1 (if any), on its immediate right and attach them to v in order from left to right so that the detached siblings of v appear on the immediate right of the initial children of v. Relabel these detached siblings of v as r_1. Moreover, relabel the initial children of v labelled r_1 (respectively, r_2) as l_1 (respectively, l_2). The vertex v becomes a non-source vertex labelled l_2.

(II) If a sink v, labelled r_2, is encountered, detach the subtrees rooted at siblings, labelled r_1 (if any), on its immediate right and attach them to v in order from left to right so that the detached siblings of v appear on the immediate right of the initial children of v retaining their labels. By this procedure, vertex v becomes a non-sink vertex labelled r_2.

We remove the orientations of the edges. By this procedure, we obtain a nondecreasing 2-noncrossing trees on n vertices with the root labelled 1. The bijection is illustrated in Figure 6.1. □

6.2. Complete ternary trees

In [14], Okoth and Wagner obtained the following result:

Proposition 6.1 (Okoth and Wagner, 2015)**.** *There is a bijection between the set of locally oriented noncrossing trees on n vertices with k sources and ℓ sinks, and the set of ternary trees on $n-1$ vertices with $k-1$ middle-edges and $\ell-1$ left-edges.*

In the bijection, vertex 1 (a source) and vertex n (a sink) are not used in determining the position of the edges. By Theorem 6.1, vertex n is a sink labelled 1 or 2. So, we use this label to assign a colour to the root of a ternary tree to obtain the next result.

Proposition 6.2. *There is a bijection between the set of nondecreasing 2-noncrossing trees on n vertices with the root labelled 1 and the set of ternary trees with $n-1$ vertices such that all left-edges and middle-edges come in two colours and the root also comes in two colours.*

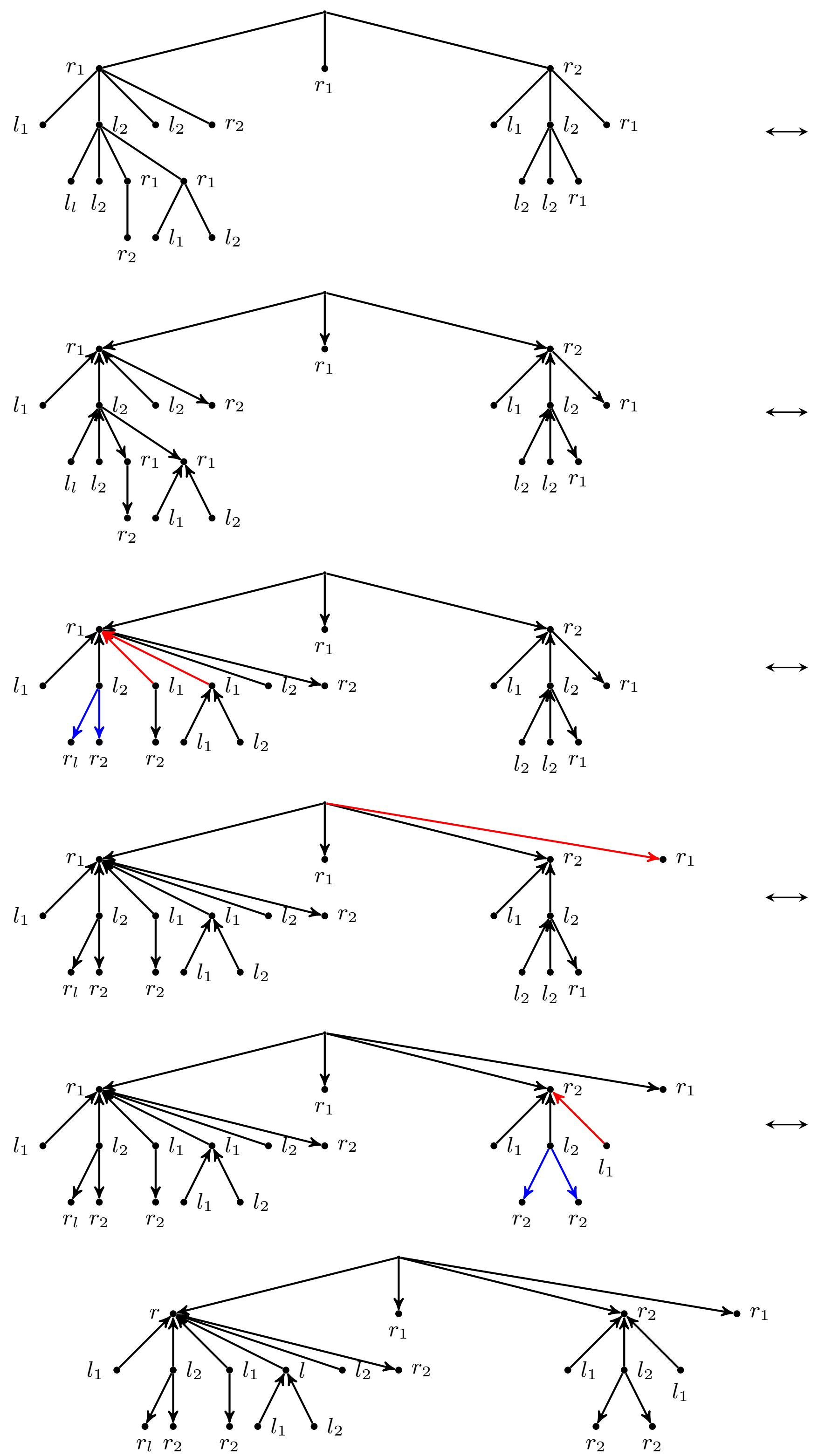

Figure 6.1: Obtaining a locally oriented noncrossing tree in which non-root sources and sinks come in two colours from a nondecreasing 2-noncrossing tree with the root labelled 1 and vice versa.

In Drake's thesis [3, Example 1.6.9], he showed that the number of complete ternary trees with $2n-1$ leaves (i.e. the tree has $n-1$ internal vertices), such that all internal vertices come in two colours and the rightmost child of every internal vertex is assigned a different colour from that of its parent, is given by

$$\frac{1}{n-1}\binom{2n-1}{i}\binom{n-1}{i+1}2^{i+1}.$$

Therefore, we prove the next result.

Theorem 6.2. *There exists a bijection between the set of nondecreasing 2-noncrossing trees on n vertices with the root labelled 1 and the set of complete ternary trees with $2n-1$ leaves such that all internal vertices are coloured 1 or 2 and the right child of every internal vertex is assigned a different colour from that of its parent.*

Proof. From Proposition 6.2, we have a bijection between the set of nondecreasing 2-noncrossing trees on n vertices with the root labelled 1 and the set of ternary trees with $n-1$ vertices such that left- and middle-edges are coloured 1 or 2 and the root is also coloured 1 or 2. So, the required bijection is achieved via the corresponding locally oriented noncrossing tree and root labelled ternary tree. From a ternary tree on $n-1$ vertices in which the root is coloured 1 or 2 and the left- and middle-edges are coloured 1 or 2, assign the colour of the left- and middle-edge to the vertex just below it. We obtain a ternary tree in which the first two children of every non-root vertex are coloured 1 or 2 and the root is also coloured 1 or 2. We complete the tree by attaching $2n-1$ vertices (which will be leaves) so that every internal vertex has 3 children; that is, we obtain a complete ternary tree. We remain to assign colours to the rightmost child of every internal vertex.

(i) Since a right child of every internal vertex, which is also an internal vertex, is assigned a different colour from that of its parent, there is one way of doing that; that is, if a non-root internal vertex is coloured 1 (respectively, 2), then its right child must be coloured 2 (respectively, 1).

(ii) If the colour of the root is chosen to be 1 (respectively, 2) then an internal vertex, which is a right child of the root (if any), is assigned colour 2 (respectively, 1). The remaining right children of internal vertices are then assigned colours as in (i).

For the reverse procedure, from a complete ternary tree with $2n-1$ leaves such that the root is coloured 1 or 2, every internal vertex is coloured 1 or 2 and the right child of every internal vertex is assigned a color different from that of its parent, delete the colours of right children of all internal vertices of the tree. Then assign the colour of every non-root internal vertex to the edge just above it. We obtain the required complete ternary tree from which we get a ternary tree by deleting all the leaves. This procedure is described in Figure 6.2. □

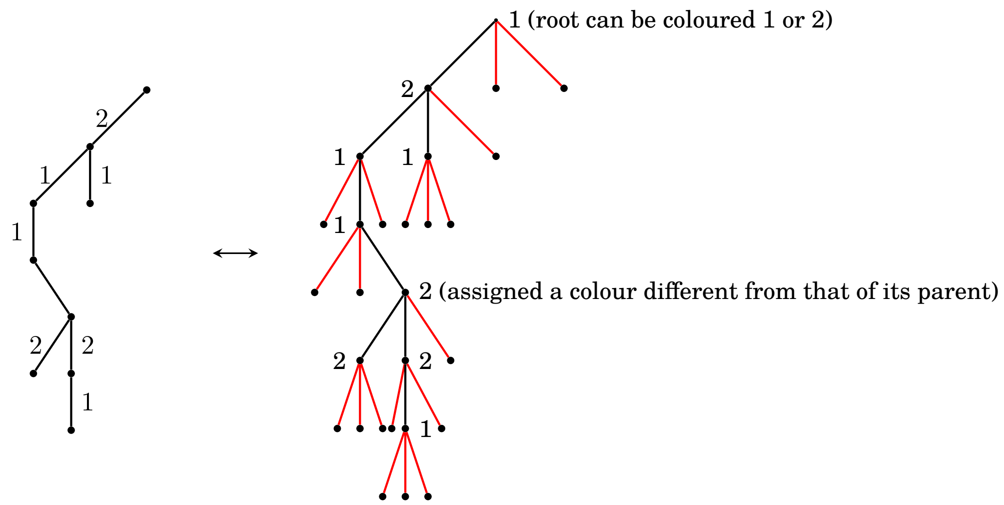

Figure 6.2: Obtaining a complete ternary tree in which the left and middle children of all internal vertices come in two colours and the colour of the right child of every internal vertex is different from that of its parent from a ternary tree in which the left-edges, middle-edges, and the root come in two colours and vice versa.

Corollary 6.1. *The number of complete ternary trees with $2n-1$ leaves, in which the root is given a specified colour, all internal vertices come in two colours, and the rightmost child of every internal vertex is assigned a different colour from that of its parent, is given by*

$$\frac{1}{n-1}\binom{2n-1}{i}\binom{n-1}{i+1}2^i.$$

Proof. The proof follows from Theorem 6.2 by assigning one colour (instead of two) to the root of the complete ternary tree in which all internal vertices come in two colours and the rightmost child of every internal vertex is assigned a different colour from that of its parent. □

6.3. 3-Schröder paths

Theorem 6.3. *There is a bijection between the set of complete ternary trees with $2n+1$ leaves such that the root comes in two colours, the first two children of all internal vertices come in two colours and the rightmost child of every internal vertex has a colour different from that of its parent, and the set of 3-Schröder paths of length n.*

Proof. We construct the bijection via Dyck paths in which the upsteps are of type $(1,2)$ and come in two colours.

Let T be a complete ternary tree with $2n+1$ leaves such that the root comes in two colours, the first two children of all internal vertices come in two colours and the rightmost child of every internal vertex has colour different from that of its parent. Let the colours be 1 and 2. We first obtain the corresponding Dyck path with n upsteps: Traversing the ternary tree in preorder, draw a $(1,2)$ step labelled 1 (respectively, labelled 2) for every internal vertex coloured 1 (respectively, coloured 2) and a $(1,-1)$ step for every leaf, except the last one. This results in a Dyck path in which the upsteps are labelled 1 or 2. Since the upsteps correspond to internal vertices, the number of upsteps is n and the downsteps are $2n$, i.e., one less than the number of leaves in the complete ternary tree.

We remark that for every upstep, there are two corresponding downsteps, i.e., the downsteps on the same level as the upstep, if one moves from left to right, without crossing any steps. Now, we obtain the 3-Schröder paths of length n from the Dyck path as follows: For every upstep labelled 2, create a $(2,1)$ step and delete a corresponding downstep which is at a higher level. Finally, remove all labels in the Dyck path. When a $(2,1)$ step is created, the length of the path increases by 1. Also, as a downstep is deleted, the length of the path decreases by 1. So, the initial length of the Dyck path is retained. The resultant structure is thus a 3-Schröder path of length n. Note that if the root of the complete ternary tree is coloured 1 (respectively, 2) then the first step in the 3-Schröder path is of type $(1,2)$ (respectively, $(2,1)$). This procedure is illustrated in Figure 6.3.

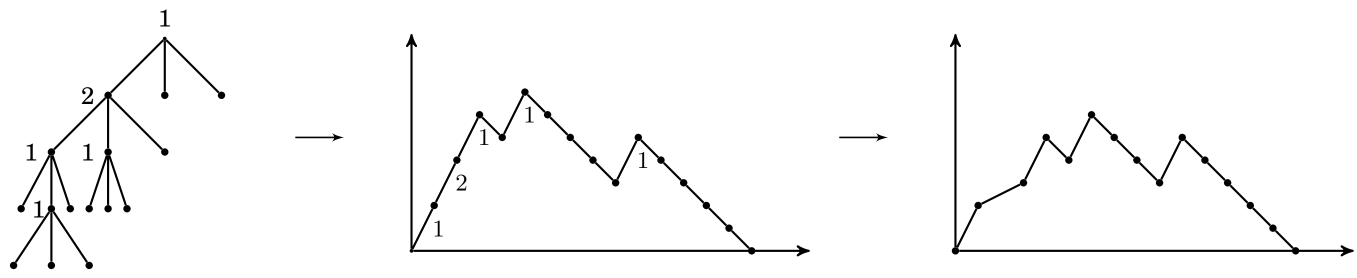

Figure 6.3: Obtaining a 3-Schröder path of length 5 from a complete ternary tree with 5 internal vertices in which the root and the first two children of all internal vertices come in two colours.

Conversely, let S be a 3-Schröder path of length n. Label all the $(1,2)$ steps as 1 and for every $(2,1)$ step, draw a $(1,2)$ step labelled 2 and introduce a corresponding downstep. The resulting figure is a Dyck path in which the upsteps are of type $(1,2)$ labelled 1 or 2. We then build a complete ternary tree from left to right as follows: Starting at the origin of the Dyck path, draw a root coloured 1 (respectively, 2), of the corresponding tree if the first upstep is labelled 1 (respectively, 2). Then, move to the next step of the Dyck path. If an upstep labelled 1 (respectively, 2) is encountered then introduce a new vertex attached to the previous vertex in the tree and colour it 1 (respectively, 2). If a downstep is encountered, then draw a leaf and move back to the parent of the leaf. Continue until all the steps in the Dyck path are considered. Finally, introduce a last leaf so that to complete the tree. The resultant combinatorial object is a complete ternary tree in which the root is given the label of the initial step in the Dyck path (or type of the initial upstep in the 3-Schröder path), the first two children of every internal vertex come in two colours and the rightmost child of every internal vertex has a colour different from that of its parent. A depiction of this process is given in Figure 6.4. □

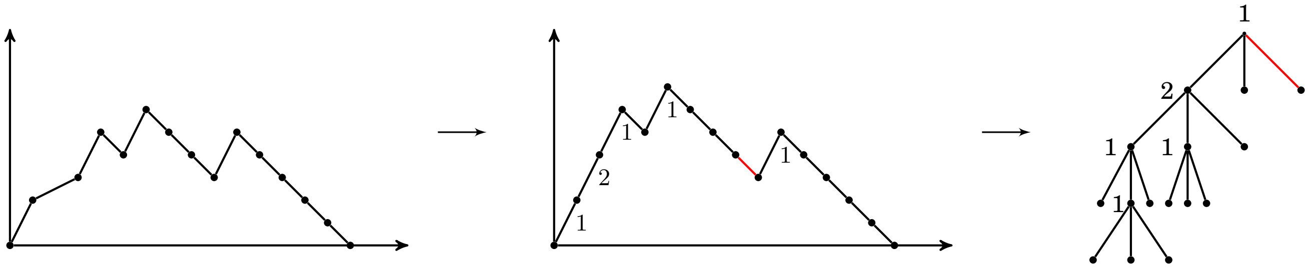

Figure 6.4: Obtaining a complete ternary tree with 5 internal vertices, in which the root and the first two children of all internal vertices come in two colours, from a 3-Schröder path of length 5.

Corollary 6.2. *The set of complete ternary trees with $2n-1$ leaves with the root coloured 2 is enumerated by*

$$\frac{1}{n-1}\sum_{i=0}^{n-1}\binom{2n-2}{i}\binom{n-1}{i+1}2^i.$$

By Theorems 6.2 and 6.3, we have:

Corollary 6.3. *There is a bijection between the set of nondecreasing 2-noncrossing trees on $n+1$ vertices with the root labelled 1 and the set of 3-Schröder paths of length n.*

Corollary 6.4. *There is a bijection between the set of nondecreasing 2-noncrossing trees on $n+1$ vertices with the root labelled 2 and the set of small 3-Schröder paths of length n.*

7. Conclusion and future work

In this paper, we have introduced nondecreasing 2-noncrossing trees and enumerated them by their number of vertices, label of the root, root degree, and number of forests. It would be interesting to study nondecreasing k-noncrossing trees and establish results similar to the ones obtained in this paper. Moreover, we have also introduced nondecreasing 2-noncrossing increasing trees and counted them by their number of vertices, label of the leftmost child of the root, root degree, and number of forests. These results can be generalised by considering k-noncrossing increasing trees introduced by Okoth in [11]. We also have constructed various bijections in relation to nondecreasing 2-noncrossing increasing trees. In some instances, we constructed bijections via complete ternary trees. It would be interesting to obtain these bijections directly. Furthermore, there exist combinatorial structures in Hopf algebras and phylogenetics. It seems to be interesting to investigate connections between these structures and nondecreasing 2-noncrossing trees.

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