SYMMETRIC OPERATOR EXTENSIONS OF COMPOSITES OF HIGHER ORDER DIFFERENCE OPERATORS

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ABSTRACT. In this paper we have considered two higher order difference operators generated by two higher order difference functions on the Hilbert space of square summable sequences. By allowing the leading coefficients to be unbounded and the other coefficients as constant functions, we have shown that the composites of two higher order difference operators are symmetric if the leading coefficients are scalar multiple of each other and the common divisor of their orders is 1. Using examples, we have shown that these conditions of symmetry cannot be weakened. Furthermore, we have shown that the deficiency index of the composite is equal to the sum of the deficiency indices of the individual operators and that the spectra of the self-adjoint operator extensions is the whole of the real line.

1. INTRODUCTION

Consider the even higher order symmetric difference equations of $2n^{th}$-order and $2m^{th}$-order given by the following

$$
\begin{aligned}
\mathcal{L}_1 y(t) &= \sum_{k=0}^{n} (-1)^k \Delta^k (p_k(t) \bigtriangledown^k y(t)) \\
&= \sum_{k=0}^{n} (-1)^k \Delta^k (p_k(t) \Delta^k y(t-k))
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
\mathcal{L}_2 y(t) &= \sum_{j=0}^{m} (-1)^j \Delta^j (q_j(t) \bigtriangledown^j y(t)) \\
&= \sum_{j=0}^{m} (-1)^j \Delta^j (q_j(t) \Delta^j y(t-j))
\end{aligned}
\tag{1.2}
$$

both defined on $H = \ell^2(\mathbb{N})$. Here, $y = y(t)$, $p_k = p_k(t)$, $q_j = q_j(t)$, $t \in \mathbb{N}$, $k = 0, 1, 2, ..., n$ and $j = 0, 1, 2, ..., m$. The operator Δ is a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$, $\forall t \in \mathbb{N}$. The spectral theory of difference operators generated by $\mathcal{L}_1(y)$ and $\mathcal{L}_2(y)$ have been analysed independently by different authors under various growth and decay conditions. For an overview of these results see [3, 4]. Recently, the attention has been shifted to the composites of the continuous counterparts of equations (1.1) and (1.2) in the case of unbounded coefficients. Two of the authors [8] deduced that the operators commute if they are of the same order and the absolutely continuous spectrum of their self-adjoint extensions is the whole real line. This was done by constructing the appropriate comparison algebras of the corresponding self-adjoint operator extensions and application of asymptotic integration under suitable decay and growth conditions. For the commutativity of the composites of differential equations, counterparts of (1.1) and (1.2), see the papers by Amitsur [1], Mironov [6, 7] and the references cited therein. Such like analysis are not available for the composites of (1.1) and (1.2).

However, it is a well known fact that results for symmetric differential operators are normally comparable to those of their discrete counterparts under similar growth and decay conditions. For such treaties see [4]. Therefore, in this study we have analysed the symmetries of the composites of (1.1) and (1.2), the existence of the symmetric extensions of their composites in addition to the spectral theory of these composites under some suitable decay and growth conditions.

This research is motivated by the researches in electromagnetism, electrodynamics, fluid mechanics, quantum physics and many branches of engineering which arises from the applications of differential and difference equations. Some of these equations lead to systems that are used for modeling and analysis of real engineering problems. For example in systems and control theory which is an interdisciplinary branch of electric-electronic engineering, a lot of difference and differential equations are used.

Our main interest is under what conditions shall $\mathcal{L}_1$ commute with $\mathcal{L}_2$ and whether the composite $\mathcal{L}_1\mathcal{L}_2$ has self-adjoint operator extensions. Our results show that $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ if they are of the same order, the leading coefficients are scalar multiple of each other and the other coefficients are constant functions. Using Cayley-transforms and von-Neumann theorems, we have derived the symmetric and self-adjoint operator extensions of the minimal operator associated with (generated by) the composite of $\mathcal{L}_1$ and $\mathcal{L}_2$. In the case of second order symmetric difference operators, we have shown that the deficiency index of the composite is equal to the sum of deficiency indices of the individual operators. The analysis of the first order systems of the composites of (1.1) and (1.2) and the existence of their self-adjoint extensions are done using asymptotic summation which is anchored on Levinson's Benzaid-Lutz theorem as stated below

Theorem 1.1. *(Levinson's Benzaid-Lutz Theorem* [5]*)*
Let $\Lambda(t, z) = diag\{\lambda_1(t, z), ..., \lambda_{2s}(t, z)\}$ for $t \geq t_0$. Assume

(i)$\lambda_i(t, z) \neq 0$ *for all $1 \leq i \leq 2s$ and $t \geq t_0$*

(ii) $R(t, z)$ is a $2s \times 2s$ matrix defined for all $t \geq t_0$,
satisfying $\Sigma_{t=0}^{\infty} |\frac{1}{\lambda_i(t,z)}| \|R(t,z)\| < \infty$, for all $i = 1, 2, ..., 2s$

(iii) $\Lambda(t, z)$ satisfies the following uniform dichotomy condition. For any pair of indices i and j, such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t, z)| \geq \delta$ for all $t \geq t_0$. Then, either $|\frac{\lambda_i(t)}{\lambda_j(t)}| \geq 1$ or $|\frac{\lambda_i(t)}{\lambda_j(t)}| \leq 1$ for a large t.

Then the linear system

$$Y(t + 1, z) = [\Lambda(t, z) + R(t, z)]Y(t, z) \tag{1.3}$$

has a fundamental matrix satisfying,

$$Y(t, z) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l, z) \ as \ t \to \infty.$$

Explicitly for the eigensolutions, we have

$$y_k(t, z) = (e_k(t, z) + r_{kk}(t, z)) \prod_{0}^{t-1} (\lambda_k(l, z)), \tag{1.4}$$

where, $e_k(t, z)$ is the normalized eigenvector and $r_{kk}(t, z) \to 0$ as $t \to \infty$.

This paper is organised as follows: 1. Introduction, 2. Symmetries and extensions of the composites and 3. Composites of order Two and Four.

2. **Symmetries and Extensions of the Composites**

In this section, we give the necessary and sufficient conditions for the composites of the symmetric difference operators generated by (1.1) and (1.2) to be symmetric and have symmetric self-adjoint operator extensions. Suppose in (1.1) and (1.2), $n = m$, $q_m(t) = \alpha p_n(t)$ with $\alpha \neq 0$, a real constant and all other coefficients are constant valued functions, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ and their composites are symmetric as shown below.

Theorem 2.1. *Suppose that $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ are symmetric difference equations given in (1.1) and (1.2) respectively with $p_n, q_m \neq 0$, $q_m(t)$ is a constant multiple of $p_n(t)$, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ if the common divisor of m and n is 1 and all the other coefficients are constants or when the rank of $\mathcal{L}_1$ and $\mathcal{L}_2$ is 2, $m = n$, q_m is a constant multiple of p_n while all other coefficients q_j, $j = 0, 1, 2, ..., m-1$ and p_k, $k = 0, 1, 2, ..., n-1$ are all constants. In particular,*

$$\begin{aligned}
\mathcal{L}_1(\mathcal{L}_2 y(t)) &= \mathcal{L}_2(\mathcal{L}_1 y(t)) \\
&= \sum_{k=0}^{n} (-1)^k \sum_{j=0}^{m} (-1)^j \{\Delta^k (p_k \Delta^j (q_j \Delta^{j+k} y(t - j - k)))\} \tag{2.1}
\end{aligned}$$

Proof. In the proof, we shall write $\mathcal{L}_1$ and $\mathcal{L}_2$ to imply $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ respectively. $\mathcal{L}_1$ and $\mathcal{L}_2$ are of orders $2n$ and $2m$ respectively and since by assumption the common divisors of m and n is 1, it implies that the common divisors of their order is 2. By the results of Mironov [6, 7] and those of Amitsur [1], the dimension of the space of common eigenfunctions of $\mathcal{L}_1$ and $\mathcal{L}_2$ is 2 and hence

$\mathcal{L}_1$ and $\mathcal{L}_2$ commute. On the other hand, if we assume $m = n$ and all other coefficients apart from p_n and q_m are constants, then by expanding $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\mathcal{L}_2(\mathcal{L}_1 y(t))$ using the definition of Δ or Δ as forward or backward difference operators and comparing all the terms of these expansions, it suffices to check the equality on the coefficients of the terms $y(t+2n-1)$ which is similar to those of the term $y(t-2n+1)$ albeit with some shift in the independent coefficients as a result of successive applications of shift operators since $\Delta = E - I$ where E is a forward shift, $E(t) = t+1$ and I is the appropriate identity. Here we always note that $q_m(t) = \alpha p_n(t)$ for all $t \in \mathbb{N}$. The coefficients of the term $y(t+2n-1)$ for both composites are given by
$(-1)^n\{q_{n-1}p_n(t+n-1) + \alpha[p_{n-1}p_n(t+n-1) + p_n(t)p_n(t+n-1) + 2(p_n(t+n-1))^2 + p_n(t+n-1)p_n(t+n)]\}$,
where q_{n-1} and p_{n-1} are constants as per the assumptions made. Now by mathematical induction, it follows that the composite can be written as claimed in the theorem. □

We now demonstrate using examples that the results of Theorem 2.1 are true and that some of the conditions given in Theorem 2.1 cannot be weakened.

Example 2.2. This example verifies the results of Theorem 2.1. We assume that all the conditions as outlined in Theorem 2.1 are satisfied and consider second order difference equations of the form:

$$\begin{aligned} \mathcal{L}_1(y) &= -\Delta(p(t)\Delta y(t-1)) + \beta_1 y(t) \\ \mathcal{L}_2(y) &= -\Delta(\alpha p(t)\Delta y(t-1)) + \beta_2 y(t) \end{aligned}$$

where α, β_1 and β_2 are real valued constants, $\alpha \neq 0$.

By expansion and comparing the coefficients of the leading terms in the composites $\mathcal{L}_1(\mathcal{L}_2 y)$ and $\mathcal{L}_2(\mathcal{L}_1 y)$, we have that for the term $y(t+2)$, the coefficients are given as $\alpha p(t+1)p(t+2)$ for $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\alpha p(t+1)p(t+2)$ for $\mathcal{L}_2(\mathcal{L}_1 y(t))$. The term $y(t+1)$ has the coefficients given as $-\beta_2 p(t+1) - \alpha[\beta_1 p(t+1) + p(t)p(t+1) + 2\{p(t+1)p(t+1)\} + p(t+1)p(t+2)]$ for both composites. For the term $y(t)$, the coefficients of the two composites are given by $\beta_1 \beta_2 + \beta_2(p(t)p(t+1)) + \alpha[2\{p(t)p(t)\} + 2\{p(t)p(t+1)\} + 2\{p(t+1)p(t+1)\} + \beta_1(p(t)p(t+1))]$ while for the term $y(t-1)$, we have, $-\beta_2 p(t) - \alpha[\beta_1 p(t) + p(t)p(t-1) + 2\{p(t)p(t)\} + p(t+1)p(t)]$ as the coefficients and for the term $y(t-2)$ the coefficients are given by; $\alpha p(t)p(t-1)$ in both composites. These clearly show that $\mathcal{L}_1(\mathcal{L}_2 y)$ is equal to $\mathcal{L}_2(\mathcal{L}_1 y)$

Example 2.3. In this example, we confirm that some of the conditions imposed on the arbitrary leading coefficients in Theorem 2.1 are necessary. Assume that $n = m = 2$ with $p_2 = p$, $q_2 = q$, $p_1 = q_1 = 0$ $\forall t \in \mathbb{N}$ and let $p_0 = r_1$, $q_0 = r_2$ where r_i, $i = 1, 2$ are non-zero constants. By expansion of the composites $\mathcal{L}_1(\mathcal{L}_2 y(t))$ and $\mathcal{L}_2(\mathcal{L}_1 y(t))$, the coefficients of $y(t \pm l)$, $l = 0, 1, 2, 3, 4$ shows some symmetry with the coefficients of $y(t+l)$ matching those of $y(t-l)$ with some kind of a shift which is as a result of successive application of shift operator, $Ef(t) = f(t+1)$. Note that the forward difference operator Δ can be re-written as $\Delta = E - I$ where I is the identity operator. On checking the coefficients of $y(t+4)$ for the composites, we obtain either $p(t+2)q(t+4)$ or $q(t+2)p(t+4)$ and

for $y(t-4)$ either the coefficients are $p(t)q(t-2)$ or $q(t)p(t-2)$. The coefficients of the term $y(t+3)$ are given by $-[2p(t+1)q(t+3)+4p(t+2)q(t+3)+2p(t+2)q(t+4)]$ or $-[2q(t+1)p(t+3)+4q(t+2)p(t+3)+2q(t+2)p(t+4)]$. The coefficients of the other middle terms, that is, $y(t\pm l)$, $l=0,1,2$ have similar patterns showing that $\mathcal{L}_1$ commutes with $\mathcal{L}_2$ if and only if $q(t)=\alpha p(t)$ for some non-zero constant α and for all $t\in\mathbb{N}$. This confirms the requirement in Theorem 2.1 that the leading coefficients must be constant multiple of each other.

Remark 2.4. We point out that if $m\neq n$, $q_m(t)$ and $p_n(t)$ are not constants, then even if $q_m(t)$ is a constant multiple of $p_n(t)$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are not commutative as shown in the next example.

Example 2.5. In this example we show that if the orders of the difference equations in (1.1) and (1.2) are not equal, then even if the leading coefficients are scalar multiple of each other, the results in Theorem 2.1 will not hold. For simplicity, we consider the following order two and four symmetric difference equations with leading coefficients as scalar multiple of the other given by

$$\mathcal{L}_1(y)=\Delta^2(p(t)\Delta^2 y(t-2))$$

and

$$\mathcal{L}_2(y)=-\Delta(\alpha p(t)\Delta y(t-1)).$$

By expansion and comparing the coefficients of the leading terms of their composites, we have for the term $y(t+2)$ as $\alpha\{p(t)p(t+2)+3p(t+1)p(t+2)+2p(t+1)p(t+3)\}$ for $\mathcal{L}_1(\mathcal{L}_2 y)$ and $\alpha\{2p(t+1)p(t+2)+3[p(t+2)]^2+p(t+2)p(t+3)\}$ for $\mathcal{L}_2(\mathcal{L}_1 y)$ which shows that $\mathcal{L}_1(\mathcal{L}_2 y)$ is not equal to $\mathcal{L}_2(\mathcal{L}_1 y)$.

Now suppose that L_1 and L_2 are minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ respectively, by the results of G. Ren and Y. Shi [12], they had shown that the respective maximal difference operators risk being multi-valued and not densely defined. In order to rescue such a scenario, G. Ren and Y. Shi [13, 14] had reverted to using subspace theory to analyse the deficiency indices and spectrum of such difference equations with the only option of reverting to operator theory approach if and only if the respective maximal difference operators are densely defined and single-valued. This situation can be navigated by working on a subspace of $\ell^2(\mathbb{N})$ since as outlined in [13] and eventually applied by Behncke and Nyamwala [4], there exists a finite interval $I_1\subset\mathbb{N}$ such that for any non-trivial solution $y(t)$ of (1.1) and (1.2), one has

$$\sum_{t\in I_1}Ry^*(t)Ry(t)>0.$$

Here, $Ry(t)$ is a partial shift operator for which if $y(t)=[y_1(t),y_2(t),...,y_{2n}(t)]^{tr}$, then $Ry(yt)=[y_1(t+1),...,y_n(t+1),y_{n+1}(t),...,y_{2n}(t)]^{tr}$. Now the deficiency indices and spectral results obtained in $\ell^2(I_1)$ can then easily be extrapolated to $\ell^2(\mathbb{N})$ using Remling's results [10] which were later validated in the discrete setting by Y. Shi [16]. With these clarifications, we therefore show that the operators

generated by the composites of $\mathcal{L}_1\mathcal{L}_2$ are symmetric and have symmetric operator extensions when the conditions outlined in Theorem 2.1 are satisfied.

Theorem 2.6. *Assume that all the conditions of Theorem 2.1 are satisfied, then the minimal operator generated by* (2.1) *is symmetric and has symmetric operator extensions.*

Proof. Suppose all the conditions of Theorem 2.1 are satisfied, then $\mathcal{L}_1$ commutes with $\mathcal{L}_2$, and therefore, for all $y(t), y_1(t) \in \ell^2(\mathbb{N})$, we have

$$
\begin{aligned}
\langle \mathcal{L}_1\mathcal{L}_2 y(t), y_1(t) \rangle &= \langle \mathcal{L}_2 y(t), \mathcal{L}_1 y_1(t) \rangle = \langle y(t), \mathcal{L}_2\mathcal{L}_1 y_1(t) \rangle \\
&= \langle y(t), \mathcal{L}_1\mathcal{L}_2 y(t) \rangle = \langle (\mathcal{L}_1\mathcal{L}_2)^* y(t), y_1(t) \rangle.
\end{aligned}
$$

This implies that $D(\mathcal{L}_1\mathcal{L}_2) \subseteq D(\mathcal{L}_1\mathcal{L}_2)^*$, and hence the composites $\mathcal{L}_1\mathcal{L}_2$ and $\mathcal{L}_2\mathcal{L}_1$ are symmetric. We now construct the respective minimal and maximal symmetric operators generated by the composite $\mathcal{L}_1\mathcal{L}_2 y(t)$.

In line with the quasi-differences given by Y. Shi [16], we construct the quasi-differences for the composites $\mathcal{L}_1\mathcal{L}_2 y$ as follows:

$$
\begin{aligned}
x_j(t) &= q_j(t)\Delta^{j-1}y(t-j) \qquad 1 \leq j \leq m \\
x_{m+1}(t) &= q_m(t)\Delta^m(y(t-m)) \\
x_{m+k}(t) &= -\Delta(q_m(t)\Delta^m y(t-m)) - p_{n-k}(t)(q_{n-k}(t)\Delta^{n-k}y(t-n+k)) \\
&\qquad 2 \leq k \leq n \\
x_{n+m+r}(t) &= -(\Delta(x_{n+m+r-1}(t)) + p_{n-r}(t)x_{m+n-r}(t)) \qquad 2 \leq r \leq n+m-1
\end{aligned}
\tag{2.2}
$$

In order to apply the quasi-differences above to convert $\mathcal{L}_1\mathcal{L}_2 y(t)$ into its first order form, we define the vector valued functions $Y(t) = [x_1(t), x_2(t), ...x_{2(n+m)}(t)]^{tr}$. Just like in the case of non-composites given in [16] and applied in [3], we also use symplectic matrix defined by:

$$
\mathcal{J} = \begin{bmatrix} 0_{n+m} & -I_{n+m} \\ I_{n+m} & 0_{n+m} \end{bmatrix}.
$$

It follows that

$$
\mathcal{J}\Delta Y(t) = P(t)RY(t),
$$

where $P(t)$ is a $2(n+m) \times 2(n+m)$ matrix that can be written in a block form

$$
P(t) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}
$$

with A, B, C as $(n+m) \times (n+m)$ matrices with non-zero entries given by

$$
A_{k,k+1} = 1 \qquad\qquad B_{n+m,n+m} = (p_n q_m)^{-1}
$$
$$
C_{k,k} = \sum_{k=1}^{n}\sum_{j=1}^{m}(p_{k-1}q_{j-1} + p_{n-1}q_{m-1}).
$$

Always note that by construction, $n = m$, p_n and q_m are scalar multiple of each other with the other coefficients as constants. $p_n(t), q_m(t) \neq 0$ for all $t \in \mathbb{N}$. R as mentioned earlier is a partial shift operator and in this case,

$$
RY(t) = [x_1(t+1), x_2(t+1), ...x_{n+m}(t+1), x_{n+m+1}(t), ...x_{2(n+m)}(t)]^{tr}
$$

. Thus for the Hilbert space $\ell^2(\mathbb{N})$, the space is now defined by

$$\ell^2(\mathbb{N}) \quad = \quad \{y(t) : y(t) = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C} \text{ and } \sum_{t=0}^{\infty} R(y^*(t))(Ry(t)) < \infty\}.$$

As before, there exists a finite interval $I_1 \subset \mathbb{N}$ such that

$$\sum_{t \in I_1} Ry^*(t) Ry(t) > 0,$$

so that we define $\ell^2(I_1)$ as a subspace of $\ell^2(\mathbb{N})$ by

$$\ell^2(I_1) = \{y(t) \in \ell^2(\mathbb{N}) : \sum_{t \in I_1} Ry^*(t) Ry(t) > 0\}.$$

The scalar product is defined in the usual way. Therefore, one defines the maximal difference operator L^* on $\ell^2(I_1)$, the subspace of $\ell^2(\mathbb{N})$ generated by (2.1) by its domain given by

$$
\begin{aligned}
D(L^*) \quad = \quad & \{y \in \ell^2(I_1) : \text{there exists } f \in \ell^2(I_1) \text{ such that} \\
& \mathcal{J}\Delta y(t) - P(t)Ry(t) = f(t)\}, \\
& \text{L}^*y = f \text{ if and only if } \mathcal{J}\Delta y(t) - P(t)Ry(t) = f(t).
\end{aligned}
$$

The pre-minimal operator generated by (2.1) is now defined by

$$
\begin{aligned}
D(L') \quad = \quad & \{y \in D(L^*) : \text{there exists } n \in \mathbb{N} \text{ such that} \\
& y(0) = y(t) = 0 \ \forall t \geq n+1\}, \\
& L^*y = L'y.
\end{aligned}
$$

Here, L' is densely defined, symmetric but not necessarily closed. Since densely defined symmetric operators defined on Hilbert spaces are closable, if L' is not closed, then we can take its closure which now gives the minimal operator generated by $\mathcal{L}_1\mathcal{L}_2$ on $\ell^2(\mathbb{N})$. This we now denote by L. The minimal symmetric operator L is densely defined symmetric operator, and therefore, has symmetric operator extensions, see the results in ([15], chapter 13). In order to have properly defined operators, we impose some boundary conditions at the left regular end point [4, 16]. We therefore define two $s \times s$ matrices (α_1, α_2) with $\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I$ and $\alpha_1\alpha_2^* = \alpha_2\alpha_1^*$ so that

$$(\alpha_1, \alpha_2) \begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = 0. \tag{2.3}$$

Usually, the left regular end point is taken at t_0 for $t_0 > 1$ and $t_0 \in \mathbb{N}$. The results of Remling [10] can then be used to extrapolate the deficiency results to $t_0 = 0$. Thus, to construct the symmetric operator extension, we note that the operator L has an isometric Cayley-transform V_L because L is linear and densely defined symmetric operator on $\ell^2(\mathbb{N})$. Hence, for any $z \in \mathbb{C}$ such that $Imz > 0$, we have $V_L = (L - zI)(L - \bar{z}I)$ with $D(V_L) = \mathcal{R}(L - \bar{z}I)$ so that $V_L(L - \bar{z}I)y = (L - zI)y$ for all $y \in D(L)$. Thus one can also define the inverse Cayley-transform by $(zI - \bar{z}V_L)(I - V_L)^{-1}$ with its domain as $\mathcal{R}(I - V_L)$. Therefore, the extension of V_L to an everywhere defined bounded operator on $\ell^2(\mathbb{N})$ denoted by $V\tilde{y} = 0$ for all $\tilde{y} \in D(V_L)^{\perp}$ is a partial isometry with $D(V) = \mathcal{R}(L - \bar{z}I)$ and

$\mathcal{R}(V) = \mathcal{R}(L - zI)$. Hence, the Cayley-transform $L \to V$ is a bijective mapping of the densely defined closed linear operators on $\ell^2(\mathbb{N})$ onto the set of partial isometries V on $\ell^2(\mathbb{N})$. The inverse Cayley-transform $\tilde{L} = (zI - \bar{z}V)(I - V)^{-1}$ is the required symmetric operator extension of L. In this case, $D(L) \subseteq D(\tilde{L}) \subseteq D(L^*)$, where $Ly = \tilde{L}y = L^*y$, for all $y \in D(L)$ with $\langle \tilde{L}y, y \rangle = \langle y, \tilde{L}y \rangle$. $\qquad\square$

The minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ can be constructed in a similar fashion [3, 4, 16]. Their symmetric operator extensions can be described via similar argument. It would be interesting if the operator extension of L can be expressed as composite of symmetric extension of $\mathcal{L}_1$ and $\mathcal{L}_2$.

In our subsequent results which are on the existence of the self-adjoint operator extension of the minimal operator generated by $\mathcal{L}_1\mathcal{L}_2$ and the absolutely continuous spectrum of the self-adjoint extension, we shall assume the following growth conditions on the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$.

$$p_k q_j = o(p_n^2 \cdot p_0 q_0)^{\frac{1}{2n}}, \quad k = 1, 2, ..., n-1, \quad j = 1, 2, ..., m-1, \quad n = m. \qquad (2.4)$$

Theorem 2.7. *Assume that the coefficients of $\mathcal{L}_1$ and $\mathcal{L}_2$ in* (1.1) *and* (1.2) *satisfy* (2.4) *in addition to the conditions in Theorem* 2.1, *then the deficiency index of the minimal difference operator generated by $\mathcal{L}_1\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ is (l, l), where $2n \le l \le 4n$ and the absolutely continuous spectrum of the self-adjoint operator extension is non-empty if $|p_n|^{\frac{-1}{n}}$ is not absolutely summable and has spectral multiplicity equal to the number of pairs of eigenvalues with magnitude one.*

Proof. In this case we use the simplifications in Theorem 2.6 which gives the first order form of $\mathcal{L}_1\mathcal{L}_2 y(t)$, that is, $\mathcal{J}\Delta Y(t) = P(t)RY(t)$. We now introduce a spectral parameter z, $z \in \mathbb{R}$. The operator L generated by $\mathcal{L}_1\mathcal{L}_2$ on $\ell^2(\mathbb{N})$ is symmetric by the results of Theorem 2.6 and therefore, its spectrum $\sigma(L)$, is a subset of $\mathbb{R}$. We now solve the equation $\mathcal{L}_1\mathcal{L}_2 y(t) = zy(t)$, $z \in \mathbb{R}$, or $(L - zI)y = 0$. We shall absorb z into $p_0 q_0$, that is, $p_0 q_0$ will be interpreted as $p_0 q_0 - z$. Therefore, using techniques of Shi [16], this first order can be expressed as

$$Y(t + 1, z) = S(t, z)Y(t, z) \qquad (2.5)$$

where,

$$S(t, z) = \begin{pmatrix} E & EB \\ CE & I - A^* + CEB \end{pmatrix}$$

$E = (I_{2n} - A)^{-1}$.

The characteristic polynomial of the $2n \times 2n$ matrix $S(t, z)$ in (2.5) can be computed explicitly. This is determined by $\mathcal{P}(t, \lambda, z) = \det(S(t, z) - \lambda . I_{2n})$. Multiplying $\mathcal{P}(t, \lambda, z)$ by $\frac{p_n q_n}{\lambda^{2n}}$ in order to obtain a simpler version and assuming that $(1 - \lambda)(1 - \lambda^{-1}) = \gamma$, we obtain a Fourier polynomial $\mathcal{F}(\gamma, t, z)$ given by

$$\mathcal{F}(\gamma, t, z) = \frac{p_n q_n}{\lambda^{2n}} \mathcal{P}(t, \lambda, z) = \sum_{k=0}^{n} \sum_{j=0}^{n} (p_k \gamma^k)(q_j \gamma^j) = \sum_{k,j=0}^{n} p_k q_j \gamma^{k+j}.$$

From our construction, note that $q_n = \alpha p_n$ for some constant $\alpha \neq 0$. Similarly, $p_{n-1}, p_{n-2}, ..., p_0$ and $q_{n-1}, q_{n-2}, ..., q_0$ are real valued constants. Suppose $|p_n(t)| \nearrow \infty$ as $t \to \infty$ and the other coefficients are such that $p_k q_j = o(p_n^2 \cdot p_0 q_0)^{\frac{1}{2n}}$ as assumed in (2.4), then the $\gamma - roots$ of the Fourier polynomial are approximately

$$\gamma \approx |\frac{p_0 q_0}{\alpha p_n^2}|^{\frac{1}{2n}} \; sign \arg(-\frac{p_0 q_0}{\alpha p_n^2}) \tag{2.6}$$

Here, $\gamma = \gamma(t, z)$ are analytical functions of t and z. One needs to show that the $\gamma - roots$ of $\mathcal{F}(\gamma, t, z)$ are given by the relation in (2.6). In order to see this, consider a polynomial of the form $\mathcal{F}(t, z, \gamma) = \alpha p_n^2 \gamma^{2n} + p_{n-1} q_{m-1} \gamma^{2n-1} + ... + p_1 q_{m-1} \gamma + p_1 q_{m-1} \gamma^{2m-1} ... + p_1 q_1 \gamma + p_0 q_0$. By application of Kummer-Liouville transforms and scaling the coefficients, one can easily transform the Fourier polynomial $\mathcal{F}(t, z, \gamma,)$ into a polynomial of the form $P(\gamma) = a_{2n} \gamma^{2n} + a_{2-1} \gamma^{2n-1} + ... + a_1 \gamma + a_0$ such that $|a_0|, |a_{2n}| = 1$, $a_i = p_k q_j$, $(k, j = 1 \ldots, 2n)$ and a_i, $(1 \leq i \leq 2n - 1)$ are bounded. This can be achieved because we have the freedom of choice of α. It follows that by taking $C = \max\{a_i : i = 0, 1, ..., 2n\}$, there exists a positive constant M, $M > C + 1$ such that magnitude of all the $\gamma - roots$ are bounded between M^{-1} and M. In so doing, the roots of $\mathcal{F}(t, z, \gamma)$ shall all be bounded within a certain interval. Once the $\gamma - roots$ have been obtained for the Kummer-Liouville transformed polynomial, the $\gamma - roots$ of the original polynomial $\mathcal{F}(\gamma, t, z)$ can be obtained via backward transformation that involves the coefficients. It remains therefore to show that any $\gamma - root$ given by (2.6) is asymptotically equal to the $\gamma - roots$ of $\mathcal{F}(t, z, \gamma)$. It suffices therefore to show that if the particular $\gamma - root$ is substituted into the polynomial, then all the middle terms tend to zero as $t \to \infty$. We only do this for the term $p_n.q_{n-1} \gamma_k^{2n-1}$ since the others are done in a similar way. It is worth noting that even though in the polynomial $\mathcal{F}(\gamma, t, z)$, when expanded, apart from the leading term, will have some coefficients with p_n. In these cases, p_n will be of power one. Thus comparatively, we have $p_n = o(p_n^2)$ and this can easily be seen in the case of power coefficients since in the limit sense $t = o(t^2)$ as $t \to \infty$.

In order to transform (2.5) into Levinson's Benzaid-Lutz (LBL) form (1.3), we will need two diagonalisations. The appropriate eigenvectors can be evaluated directly from the quasi-differences by replacing Δ by $(\lambda - 1)$ and $y(t - k)$ by λ^{-k}. Thus from (2.2) we have,

$$\begin{aligned} x_j(t) &= q_j(t)(\lambda - 1)^{j-1} \lambda^{-j} \quad 1 \leq j \leq m \\ x_{m+1}(t) &= q_m(t)(\lambda - 1)^m \lambda^{-m} \\ x_{m+k}(t) &= -(\lambda - 1)(q_m(t)(\lambda - 1)^m \lambda^{-m}) - p_{n-k}(t)(q_{n-k}(t)(\lambda - 1)^{n-k} \lambda^{-(n+k)}) \\ &\quad 2 \leq k \leq n \\ x_{n+m+r}(t) &= -((\lambda - 1)(x_{n+m+r-1}(t)) + p_{n-r}(t)x_{m+n-r}(t)) \quad 2 \leq r \leq n + m - 1 \end{aligned}$$

The matrix $T(t, z)$ formed with the eigenvectors as columns will diagonalise $S(t, z)$ [9]. Thus one transforms the system by $\chi(t, z) = T(t, z)Y(t, z)$ which results into a system of the form,

$$\begin{aligned} \chi(t+1,z) &= T^{-1}(t+1,z)S(t,z)T(t,z)\chi(t,z) \\ &= [\Lambda(t,z) + \mathfrak{R}(t,z)]\chi(t,z) \end{aligned}$$

where,

$$\mathfrak{R}(t,z) = -T^{-1}(t+1,z)\Delta T(t,z)\Lambda(t,z) \qquad (2.7)$$

Here, $\mathfrak{R}(t,z)$ consists of ℓ^1 and ℓ^2 terms. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t,z)$, $k=1,2$. In (2.7), $\Delta T(t,z) = T(t+1,z) - T(t,z)$. The second diagonalisation is done by use of the eigenvectors of the matrix $[\Lambda(t) + \mathfrak{R}(t,z)]$. By applying the results of Behncke and Hinton [2], we shall require a matrix of the form $[I + B(t,z)]$ having

$$B_{kk}(t,z) = 0, \qquad B_{kj}(t,z) = (\lambda_j - \lambda_k)^{-1}\mathfrak{R}_{kj},$$

$$k \neq j, \qquad k, j = 1, \dots, s.$$

The second diagonalisation will thus result into correction terms added to the diagonals given by $(\Lambda_2)_{kk} = diag((\mathfrak{R}B)_{kk})$. Thus the second diagonalisation is done using the transformation

$$\psi(t,z) = [I + B(t,z)]\chi(t,z)$$

which results into a system of the form

$$\psi(t+1) = \{[\Lambda(t,z) + \Lambda_2(t,z)] + [I + B(t+1,z)]^{-1}\mathfrak{R}(t)[I + B(t,z)]\}\psi(t,z)$$

in which $(\Lambda_2) = diag(\mathfrak{R}B)_{kk}$.
One therefore obtains LBL form (1.3) to which we apply Theorem 1.1 to obtain the solutions of the form

$$y_k(t,z) = (e_k(t,z) + r_{kk}(t,z))\prod_{l=t_0}^{t-1}\lambda_k(l,z)$$

The asymptotics and summability of $y_k(t,z)$ will depend on $\prod_{l=t_0}^{t-1}\lambda_k(l,z)$. These are analysed off the real axis based on the magnitude of the corresponding eigenvalues since all the $\gamma - roots$ are such that $|\gamma| \approx |\frac{p_0 q_0}{\alpha p_n^2(t)}|^{\frac{1}{2n}}$ and as $t \to \infty$, it implies that $|\gamma_r| < 2 \ \forall \ r = 1, 2, ..., 2n$ and thus we have $2n$ pairs of eigenvalues with absolute value almost equal to 1. Thus by application of LBL theorem, the dichotomy condition can only be established off the real axis. Therefore, the number of roots that will have a magnitude greater than 1 off the real axis will be half the roots while the others will have a magnitude less than 1. Now let $z = z_0 + i\eta$, z_0, $\eta \in \mathbb{R}$, $\eta > 0$ and small, then, for $|\lambda_{r+}(t,z) + \lambda_{r-}^{-1}(t,z)| < 2$ as $t \to \infty$, which is the required uniform dichotomy condition, it follows that if $|p_n|^{\frac{-1}{n}}$ is absolutely summable since by construction, α, p_0, q_0 are constants, then all the associated eigenfunctions are z-uniformly square summable and $def\mathcal{L} = (4n, 4n)$ (limit circle case). On the other hand if $|p_n|^{\frac{-1}{n}}$ is not absolutely summable, then half of the associated eigenfunctions will lose their square summability and $def\mathcal{L} = (2n, 2n)$ (limit point case). The minimal operator generated by

$(\mathcal{L}_1 \mathcal{L}_2 - z)y(t) = 0$ has a self-adjoint operator extension $\mathcal{H}$ by application of von-Neumann theorems ([17], Theorem 8.11 and 8.12) which can be described via its domain $D(\mathcal{H})$ as follows

$$D(H) \;=\; D(L) \dotplus N(L^* - z) \dotplus N(L^* + z) \; or \tag{2.8}$$

$$D(H) = D(L) \dotplus \{y + Vy : y \in N(L^* - z)\}$$

The roots that off the real axis have magnitude greater than 1 lead to the eigensolutions that lose their square summability as $\eta \to 0^+$ and hence contributes to the continuous spectrum. The spectral multiplicity can thus be analysed via the M-matrix.

The M-matrix of L is the Borel-transform of the spectral measure μ. The density of the absolutely continuous spectrum of H is given by

$$(\frac{1}{\pi}) \lim_{\epsilon \to 0^+} M(\mu + i\epsilon) = (\frac{1}{\pi}) M(\mu_+) = \varrho(\mu).$$

The spectrum is absolutely continuous if M has finite limits $M(\mu_+)$. The eigenvalues of H will correspond to the poles of M. The M-matrix $M(z)$ is determined off the real axis and can be constructed from the eigenfunctions of H that are square summable.

Assume that $V(t, z) = \begin{pmatrix} V_1(t, z) \\ V_2(t, z) \end{pmatrix}$ is the set of square summable eigenfunctions of L with Dirichlet boundary conditions given in (2.3) with

$$\tilde{\alpha}_1 \tilde{\alpha}_1^* + \tilde{\alpha}_2 \tilde{\alpha}_2^* = I_{n+m}, \quad \tilde{\alpha}_1 \tilde{\alpha}_2^* - \tilde{\alpha}_2 \tilde{\alpha}_1^* = 0.$$

Then the square summable eigenfunctions are given by $\begin{pmatrix} Y_{t_0} \end{pmatrix} (t) \begin{pmatrix} I_{n+m} \\ M_{t_0}(z) \end{pmatrix}$.

and one can therefore show that

$$\begin{pmatrix} M_{t_0}(z) \end{pmatrix} = V_2(t_0, z) V_1^{-1}(t_0, z).$$

$V_1^{-1}(t_0, z)$ exists boundedly because $V_1(t_0, z)$ is the Vandemonde's matrix. $(M_{t_0})(z)$ is continuous for all z within the region of consideration and the spectrum is discrete at most, otherwise, we have $\sigma_{ac}(H) = \mathbb{R}$ if $p_n(t) < 0$ and if $p_n(t) > 0$, $\sigma_{ac}(H) \subseteq [\bar{c}, \infty)$ where $\bar{c} = \limsup \frac{p_0 q_0}{\alpha p_n(t)}$, in both cases the spectral multiplicity is $2n$.

$\square$

3. Composites of order Two and Four Operators

In this section we determine the deficiency indices and spectral multiplicities of composites of order two and four operators that are generated by (2.1) as a justification of the claim in Theorem 2.7. Assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative so that the corresponding difference expression is given by (2.1). In order to obtain the comparative analysis of the deficiency indices and spectral results of the associated difference operators, we solve the equations $(\mathcal{L}_1 y) = zy$, $(\mathcal{L}_2 y) = zy$ and $\mathcal{L}_1(\mathcal{L}_2 y) = zy$ as illustrated in the following examples.

Example 3.1. We consider second order difference equations of the form:

$$\begin{aligned} \mathcal{L}_1(y) &= -\Delta p(t)\Delta y(t-1) + b_0 y(t) \\ \mathcal{L}_2(y) &= -\Delta \alpha p(t)\Delta y(t-1) + c_0 y(t) \end{aligned}$$

where α is a real-valued constant, $\alpha \neq 0$. In this example we have that $p(t) \nearrow \infty$. We compute the deficiency indices of the minimal operator generated by the composite of the above difference equations.

The composite $\mathcal{L}_1 \mathcal{L}_2 y(t)$ is given by,

$$\mathcal{L}_1 \mathcal{L}_2 y(t) = \triangle^2 (\alpha p^2 \triangle^2 y(t-2)) - \triangle p(b_0 + c_0) \triangle y(t-1) + b_0 c_0 y(t) \tag{3.1}$$

Reducing (3.1) to first order, obtaining its characteristic polynomial and assuming that $(1-\lambda)(1-\lambda^{-1}) = \gamma$ then scaling by $\frac{\alpha p^2}{\gamma^2}$ we obtain the Fourier polynomial

$$\mathcal{F}(\gamma, t) = \alpha p^2 \gamma^2 + (c_0 p + \alpha b_0 p)\gamma + b_0 c_0 \tag{3.2}$$

Solving (3.2) above explicitly, we obtain,

$$\begin{aligned} \gamma_1 &= -(\frac{c_0 + \alpha b_0}{2\alpha p}) + \frac{1}{2\alpha p}(c_0 - \alpha b_0) = \frac{-b_0}{p} \\ \gamma_2 &= -(\frac{c_0 + \alpha b_0}{2\alpha p}) - \frac{1}{2\alpha p}(c_0 - \alpha b_0) = \frac{-c_0}{\alpha p} \end{aligned}$$

Since $(1-\lambda)(1-\lambda^{-1}) = \gamma$, substituting and solving for λ we have that,

$$\begin{aligned} \lambda_{1/2} &= \lambda^2 + (\frac{-b_0}{p} - 2)\lambda + 1 \\ &= \lambda^2 - (\frac{b_0}{p} + 2)\lambda + 1 \end{aligned}$$

and

$$\begin{aligned} \lambda_{3/4} &= \lambda^2 + (\frac{-c_0}{\alpha p} - 2)\lambda + 1 \\ &= \lambda^2 - (\frac{c_0}{\alpha p} + 2)\lambda + 1 \end{aligned}$$

Which results to

$$\lambda_1 \approx 1 + |\frac{b_0}{P}|^{\frac{1}{2}} + \frac{b_0}{2p} + \frac{b_0}{8p}|\frac{b_0}{P}|^2 - \frac{b_0^2}{128p^2}|\frac{b_0}{P}|^2 + O(p^{-3})$$

$$\lambda_2 \approx 1 - |\frac{b_0}{P}|^{\frac{1}{2}} + \frac{b_0}{2p} - \frac{b_0}{8p}|\frac{b_0}{P}|^2 + \frac{b_0^2}{128p^2}|\frac{b_0}{P}|^2 + O(p^{-3})$$

$$\lambda_3 \approx 1 + |\frac{c_0}{\alpha P}|^{\frac{1}{2}} + \frac{c_0}{2\alpha p} + \frac{c_0}{8\alpha p}|\frac{c_0}{\alpha P}|^2 - \frac{c_0^2}{128\alpha p^2}|\frac{c_0}{\alpha P}|^2 + O(p^{-3})$$

$$\lambda_4 \approx 1 - |\frac{c_0}{\alpha P}|^{\frac{1}{2}} + \frac{c_0}{2\alpha p} - \frac{c_0}{8\alpha p}|\frac{c_0}{\alpha P}|^2 + \frac{c_0^2}{128\alpha p^2}|\frac{c_0}{\alpha P}|^2 + O(p^{-3})$$

By replacing Δ by $(\lambda-1)$ and $y(t-k)$ by λ^{-k} in the quasi-differences and normalizing the first component through multiplication by λ, we obtain the eigenvectors given as:

$$v_k = \{1,\ (\lambda_k - 1)\lambda_k^{-1},\ (c_o p + \alpha b_0 p)(\lambda_k - 1)^3 \lambda_k^{-1},\ \alpha p^2 (\lambda_k - 1)^2 \lambda_k^{-1}\}^{tr}$$

Approximating the eigenvectors using the leading terms only, we obtain the eigenvectors of the form:

$$v_1 = [1,\ (\frac{b_0}{p})^{\frac{1}{2}} + (\frac{b_0}{p}),\ (\frac{c_o^2 b_0^3}{p})^{\frac{1}{2}} + (\frac{b_0^5}{p})^{\frac{1}{2}},\ \alpha p b_0 + \alpha(\frac{b_0^3}{p})^{\frac{1}{2}}]^{tr}$$

$$v_2 = [1,\ (\frac{b_0}{p})^{\frac{1}{2}} - (\frac{b_0}{p}),\ (\frac{c_o^2 b_0^3}{p})^{\frac{1}{2}} - (\frac{b_0^5}{p})^{\frac{1}{2}},\ \alpha p b_0 - \alpha(\frac{b_0^3}{p})^{\frac{1}{2}}]^{tr}$$

$$v_3 = [1,\ (\frac{c_0}{p})^{\frac{1}{2}} + (\frac{c_0}{p}),\ (\frac{b_0^2 c_0^3}{p})^{\frac{1}{2}} + (\frac{c_0^5}{p})^{\frac{1}{2}},\ \alpha p b_0 - \alpha(\frac{b_0^3}{p})^{\frac{1}{2}}]^{tr}$$

$$v_4 = [1,\ (\frac{c_0}{p})^{\frac{1}{2}} - (\frac{c_0}{p}),\ (\frac{b_0^2 c_0^3}{p})^{\frac{1}{2}} - (\frac{c_0^5}{p})^{\frac{1}{2}},\ \alpha p b_0 + \alpha(\frac{b_0^3}{p})^{\frac{1}{2}}]^{tr}$$

Thus the matrix of transformation $T(t, \lambda, z)$ is given by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) \\ O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) & O(p^{\frac{-1}{2}}) \\ O(p) & O(p) & O(p) & O(p) \end{pmatrix}$$

As $t \to \infty$, the absolute value of λ_k tends to 1 and by application of Levinson's Benzaid-Lutz theorem, the dichotomy conditions can only be established off the real axis. $|\lambda_1|,\ |\lambda_3| > 1$ while $|\lambda_2|,\ |\lambda_4| < 1$ as $t \to \infty$ which is the required dichotomy condition. Performing the first diagonalisation as outlined Theorem 2.7, equation (2.7) results to

$$\mathfrak{R}(t, z) = \begin{pmatrix} O(\Delta p) & O(\Delta p) & O(\Delta p) & O(\Delta p) \\ O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) \\ O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) & O(p^{\frac{1}{2}}\Delta p) \\ O(p^{-1}\Delta p) & O(p^{-1}\Delta p) & O(p^{-1}\Delta p) & O(p^{-1}\Delta p) \end{pmatrix}$$

The correction terms of are of the form:

$$\mathfrak{R}_{k,k}(t, z) = \{O(p^{\frac{-1}{2}}) + O(\Delta p),\ O(p^{\frac{-1}{2}}) + O(p^{\frac{1}{2}}\Delta p),$$
$$O(p^{\frac{-1}{2}}) + O(p^{\frac{1}{2}}\Delta p),\ O(p^{\frac{-1}{2}}) + O(p^{-1}\Delta p)\}$$

Therefore, since p is unbounded, $\Delta p \to 0$ as $t \to \infty$ so that the correction terms after the first diagonalisation are bounded and the off diagonal terms given by $\{O(\Delta p),\ O(p^{\frac{1}{2}}\Delta p)$ and $O(p^{-1}\Delta p)\}$ are ℓ^2 terms. After the second diagonalisation, the correction terms are of the form:

$$\mathfrak{R}B_{k,k}(t, z) = \{O(p^{\frac{-1}{2}}) + O(\Delta p) + O(p(\Delta p)^2),\ O(p^{\frac{-1}{2}}) + O(p^{\frac{1}{2}}\Delta p) + O(p^{\frac{3}{2}}(\Delta p)^2),$$
$$O(p^{\frac{-1}{2}}) + O(p^{\frac{1}{2}}\Delta p) + O(p^{\frac{3}{2}}(\Delta p)^2),\ O(p^{\frac{-1}{2}}) + O(p^{-1}\Delta p) + O((\Delta p)^2))\}$$

in which both $(\Delta p) \to 0$ and $(\Delta p)^2 \to 0$ as $t \to \infty$ so that the correction terms can remain bounded. The off diagonal terms $\{O(p(\Delta p)^2),\ O(p^{\frac{3}{2}}(\Delta p)^2)$ and $O((\Delta p)^2)\}$

are ℓ^1 terms. Thus with two diagonalisations, (2.5) is converted into LBL form and an explicit computation shows that

$$\lim_{\eta \to 0} \langle y_k(t, z), y_k(t, z) \rangle \approx \prod_{l=t_0}^{t-1} \lambda_k(l, z)$$

hence if $|O(p)^{\frac{-1}{2}}|$ is absolutely summable, then all the four solutions are square summable and $defL = (4, 4)$, thus a limit circle case and the spectrum is discrete. Conversely, if $|O(p)^{\frac{-1}{2}}|$ is not absolutely summable, then $y_1(t)$ and $y_3(t)$ lose their square summability as $\eta \to 0^+$ and we have that $defL = (2, 2)$, a limit point case. We now compute the M-matrix using the eigenvectors associated with $\lambda_2(t, z)$ and $\lambda_4(t, z)$ whose eigensolutions remain square summable. Their eigensolutions are of the form:

$$y_k(t, z) = \rho_k(t, z)[e_k(t, z) + \circ(1)] \prod_{l=t_0}^{t-1} \lambda_k(l)$$

where $\rho_k(t, z)$ are the diagonal terms of $T(t, z)[I + B(t, z)]$ which are bounded. For $z = z_0 + i\eta$, $\eta > 0$, small and $z_0 \in \mathbb{R}$, then from the relation

$$ImM(t, z) = \lim_{\eta \to 0^+} \langle F(t, z), F(t, z) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|\alpha p_n(l)|^{\frac{1}{2}}} \approx -2\eta \int_{t_0}^{t-1} |\alpha p_n(l)|^{-\frac{1}{2}} dl.$$

The one for $y_4(t)$ follows in a similar fashion. These eigenfunctions $y_1(t)$ and $y_3(t)$ that lose their square summability contribute to the absolutely continuous spectrum. We note that the leading term associated with spectral parameter z is given by $|\frac{p_0 q_o - z}{\alpha p_n}|^{\frac{1}{2}}$ so that as $p_n(t) \nearrow \infty$ and $\bar{c} < z < \infty$ where $\bar{c} = \lim \sup(\frac{p_0 q_0}{\alpha p_n})$. hence, the absolutely continuous spectrum is a subset of $[\bar{c}, \infty)$ and of spectral multiplicity 2.

Finally, we compute the deficiency indices of a minimal difference operator generated by the composites of order four difference equations.

Example 3.2. In this example we consider the fourth order difference equations of the form:

$$\begin{aligned} \mathcal{L}_1(y) &= \Delta^2(t^\epsilon \Delta^2 y(t-2)) + c_1 y(t) \\ \mathcal{L}_2(y) &= \Delta^2(\alpha t^\epsilon \Delta^2 y(t-2)) + c_2 y(t) \end{aligned}$$

where α is a real-valued constant, $\alpha \neq 0$ and $\epsilon > 0$.

The composite of the above equations as given by:

$$\mathcal{L}_1 \mathcal{L}_2 y = \Delta^4(\alpha t^{2\epsilon} \Delta^4 y(t-4)) - \Delta^2(b_0 t^\epsilon + c_0 t^\epsilon)\Delta^2 y(t-2) + b_0 c_0 y(t) \quad (3.3)$$

and the quasi-differences of (3.3) above are given as:

$$
\begin{aligned}
x_1(t) &= y(t-1) & x_3 &= \Delta^2 y(t-3) \\
x_2 &= \Delta y(t-2) & x_4 &= \Delta^3 y(t-4) \\
u_1(t) &= -\Delta(c_2 t^\epsilon + \alpha c_1 t^\epsilon)\Delta^2 y(t-2) - \Delta^3 \alpha t^{2\epsilon}\Delta^4 y(t-4) \\
u_2(t) &= (c_2 t^\epsilon + \alpha c_1 t^\epsilon)\Delta^2 y(t-2) + \Delta^2 \alpha t^{2\epsilon}\Delta^4 y(t-4) \\
u_3(t) &= -\Delta \alpha t^{2\epsilon}\Delta^4 y(t-4) & u_4(t) &= \alpha t^{2\epsilon}\Delta^4 y(t-4)
\end{aligned}
$$

Reducing (3.3) to first order system using the above quasi-differences, obtaining the characteristic polynomial and thereafter performing the right scaling and taking $(1-\lambda)(1-\lambda^{-1}) = \gamma$, we obtain the fourier polynomial given by:

$$
\mathcal{F}(\gamma, t) = \alpha t^{2\epsilon}\gamma^4 + (c_2 t^\epsilon + \alpha c_1 t^\epsilon)\gamma^2 + c_1 c_2 \tag{3.4}
$$

solving the zeros of (3.4) we have that:

$$
\begin{aligned}
\gamma_1 &\approx i \,|\, \frac{c_1}{t^\epsilon} \,|^2 \\
\gamma_2 &\approx -i \,|\, \frac{c_1}{t^\epsilon} \,|^2 \\
\gamma_3 &\approx i \,|\, \frac{c_2}{t^\epsilon} \,|^2 \\
\gamma_4 &\approx -i \,|\, \frac{c_2}{t^\epsilon} \,|^2
\end{aligned}
$$

and since $(1-\lambda)(1-\lambda^{-1}) = \gamma$ it implies that:

$$
\begin{aligned}
\lambda_{1/2} &= 1 - \frac{i}{2} \,|\, \frac{c_1}{t^\epsilon} \,|^{\frac{1}{2}} \pm \{\frac{c_1}{4t^\epsilon} - i \,|\, \frac{c_1}{t^\epsilon} \,|^{\frac{1}{2}}\}^{\frac{1}{2}} \\
\lambda_{3/4} &= 1 + \frac{i}{2} \,|\, \frac{c_1}{t^\epsilon} \,|^{\frac{1}{2}} \pm \{\frac{c_1}{4t^\epsilon} + i \,|\, \frac{c_1}{t^\epsilon} \,|^{\frac{1}{2}}\}^{\frac{1}{2}} \\
\lambda_{5/6} &= 1 - \frac{i}{2} \,|\, \frac{c_2}{t^\epsilon} \,|^{\frac{1}{2}} \pm \{\frac{c_2}{4t^\epsilon} - i \,|\, \frac{c_2}{t^\epsilon} \,|^{\frac{1}{2}}\}^{\frac{1}{2}} \\
\lambda_{7/8} &= 1 + \frac{i}{2} \,|\, \frac{c_2}{t^\epsilon} \,|^{\frac{1}{2}} \pm \{\frac{c_2}{4t^\epsilon} + i \,|\, \frac{c_2}{t^\epsilon} \,|^{\frac{1}{2}}\}^{\frac{1}{2}}
\end{aligned}
$$

Which results to:

$$\lambda_1 \approx 1 - \frac{i}{2} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_2 \approx 1 - \frac{i}{2} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_3 \approx 1 + \frac{i}{2} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_4 \approx 1 + \frac{i}{2} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \mid \frac{c_1}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_5 \approx 1 - \frac{i}{2} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_6 \approx 1 - \frac{i}{2} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_7 \approx 1 + \frac{i}{2} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{2}} - \frac{i}{\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{4}} + \frac{i}{8\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

$$\lambda_8 \approx 1 + \frac{i}{2} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{2}} + \frac{i}{\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{1}{4}} - \frac{i}{8\sqrt{2}} \mid \frac{c_2}{t^\epsilon} \mid^{\frac{3}{4}} + O(t^{\frac{-5}{4}\epsilon})$$

By obtaining the transforming matrix as outlined in example 3.1 above and performing the two diagonalisations to convert (2.5) into LBL form and by mathematical induction, we have that:

$$\lim_{\eta \to 0} \langle y_k(t,z), y_k(t,z) \rangle \approx \prod_{l=t_0}^{t-1} \lambda_k(l,z)$$

hence if $|O(t^\epsilon)^{\frac{-1}{4}}|$ is absolutely summable, then all the eight solutions are square summable and $defL = (8,8)$, thus a limit circle case and the spectrum is discrete. Conversely, if $|O(t^\epsilon)^{\frac{-1}{4}}|$ is not absolutely summable, then $y_1(t)$ and $y_4(t)$, $y_5(t)$ and $y_8(t)$ lose their square summability as $\eta \to 0^+$ and we have that $defL = (4,4)$, a limit point case. The M-matrix can then be computed using the eigenvectors associated with $\lambda_2(t,z)$, $\lambda_3(t,z)$, $\lambda_6(t,z)$ and $\lambda_7(t,z)$ whose eigensolutions remain square summable. Their eigensolutions are of the form:

$$y_k(t,z) = \rho_k(t,z)[e_k(t,z) + \circ(1)] \prod_{l=t_0}^{t-1} \lambda_k(l)$$

where $\rho_k(t,z)$ are the diagonal terms of $T(t,z)[I + B(t,z)]$ which are bounded. Again for $z = z_0 + i\eta$, $\eta > 0$, small and $z_0 \in \mathbb{R}$, and from the relation

$$ImM(t,z) = \lim_{\eta \to 0^+} \langle F(t,z), F(t,z) \rangle,$$

it follows that $y_2(t)$ remains square summable and hence by Euler's formula:

$$\ln \prod_{l=t_0}^{t-1} |\lambda_2(l)|^2 \approx -2\eta \sum_{t_0}^{t-1} \frac{1}{|t^\epsilon(l)|^{\frac{1}{4}}} \approx -2\eta \int_{t_0}^{t-1} |t^\epsilon(l)|^{-\frac{1}{4}} dl.$$

The one for $y_3(t)$, $y_6(t)$, and $y_7(t)$ are done in a similar fashion. These eigenfunctions $y_1(t)$, $y_4(t)$, $y_5(t)$ and $y_8(t)$ that lose their square summability contribute to the absolutely continuous spectrum. The leading term associated with spectral parameter z is given by $|\frac{c-z}{t^\epsilon}|^{\frac{1}{4}}$ so that as $t^\epsilon \nearrow \infty$ and $\overline{q} < z < \infty$ where $\overline{q} = \limsup(\frac{c}{t^\epsilon})$. Hence, the absolutely continuous spectrum is a subset of $[\overline{q}, \infty)$ and of spectral multiplicity 4.

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