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SPECTRAL THEORY OF COMMUTATIVE HIGHER ORDER DIFFERENCE OPERATORS WITH UNBOUNDED COEFFICIENTS

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Abstract. We have established the necessary and sufficient conditions for any two even higher order symmetric difference maps to generate commuting minimal difference operators. We have done this through construction of appropriate comparison algebras of the self-adjoint operator extensions of the minimal operators generated and application of asymptotic summation. The results show that if the first difference on the coefficients tends to zero whenever the coefficients are allowed to be unbounded and that the difference maps considered have the same order, then they generate minimal operators that commute and the corresponding self-adjoint operators commute too. We have further shown that the self-adjoint operator extensions of the respective minimal operators can be expressed as the composite of the independent self-adjoint operator extensions if the generated minimal difference operators have closed ranges. Finally, we have shown that the spectra of these self-adjoint operator extensions are the whole of the real line if the coefficients are unbounded. These results therefore, extend the existing results in the continuous case to discrete setting.

1. Introduction

We consider two symmetric higher order difference functions $\mathcal{L}_1$ and $\mathcal{L}_2$ defined by

$$\mathcal{L}_1 y(t) = \sum_{k=0}^{n}(-1)^k \Delta^k (p_k \Delta^k y(t-k)), \quad \mathcal{L}_2 y(t) = \sum_{j=0}^{m}(-1)^j \Delta^j (q_j \Delta^j y(t-j)), \tag{1.1}$$

on $\ell^2(\mathbb{N})$. Here, $p_k = p_k(t)$, $q_j = q_j(t)$, $t \in \mathbb{N}$, $k = 0, 1, 2, ..., n$ and $j = 0, 1, 2, ..., m$, are real valued functions which are assumed to have second forward difference with the assumption that $\Delta(p_k(t))$, $\Delta(q_j(t)) \to 0$ as $t \to \infty$. The operator Δ is a forward difference operator defined by $\Delta f(t) = f(t+1) - f(t)$ for all $t \in \mathbb{N}$. The difference maps $\mathcal{L}_1$ and $\mathcal{L}_2$ are ordinarily defined on weighted Hilbert spaces $\ell^2_w(\mathbb{N})$ or $\ell^2_w(\mathbb{Z})$ where $w = w(t) > 0$ is a weight function. In this case, the inner

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products for any $\{y_1(t)\}_{t=t_0}^{\infty}$ and $\{y_2(t)\}_{t=t_0}^{\infty}$ in $\ell_w^2(\mathbb{N})$ are defined by

$$\langle y_1(t), y_2(t)\rangle = \sum_{t=t_0}^{\infty} y_1(t)w(t)\overline{y_2(t)}$$

where t_0 is some initial integral value. In this paper, we have studied the solutions of the equations

$$\mathcal{L}_1(\mathcal{L}_2 y(t)) = zy(t), \qquad \mathcal{L}_2(\mathcal{L}_1 y(t)) = zy(t), \tag{1.2}$$

where $z \in \mathbb{C}$ is the spectral parameter. In particular, we have been interested in cases where in (1.2) above, $\mathcal{L}_1$ commutes with $\mathcal{L}_2$. In order to simplify the computations, we have assumed that $w(t) = 1$, for all $t \in \mathbb{N}$ or $t \in \mathbb{Z}$. Thus the analysis can be done on $\ell^2(\mathbb{N})$ which is isomorphic to $\ell_w^2(\mathbb{N})$ when $w(t) = 1$. The deficiency indices and the spectral results in the two spaces are similar if $w(t) = 1$ for all $t \in \mathbb{N}$. The interest in this case is the necessary and sufficient conditions for the commutativity of the minimal and maximal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ and if so, the required conditions for the self-adjoint operator extension H generated by $\mathcal{L}_2\mathcal{L}_1$ to be expressed as H_2H_1. Here, H_1 and H_2 are the self-adjoint operator extensions of minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. The solutions of (1.2) have been obtained by constructing the Hamiltonian systems of (1.2), construction of the equivalent first order systems and the appropriate maximal and minimal operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\ell^2(\mathbb{N})$. Thus by constructing appropriate comparison algebras and applying asymptotic summation, we have established the existence of self-adjoint operator extension of the minimal operator generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ as well as that of the composite $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$. Moreover, using the same approach, we have determined the necessary and sufficient conditions for these self-adjoint operator extensions to commute. Finally, we have also determined the location of the absolutely continuous spectrum and the spectral multiplicity of the respective self-adjoint operator extensions.

This research is motivated by the invent of researches in electric-electronics where a system is organized as a chain of subsystems where the output of one subsystem is the input of the subsequent subsystem. It is worth noting that main applications of differential equations arise naturally in electrodynamics, quantum mechanics and other branches of engineering. In engineering, this occurs in control theory and in particular, in electrical engineering. When the cascade of a system is connected in form of subsystems, the engineering ingenuity requires that the order of the subsystems can be changed without interfering with the quality and properties of the final product, output. The conditions necessary and sufficient for this achievement is the concept behind the theory of commutativity in Mathematics and Control theory for engineers. In the sequel, we formulate the first order systems and construct maximal and minimal operators generated by $\mathcal{L}_r$, $r = 1, 2$.

In order to solve the equations $\mathcal{L}_r y = zy$, $r = 1, 2$, we need to convert the equations into their first order systems. To do that, we require quasi-differences [4, 21]. These are the discretised version of quasi-derivatives as given in [15] in equation (1.2). These are given by

$$x_i(t) = \Delta^{i-1} y(t-i), \qquad u_n(t) = p_n(t)\Delta^n y(t-n),$$
$$u_k(t) = \sum_{l=k}^{n} (-1)^{l-k} (p_l(t)\Delta^l y(t-l)).$$

for $\mathcal{L}_1$. The quasi-differences for $\mathcal{L}_2$ is done in a similar way and hence using the $2s$, $s = n, m$, vector valued function, $Y(t,z) = [x(t,z), u(t,z)]^{tr}$, where $x(t,z) = [x_1(t,z), ..., x_s(t,z)]^{tr}$ and $u(t,z) = [u_1(t,z), ..., u_s(t,z)]^{tr}$, we obtain the required first order system given by

$$\Delta \begin{bmatrix} x(t,z) \\ u(t,z) \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \begin{bmatrix} x(t+1,z) \\ u(t,z) \end{bmatrix} \tag{1.3}$$

Here A, B and C are $s \times s$ matrices with non-zero entries as

$$A_{l,l+1} = 1, \qquad B_{s,s} = h_s^{-1}, \qquad C_{l,l} = h_{l-1},\ l = j, k = 1, \ldots, s.$$

$h_0 = h_0(t) - z$, $h = p_k, q_j$. In a more precise way, we can introduce a symplectic matrix $\mathcal{J}$ and write the above first order form in its Hamiltonian system

$$\mathcal{J}\Delta \begin{bmatrix} x(t,z) \\ u(t,z) \end{bmatrix} = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix} \begin{bmatrix} x(t+1,z) \\ u(t,z) \end{bmatrix}, \quad \mathcal{J} = \begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix} \tag{1.4}$$

This leads to an equivalent first order of the form

$$\mathcal{J}\Delta Y(t,z) - PKY(t,z) = F(t),$$

where K is a partial shift operator defined by

$$K([x(t,z), u(t,z)]^{tr}) = [x(t+1,z), u(t,z)]^{tr}$$

,

$$P = \begin{bmatrix} -C & A^* \\ A & B \end{bmatrix}, \;\; F(t) \;\; 2s- \;\text{vector} function$$

with $F(t) = [f(t), 0, ..., 0]^{tr}$, $f(t) \in \ell^2(\mathbb{N})$.
The first order form can be written as

$$\begin{bmatrix} x(t+1,z) \\ u(t+1,z) \end{bmatrix} = S(t,z) \begin{bmatrix} x(t,z) \\ u(t,z) \end{bmatrix}. \tag{1.5}$$

where $S(t,z)$ has the block form of

$$S(t,z) = \begin{bmatrix} E & EB \\ CE & I - A^* + CEB \end{bmatrix}$$

and $E = (I - A)^{-1}$.

In order to properly define the maximal and minimal difference operators generated by $\mathcal{L}_r$ in (1.1), we need some regularity conditions just like in the continuous case. The space $\ell^2(\mathbb{N})$ is described as follows using the regularity conditions.

$$\begin{aligned} \ell^2(\mathbb{N}) \;\; = \;\; & \{y(t) : y(t) = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C} \\ & \left. \sum_{t=0}^{\infty} K(Y^*(t))(KY(t)) < \infty \right\}. \end{aligned}$$

Similarly, there exists an integral domain I_1, $I_1 \subset \mathbb{N}$ such that

$$\sum_{t\in I_1} K(Y^*(t))(KY(t)) > 0,$$

$\ell^2(I_1) \subset \ell^2(\mathbb{N})$ and

$$\ell^2(I_1) = \left\{ y(t) \in \ell^2(\mathbb{N}) : \sum_{t\in I_1} K(Y^*(t))(KY(t)) > 0 \right\}.$$

By doing so, the scalar products are therefore defined by

$$\langle y(t), y_1(t)\rangle = \sum_{t\in I_1} y(t)\overline{y_1(t)}.$$

In addition, for any vector $F(t)$, we need

$$\mathcal{J}\Delta Y(t,z) - PKY(t,z) = F(t), \text{ with } \|Y(t,z)\| = 0, \text{ if, } F(t) = 0.$$

This is the condition that is missing in [21] which makes the definition of maximal operators in her work just mere difference relations and not properly defined maximal operators.
Next we define maximal and the minimal difference operators generated by $\mathcal{L}_r$. Define a map L_r^* via its domain $D(L_r^*)$ given by

$$\begin{aligned} D(L_r^*) &= \{y(t) \in \ell^2(I_1) : \text{there exists } f(t) \in \ell^2(I_1) \text{ such that} \\ &\quad \mathcal{J}\Delta Y(t) - PKY(t) = F(t)\} \end{aligned}$$

and

$$L_r^* y(t) = f(t) \text{ if and only if } \mathcal{J}\Delta Y(t) - PKY(t) = F(t).$$
$$\mathcal{L}_r y(t) = L_r^* y(t), \ \forall y(t) \in D(L_r^*).$$

Then $D(L_r^*)$ is the largest possible domain in which L_r^* can be defined. L_r^* is the maximal difference operator generated by $\mathcal{L}_r$ on $\ell^2(\mathbb{N})$. L_r^* is symmetric, closed and densely defined. A restriction of L_r^* to a smaller domain $D(L_{r_0})$ defined by

$$\begin{aligned} D(L_{r_0}) &= \{y(t) \in D(L_r^*) : \text{ there exists } n \in \mathbb{N} \\ &\quad \text{such that } y(0) = y(t) = 0 \text{ for all } t \geq n+1\}. \end{aligned}$$
$$L_r^* y(t) = L_{r_0} y(t), \qquad \forall y(t) \in D(L_{r_0}),$$

results into a pre-minimal difference operator generated by $\mathcal{L}_r$. Here, L_{r_0} is densely defined, symmetric but not necessarily closed. Since in our analysis we require closed linear operators and because densely defined operators are closable, we will take the closure of pre-minimal operator $\overline{L_{r_0}}$ to be our minimal operator. This will be denoted by L_r. In order to have properly defined operators, we impose some boundary conditions at the left regular end point [6, 21]. Define two $s \times s$ matrices (α_1, α_2) with $\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I$ and $\alpha_1\alpha_2^* = \alpha_2\alpha_1^*$ so that

$$(\alpha_1, \alpha_2) \begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = 0. \tag{1.6}$$

In most cases, the left regular end point is taken as t_0 for $t_0 > 1$ and $t_0 \in \mathbb{N}$. The results of Remling [19] can then be used to extrapolate the deficiency results to $t_0 = 0$.

Let z be spectral parameter with $\text{Im}z > 0$. We now define the pair (N_+, N_-) as the deficiency indices of L_r, if $N_+ = \dim \mathcal{N}_{L_r^*-\bar{z}}$ and $N_- = \dim \mathcal{N}_{L_r^*-z}$, where $\mathcal{N}_{L_r^*-\bar{z}}$ and $\mathcal{N}_{L^*-z}$ are taken as the null spaces of $L_r^* - \bar{z}I$ and $L_r^* - zI$ respectively. If $N_- = N_+ = 0$, then L_r is self-adjoint. On the other hand if $N_- = N_+ \neq 0$ then by von Neumann theorems [20], L_r has self-adjoint operator extension H_r defined by;

$$D(H_r) = D(L_r) \dot{+} \{y + V_r y \; y \in \mathcal{N}(L_r^* - zI)\}, \quad H_r y = L_r y = \mathcal{L}_r y, \; \forall y \in D(L_r). \tag{1.7}$$

V_r is a uniquely determined isometric mapping such that $V_r : \mathcal{N}(L_r^* - zI) \to \mathcal{N}(L_r^* + zI)$. In the case of non-limit point, added boundary conditions are imposed at infinity in order to obtain the basis of the set of solutions of $(\mathcal{L}_r - zI)y = 0$ that are uniformly square summable.

The analysis of the first order systems of (1.1) and the existence of H_r are done using asymptotic summation which is anchored on the discretized version of Levinson's Theorem as stated below.

Theorem 1.1. *(Levinson's-Benzaid-Lutz Theorem [7])*
Let $\Lambda(t, z) = diag\{\lambda_1(t, z), ..., \lambda_{2s}(t, z)\}$ for $t \geq t_0$. Assume

(i) *$\lambda_i(t, z) \neq 0$ for all $1 \leq i \leq 2s$ and $t \geq t_0$*
(ii) *R(t, z) is a $2s \times 2s$ matrix defined for all $t \geq t_0$, satisfying $\Sigma_{t=0}^{\infty} |\frac{1}{\lambda_i(t,z)}| \|R(t, z)\| < \infty$, for all $i = 1, 2, ..., 2s$*
(iii) *$\Lambda(t, z)$ satisfies the following uniform dichotomy condition. For any pair of indices i and j, such that $i \neq j$, assume there exists δ with $0 < \delta < 1$ such that $|\lambda_i(t, z)| \geq \delta$ for all $t \geq t_0$. Then, either $|\frac{\lambda_i(t)}{\lambda_j(t)}| \geq 1$ or $|\frac{\lambda_i(t)}{\lambda_j(t)}| \leq 1$ for a large t.*

Then the linear system

$$Y(t + 1, z) = [\Lambda(t, z) + R(t, z)]Y(t, z) \tag{1.8}$$

has a fundamental matrix satisfying,

$$Y(t, z) = [I + o(1)] \prod_{l=t_0}^{t-1} \Lambda(l, z) \; \text{as} \; t \to \infty.$$

Explicitly for the eigensolutions, we have

$$y_k(t, z) = (e_k(t, z) + r_k(t, z)) \prod_{0}^{t-1} (\lambda_k(l, z)). \tag{1.9}$$

Here, $e_k(t, z)$ is the normalized eigenvector and $r_k(t, z) \to 0$ as $t \to \infty$.
Asymptotic summation by now is a straight forward method where we compute the eigenvalues of the matrix $S(t, z)$, establishing the uniform dichotomy condition and finally transforming (1.5) into the almost diagonal system depending on the decay conditions of the coefficients. This shall require computation of eigenvalues of $S(t, z)$, the corresponding eigenvectors and some diagonalizations. Thus for the eigenvalues, we need the characteristic polynomial $\mathcal{P}(t, \lambda, z) = det[S(t, z) - I_{2n}]$. If we multiply $\mathcal{P}(t, \lambda, z)$ by $(-1)^s h_s \lambda^{-s}$,

$h_s = p_n, q_m,\ \ s = n, m$, we obtain the fourier polynomial

$$\mathcal{F}(t, \gamma, z) = (-1)^s h_s \lambda^{-s} \mathcal{P}(t, \lambda, z) = \sum_{l=0}^{s} h_l \gamma^l, \tag{1.10}$$

where $\gamma = (1-\lambda)(1-\lambda^{-1})$. The appropriate eigenvectors can be evaluated directly from the quasi-differences by replacing Δ by $(\lambda - 1)$ and $y(t-l)$ by λ^{-l}.

The matrix $T(t, z) = [v_1, \ldots, v_s]$ is now used to diagonalise equation (1.5). We make the transformation $\chi(t, z) = T(t, z)Y(t, z)$ which results into

$$\begin{aligned} \chi(t+1, z) &= T^{-1}(t+1, z)S(t, z)T(t, z)\chi(t, z) \\ &= [\Lambda(t, z) + \mathfrak{R}(t, z)]\chi(t, z) \end{aligned}$$

where,

$$\mathfrak{R}(t, z) = -T^{-1}(t+1, z)\Delta T(t, z)\Lambda(t, z) \tag{1.11}$$

and

$$\Lambda(t, z) = diag\left(\lambda_l(t, z)\right), l = j, k = 1, \ldots, s.$$

Here, $\mathfrak{R}(t, z)$ consists of terms in $\ell^2(\mathbb{N})$ and $\ell^1(\mathbb{N})$. The correction terms as a result of the first transformation are given by $\mathfrak{R}_{kk}(t, z)$, $k = 1, 2$. In (1.11), $\Delta T(t, z) = T(t+1, z) - T(t, z)$. The second diagonalisation is done by use of the eigenvectors of the matrix $[\Lambda(t) + \mathfrak{R}(t, z)]$. Using the results of Behncke and Hinton [2], a matrix of the form $[I + B(t, z)]$ with

$$B_{kk}(t, z) = 0, \qquad B_{kj}(t, z) = (\lambda_j - \lambda_k)^{-1}\mathfrak{R}_{kj},$$

$$k \neq j, \qquad k, j = 1, \ldots, s., \qquad t \geq t_0,$$

will be required for the second diagonalisation. The second diagonalisation results into correction terms added to the diagonals given by $(\Lambda_2)_{kk} = diag((\mathfrak{R}B)_{kk})$. The second diagonalisation is thus done using the transformation

$$\psi(t, z) = [I + B(t, z)]\chi(t, z)$$

and which results into a system of the form

$$\psi(t+1) = \{[\Lambda(t, z) + \Lambda_2(t, z)] + [I + B(t+1, z)]^{-1}\mathfrak{R}(t)[I + B(t, z)]\}\psi(t, z)$$

where $(\Lambda_2) = diag(\mathfrak{R}B)_{kk}$.

One therefore obtains Levinson-Benzaid-Lutz (LBL) form of

$$\psi(t+1, z) = [\Lambda(t, z) + R(t, z)]\psi(t, z)$$

to which we apply Theorem 1.1 to obtain the solutions of the form

$$Y(t, z) = T(t, z)[I + B(t, z)][E(t, z) + \circ(I_s)] \prod_{t_0}^{t-1} \Lambda(t, z)$$

if we apply backward transformations. Note that the matrices $T(t,z)[I+B(t,z)]$ shall be bounded and the asymptotics of the eigensolutions shall depend on the matrix $\prod_{t_0}^{t-1}\Lambda(t,z)$. Explicitly, we have

$$y_k(t,z)=\rho_k(t,z)[e_k(t,z)+r_{kk}(t,z)]\prod_{t_0}^{t-1}\lambda_k(t,z)$$

where $\rho_k(t,z)$ is a bounded function given by $(T(t,z)[I+B(t,z)])_{kk}$, $e_k(t,z)$ is a normalised eigenvector vector while $r_{kk}(t,z)\in\ell^1(\mathbb{N})$ and tends to zero as $t\to\infty$. The spectral results of H_r are now obtained via the M-matrix [12, 19, 21]. In higher dimensions, the M-matrix is equivalent to the classical Titmarsh-Weyl m-function. In order to construct the M-matrix associated with the first order system, we let

$$\Omega_r(t,z)=\begin{bmatrix}\Omega_{r_1}(t,z)\\ \Omega_{r_2}(t,z)\end{bmatrix}$$

be the set of square summable solutions satisfying boundary conditions (1.6) at $t_0=0$ with $\alpha_1=I_s$, the identity matrix of order s, and $\alpha_2=0_s$. Then applying boundary conditions at the left regular endpoint at t_0, we have

$$M_r(t_0,z)=\Omega_{r_2}(t_0,z)\Omega_{r_1}^{-1}(t_0,z).$$

Here, $\Omega_{r_1}(t_0,z)$ is the Vandermonde's matrix for the eigenvalues of square summable solutions and it is easy to show that $\Omega_{r_1}^{-1}(t_0,z)$ exists and $M_r(t_0,z)$ is continuous and bounded. The $M_r(t_0,z)$ is the Herglotz function. This implies that the spectrum is not singular continuous apart from at the isolated poles and hence the density of spectral measure of H_r is absolutely continuous. In order to determine the continuity of $M_r(t_0,z)$, we compute the scalar product of square summable solutions with $z=z_0+i\eta$, $z_0,\eta\in\mathbb{R}$, $\eta>0$, small, while taking limits as $\eta\to 0^+$ as given below

$$\lim_{\eta\to 0^+}\langle y_l(t_0,z),y_l(t_0,0)\rangle.$$

The M-matrix can then be shown to be bounded absolutely since the above limit exists boundedly and this allows one to represent the resolvent of any arbitrary self-adjoint operator extension H_r of L_r using Green's function as

$$(H_r^{-1}-zI)y(t)=\sum_{0}^{\infty}G_r(t,t_1,z)y(t_1),\forall y(t)\in\mathcal{H},$$

where the Green function $G_r(t,t_1,z)$ is defined by:

$$G_r(t,t_1,z)=\left\{\begin{array}{cc}\chi_r(t,z)R(\Phi)_r^*(s,\bar{z}): & 0\le s\le t-1\\ \Phi_r(t,z)R(\chi_r)^*(s,\bar{z}): & t\le s<\infty\end{array}\right\}.$$

$\Phi_r(t,z)$ are the fundamental solutions of the first order systems and $\chi_r(t,z)$ are the $s\times 2s$ matrices consisting of those solutions satisfying boundary conditions $\alpha(\alpha_1,\alpha_2)$ with $\alpha_1=I_n$ and $\alpha_2=0$ at the regular point $t=0$ or t_0 and are z-uniformly square summable as $t\to\infty$.

The rank of the M-matrix is therefore the spectral multiplicity of the absolutely continuous spectrum of H_r and this loosely translates to the number of solutions that lose their square summability as $\eta \to 0^+$. For our results in Section 2, we apply the concept of Greens function in order to construct the appropriate isometric isomorphisms corresponding to H_r and comparison algebras. This is the discretised version of H_r and W_r in Theorem 2.1 of [15]. This is constructed as follows:

$$H_r^{-1}y(t) = \sum_0^\infty G_r(t,t_1)y(t_1), \quad y(t) \in \ell^2(I_1).$$

and the corresponding isometric isomorphism operator W_r by:

$$W_r = H_r^{-\frac{1}{2n}} = \sum_0^\infty (-1)\begin{pmatrix} \frac{1}{2n} \\ l \end{pmatrix}(I - H_r^{-1})^l, \quad r = 1,2, \quad n = m. \tag{1.12}$$

Let $\sigma(H_r)$, $\sigma_{ac}(H_r,l)$, $\sigma(H)$ and $\sigma_{ac}(H,l)$ denote the spectrum of H_r, absolutely continuous spectrum of H_r with spectral multiplicity of l, spectrum of H and absolutely continuous spectrum of H with spectral multiplicity of l respectively. Our results shows that if $\Delta(h(t)) \to 0$ as $t \to \infty$, $h(t) = p_k(t),\ q_j(t)$ and $n = m$, then $\mathcal{L}_1$ and $\mathcal{L}_2$ generate minimal difference operators L_1 and L_2 respectively that commute. In the case of unbounded coefficients with the conditions $h_{l-1} \cdot h_{l+1} = o(h_l^2)$, $h_l = p_k,\ q_j$, the self-adjoint operator extensions H_1 and H_2 corresponding to minimal operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ commute also. The self-adjoint operator extension of minimal operator generated by $\mathcal{L}_1\mathcal{L}_2$, H, can be expressed as a composite of H_1 and H_2. The absolutely continuous spectrum of H_1, H_2 and H is the whole of $\mathbb{R}$ with absolutely continuous spectrum of H having spectral multiplicity similar to the sum of that of H_1 and H_2. This paper therefore is the discretised version of [15]. However, we have included detailed analysis of the spectral properties of the composite operators with robust examples.

The remaining part of this paper is organized as follows: 1 Introduction, 2 Commutativity of Generated Operators, 3 Spectral Analysis of L_1, L_2 and L with unbounded coefficients.

2. Commutativity of Generated Operators

The results of this section are the discretised version of the results in [15], especially the proof of Theorem 2.2 follows closely from that of Theorem 2.1 in [15]. For any real valued function $h(t)$, we shall define the commutator $[\Delta, h(t)]$ to be the relation

$$[\Delta, h(t)]y(t) = \Delta(h(t)y(t)) - h(t)(\Delta y(t)),\ \forall t \in \mathbb{N}$$

The commutator vanishes identically to zero if and only if $\Delta(h(t)) \to 0$ as $t \to \infty$. It is therefore true like in the continuous case, that in the discrete case we replace "derivative" with Δ to obtain our commutator. For $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$, there are many examples that satisfy this condition. For example, constant functions, almost constant functions, that is, the functions of the form $h(t) = c + g(t)$ such that $g(t) \to 0$ as $t \to \infty$. However, there exists also unbounded functions that satisfy the condition. For example, take $h(t) = \ln t$, $t \in \mathbb{N}$, it follows that

$\Delta h(t) = \ln(1+\frac{1}{t}) \to 0$ as $t \to \infty$. These show that there exist non-trivial functions that satisfy the commutator condition. Therefore, we work with the following decay conditions:

$$p_n, q_m > 0, \ \ \Delta h \to 0, \qquad \text{as } t \to \infty, \qquad h = p_k, q_j, \ \ k, j = 1, 2, ... \max(n, m). \tag{2.1}$$

and

$$\frac{\Delta h_l}{h_l} \in \ell^2(I_1), \qquad (\frac{\Delta h_l}{h_l})^2, \ \frac{\Delta^2 h_l}{h_l} \in \ell^1(I_1), \qquad h_l = p_l, q_l, \ \ l = k, j. \tag{2.2}$$

In the theorem below, we have shown that at limiting point, the spectrum of H_r, the self-adjoint operator extension in (1.7), has no singular continuous spectrum and the absolutely continuous spectrum agree with that of constant coefficient limiting operators.

Theorem 2.1. *Consider the minimal difference operators L_1 and L_2 generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ in (1.1) respectively and assume (2.1) and (2.2) are satisfied. Then, L_1 and L_2 are at limit points at infinity if the boundary conditions defined by matrices α_r (1.6) at $t_0 = 0$ are satisfied. The corresponding self-adjoint operator extension H_r has no singular continuous spectrum, $\sigma_{sc}(H_r) = \emptyset$. The absolutely continuous spectra of the self-adjoint operator extensions agree with those of constant coefficient limiting operators. In particular, the spectral measure μ for H_r belongs to absolutely continuous spectra having multiplicities equal to half the number of roots of the corresponding characteristic polynomials with unit magnitude.*

Proof. The proof follows closely from that of Theorem 4.7 of Behncke and Nyamwala [3] with some modification in line with the formulation of $\mathcal{L}_1$ and $\mathcal{L}_2$. Here, we define $\mathcal{L}_1$ and $\mathcal{L}_2$ on the Hilbert space $\ell^2_w(I_1) \cong \ell^2(I_1)$, $w = w(t) = 1$, for all $t \in \mathbb{N}$. Each of the difference equations is converted into their first order system. The respective propagator form (1.5) can be computed explicitly and the associated eigenvalues determined. Uniform dichotomy condition can then be established uniformly for those eigenvalues $\lambda(t,z)$ of unit magnitude since the other eigenvalues shall satisfy uniform dichotomy condition irrespective of the value of spectral parameter z [3]. Thereafter, the transforming matrix $T(t,z)$ can be determined from the eigenvectors and the first order system diagonalized repeatedly to obtain Levinson-Benzaid-Lutz (LBL) form as explained in Section 1. The assumptions in (2.2) imply that two diagonalizations will bring the first order form into the required Levinson-Benzaid-Lutz form (1.8). Application of the discretized version of Levinson's Theorem now gives the form of solutions (1.9) that are required for analysis. We consider the zeros of $\mathcal{F}(t,\gamma,z)$ in (1.10). The roots of the polynomial in (1.10) will be considered within a set $\mathcal{K} = \{z \in \mathbb{C} \mid z - z_0 \mid < \epsilon\}$ with relevant spectral parameters which can be adjusted until $\mathcal{P}(\lambda,t,z)$ has distinct roots. Suppose the roots of the characteristic polynomial $\mathcal{P}_r(\lambda,t,z)$ are $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_{2l}, \tilde{\lambda}_{2l+1}, \tilde{\lambda}_{2l+2} + ... + \tilde{\lambda}_{2s}$, $s = n, m$, $l \leq s$. Here $\lambda_l = \lambda_l(t,z), \tilde{\lambda}_l = \tilde{\lambda}_l(t,z)$ where $\mid \lambda_k \mid \neq 1$, $k = 1, 2, ..., 2l$, $\mid \tilde{\lambda}_k \mid = 1$, $k = 2l+1, ..., 2s$, then by the results of Behncke [1] and Behncke and Nyamwala

[3], the dichotomy condition is only required for those characteristic roots (eigenvalues of $S(t,z)$ with magnitude 1, that is $\tilde{\lambda}_k(t,z)$). Using complex analysis, we represent these eigenvalues by $\tilde{\lambda}_k(t,z) = \exp(i\beta_k)$ for some angle β_k. It follows that $\tilde{\lambda}_k(t,z) = \exp(-i\beta_k)$ will also be a root. Then off the real axis and taking $z = z_0 + i\epsilon$, $\epsilon > 0$, z_0, $\epsilon \in \mathbb{R}$, $\epsilon > 0$ is small, we have,

$$\tilde{\lambda}_k(z_0 + i\epsilon) + \overline{\tilde{\lambda}}_k(z_0 + i\epsilon) = 2\cos\beta_k \quad k = l+1, ..., (s-l).$$

Thus

$$\tilde{\lambda}_k(z_0 + i\epsilon) = \tilde{\lambda}_k(z_0) + \frac{i\epsilon}{\partial_{\lambda_k}\mathcal{P}_r(\lambda, t, z)}.$$

It is the correction term $i\epsilon(\partial_{\lambda_k}\mathcal{P}_r(\lambda,t,z))^{-1}$ which will lead to either $\tilde{\lambda}_k(z_0 + i\epsilon)$ having magnitude greater than 1 or less than 1 off the real axis. The number of roots with magnitude almost one at limiting point but with magnitude greater than 1 off the real axis will be half the roots $\tilde{\lambda}_l(t,z)$ while the others will have magnitude less than 1 off the real axis. This is the required uniform dichotomy condition on the eigenvalues of $\mathcal{P}_r(\lambda,t,z)$. The roots $\lambda_l(t,z)$ that off the real axis have magnitude greater than 1 lead to eigensolutions that lose their square summability as $\epsilon \to 0^+$ and hence contributes to absolutely continuous spectrum. The spectral multiplicity can thus be analysed via the M-matrix.
The Titchmarsh-Weyl functions for the minimal difference operators L_1 and L_2 are the respective M-matrices $M_r(z) = M_r$, $r = 1,2$ and which are Borel-transforms of the spectral measures μ_r, $r = 1,2$. The density of the absolutely continuous spectrum of each H_r is given by

$$(\frac{1}{\pi})\lim_{\epsilon\to 0^+} M_r(\mu_r + i\epsilon) = (\frac{1}{\pi})M_r(\mu_+) = \varrho(\mu_r).$$

The spectrum is absolutely continuous if $M_r(\mu_r + i\epsilon)$ has finite limits $M_r(\mu_{r_+})$. The eigenvalues of H_r correspond to the poles of M_r. The $M_r(z)$ are determined off the real axis and can be constructed from the eigenfunctions of H_r that are square summable. In this case, note that the deficiency indices for L_1 and L_2 are (n,n) and (m,m) respectively at the limiting point with $\alpha-$boundary conditions also imposed as $t \to \infty$. Application of von-Neumann theorems now show that L_1 and L_2 have self-adjoint operator extensions that are described by $D(H_r)$ in (1.7).
Assume that

$$V_r(t,z) = \begin{pmatrix} V_{r_1}(t,z) \\ V_{r_2}(t,z) \end{pmatrix},$$

it follows that

$$V_r(t_0,z) = \begin{pmatrix} Y \end{pmatrix}_r (t_0) \begin{pmatrix} I_s \\ M_{t_0}(z) \end{pmatrix}$$

satisfy boundary conditions at t_0 and also at infinity. The boundary conditions are those given in (1.6) and hence

$$\begin{pmatrix} M_{t_0} \end{pmatrix}_r (z) = V_{r_2}(t_0,z)V_{r_1}^{-1}(t_0,z)$$

In this case, $V_{r_1}(t_0, z)$ are the Vandermonde matrices for eigenvalues of the square summable solutions. $V_{r_1}^{-1}(t_0, z)$ exists boundedly and hence $(M_{t_0})_r(z)$ are continuous for all z within the region of consideration. The continuous spectrum for H_r is not singular continuous spectrum but absolutely continuous spectrum. The left end regular point t_0 can then be extended to $t_0 = 0$ using the results of Remling [19]. □

Suppose that $p_k(t)$ and $q_j(t)$ are almost constant coefficients and $m = n$, then as shown in the next result, the set of commutators generated by the self-adjoint operator extensions of L_1 and L_2 contain the compact operators on $\ell^2(\mathbb{N})$, that is, $\mathcal{K}(\ell^2(\mathbb{N})) \subseteq W_1^n[H_1, H_2]W_2^n$, where $\mathcal{K}(\ell^2(\mathbb{N}))$ is the set of compact operators defined on $\ell^2(\mathbb{N})$. Finally, we show that H_1 and H_2 commute.

Theorem 2.2. *Let L_1 and L_2 be the minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively in (1.1). Assume (2.1) and (2.2) are satisfied in addition to the following conditions*

$$m = n, \qquad p_k(t) \to c_k, \;\; q_j(t) \to d_j,$$

where, c_k and d_j are constants. Then:

(i). *At limiting point, the deficiency indices of L_1 and L_2 are (n, n). L_1 and L_2 have self-adjoint operator extensions; H_1 and H_2, respectively.*
(ii). *The set of commutators $W_1^n[H_1, H_2]W_2^n$ contains the set of compact operators defined on $\ell^2(\mathbb{N})$.*
(iii). *The self-adjoint operator extensions H_1 and H_2 commutes.*

Proof. The proof of this theorem is the discretised version of the proof of Theorem 2.1 of [15] and some steps have been included for completeness.
(i) Follows immediately from Theorem 2.1.
(ii) In order to construct the set of compact operators from the set of commutators $W_1^n[H_1, H_2]W_2^n$ with W_1 and W_2 as defined in (1.12), we apply the techniques of comparison algebras as constructed by H. O. Cordes [9] and applied in the work of S. T. Melo [14] to construct self-adjoint operator extensions with common domain and show that the set of commutators contain the set of compact operators on $\mathcal{H} = \ell^2(\mathbb{N})$. Note that the operators $W_r^n H_r W_r^n$ are bounded in $\mathcal{H}$. The boundedness properties of these operators are required since the theory of comparison algebras are embedded in the theory of C^*-algebras.

Let $\{H_r, \mathbb{N}, \mu\}$ be a comparison triple for each self-adjoint operator extension H_r. It is this triple that we use to construct the H_r-comparison algebras. The comparison algebras are actually C^*-algebras. Here, μ is the counting measure in $\mathbb{N}$. Similarly, let $\mathcal{C}_0^\infty(\mathbb{N})$ be the class of all bounded complex-valued functions which includes also constant functions. This implies that any function $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ can be thought of as $h(t) = h_0(t) + c$ such that $h_0(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ and $c \in \mathbb{C}$. In our case, the function $h(t)$ represents $p_k(t)$ and $q_j(t)$, $k, j = 0, 1, \ldots, n$. On the other hand, assume that $\mathcal{W}$ is the class of all linear symmetric difference operators of order $2n$ generated by $\mathcal{L}_r$ such that L_r, L_r^*, H_r and those generated by complex conjugates of $\mathcal{L}_r$ all having compact support are contained in

$\mathcal{W}$. The interest in this case is the algebra generated by classes of operators of the form $T = W_1^n H_r W_2^n$ together with the classes of multiplication operators $Ty(t) = h(t)y(t)$ in addition to the commutators of two generating operators $[W_1^n H_r W_2^n, h(t)H_r]$. Thus the required H_r-comparison algebras are obtained by norm-closing the algebra generated by the function $h(t)$ and the operator $W_1^n H_r W_2^n$ for all functions $h(t) \in \mathcal{C}_0^\infty(\mathbb{N})$ and $W_1^n H_r W_2^n \in \mathcal{W}$ having compact support. This results into minimal comparison algebra of H_r which we will denote by $\mathcal{C}$. The comparison algebra is a non-unital C^*-algebra since $\mathbb{N}$ is non-compact. It can be made unital by adjoining $\{e\}$ to it, e is the identity element. Every minimal algebra, that is, every comparison algebra of self-adjoint operator H_r contain the entire ideal of compact operators $\mathcal{K}(\mathcal{H})$ defined on $\mathcal{H}$ [9]. We now show that every element in $\mathcal{K}(\mathcal{H})$ is in the set of commutator and that the commutator set is non-empty. Suppose that $f(t) \in \mathcal{C}_0^\infty(\mathbb{N})$, $h(t) = p_k(t), \quad q_j(t)$ with $h(t)$ not vanishing to zero identically as $t \to \infty$, then one can consider the operator defined by $h(t)W_r = h(t)H_r^{-\frac{1}{2n}}$. This operator is compact and is in the set of commutator since in general any operator of the form Δ generates an operator of the form $H_r^{-\frac{1}{2n}}$ and $[h(t), \Delta]$ is in the set of commutator if and only if $\Delta h(t) \to 0$ as $t \to \infty$ and this is one of the assumptions made. Thus $h(t)W_r \neq 0$. It follows, therefore, by construction that $H_r W_2^n = W_1^n H_r$ and they are defined on the space $H_r(\mathcal{C}_0^\infty(\mathbb{N}))$. Moreover, there exists compact operators C_1 and C_2 defined on $\mathcal{H}$ such that $W_1^n C_1 W_2^n = W_1^n C_2 W_2^n$.

We now show that the operator $H_1 W_1^n$ commutes with $H_2 W_2^n$. First note that by Gelfand-Naimark theorem, the H_1-comparison algebra is isomorphic to H_2-comparison algebra and since W_1 and W_2 are isometric isomorphisms constructed from the same underlying manifold, they are commutative since isometries of the same manifold form abelian group under composition. Secondly, by construction, the operators $H_r W_2^n$ and $W_1^n H_r$ are commutative and since they are compact, there exists a compact operator C_0 such that $H_r W_2^{2n} - W_1^n H_r W_2^n = C_0 W_2^n$, where $C_0 = H_r W_2^n - W_1^n H_r$ and this implies that for any function $y(t) \in D(W_r^{-n})$, we have $H_r W_2^n y(t) = W_1^n H_r y(t) + C_0 y(t)$ so that $W_1^n H_r$ with domain $D(W_r^{-n})$ has a continuous extension $(W_r^n H_r)^{**}$ which is a linear continuous operator in $\mathcal{H}$. Similarly, $H_r W_2^n = (W_1^n H_r)^{**} + C_0$. Computing the adjoints of both sides of the last relation, we get $(H_r W_2^n)^* = H_r^* W_2^n + C_0^*$ which implies that $(H_r W_2^n)^* - H_r^* W_2^n$ and $H_r W_2^n - (W_1^n H_r)^{**}$ are compact. Using similar arguments, there exists a compact operator C_1 such that $C_1 = W_1^n [H_1, H_2] W_2^n$. This is because the operator $W_1^n [H_1, H_2] W_2^n$ is bounded in $\mathcal{H}$. Applying the fact that $H_r W_2^n - (W_1^n H_r)^{**}$ is compact, for $y(t) \in H_r \mathcal{C}_0^\infty(\mathbb{N})$, we obtain

$$\begin{aligned}
H_1 W_1^n H_2 W_2^n y(t) &= (W_1^n H_1)^{**} H_2 W_2^n y(t) + C_0 y(t) \\
&= C_0 y(t) + W_1^n H_1 H_2 W_2^n y(t) \\
&= C_1 y(t) + W_2^n H_2 H_1 W_1^n y(t) + C_2 y(t) \\
&= C_3 y(t) + (W_2^n H_2)^{**} H_1 W_1^n y(t) \\
&= H_2 W_2^n H_1 W_1^n y(t) + C_4 y(t),
\end{aligned}$$

for some C_2, C_3, C_4 compact operators in $\mathcal{H}$. This implies that the set of commutator $W_1^n[H_1, H_2]W_2^n$ contains the set of compact operators on $\mathcal{H}$ and it is non-empty.

(iii) We apply Zagorodnyuk results [23] to prove that H_1 and H_2 commute. From the construction of the commutator sets $W_1^n[H_1, H_2]W_2^n$ in (ii) above, it follows that $H_1 W_1^n$ commutes with $H_2 W_2^n$ and hence we need to show that $D(H_1) = D(H_2)$ so that H_1 commutes with H_2. From the assumptions in (2.1) and the comparison algebra $\mathcal{C}$ constructed in (ii), it follows that H_1 and H_2 have common domain, that is, $D(H_1) = D(H_2)$ and since H_r, $r = 1, 2$ is self-adjoint, $D(H_r)$ is densely defined, $\overline{D(H_r)} = \mathcal{H} = \ell^2(\mathbb{N})$. It is also true that $H_r(D(H_r)) \subseteq D(H_r)$ and the restriction of H_2 to $(H_1 - iI)D(H_1)$ or H_1 to $(H_2 - iI)D(H_2)$ is not only essentially self-adjoint but also self-adjoint. Because $n = m$ by assumption, the symplectic matrix $\mathcal{J}$ in (1.4) results into $\mathcal{J}H_r = H_r\mathcal{J}$. By the results of [23], it follows that H_1 commutes with H_2 and so are their respective minimal and maximal difference operators.

This implies that the commutator algebra generated by H_r contains the set of compact operators in $\mathcal{H}$ and that the operators H_1 and H_2 commute and so are the respective minimal difference operators. □

The example below verifies the results of Theorem 2.2. This result gives the necessary and sufficient conditions on the coefficients of difference operators of the same order to commute. The condition that $\Delta(h(t)) \to 0$ as $t \to 0$ is necessary and cannot be omitted as the example also shows.

Example 2.3. As an immediate example, consider the operators generated by difference equations

$$\mathcal{L}_1 y(t) = -\Delta(p(t)\Delta y(t-1)), \qquad \mathcal{L}_2 y(t) = -\Delta(q(t)\Delta y(t-1)),$$

$$\Delta p(t), \;\; \Delta q(t) \to 0 \qquad \text{as } \; t \to \infty.$$

It is easy to check that $\mathcal{L}_1(\mathcal{L}_2 y(t)) - \mathcal{L}_2(\mathcal{L}_1 y(t)) \to 0$ as $t \to \infty$ and hence the minimal operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ as well as their respective self-adjoint operator extensions are commutative. Similarly, assume that $p(t)$ and $q(t)$ are almost constant coefficients, then commutativity shall be achieved. On the other hand, if we assume for simplicity, that $p(t) = 1$ for all $t \in \mathbb{N}$ and $q(t) = \cos\frac{\pi t}{2}$, then $\Delta q(t)$ shall be 1 or -1 depending on whether t is odd or even. A simple computation shows that $\mathcal{L}_1 y(t)$ does not commute with $\mathcal{L}_2 y(t)$. Hence the condition that $\Delta h(t) \to 0$ as $t \to \infty$ in (2.1) is necessary.

In the remaining Section, we will assume that (2.1) and (2.2) hold and in addition, we will also assume that $n = m$. We now develop a similar analysis to that of $\mathcal{L}_1 y(t)$ and $\mathcal{L}_2 y(t)$ in Section 1 and so far in Section 2 for the composite function $\mathcal{L}_1(\mathcal{L}_2 y(t))$. Thus we now consider the composite of $\mathcal{L}_1$ and $\mathcal{L}_2$, denoted by $\mathcal{L}$ defined on $\ell^2(\mathbb{N})$. The difference equation for $n = m$ is a $4n$th order symmetric difference equation given by

$$\begin{aligned}\mathcal{L}y &= \mathcal{L}_1(\mathcal{L}_2 y) = \mathcal{L}_2(\mathcal{L}_1 y) \\ &= \sum_{k=0}^{n}(-1)^k\left\{\sum_{j=0}^{m}(-1)^j\Delta^k(p_k\Delta^{k+j}(q_j\Delta^j y(t-j)))\right\} \\ &= \sum_{j=0}^{m}(-1)^j\left\{\sum_{k=0}^{n}(-1)^k\Delta^j(q_j\Delta^{j+k}(p_k\Delta^k y(t-k)))\right\},\end{aligned} \tag{2.3}$$

defined on $\ell^2(\mathbb{N})$ with $\sum_{k=0}^{n}(\sum_{j=0}^{m}(p_kq_j)) > 0$. The interest in this analysis is to develop structural properties of the minimal operator generated in the Hamiltonian form as well as the first order form of $\mathcal{L}_1(\mathcal{L}_2 y) = zy$ for some spectral parameter z. The coefficients p_k and q_j are assumed to satisfy similar growth and decay conditions that have been set in (2.1) and (2.2) so that at the end, we determine all the necessary and sufficient conditions for $H = H_1H_2$ where H is the self-adjoint operator extension generated by minimal operator L corresponding to $\mathcal{L}_1(\mathcal{L}_2)$. We thus solve the equation $\mathcal{L}y = zy$, where z is the spectral parameter. First, we need to reduce $\mathcal{L}y = zy$ into its first order form using quasi-differences as given in [8]. These are of the form:

$$x_j(t) = q_j(t)\Delta^{j-1}y(t-j), \qquad 1 \le j \le m$$

$$x_{m+1}(t) = q_m(t)\Delta^m(y(t-m))$$

$$x_{m+k}(t) = -\Delta(q_m(t)\Delta^m y(t-m)) - p_{n-k}(t)(q_{n-k}(t)\Delta^{n-k}y(t-n+k)), \quad 2 \le k \le n$$

$$x_{n+m+r}(t) = -(\Delta(x_{n+m+r-1}(t)) + p_{n-r}(t)x_{m+n-r}(t)), \qquad 2 \le r \le n+m-1.$$

In this case, we define $2(n+m)$-dimensional vector

$$Y(t,z) = [x_1(t), x_2(t), \ldots x_{2(n+m)}(t)]^{tr}.$$

Just like in Section 1, we use symplectic matrix

$$\mathcal{J} = \begin{bmatrix} 0_{n+m} & -I_{n+m} \\ I_{n+m} & 0_{n+m} \end{bmatrix}.$$

Therefore, the Hamiltonian system for (2.3) is given by

$$\mathcal{J} \triangle Y(t) = P(t)K_1Y(t), \tag{2.4}$$

where K_1 is the forward partial shift operator and $P(t)$ is a $2(n+m)\times 2(n+m)$ matrix that can be written in a block form

$$P(t) = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$$

with A, B, C as $(n+m)\times(n+m)$ matrices having non-zero entries given by

$$A_{l,l+1} = 1 \qquad\qquad B_{n+m,n+m} = (p_nq_m)^{-1},$$

$$C_{l,l} = f_l = \sum_{k=0}^{l}\left(\sum_{j=0}^{l}(p_kq_j)\right), \qquad\qquad l = 0,1,2,\cdots,n+m.$$

Here, the term $p_0 q_0$ should be interpreted as $p_0 q_0 - z$. The first order can now be written explicitly as

$$\begin{aligned} Y(t+1,z) &= \begin{bmatrix} E & EB \\ CEB & I - A^* + CEB \end{bmatrix} Y(t,z) \\ &= S(t,z)Y(t,z). \end{aligned} \tag{2.5}$$

Here, we note that $E = (I_{n+m} - A)^{-1}$.

In order to properly define the maximal and minimal difference operators generated by $\mathcal{L}$ in (2.3), we still need some regularity conditions just like in Section 1. These are

$$\begin{aligned} \ell^2(\mathbb{N}) &= \{y(t) : y(t) = \{y(t)\}_{t=0}^{\infty} \subset \mathbb{C} \\ & \left. \sum_{t=0}^{\infty} K_1(Y^*(t))K_1(Y(t)) < \infty \right\}. \end{aligned}$$

in addition, there exists an integral domain $I_2 \subset \mathbb{N}$ such that

$$\sum_{t \in I_2} K_1 Y^*(t) K_1 Y(t) > 0,$$

$\ell^2(I_2) \subset \ell^2(\mathbb{N})$ and

$$\ell^2(I_2) = \left\{ y(t) \in \ell^2(\mathbb{N}) : \sum_{t \in I_2} K_1 Y^*(t) K_1 Y(t) > 0 \right\}.$$

we thus define the scalar products for $y(t),\ y_1(t) \in \ell^2(I_2)$ by

$$\langle y(t), y_1(t) \rangle = \sum_{t \in I_2} y(t) \overline{y_1(t)}.$$

From the results of Theorem 2.2, it follows that $I_2 \subseteq I_1$ and hence $\ell^2(I_2)$ is a subspace of $\ell^2(I_1) \subset \ell^2(\mathbb{N})$.
Now if $F_1(t)$ is $(2(n+m))$-dimensional vector, then we also need a similar condition like that stated in section 1 in order for the operators generated to be properly defined.

$$\mathcal{J}\Delta Y(t,z) - PK_1 Y(t,z) = F_1(t), \ \text{ with } \ \|Y(t,z)\| = 0, \text{ if, } F_1(t) = 0.$$

The construction of the associated maximal and minimal difference operators via their domains is done as follows. The maximal difference operator L^* defined via its domain $D(L^*)$ is given by

$$\begin{aligned} D(L^*) &= \{ y(t) \in \ell^2(I_2) : \text{there exists } F_1(t) \in \ell^2(I_2) \text{ such that} \\ & \mathcal{J}\Delta Y(t) - PK_1 Y(t) = F_1(t) \} \end{aligned}$$

and

$$L^* Y(t,z) = F_1(t) \ \text{ if and only if } \ \mathcal{J}\Delta Y(t,z) - PK_1 Y(t,z) = F_1(t).$$

The corresponding pre-minimal operator is now given by

$$\begin{aligned} D(L_0) &= \{y(t) \in D(L^*) : \text{ there exists } n \in \mathbb{N} \\ &\quad \text{such that } y(0) = y(t) = 0 \text{ for all } t \geq n+1\}. \end{aligned}$$

$$L^* y(t) = L_0 y(t), \qquad \forall y(t) \in D(L_0).$$

L_0 is densely defined, symmetric but not necessarily closed. Since we need closed operators, we take the closure of this pre-minimal operator to be the minimal difference operator which we denote by L. The boundary conditions at the regular left end point [6, 21] can now be defined using $(n+m)\times(n+m)$ matrices $(\tilde{\alpha_1}, \tilde{\alpha_2})$ with $\tilde{\alpha_1}\tilde{\alpha_1}^* + \tilde{\alpha_2}\tilde{\alpha_2}^* = I_{n+m}$ and $\tilde{\alpha_1}\tilde{\alpha_2}^* = \tilde{\alpha_2}\tilde{\alpha_1}^*$ so that at the regular left end point t_0,

$$(\tilde{\alpha_1}, \tilde{\alpha_2}) \left[\ Y(t_0)\ \right] = 0. \qquad where\ \ Y(t) = [x_1(t), x_2(t), ..., x_{2(n+m)}]^{tr} \tag{2.6}$$

In solving the equation $\mathcal{L}y = zy$, we shall apply techniques of Eastham [10] which have been applied extensively by Nyamwala and other authors, see [6, 5, 3, 16, 17, 18] for details. We assume the following growth conditions for the remaining part of the analysis:

$$h_{l-1}h_{l+1} = o(h_l^2), \qquad h_l = p_l, q_l, \quad l = 1, 2, \ldots, n = m. \tag{2.7}$$

We note now that the growth conditions in (2.7) results into $f_{l-1}f_{l+1} = o(f_l^2)$, where

$$f_l = \sum_{k,j=0}^{l} p_k q_j, \quad l = 0, 1, 2, \ldots, n+m. \tag{2.8}$$

In order to transform (2.3) into the required first order form , we need the eigenvalues of the matrix $S(t, z)$ which we obtain from the zeroes of the characteristic polynomial $\mathcal{P}(t, \lambda, z) = \det(S(t, \lambda, z) - \lambda \cdot I_{2(n+m)})$. By multiplying $\mathcal{P}(t, \lambda, z)$ by $(-1)^{n+m}p_n q_m \lambda^{-(n+m)}$ and taking $\gamma = 2 - (\lambda + \lambda^{-1})$, we obtain a reduced polynomial of the form

$$\begin{aligned} \mathcal{F}(t, \gamma, z) &= (-1)^{n+m} p_n q_m \lambda^{n+m} \mathcal{P}(t, \lambda, z) \\ &= \sum_{l=0}^{n+m} f_l \gamma^l. \end{aligned} \tag{2.9}$$

Before we proceed with our analysis, we start by proving that the basis of the homogeneous equation $\mathcal{L}_1\mathcal{L}_2 y = 0$ can be determined from the bases of solutions of equations $\mathcal{L}_1 y = 0$ and $\mathcal{L}_2 y = 0$.

Theorem 2.4. *Suppose $\mathcal{L}_1\mathcal{L}_2 y = 0$, $n = m$, then the basis of the solutions of this equation can be determined from the bases of the solutions of $\mathcal{L}_1 y = 0$ and $\mathcal{L}_2 y = 0$.*

Proof. The proof in this case follow from the results of [13] with obvious modification to suit difference equations. Suppose that $\{y_1, y_2, \ldots, y_{2n}\}$ is a basis of solutions of $\mathcal{L}_2 y = 0$ and $\{\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{2n}\}$ is the basis of solutions of $\mathcal{L}_1 y = 0$, then the basis of the solutions of the composite is obtained using the Green's function $G_2(t, t_1)$ associated with $\mathcal{L}_2 y = 0$, and the corresponding Wronskian determinant. For a fixed left regular end point $t_0 \in \mathbb{N}$, define functions

$$y_{2n+s}(t) = \sum_{t_1=t_0}^{t-1} G_2(t, t_1)\tilde{y}_s(t_1) \quad s = 1, 2, \ldots, 2n.$$

Then the set $\tilde{Y} = \{y_1, y_2, \ldots, y_{2n}, y_{2n+1}, y_{2n+2}, \ldots, y_{4n}\}$, is a basis of solutions of the composite equation $\mathcal{L}_1\mathcal{L}_2 y = 0$ for $t \in \mathbb{N}$. To prove this, consider the equations $\mathcal{L}_2 y_k = 0$, $k = 1, 2, \ldots, 2n$, $\mathcal{L}_2 y_k = \tilde{y}_k$, $k = 2n+1, \ldots, 4n$ and $\mathcal{L}_1\mathcal{L}_2 y_k = 0$, $k = 2n+1, \ldots, 4n$. This implies that

$$(\mathcal{L}_1\mathcal{L}_2)y_k) = \mathcal{L}_1(\mathcal{L}_2 y_k) = \mathcal{L}_1\tilde{y}_k = 0, \quad k = 2n+1, \ldots, 4n.$$

The rest of the proof now follows from the usual linear algebra techniques. □

In the next result, we now prove that if $\mathcal{L}_1$ and $\mathcal{L}_2$ commute, then their composite is symmetric and so is the operator L_1L_2.

Theorem 2.5. *Assume that* (2.1) *is satisfied such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative. Then the minimal difference operators L_1 and L_2 are commutative and the composite $\mathcal{L}_1\mathcal{L}_2$ is symmetric as well as L_1L_2.*

Proof. From Theorem 2.2, we have that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative and so are L_1 and L_2. Similarly, the operators L_1^* and L_2^* commute and to see this, take $y, y_1 \in D(L_2)$ and we have

$$\langle y, L_1^* L_2^* y_1\rangle = \langle y, (L_2L_1)^* y_1\rangle = \langle y, (L_1L_2)^* y_1\rangle = \langle y, L_2^* L_1^* y_1\rangle.$$

Hence $L_1^*L_2^* = L_2^*L_1^*$.
It remains to show that $D(L_1L_2) \subseteq D((L_1L_2)^*)$. Since L_1 and L_2 are closed symmetric operators, it follows that $\overline{D(L_2)} = \mathcal{H}$ and since L_1 commutes with L_2, we have $D(L_1L_2) = D(L_2)$ such that $D(L_1) \subseteq \mathcal{R}(L_2)$. Suppose now that $\tilde{y} \perp D(L_2)$ such that $\langle \tilde{y}, y\rangle = 0$, for all $y \in D(L_2)$, then for any solution $\hat{y}$ of $\mathcal{L}_1\mathcal{L}_2\hat{y} = \tilde{y}$ and since $D((L_1L_2)^*) = D(L_1^*L_2^*) = D(L_2^*)$, it implies that

$$\langle \hat{y}, \mathcal{L}_1\mathcal{L}_2 y\rangle = \langle L_2^*L_1^*\hat{y}, y\rangle = \langle L_1^*L_2^*\hat{y}, y\rangle = \langle \mathcal{L}_1\mathcal{L}_2\hat{y}, y\rangle = \langle \tilde{y}, y\rangle = 0.$$

This implies that $y \in \mathcal{R}(L_1L_2)^\perp = \mathcal{N}(L_2^*L_1^*) = \mathcal{N}((L_1L_2)^*)$ and hence $\tilde{y} = \mathcal{L}_1\mathcal{L}_2\hat{y} = (L_1L_2)^*\hat{y} = 0$. The operator L_1L_2 is symmetric and so is the function $\mathcal{L}_1\mathcal{L}_2$ that generates it on $\mathcal{H} = \ell^2(\mathbb{N})$. □

The composite operator L_1L_2 is now symmetric but not necessarily densely defined. We now state the necessary and sufficient conditions for the composite to be densely defined.

Theorem 2.6. *Let L_1 and L_2 be minimal difference operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively. Assume (2.1) is satisfied and that $\mathcal{R}(L_1)$, $\mathcal{R}(L_2)$ are closed. Then the operator L_1L_2 is densely defined.*

Proof. By construction of L_1 and L_2 in Section 1, the two operators are closed. This implies that their ranges are closed and hence there exists a closed subspace $\mathcal{D}$ of $\mathcal{H}\times\mathcal{H}$ defined by

$$\mathcal{D} = \{(y, L_1L_2y) : \quad y \in D(L_1L_2)\}.$$

By injectivity and linearity of L_1 and L_2 which imply that of L_1L_2, $\mathcal{D}$ is graph of L_1L_2 because for any $(0,\tilde{y}) \in \mathcal{D}$, it follows that $\tilde{y} = 0$. On the other hand, for any $y \in D(L_1L_2)$, there is at most one $\hat{y}$ such that $(y,\hat{y}) \in \mathcal{D}$ and $\hat{y} = L_1L_2y$. The operator L_2^{-1} can now be defined on $\mathcal{R}(L_2)$ because $\mathcal{R}(L_2)$ is closed and L_2 is injective hence $D(L_1) = \mathcal{R}(L_2)$. For the operator L_2L_1, now consider $D(L_1L_2)$ which can be expressed as

$$D(L_1L_2) = D(L_2) \cap L_2^{-1}D(L_1).$$

Since $D(L_1)$ and $D(L_2)$ are dense in $\mathcal{H}$ by construction, $L_2^{-1}D(L_1)$ is a dense subspace of $\mathcal{H}$ and thus by Baires Category theorem, the subspace $D(L_2)\cap L_2^{-1}D(L_1)$ is dense in $\mathcal{H}$. The operator L_1L_2 is densely defined. $\square$

The operator L_1L_2 together with conditions in Theorem 2.6 is now symmetric and densely defined but not necessarily closed. We may take its closure, that is, $\overline{L_1L_2}$ and this we shall denote by L. This allows for the analysis of spectral theory of self-adjoint operator extension of the composite of operators L_1 and L_2. Note that by Closed-Range Theorem, if all the conditions in Theorem 2.6 are satisfied, then the range of L^* is closed. Thus by application of Rank-Nullity theorem we have

$$\dim \mathcal{N}(L^* - zI) = \dim \mathcal{N}(\overline{L_1L_2}^* - zI) = \dim \mathcal{N}(L_1^* - zI) + \dim \mathcal{N}(L_2^* - zI).$$

This implies that $\mathrm{def}L = \mathrm{def}L_1 + \mathrm{def}L_2$.

3. Spectral Analysis of L_1, L_2 and L with unbounded coefficients.

In this section, we will assume that (2.1), (2.2) and (2.7) are satisfied in addition to those conditions in Theorem 2.6 so that $\mathcal{L}_1$ and $\mathcal{L}_2$ are commutative and the operators L_1 and L_2 are commutative as well. As before, we will take the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N})$ and for simplicity take $n = m$ though the analysis can still be carried out even if $n \neq m$. We will also allow the coefficients p_k and q_j to be unbounded. The reader is cautioned that the set up in [16] is slightly different from ours since in the mentioned reference, the author had non-zero odd order coefficients. We now formulate results which are extensions of those in Theorem 2.1 under unbounded coefficients. We begin with the approximation of the eigenvalues if (2.7) is assumed.

Lemma 3.1. *Suppose that (2.7) is satisfied, then the eigenvalues of the first order system (1.5), computed from the zeros of the polynomial $\mathcal{F}(t,\gamma,z)$ are approximately given by*

$$\lambda_l(t,z) \approx 1 + \frac{h_{l-1}}{2h_l} \pm \left\{ \frac{h_{l-1}}{h_l} + \frac{h_{l-1}^2}{4h_l^2} \right\}^{\frac{1}{2}} + o(1), \quad h_l = p_l, q_l.$$

Proof. In this case, we consider the polynomial $\mathcal{F}(t,\gamma,z)$ in (1.10) and without loss of generality, we may assume that $h_l = p_l$ since the proof using the coefficients q_l is done in a similar way and will thus follow at once. Now we can equate $\mathcal{F}(t,\gamma,z)$ to zero and by right scaling, we multiply all through by γ^{-l+1} to obtain a reduced polynomial of the form

$$p_l\gamma + p_{l-1} + \mathcal{R}(t,\gamma,z), \tag{3.1}$$

where

$$\mathcal{R}(t,\gamma,z) = \gamma^{-l+1}\left\{ \sum_{\nu=l+1}^{n} p_\nu\gamma^\nu + \sum_{\nu=0}^{l-2} p_\nu\gamma^\nu \right\}.$$

We need, therefore, to show that $\mathcal{R}(t,\gamma,z) \to 0$ as $t \to \infty$. In order to show this, we invoke the coefficient growth conditions in (2.7) and only show this for the coefficients $\nu = l+1$ and $\nu = l-2$ since as one moves further away from the l index, the summands tend to zero rapidly faster as $t \to \infty$. Assume, therefore, that $\nu = l+1$ then we have the magnitude of leading terms for $\mathcal{R}(t,\gamma,z)$ as either $O(\mid \frac{p_{l-1}}{p_l} \mid^2 \mid p_{l+1} \mid)$ or $O(\mid p_{l-2} \mid\mid \frac{p_l}{p_{l-1}} \mid)$. These terms go to zero as $t \to \infty$ as shown here.

$$\mid \frac{p_{l-1}}{p_l} \mid^2 \mid p_{l+1} \mid \approx \mid \frac{p_{l-1}p_{l+1}}{p_l^2} \mid\mid p_{l-1} \mid = o(1).$$

$$\mid p_{l-2} \mid\mid \frac{p_l}{p_{l-1}} \mid \approx \mid p_{l-2}\frac{p_{l-1}}{p_{l-1}}\frac{p_l}{p_{l-1}} \mid = \mid \frac{p_{l-2}p_l}{p_{l-1}^2} \mid\mid p_{l-1} \mid = o(1).$$

Now substitute the value of $\gamma = 2 - (\lambda_l(t,z) + \lambda_l^{-1}(t,z))$ in (3.1) and apply completing square formula. Finally, one can easily show that these λ-roots are actually the approximate value of the eigenvalues of $S(t,z)$ in (1.5). For this, take $\lambda_l(t_0,z)$, t_0 left regular end point, as the initial approximate of the zeros of (1.10) and within a suitably restricted region, perform infinite iterations and since the underlying space is a Hilbert space, application of Banach fixed point theorem will complete the proof. For more details, see [5] and the references cited therein.

□

The next result gives the spectral results of the operators generated by $\mathcal{L}_1$ and $\mathcal{L}_2$ under unbounded coefficients and thus could be considered as extension of Theorem 2.1. In this case, the location of the absolutely continuous spectrum is explicit unlike in Theorem 2.1.

Theorem 3.2. *Assume that (2.1), (2.2) and (2.7) are satisfied and $n = m$. Then the minimal difference operators L_1 and L_2 are commutative, have self-adjoint operator extensions H_1 and H_2 with absolutely continuous spectrum $\mathbb{R}$ of* *various multiplicities. Moreover, the operators H_1 and H_2 are commutative and defined via separated boundary conditions at infinity.*

Proof. The proof of this theorem follows closely from Theorem 2.1, Theorem 2.2. Lemma 3.1 and the main results of [16]. For completion, we will therefore give an outline of the proof and detailed computations where required. Commutativity of the generated minimal difference operators are immediate from Theorem 2.2. By the results of Lemma 3.1 above, the eigenvalues of each of the first order systems generated by $\mathcal{L}_r y = zy$ can be approximated. Now one needs to show that these eigenvalues satisfy the z-uniform dichotomy condition. For the eigenvalues generated from the same root γ_l, the dichotomy condition is established via the correction term $z \mid \frac{\partial \mathcal{F}(t,\gamma_l,z)}{\partial \gamma} \mid^{-1}$ for $z = z_0 + i\eta$, $z_0, \eta \in \mathbb{R}$, $\eta > 0$, small. For the coefficients of $\mathcal{L}_1 y = zy$, explicit approximation of these λ-roots lead to the following approximations as $t \to \infty$

$$\lambda_{l\pm}(t,z) \approx 1 \mp (z_0 + i\eta) \mid p_l \mid^{l-2} \mid p_{l-1} \mid^{-l+1}$$

and hence at the limit point, we have $\mid \lambda_{l+}(t,z) \mid < 1$ and $\mid \lambda_{l-}(t,z) \mid > 1$. The analysis for the roots of $\mathcal{L}_2 y = zy$ is done in a similar way. This will separate the magnitude of $\lambda_{l+}(t,z)$ and $\lambda_{l-}(t,z)$ such that one root has magnitude greater than one and the other less than one. For the eigenvalues belonging to different γ-roots, say, γ_l and $\gamma_{l\pm\nu}$, then note that by application of (2.7) we have the relation

$$\mid \frac{\partial \mathcal{F}(t,\gamma_{l-\nu},z)}{\partial \gamma} \mid^{-1} \gg \mid \frac{\partial \mathcal{F}(t,\gamma_{l},z)}{\partial \gamma} \mid^{-1} \gg \mid \frac{\partial \mathcal{F}(t,\gamma_{l+\nu},z)}{\partial \gamma} \mid^{-1}.$$

This will result into correction terms with different magnitudes and hence will lead to distinct asymptotic behaviour of eigensolutions as required. The two arguments now settle the uniform dichotomy condition.

The first order systems can then be diagonalized appropriately and be transformed into the Levinson-Benzaid-Lutz form like in Section 1 and thus the solutions of the respective first order are given by (1.9). The deficiency indices, here, will depend on the nature of the γ_l-roots. The two solutions associated to the γ_l-roots such that $\mid \gamma_l \mid \to 2$ as $t \to \infty$ will both be square summable if $\mid h_l \mid^{l-2} \mid h_{l-1} \mid^{-l+1}$ is summable. In this particular case, the operators L_r will be at non-limit point at infinity. The self-adjoint operator extensions can only be described by separated boundary conditions. We will do this for the operator L_1 since the analysis for the second operator L_2 is done in a similar way. For simplicity, we may assume that $\mathrm{def} L_1 = (\zeta, \zeta)$, $n \le \zeta \le 2n$. This implies that the set of all square summable solutions of $L_1 y = zy$ is linearly dependent. In order to obtain the basis of this set, we let $\omega_1, \ldots, \omega_{\zeta - n}$ be the set of solutions which are linearly independent modulo $D(L_1)$ at infinity and these we may choose as the eigensolutions of $L_1^* \omega_k^* = z\omega_k$, $k = 1, 2, \ldots, \zeta - n$. Thus we set $\zeta - n$ extra boundary conditions at infinity by demanding that

$$\lim_{t \to \infty} \omega_k^*(t) \mathcal{J} \omega_{k_1}(t) = 0, \qquad \text{for } \; k, k_1 = 1, 2, \ldots, \zeta - n.$$

This will result into a set of linearly independent square summable solutions for the equation $\mathcal{L}_1 y = zy$. The self-adjoint operator extension of the operator L_1 is defined using separated boundary conditions given here below

$$\begin{aligned} D(H_{1\infty}) &= \{y(t) \in D(L_1^*) | (\alpha_1, \alpha_2) y(0) = 0, \\ &\qquad \lim_{t\to\infty} \omega_k^*(t) \mathcal{J} y(t) = 0, \qquad \text{for } \ k = 1, 2, \ldots, \zeta - n. \Big\} \end{aligned}$$

It follows, therefore, that for any self-adjoint operator extension of L_1 defined this way, we obtain the relation,

$$\dim\left(D(H_{1\infty})/D(L_1)\right) = \dim \mathcal{N}(L_1^* - zI) = n$$

so that $H_{1\infty}$ is n-dimensional operator extension of L_1. One has, therefore, $L_1 \subset H_1 = H_{1\infty} = L_1^*$ in the operator sense. Here, the operator H_1 is the self-adjoint operator extension of L_1 at limit point which is defined in Section 1 and also similar to that constructed under conditions imposed on the coefficients in Theorem 2.2. The analysis and construction of the self-adjoint operator extension of L_2, $H_{2\infty}$, when $| \ q_j \ |^{j-2} | \ q_{j-1} \ |^{-j+1}$ are summable is done in a similar way.

Those eigensolutions of L_r that lose their square summability as $\eta \to 0^+$ contribute to absolutely continuous spectrum. Just like in the case of differential operators, as well as in Theorem 2.1, we need to check that the associated M-matrix is bounded so that the spectral measure is continuous. For that, we apply the relation for computing $\text{Im}M(t,z)$ of the M-matrix as used in [4, 19, 18, 21]. Here, we do it for the solution associated with the eigenvalue $\lambda_{l+}(t,z)$.

$$ImM(t,z) = \lim_{\eta\to 0^+} \eta \langle y_{l+}(t,z), y_{l+}(t,z)\rangle = \lim_{\eta\to 0^+} \eta \mid \rho_{l+}(t,z) \mid^2 \prod_{s=t_0}^{t} \mid \lambda_{l+}(s,z) \mid^2$$

and by application of Euler logarithmic relation, this can be approximated by

$$\lim_{\eta\to 0^+} \int_{t_0}^{\infty} \mid \rho_{l+}(t,z) \mid^2 \exp\left(-2\eta \int_{t_0}^{t} \mid h_l \mid^{l-2} \mid h_{l-1} \mid^{-l+1} (s,t) ds\right) dt,$$

which is bounded and the limit exists non-trivially. Here, $\rho_{l+}(t,z)$ is the normalized eigenvector. Since the coefficients are allowed to be unbounded, the absolutely continuous spectrum will be the whole of $\mathbb{R}$. The spectral multiplicity will be the number of solutions with such asymptotic behaviour.

The case when $| \ p_k \ |^{k-2} | \ p_{k-1} \ |^{-k+1}$ and $| \ q_j \ |^{j-2} | \ q_{j-1} \ |^{-j+1}$ are summable for all $k = 1, 2, \ldots, n$ and for all $j = 1, 2, \ldots, m$ is similar to that in Section 2 in Theorem 2.1, since both the operators L_1 and L_2 will be at limit point with discrete spectrum at most and the self-adjoint operator extension given by (1.7). □

We now extend the results of Lemma 3.1 to the case of the composite operators by obtaining the expression for approximating the roots of the polynomial in (2.9).

Lemma 3.3. *Suppose that (2.7) is satisfied and that $\Delta f_l \to 0$ as $t \to \infty$ as required by (2.1), f_l defined in (2.8), then the γ-roots of $\mathcal{F}(t,\gamma,z)$ in (2.9) can be approximated from the relationship*

$$f_l\gamma + f_{l-1} + \mathcal{R}(t,\gamma,z), \tag{3.2}$$

where

$$\mathcal{R}(t,\gamma,z) = \gamma^{-l+1}\left\{\sum_{\nu=l+1}^{n+m} f_\nu\gamma^\nu + \sum_{\nu=0}^{l-2} f_\nu\gamma^\nu\right\}.$$

Proof. Just like in Lemma 3.1, it suffices to show that $\mathcal{R}(t,\gamma,z) \to 0$ as $t \to \infty$. This can be shown for the coefficients $\nu = l+1$ and $\nu = l-2$. Assume, therefore, that $\nu = l+1$ then we have that the leading term for $\mathcal{R}(t,\gamma,z)$ is either $O(\mid \frac{f_{l-1}}{f_l} \mid^2\mid f_{l+1} \mid)$ or $O(\mid f_{l-2} \mid\mid \frac{f_l}{f_{l-1}} \mid)$. It is easy to show that these terms go to zero as $t \to \infty$ using similar approach like in Lemma 3.1. □

The eigenvalues of $\mathcal{L}y = zy$ resulting from polynomial (2.9) can then be approximated using (3.3). The dichotomy conditions and the form of solutions follow closely from the results in Section 2 with the obvious change of the appropriate coefficients. The continuity of the M-matrix follows immediately from the results of Theorem 2.1 in the case of approximate constant coefficients and Theorem 3.2 in the case of unbounded coefficients. Only those γ values such that the associated λ-roots satisfy the relation $\mid \lambda(t,z) + \lambda^{-1}(t,z) \mid \leq 2$ shall lead to some solutions that lose their square summability and hence contribute to absolutely continuous spectrum. These can now be summarised in the following two theorems. Theorem 3.4 extends the results of Theorem 2.1 to the composite case while Theorem 3.5 extends the results of Theorem 3.2 to the composite case.

Theorem 3.4. *Consider the minimal difference operator L generated by $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$ in (2.3) and assume that (2.1)- (2.2) in addition to the conditions in Theorem 2.6 are satisfied. Then L is at limit point at infinity if the boundary conditions defined by the matrix $\tilde{\alpha}$ at t_0 are satisfied. The corresponding self-adjoint operator extension H has no singular continuous spectrum and the absolutely continuous spectrum agrees with that of the constant coefficient limiting operator with the spectral multiplicity equals to half the number of eigenvalues with magnitude 1.*

Proof. The proof follows closely from that of Theorem 2.1 with some modification in line with the composite L_1L_2. Here, we define $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$ on the Hilbert space $\ell^2(\mathbb{N}) \cong \ell^2_w(\mathbb{N})$, $w = w(t) = 1$, for all $t \in \mathbb{N}$. The first order system of the composite is obtained as in (2.5) and the corresponding characteristic polynomial $\mathcal{P}(\lambda,t,z)$ computed. The roots of the polynomial in (2.9) will be considered within a set $\mathcal{K} = \{z \in \mathbb{C} \mid z - z_0 \mid < \epsilon\}$ with relevant spectral parameters which can be adjusted until $\mathcal{P}(\lambda,t,z)$ has distinct roots. The roots are then approximated using (3.3) and one can apply $\gamma = 2 - (\lambda + \lambda^{-1})$ to obtain λ-roots from the corresponding γ-roots. By superimposition principle, the roots of $\mathcal{P}(\lambda,t,z)$, the characteristic polynomial generated by $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$, will have $\lambda_1(t,z), \lambda_2(t,z), ..., \lambda_{4s}, \tilde{\lambda}_{4s+1}, \tilde{\lambda}_{4s+2} + ... + \tilde{\lambda}_{2(m+n)}$, where $\mid \lambda_l(t,z) \mid \neq 1$,

$l = 1, 2, ..., 4s$, $\mid \tilde{\lambda}_l(t,z) \mid= 1$, $l = 4s+1, ..., 2(m+n)$. Just like in Theorem 2.1, the dichotomy condition is only required for those characteristic roots (eigenvalues of $S(t,z)$) with magnitude 1, that is $\tilde{\lambda}_l(t,z)$. Using complex analysis, represent these eigenvalues by $\tilde{\lambda}_l(t,z) = \exp(i\beta_l)$ for some angle β_l. It follows that $\tilde{\lambda}_l(t,z) = \exp(-i\beta_l)$ will also be a root. Then off the real axis and taking $z = z_0 + i\epsilon$, $\epsilon > 0$, z_0, $\epsilon \in \mathbb{R}$, $\epsilon > 0$ is small, we have,

$$\tilde{\lambda}_l(z_0 + i\epsilon) + \overline{\tilde{\lambda}}_l(z_0 + i\epsilon) = 2\cos\beta_l \quad l = 2s+1, ..., 2(s-l).$$

Thus

$$\tilde{\lambda}_l(z_0 + i\epsilon) = \tilde{\lambda}_l(z_0) + \frac{i\epsilon}{\partial_{\lambda l}\mathcal{P}(\lambda, t, z)}(z_0, \lambda_l).$$

It is the correction term $i\epsilon(\partial_{\lambda k}\mathcal{P}(\lambda,t,z))^{-1}$ which will lead to either $\tilde{\lambda}_l(z_0 + i\epsilon)$ having magnitude greater than 1 or less than 1 off the real axis. As before, the number of roots that will have magnitude greater than 1 off the real axis will be half the roots $\tilde{\lambda}_l(t,z)$ while the others will have magnitude less than 1 which establishes the uniform dichotomy condition.

The M-matrix of L again is the Borel-transform of the spectral measure μ. The density of the absolutely continuous spectrum of H is given by

$$(\frac{1}{\pi}) \lim_{\epsilon \to 0^+} M(\mu + i\epsilon) = (\frac{1}{\pi}) M(\mu_+) = \varrho(\mu).$$

The spectrum is absolutely continuous if M has finite limits $M(\mu_+)$. The eigenvalues of H correspond to the poles of M. The $M(z)$ is determined off the real axis and can be constructed from the eigenfunctions of H that are square summable. Thus one has to estimate the zeros of the characteristic polynomial $\mathcal{P}(\lambda,t,z)$ and then apply the results of Theorem 1.1 in order to obtain the right form of solutions as in (1.9) which can be summed over the integral domain I_2 given in Section 1 and with $r_{kk}(t,z) \to 0$ as $t \to \infty$.
In this case note that the deficiency indices for L are $(n+m, n+m)$ at the limiting point with $\alpha-$boundary conditions also imposed as $t \to \infty$.
Assume that $V(t,z) = \begin{pmatrix} V_1(t,z) \\ V_2(t,z) \end{pmatrix}$ is the set of square summable eigenfunctions of L with Dirichlet boundary conditions given in (2.6) with

$$\tilde{\alpha}_1\tilde{\alpha}_1^* + \tilde{\alpha}_2\tilde{\alpha}_2^* = I_{n+m}, \quad \tilde{\alpha}_1\tilde{\alpha}_2^* - \tilde{\alpha}_2\tilde{\alpha}_1^* = 0.$$

Then the square summable eigenfunctions are given by $\left(Y_{t_0} \right)(t) \begin{pmatrix} I_{n+m} \\ M_{t_0}(z) \end{pmatrix}$. and one can then show that

$$\left(M_{t_0} \right)(z) = V_2(t_0, z)V_1^{-1}(t_0, z).$$

$V_1^{-1}(t_0,z)$ exists boundedly because again $V_1(t_0,z)$ is the Vandemonde's matrix. $(M_{t_0})(z)$ is continuous for all z within the region of consideration. The continuous spectrum for H is not singular continuous spectrum but absolutely continuous spectrum. The left end regular point t_0 can then be extended to $t_0 = 0$ using the results of Remling [19]. □

We now prove a result that extends those of Theorem 3.2 to the composite case. This result therefore completes the spectral analysis of operator L_1, L_2 and their respective composite L.

Theorem 3.5. *Let L be the difference operator generated by the difference function $\mathcal{L}$ in (2.3). Assume that (2.1), (2.2) and (2.7) are satisfied in addition to conditions in Theorem 3.2. Then,*

(i) *If the coefficients p_k and q_j are almost constant coefficients and $n = m$, then at limit point $defL = (n+m, n+m)$ and the spectrum of H coincides with that of the constant coefficients case.*
(ii) *Suppose that $n = m$ and (2.7) is satisfied, moreover, if $| f_{l-1} |^{-l+1} | f_l |^{l-2}$, for all $l = 1, 2, \ldots, n+m$, is not summable, then $defL = (n+m, n+m)$ and the absolutely continuous spectrum of H is the whole of real line, $\sigma_{ac}(H) = \mathbb{R}$.*

Proof. (i) The proof follows immediately from Theorems 2.1, 2.2, 3.2 and 3.4. We note here, that by construction, the operator H is $n+m$-extension of the operator L and it is constructed as

$$D(H) = D(L)\dot{+}\{y + Vy : \ y \in \mathcal{N}(L^* - zI)\},$$

where $V = (zI - L)(\bar{z}I - L)^{-1}$. (ii) We note that if $| f_{l-1} |^{-l+1} | f_l |^{l-2}$ is not summable, then each of the $\gamma - roots$ of $\mathcal{F}(\gamma, t, z)$ in (2.9) will have $\lambda - roots$ with magnitude almost equal to one as explained in Theorem 3.4. It is half of the roots that lose their square summability as $t \to \infty$ and $\eta \to 0^+$ that contributes to absolutely continuous spectrum. Since the coefficients are unbounded, the spectrum is the whole of $\mathbb{R}$. □

Finally we relate the operator H and the composites of H_1 and H_2.

Theorem 3.6. *Let L_1, L_2 and L be minimal difference operators generated by $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ respectively and defined on $\ell^2(\mathbb{N})$. Assume that (2.1), (2.2) and (2.7) are satisfied. Moreover, assume that L_1 and L_2 are injective with closed ranges, then*

(i) *$H = H_1H_2$ with $\sigma_{ac}(H) = \mathbb{R}$ if $| p_k |^{k-2} | p_{k-1} |^{-k+1}$ and $| q_j |^{j-2} | q_{j-1} |^{-j+1}$ are not summable for all $k, j = 1, 2, \ldots, n$.*
(ii) *$H = H_1H_2$ with $\sigma(H)$ discrete if $| p_k |^{k-2} | p_{k-1} |^{-k+1}$ and $| q_j |^{j-2} | q_{j-1} |^{-j+1}$ are summable for all $k, j = 1, 2, \ldots, n$.*

Proof. The proof of this theorem now follows from those of Theorems 2.2, 3.2, 3.4 and 3.5. We only note that by Rank-Nullity theorem, $\mathrm{def}\tilde{L} = \mathrm{def}L_1 + \mathrm{def}L_2$ and in the two cases considered in the theorem, L has self-adjoint operator extension which we shall denote by H. This is a $2n$-dimensional extension of L. From the results of Theorem 3.4, without loss of generality, we may even take $H = H_1H_2$. The rest of the proof are now straight forward.

□

The example below now verifies the results of Theorems 3.4 and 3.5 by computing the composite of two second order symmetric difference operators with unbounded coefficients and then analysing their deficiency indices and spectra.

Example 3.7. We consider two second order difference operators generated by Sturm-Liouville equation with potential $p(t) = \ln(t)$ and a second order square well given by

$$\mathcal{L}_1 y(t) = -\Delta^2 y(t-1) + (\ln t) y(t), \qquad \mathcal{L}_2 y(t) = -\Delta^2 y(t-1) + t^{-\epsilon} y(t), \ \epsilon > 0 \quad (3.3)$$

defined on $\ell^2(\mathbb{N})$. The functions $\mathcal{L}_1$ and $\mathcal{L}_2$ at limit point commute and their composite can be written as

$$\mathcal{L}y(t) = \Delta^4 y(t-2) - \Delta\left[(\ln t + t^{-\epsilon})\Delta y(t-1)\right] + t^{-\epsilon} \ln t y(t).$$

The conversion of the composite equation $\mathcal{L}y = zy$ into its first order form is done using quasi-differences of the form

$$x_1(t) = y(t-1), \qquad x_2(t) = \Delta y(t-2),$$
$$u_1(t) = (\ln t + t^{-\epsilon})\Delta y(t-1) - \Delta^3 y(t-2), \qquad u_2(t) = \Delta^2 y(t-2).$$

The characteristic polynomial $\det(S(t,z) - \lambda \cdot I_4)$ when multiplied by λ^{-2} and equated to zero leads to

$$\gamma^2 + (\ln t + t^{-\epsilon})\gamma + t^{-\epsilon} \ln t - z = 0,$$

where $\gamma = 2 - (\lambda + \lambda^{-1})$. Thus we fix our regular left end point at $t_0 > 1$ and hence we have approximate values of γ as

$$\gamma_1 \approx -t^{-\epsilon} + t^{-2\epsilon}(\ln t)^{-1} + O(\ln t)^{-2}$$
$$\gamma_2 \approx -\ln t - t^{-2\epsilon}(\ln t)^{-1} + O(\ln t)^{-2}.$$

Now approximation for the λ-roots for this system gives

$$\lambda_{1\pm}(t,z) \approx 1 \pm t^{-\frac{\epsilon}{2}} + O(t^{-\epsilon}(\ln t)^{-1})$$

and

$$\lambda_{2+}(t,z) \approx (\ln t)^{-1} + O(\ln t)^{-2}, \qquad \lambda_{2-}(t,z) \approx 2 + \ln t - (\ln t)^{-1} + O(\ln t)^{-2}.$$

The solutions associated with the λ-roots derived from γ_2 contribute $(1,1)$ to the deficiency indices because of the nature of the λ-roots. On the other hand, the two solutions associated with γ_1 will all be square summable if the term $t^{-\frac{\epsilon}{2}}(lnt)^{-1}$ is summable and thus contributes $(2,2)$ to the deficiency indices. All the solutions will be uniformly square summable. Thus $\text{def}L = (3,3)$ and the spectrum of H is discrete at most. Meanwhile, if the term $t^{-\frac{\epsilon}{2}}(lnt)^{-1}$ is not summable, the γ_1 also contributes $(1,1)$ to the deficiency indices with one solution losing its square summability as $\eta \to 0^+$ and hence the absolutely continuous spectrum of H is non-empty. Thus $\text{def}L = (2,2)$ and $\sigma_{ac}(H) \subset [0,\infty)$ of spectral multiplicity one.

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