Commun. Combin., Cryptogr. & Computer Sci., 2 (2024), 119–135

ISSN:2783-5456

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: http://cccs.sgh.ac.ir

On non-decreasing 2-plane trees

Yvonne Wakuthii Kariuki[a], Isaac Owino Okoth[b,*], Fredrick Oluoch Nyamwala[c]

[a]Department of Mathematics
Kibabii University, Bungoma, Kenya.

[b]Department of Pure and Applied Mathematics
Maseno University, Maseno, Kenya.

[c]Department of Mathematics, Physics and Computing
Moi University, Eldoret, Kenya.

Abstract

In this paper, we have introduced the set of non-decreasing 2-plane trees. These are plane trees whose vertices receive labels from the set $\{1, 2\}$ such that the sum of labels of adjacent vertices is at most 3 and that the labels of siblings are weakly increasing from left to right. We have obtained the formula for the number of these trees with a given number of vertices and label of the root. Further, we have obtained the number of these trees given root degrees and label of the eldest child of the root. We have also constructed bijections between the set of non-decreasing 2-plane trees with roots labelled 2 and the sets of little Schröder paths, plane trees in which leaves receive two labels, restricted lattice paths and increasing tableaux. For non-decreasing 2-plane trees with roots labelled 1, we have obtained bijections between the set of these trees and the sets of large Schröder paths and row-increasing tableaux.

Keywords: Non-decreasing 2-plane tree, Large Schröder path, Little Schröder path, Lattice path, Increasing tableau, Row-increasing tableau.

2020 MSC: 05C30, 05A19, 05C05.

1. Introduction

Plane trees, also called ordered trees, are rooted trees where the children of each vertex are ordered. In [5], plane trees are enumerated according to number of vertices, number of leaves, root degree and level of a vertex. The set of plane trees and their bijections with other combinatorial structures have been extensively investigated such as in [3, 4]. Consequently, the class of k-plane trees was introduced by Gu, Prodinger and Wagner in [10]. These are labelled plane trees where the sum of the labels of any two adjacent vertices is at most $k+1$.

Schröder numbers (also referred to as super-Catalan numbers) have received much attention as in [2, 8, 22]. These numbers are categorized into large Schröder numbers and little Schröder numbers. The large Schröder numbers count large Schröder paths; these are lattice paths from $(0,0)$ to $(2n,0)$ that comprise

*Corresponding author
Email addresses: ykariuki@kibu.ac.ke (Yvonne Wakuthii Kariuki), ookoth@maseno.ac.ke (Isaac Owino Okoth), foluoch2000@mu.ac.ke (Fredrick Oluoch Nyamwala)

Received: February 9, 2024 Revised: April 8, 2024 Accepted: April 8, 2024

of up-steps $(1,1)$, down-steps $(1,-1)$ and horizontal steps $(2,0)$ which do not go below the x-axis as in [2]. The sequence is recorded as $A006318$ in the celebrated Online Encyclopaedia of Integer Sequences (OEIS) [23]. On the other hand, little Schröder paths are like large Schröder paths but there is no horizontal step on the x-axis. They are counted by little Schröder numbers [8], represented by the sequence $A001003$ in the OEIS [23]. From [21], we find that the generating function of large and little Schröder numbers satisfy

$$L(x) = \frac{1-x-\sqrt{x^2-6x+1}}{2} \tag{1.1}$$

and

$$S(x) = \frac{1+x-\sqrt{x^2-6x+1}}{4} \tag{1.2}$$

respectively.

Many combinatorial structures are known to be enumerated by little Schröder numbers. For example, in Stanley's paper [24], the author obtained a bijection between the set of dissections of a regular convex $(n+2)$-gon and the set of standard Young tableau, a concept that has been widely used in creating further bijections such as in [6, 11, 18]. The paper [18] introduced the set of increasing tableaux, defined as a semi-standard Young tableaux where the entries of the rows and columns are strictly increasing and for every element i in the table, all the positive integers less than i are also found in the table and any entry appears at most twice. We denote the set of increasing tableaux with n columns, 2 rows and k repeated terms by $\mathrm{Inc}_k(2\times n)$. A bijection between the set $\mathrm{Inc}_k(2\times n)$ and the set of little Schröder paths with n up-steps and k horizontal steps was also obtained in the aforementioned paper. Further, a refinement of the formula counting increasing tableau was obtained in [6].

In the same paper [6], Du et.al introduced the set of row-increasing tableaux denoted as $\mathrm{RInc}_k(2\times n)$. These are increasing tableaux having strictly increasing row entries but weakly increasing columns. They showed that

$$|\mathrm{RInc}_k(2\times n)| = \frac{1}{n-k+1}\binom{2n-k}{k}\binom{2n-2k}{n-k}. \tag{1.3}$$

This formula also enumerates the set of large Schröder paths with k horizontal paths and n up-steps. In [1], Aguiar et.al created a bijection between the set of rooted planar trees with $n+1$ leaves and m internal nodes where the internal nodes have at least two children and the set of lattice paths of length $2n$ from $(0,0)$ to (n,n) with $n-m$ diagonal steps and comprising of the steps $(1,0)$, $(0,1)$ and $(1,1)$ never above the line $y=x$. In this paper, we connect their results to our counting structures. We use the symbolic method which has previously been used to find the generating functions of various structures such as in [12, 19, 20] and apply the Lagrange Inversion Formula to extract the coefficients in the generating functions.

Theorem 1.1 (Lagrange Inversion Formula [7]). The coefficient of x^n in the functional equation $T(x) = x\Phi(T(x))$ is given by $[x^n]T(x)^k = \frac{k}{n}[t^{n-k}]\phi(t)^n$ where $\Phi(0)\neq 0$.

This paper is organized as follows: In Section 2, we introduce the class of non-decreasing 2-plane trees, classify them and find the enumerative formulas. Bijections between the set of non-decreasing 2-plane trees with roots labelled 2 and the sets of other combinatorial structures are obtained in Section 3 and these bijections are extended to sets of non-decreasing 2-plane trees with roots labelled 1 in Section 4. We conclude the paper in Section 5 by exposing some problems for further research.

2. Non-decreasing 2-plane trees

Consider a plane tree P. A vertex v of P is said to be a child (resp. parent) of a vertex u if the two vertices are adjacent and vertex v is at a lower level (resp. higher level) than u. Children that share a parent are siblings. A child which appears on the far left is the eldest with the one on the far right being the youngest.

Definition 2.1. A non-decreasing 2-plane tree is a plane tree whose vertices are labelled from the set $\{1, 2\}$ such that there are no edges whose endpoints are both labelled 2 and the labels of siblings are non-decreasing from left to right.

See Figure 1 for an example of a non-decreasing 2-plane tree.

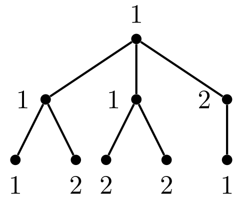


Figure 1: Non-decreasing 2-plane tree on 9 vertices with root labelled 1.

We remark that plane trees with labels in the set $\{1, 2\}$ such that the sum of labels of any two adjacent vertices does not exceed 3 was introduced in [9]. They are commonly referred to as 2-plane trees. These trees have been studied extensively in [12, 13, 16]. If the labels of the vertices of the plane trees are from the set $\{1, 2, \ldots, k\}$ such that the sum of labels of any two adjacent vertices is no more $(k+1)$, then the trees are called k-plane trees [10]. Studies on k-plane trees have been done in [14, 15, 17] among other papers.

There are two classes of non-decreasing 2-plane trees: those with roots labelled 1 and those with roots labelled 2. Let $W(x)$ and $B(x)$ be the generating function for non-decreasing 2-plane trees with roots labelled 1 and 2 respectively, where x marks a vertex. Due to the condition that any two adjacent vertices are given labels whose sum is at most 3, then when the root is labelled 2, the children of the root have to be labelled 1. This is depicted in Figure 2. The generating function $B(x)$ thus satisfies the equation

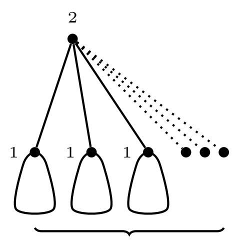


Figure 2: Non-decreasing 2-plane tree with root label 2.

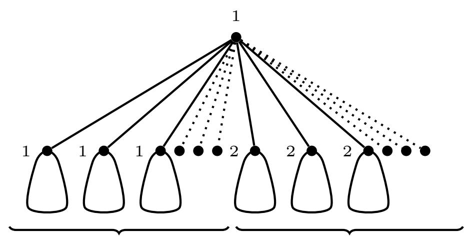


Figure 3: Non-decreasing 2-plane tree with root label 1.

$$B(x) = x \cdot \frac{1}{1 - W(x)}. \tag{2.1}$$

For the non-decreasing 2-plane trees with root labelled 1, the children of the root can either be labelled 1 or 2 where all the children labelled 2 come at the right of the ones labelled 1. This is shown in Figure 3. The generating function $W(x)$ is thus given by

$$W(x) = x \cdot \frac{1}{1 - W(x)} \cdot \frac{1}{1 - B(x)}. \tag{2.2}$$

Substituting (2.1) in (2.2), we have

$$W(x) = \frac{B(x)}{1 - B(x)} \tag{2.3}$$

and from (2.1) we get

$$W(x) = \frac{B(x) - x}{B(x)}. \tag{2.4}$$

By equating (2.3) and (2.4) we have $2B(x)^2 - (1+x)B(x) + x = 0$ which simplifies to

$$B(x) = \frac{1 + x - \sqrt{x^2 - 6x + 1}}{4}.$$

This is the generating function for the Little Schröder numbers as in (1.2).

Substituting (2.1) in (2.2) and simplifying we get $W(x)^2 + (x-1)W(x) + x = 0$. This quadratic equation is solved to get

$$W(x) = \frac{1 - x - \sqrt{x^2 - 6x + 1}}{2},$$

which is the generating function for large Schröder numbers.

2.1. Number of vertices

The number of non-decreasing 2-plane trees of n vertices with root labelled 1 create the sequence $1, 2, 6, 22, 90, \cdots$ which is the sequence of the Large Schröder numbers while those with root labelled 2 create the sequence $1, 1, 3, 11, 45, \cdots$ which is the sequence of the Little Schröder numbers.

Theorem 2.2. The number of non-decreasing 2-plane trees with roots labelled 2 on n vertices is given by

$$\frac{1}{n} \sum_{j=0}^{n-1} (-1)^{n-j-1} 2^j \binom{n}{j+1} \binom{n+j-1}{j}. \tag{2.5}$$

Proof. Since $2B(x)^2 - (1+x)B(x) + x = 0$ then

$$B(x) = \frac{x(1 - B(x))}{1 - 2B(x)}.$$

This is in a form we can apply Lagrange inversion to extract the coefficient of x^n as shown below:

$$\begin{aligned} [x^n]B(x) &= \frac{1}{n}[b^{n-1}](1-b)^n(1-2b)^{-n} \\ &= \frac{1}{n}[b^{n-1}] \sum_{i=0}^{n} \binom{n}{i} (-1)^i b^i \sum_{j=0}^{n-1} \binom{-n}{j} (-2)^j b^j \\ &= \frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{n-j-1} (-1)^{n-j-1} \binom{n+j-1}{j} 2^j. \end{aligned}$$

□

In the following theorem, we obtain an equivalent result to Theorem 2.2, when the roots of the trees are labelled 1.

Theorem 2.3. There are

$$\frac{1}{n} \sum_{j=0}^{n-1} \binom{n}{j+1} \binom{n+j-1}{j} \tag{2.6}$$

non-decreasing 2-plane trees on n vertices with roots labelled 1.

Proof. From (2.3), we have

$$B(x) = \frac{W(x)}{1+W(x)}.$$

By (2.1), we get

$$B(x) = \frac{x}{1-W(x)} = \frac{W(x)}{1+W(x)}.$$

It follows that

$$W(x) = \frac{x(1+W(x))}{1-W(x)}. \tag{2.7}$$

Applying Lagrange inversion formula, we get

$$\begin{aligned}[x^n]W(x) &= \frac{1}{n}[w^{n-1}](1+w)^n(1-w)^{-n}\\ &= \frac{1}{n}[w^{n-1}]\sum_{i=0}^{n}\binom{n}{i}w^i\sum_{j=0}^{n-1}\binom{-n}{j}(-1)^j w^j\\ &= \frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{n-j-1}\binom{n+j-1}{j}.\end{aligned}$$

This completes the proof. □

2.2. Root degree

The total number of non-decreasing 2-plane trees on n vertices can be broken down to reflect the various degrees of the root.

Theorem 2.4. The number of non-decreasing 2-plane trees on n vertices with roots of degree k labelled 2 is given by

$$\frac{k}{n-1}\sum_{j=0}^{n-k-1}\binom{n-1}{k+j}\binom{n+j-2}{j}. \tag{2.8}$$

Proof. Since the subtrees rooted at a vertex of label 2 are always rooted at vertices of label 1 then the required result is:

$$\begin{aligned}[x^n]xW^k &= \frac{k}{n-1}[w^{n-k-1}](1+w)^{n-1}(1-w)^{-(n-1)}\\ &= \frac{k}{n-1}[w^{n-k-1}]\sum_{i=0}^{n-1}\binom{n-1}{i}w^{i+j}\sum_{j=0}^{n-k-1}\binom{-(n-1)}{j}(-1)^j\\ &= \frac{k}{n-1}\sum_{j=0}^{n-k-1}\binom{n-1}{n-k-j-1}\binom{n+j-2}{j}.\end{aligned}$$

□

Setting $k=1$ in (2.8), we find that there are

$$\frac{1}{n-1}\sum_{j=0}^{n-2}\binom{n-1}{j+1}\binom{n+j-2}{j}$$

planted non-decreasing 2-plane trees on n vertices if the root is labelled 2. We remark that this formula is easily determined by attaching root vertices labelled 2 to non-decreasing 2-plane trees on $n-1$ vertices whose roots are labelled 1, i.e., we replace n with $n-1$ in (2.6).

Corollary 2.5. The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 and of degree k such that all the children of the root are labelled 1 is given by

$$\frac{k}{n-1}\sum_{j=0}^{n-k-1}\binom{n-1}{k+j}\binom{n+j-2}{j}. \tag{2.9}$$

Proof. The proof follows easily from Theorem 2.3 after changing the root labels in trees under consideration from 1 to 2. □

Theorem 2.6. The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 and of degree ℓ such that all the children of the root are labelled 2 is given by

$$\frac{\ell}{n-1}\sum_{j=0}^{n-\ell-1}\binom{n-1}{\ell+j}(-1)^{n-\ell-j-1}\binom{n+j-2}{j}2^j. \tag{2.10}$$

Proof. Since all the subtrees rooted at the root are of label 2 then the desired result is:

$$\begin{aligned}[x^n]xB(x)^\ell &= \frac{\ell}{n-1}[b^{n-\ell-1}](1-b)^{n-1}(1-2b)^{-(n-1)}\\ &= \frac{\ell}{n-1}[b^{n-\ell-1}]\sum_{i=0}^{n-1}\binom{n-1}{i}(-1)^i b^i\sum_{j=0}^{n-\ell-1}\binom{-(n-1)}{j}(-2)^j b^j\\ &= \frac{\ell}{n-1}\sum_{j=0}^{n-\ell-1}\binom{n-1}{n-\ell-j-1}(-1)^{n-\ell-j-1}\binom{n+j-2}{j}2^j.\end{aligned}$$

□

We now prove the following theorem:

Theorem 2.7. The number of non-decreasing 2-plane trees on n vertices with roots of degree d labelled 1 such that there are k children labelled 1 is given by

$$\frac{1}{n-1}\sum_{j=0}^{n-d-2}\frac{k(n-1)+j(d-k)}{n-d-j-1}\binom{n+k-d-2}{k+j}\binom{n+j-2}{j}. \tag{2.11}$$

Proof. We extract the coefficient of x^n in $xW(x)^kB(x)^{d-k}$ where $W(x)$ and $B(x)$ are the generating functions for non-decreasing 2-plane trees with roots labelled 1 and 2 respectively. Here again, x marks a vertex:

$$\begin{aligned}[x^n]xW(x)^kB(x)^{d-k} &= [x^{n-1}]W(x)^kW(x)^{d-k}(1+W(x))^{k-d} = [x^{n-1}]W(x)^d(1+W(x))^{k-d}\\ &= [x^{n-1}]W(x)^d\sum_{i\geqslant 0}\binom{k-d}{i}W(x)^i\\ &= \sum_{i\geqslant 0}\binom{k-d}{i}[x^{n-1}]W(x)^{d+i}.\end{aligned}$$

By Lagrange Inversion Formula, we get

$$\begin{aligned}[x^n]xW(x)^kB(x)^{d-k} &= \sum_{i\geqslant 0}\binom{k-d}{i}\frac{d+i}{n-1}[w^{n-d-i-1}](1+w)^{n-1}(1-w)^{-(n-1)}\\ &= \frac{1}{n-1}\sum_{i\geqslant 0}\binom{k-d}{i}(d+i)[w^{n-d-i-1}]\sum_{j\geqslant 0}\sum_{l\geqslant 0}\binom{n-1}{l}\binom{-(n-1)}{j}(-1)^j w^{l+j}\\ &= \frac{1}{n-1}\sum_{i\geqslant 0}\left[d\binom{k-d}{i}+(k-d)\binom{k-d-1}{i-1}\right]\sum_{j\geqslant 0}\binom{n-1}{n-d-i-j-1}\binom{n+j-2}{j}.\end{aligned}$$

By Vandermonde Convolution, we obtain

$$[x^n]xW(x)^kB(x)^{d-k} = \frac{1}{n-1}\sum_{j\geqslant 0}\left[d\binom{n+k-d-1}{n-d-j-1}+(k-d)\binom{n+k-d-2}{n-d-j-2}\right]\binom{n+j-2}{j}$$
$$= \frac{1}{n-1}\sum_{j\geqslant 0}\frac{k(n-1)+j(d-k)}{n-d-j-1}\binom{n+k-d-2}{k+j}\binom{n+j-2}{j}.$$

This completes the proof. □

By summing over all k in (2.11), we obtain the following corollary:

Corollary 2.8. There are

$$\frac{1}{n-1}\sum_{k=0}^{d}\sum_{j=0}^{n-d-2}\frac{k(n-1)+j(d-k)}{n-d-j-1}\binom{n+k-d-2}{k+j}\binom{n+j-2}{j}$$

non-decreasing 2-plane trees on n vertices with roots of degree d labelled 1.

2.3. Label of eldest child of the root

Since the the eldest child of the root labelled 2 is always labelled 1, it is of less significance to enumerate them by label of the eldest child. We then concentrate on non-decreasing 2-plane trees with root labelled 1.

Theorem 2.9. The number of non-decreasing 2-plane trees on n vertices with root labelled 1 such that the eldest child of the root is labelled 1 is given by

$$\frac{2}{n}\sum_{j=0}^{n-2}\binom{n}{j+2}\binom{n+j-1}{j}. \tag{2.12}$$

Proof. The decomposition of the trees in question is as shown in Figure 4.

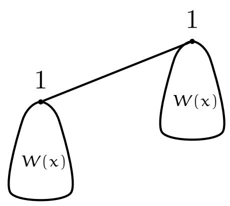


Figure 4: Decomposition of trees with root labelled 1 and eldest child of the root is labelled 1.

The generating function for these trees is $W(x)^2$ where $W(x)$ is the generating function for trees with root labelled 1. We thus have that the number of non-decreasing 2-plane trees on n vertices with root labelled 1 such that the eldest child of the root is labelled 1 is

$$[x^n]W(x)^2 = \frac{2}{n}[w^{n-2}](1+w)^n(1-w)^{-n}$$
$$= \frac{2}{n}[w^{n-2}]\sum_{i\geqslant 0}\sum_{j\geqslant 0}\binom{n}{i}\binom{-n}{j}(-1)^j w^{i+j}$$
$$= \frac{2}{n}\sum_{j=0}^{n-2}\binom{n}{n-j-2}\binom{n+j-1}{j}.$$

Thus the formula. □

Bijective proof of Theorem 2.9. Label the root as r. Let the eldest child of r be i. Delete the edge connecting r to i. Introduce a new root j labelled 2 whose children are i and r in this order from left to right. The tree has root degree 2 labelled 2 on $n+1$ vertices. The number of these trees is given by (2.8) if we set $k=2$. This is illustrated in Figure 5. □

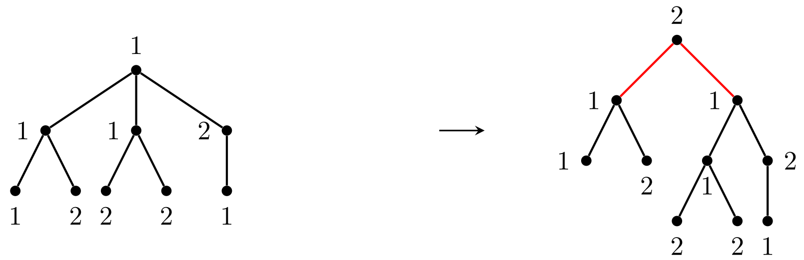


Figure 5: A non-decreasing 2-plane tree on 9 vertices with root labelled 1 and eldest child labelled 1 with its corresponding non-decreasing 2-plane tree on 10 vertices with root labelled 2.

Corollary 2.10. The number of non-decreasing 2-plane trees on n vertices whose roots are labelled 1 such that the eldest children of the roots are labelled 2 is given by

$$\frac{1}{n}\sum_{j=0}^{n-2}\frac{3j+4-2n}{n-j-1}\binom{n}{j+2}\binom{n+j-1}{j}.$$

Proof. This is the number of non-decreasing 2-plane trees with roots labelled 1 such that the eldest child of the root is labelled 2 as obtained by subtracting the number in (2.12) from (2.6), i.e.,

$$\frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{j+1}\binom{n+j-1}{j}-\frac{2}{n}\sum_{j=0}^{n-2}\binom{n}{j+2}\binom{n+j-1}{j}$$

which simplifies to the formula in the statement of the corollary. Alternatively, this can also be achieved by summing over ℓ in (2.10), i.e.,

$$\sum_{\ell=1}^{n-1}\sum_{j=0}^{n-\ell-1}\frac{\ell}{n-1}\binom{n-1}{\ell+j}(-1)^{n-\ell-j-1}\binom{n+j-2}{j}2^{j}.$$

□

3. Bijections of trees with roots labelled 2

3.1. Leaf-labelled plane trees

Plane trees are one of the many structures counted by the Catalan numbers. The number of plane trees on $n\geqslant 2$ vertices and having k leaves is given by the Narayana number,

$$\frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

So, if the leaves are given labels 1 and 2 then the number of such trees is

$$\frac{2^{k}}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

We therefore, have the total number of plane trees on $n \geqslant 2$ vertices whose leaves are labelled with integers 1 and 2 as

$$\frac{1}{n-1}\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}2^k$$

which are little Schröder numbers. We therefore prove the following result:

Theorem 3.1. There is a bijection between the set of non-decreasing 2-plane trees on n vertices with roots labelled 2 and the set of plane trees on n vertices whose leaves are labelled with integers 1 and 2.

Proof. Let T be a non-decreasing 2-plane tree on n vertices with root labelled by 2. Now, let v be a non-root internal vertex of T labelled 2. Detach the subtrees rooted at the children of v and attach them to the parent of v in order from left to right so that the children of v appear on the immediate right of v. Do this for all non-root internal vertices labelled 2. The labels of all internal vertices (inclusive of the root) are dropped and the resultant structure is a plane tree on n vertices whose leaves are labelled 1 or 2.

For the reverse procedure, label the root of the plane tree as 2 and all the internal vertices as 1. Now, let ℓ_1 and ℓ_2 (if any) be leaves labelled 2 that share a parent such that ℓ_1 is older than ℓ_2 and that there are younger siblings of ℓ_1 labelled 1 between ℓ_1 and ℓ_2 or all on the right of ℓ_1 if there is no such ℓ_2. Detach all the vertices between ℓ_1 and ℓ_2 or on the right of ℓ_1 if there is no such ℓ_2 and the subtrees rooted at them and attach the subtrees at ℓ_1 such that these vertices which were sharing a parent with ℓ_1 are now children of ℓ_1. The attachment is done in order from left to right and the labels are retained. Do this to all vertices with the same properties as ℓ_1 and ℓ_2. The structure obtained is a non-decreasing 2-plane tree on n vertices whose root is labelled 2.

A depiction of the bijection is given in Figure 6.

□

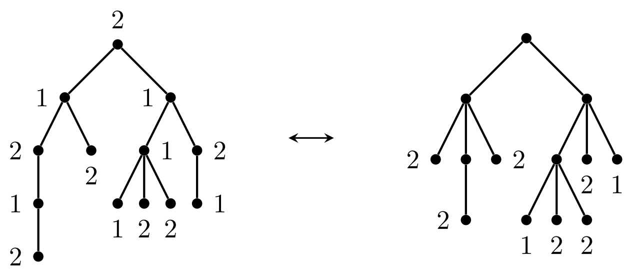


Figure 6: A a non-decreasing 2-plane tree on n vertices and its corresponding plane tree on n vertices such that the leaves labelled 1 or 2.

3.2. Little Schröder paths

Theorem 3.2. There is a bijection between the set of non-decreasing 2-plane trees on $n+1$ vertices with roots labelled 2 and the set of little Schröder paths of length n with no horizontal steps on the x-axis.

Proof. Consider a non-decreasing 2-plane tree T on $n+1$ vertices with root labelled 2. We obtain the corresponding little Schröder path of length n with no horizontal step at level zero by the following procedure:

(i) We traverse T using preorder traversal and for every vertex labelled 1 (resp. labelled 2) encountered as we move away from the root, we draw a unit up-step (resp. horizontal step of length 2).

(ii) Moreover, for every edge encountered as we move towards the root, if the initial vertex of the edge is labelled 1, then we draw a unit down-step. Otherwise, we do not draw any step.

Let the number of non-root vertices labelled 2 be k. Then the number of vertices labelled 1 is $n-k$. Since for each vertex labelled 1, we draw a unit up-step as we move away from the root and unit down-step as we move towards the root, and for each non-root vertex labelled 2, we draw horizontal step of length 2 as we move away from the root and no step as we move towards the root, then the total length of the path is $2n$. Since the children of the root are all labelled 1, then we are assured of having no horizontal path on the x-axis. So, by this procedure we obtain little Schröder paths from $(0,0)$ to $(2n,0)$ with no horizontal paths on the x-axis.

Conversely, a non-decreasing 2-plane tree on $n+1$ vertices is obtained from a little Schröder path of length $2n$ with no horizontal step on the x-axis as follows:

(i) Draw a vertex of the tree which will root of the tree. Label it 2. We then build the tree recursively.

(ii) Starting at the origin of the path, for each up-step encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 1. Note that there are no horizontal steps on the x-axis. So, the initial step is an up-step and the vertex corresponding to this vertex is labelled 1. Also, the condition that there are no horizontal steps on the x-axis guarantees that all the children of the root are labelled 1.

(iii) When a horizontal-step is encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 2. Moreover, move up the tree back to the vertex in which a new vertex has been attached.

(iv) For every down-step encountered, move up the tree to the parent of the vertex last visited. This vertex is labelled 1.

Let the number of up-steps be k. Then the number of down-steps is k. Moreover, there are $n-k$ horizontal steps since each horizontal step is of length 2. Since each up-step corresponds to a non-root vertex labelled 1 and each horizontal step corresponds to a non-root vertex labelled 1, then there are a total of $k+n-k=n$ non-root vertices. Thus there are $n+1$ vertices in the corresponding non-decreasing 2-plane tree. The coherence condition that there are no edges whose endpoints are labelled 2 is satisfied since for each horizontal step encountered a vertex labelled 2 is drawn but it is attached to a vertex which corresponds to a down-step which is labelled 1. Moving up the tree to the parent of the vertex last visited ensures this fact. This procedure stops with the root and the corresponding structure is a non-decreasing 2-plane tree. □

Table 1 provides a list of all non-decreasing 2-plane trees on 4 vertices with roots labelled 2 and their corresponding little Schrö paths of length 6 such that there are no horizontal step on the x-axis.

3.3. Restricted lattice paths

Consider lattice paths in which only horizontal steps $(1,0)$, vertical steps $(0,1)$ and diagonal steps $(1,1)$ are allowed. Moreover, let the steps lie entirely above the line $y=x$. In this paper, we coin the name restricted lattice paths for these paths. In [1], it was showed the set of these paths from $(0,0)$ and (n,n) is enumerated by little Schröder numbers,

$$\frac{1}{n}\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{j+1}\binom{n+j-1}{j}2^j.$$

This formula also counts non-decreasing 2-plane trees on $n+1$ vertices with root labelled 2 as proved in the previous section. It is thus of combinatorial importance to construct a bijection between the set of restricted lattice paths and the set of non-decreasing 2-plane trees which we obtain in the next theorem.

Theorem 3.3. There exists a bijection between the set of non-decreasing 2-plane trees on $n+1$ vertices with root labelled 2 and the set of restricted lattice paths from $(0,0)$ to (n,n).

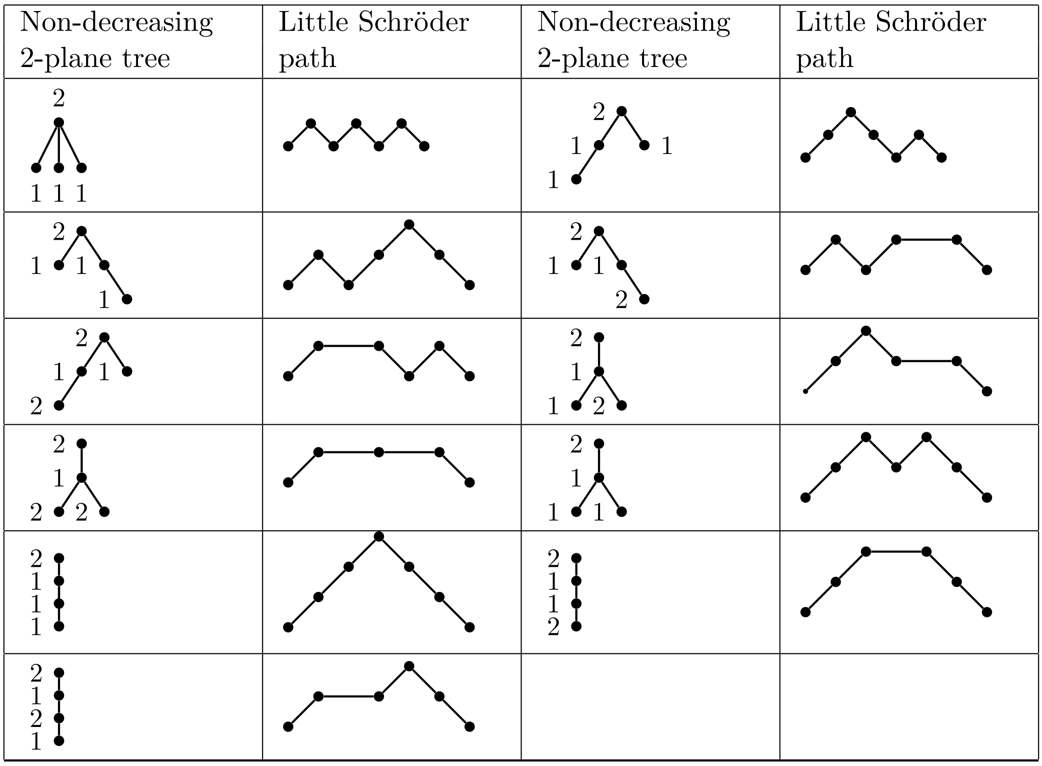


Table 1: All non-decreasing 2-plane trees on 4 vertices with roots labelled 2 and their corresponding little Schrö paths of length 6 such that there are no horizontal step on the x-axis.

Proof. Consider a non-decreasing 2-plane tree T on $n+1$ vertices with root labelled 2. We obtain the corresponding restricted lattice path from $(0,0)$ to (n,n) as follows:

(i) We use preorder traversal to visit the vertices of T and for each vertex labelled 1 (resp. labelled 2) encountered as we move away from the root, draw a $(0,1)$ step (resp. $(1,1)$ step).

(ii) Moreover, for every edge encountered as we move towards the root, if the initial vertex of the edge is labelled 1, then we draw a $(1,0)$ step. Otherwise, do not draw any step.

Let the number of non-root vertices labelled 2 be k. Then, the number of vertices labelled 1 is $n-k$. Since for each vertex labelled 1, we draw a $(0,1)$ as we move away from the root and a $(1,0)$ as we move towards the root, and for each non-root vertex labelled 2, we draw $(1,1)$ as we move away from the root and no step as we move towards the root, then the path starts at $(0,0)$ and ends at (n,n). Since the children of the root are all labelled 1, then we are assured of the path lying entirely above the line $y=x$. So, by this procedure, we obtain restricted lattice paths from $(0,0)$ to (n,n).

Conversely, a non-decreasing 2-plane tree on $n+1$ vertices is obtained from a restricted lattice path from $(0,0)$ to (n,n) by the following procedure:

(i) Draw a vertex of the tree which is the root of the tree. Label it 2. We then build the tree recursively.

(ii) Starting at the origin of the path, for each $(0,1)$ encountered, attach a vertex to the tree being constructed at the immediate vertex previously drawn and label it 1. Note that the path lies entirely above the line $y=x$. So, the initial step is a $(0,1)$ step and its corresponding vertex is labelled 1. Also, the condition that the path lies entirely above the line $y=x$ guarantees that all the children of the root are labelled 1.

(iii) When a $(1,1)$ step is encountered, attach a vertex to the tree at the immediate vertex previously drawn and label it 2. Moreover, move up the tree back to the vertex in which a new vertex has been attached.

(iv) For every $(1,0)$ step encountered, move up the tree to the parent of the vertex last visited. This vertex is labelled 1.

Let the number of $(0,1)$ steps be k. Then the number of $(1,0)$ steps is k as well. Moreover, there are $n-k$ steps of type $(1,1)$. Since each $(0,1)$ step corresponds to a non-root vertex labelled 1, each $(1,0)$ step corresponds to a non-root vertex labelled 1 and each $(1,1)$ step corresponds to a non-root vertex labelled 2, then there are $k+(n-k)$ non-root vertices in the tree. Thus there are $n+1$ vertices in the corresponding non-decreasing 2-plane tree. The coherence condition that there are no edges whose endpoints are labelled 2 is satisfied since for each $(1,1)$ step encountered, a vertex labelled 2 is drawn but it is attached to a vertex which corresponds to a $(0,1)$ step which is labelled 1. Moving up the tree to the parent of the vertex last visited ensures this condition is satisfied. Figures 7 and 8 show an illustration of the bijection. □

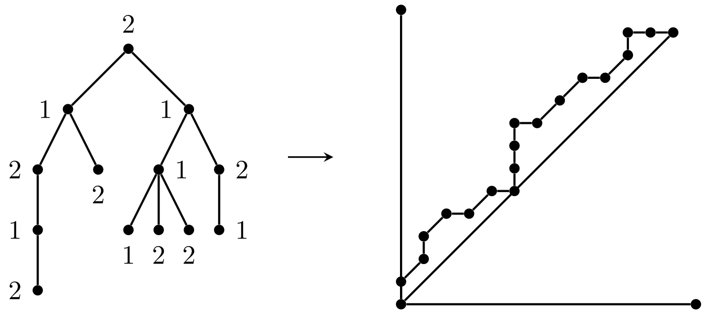


Figure 7: An example showing the procedure of obtaining a lattice path from $(0,0)$ to $(12,12)$ from a non-decreasing 2-plane tree on 13 vertices.

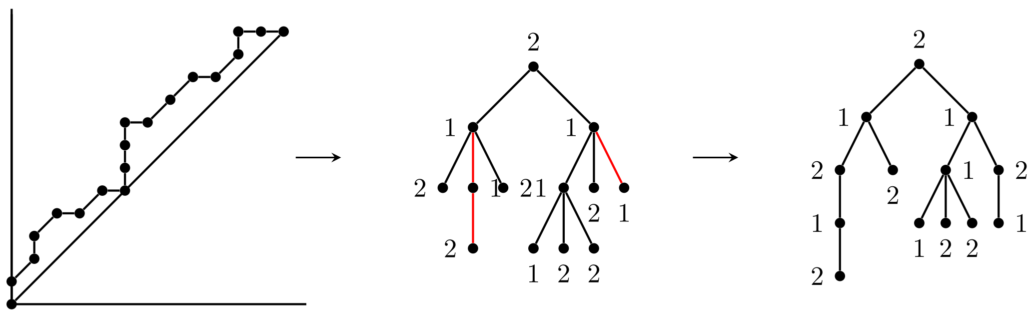


Figure 8: An example showing the procedure of obtaining a non-decreasing 2-plane tree on 13 vertices from a lattice path from $(0,0)$ to $(12,12)$.

3.4. Increasing tableaux

A bijection between the set of standard Young tableaux and the set of increasing tableaux was demonstrated in [18]. An increasing tableau, $\mathrm{Inc}_k(2\times n)$, was defined in Section 1. This is a table comprising of 2 rows and n columns where both rows and columns have strictly increasing values, such that for each value i contained in the table, all the smaller positive integer values less than i also appear in the table. Every value in the table can be repeated at most twice and k represents the number of repeated terms. We construct a bijection between the set of these structures and the set of non-decreasing 2-plane trees with root labelled 2 as below.

Theorem 3.4. There is a bijection between the set of non-decreasing 2-plane trees with root labelled 2 on $n+1$ vertices such that the number of vertices labelled 2 is $k+1$ and the set of increasing Young tableaux, $\mathrm{Inc}_k(2\times n)$.

Proof. Given a non-decreasing 2-plane tree on $n+1$ vertices, with root labelled 2 and $k+1$ vertices labelled 2, draw an empty table of 2 rows and n columns and fill the values by the below procedure.

(i) Traverse the non-decreasing 2-plane tree by preorder traversal and draw arrows along the edges of the tree as one moves away from the root and towards the root. Each entry in the tableau corresponds to exactly one arrow.

(ii) The first arrow leads to the first child of the root and for this vertex, fill in the positive integer value 1 into row 1 column 1 of the tableau and move to the next arrow.

(iii) Let i^{th} arrow point towards vertex v_i as one moves away from the root for each $i \geqslant 2$.

  (a) If v_i is labelled 1, then fill in the next positive integer value into the table in row 1.

  (b) If v_i is labelled 2, then fill in the next positive integer value into the table in both row 1 and row 2.

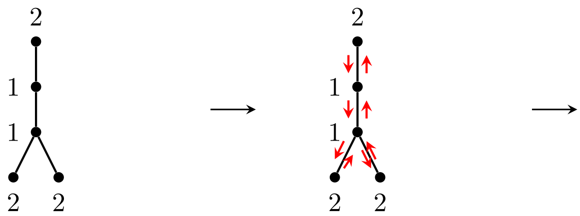
(iv) Let i^{th} arrow point away from vertex v_i as one moves towards the root for each $i \geqslant 2$.

  (a) If v_i is labelled 1, then fill in the next positive integer value into the table in row 2.

  (b) If v_i is labelled 2, then do not fill in any value.

(v) Perform steps (iii) and (iv) until the whole tree is traversed.

The vertices labelled 2 are repeated in the tableau as in step (iii)(b). So the repeated values are k. Since all children of the root are labelled 1, the entry in row 1 column 1 of the tableau is always 1. In the table filled, all the entries of the rows and columns are strictly increasing and no entry is repeated more than twice, which satisfies the properties of the set of $\mathrm{Inc}_k(2\times n)$. An illustration of this procedure is given in Figure 9.



1	2	3	4
3	4	5	6

Figure 9: An increasing tableau, $\mathrm{Inc}_2(2\times 4)$, obtained from a non-decreasing 2-plane tree on 5 vertices.

Conversely, consider increasing tableau of type $n \times 2$ where k positive integer values are repeated. The other values are thus $1, 2, \ldots, 2n-k$. We obtain a corresponding non-decreasing 2-plane tree on n vertices with root labelled 2 and having $k+1$ vertices labelled 2 by the following procedure:

(i) Draw a root of the tree and label it 2. Then draw a child of the root and label it 1. This corresponds to entry 1 of row 1 column 1 of the tableau.

(ii) Consider entries $2, 3, \ldots$ in the tableau. Let the least of the values be i.

  (a) If i is in row 1 but not in row 2 then draw a child attached to the last vertex drawn. Label it 1.

  (b) If i is not in row 1 but in row 2 then move back to the parent of the last vertex drawn.

(c) If i is in row 1 and row 2 then draw a child attached to the last vertex drawn. Label it 2 and move back to the parent of the child.

(iii) Perform step (ii) above for all values of i building the tree from left to right. The process ends at the root when the last integer in the tableau is considered.

(iv) In the tree constructed, for any vertex i labelled 2, let the label of its sibling j on the immediate right be labelled 1. Detach the subtree rooted at j and attach it as a child of i, on the right of the existing children of i (if any).

The tree obtained has all vertices labelled either 1 or 2 and the labels of any vertices sharing an edge have a sum of not more that 3. Further, the labels of any siblings are non-decreasing from left to right forming a non-decreasing 2-plane trees on $n+1$ vertices. Figure 10 represents this procedure. □

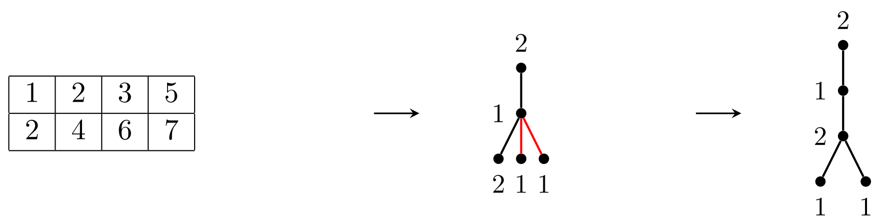


Figure 10: A procedure from an increasing tableau $\text{Inc}_1(2 \times 4)$ to a non-decreasing 2-plane tree on 5 vertices with 1 repeated value.

4. Bijections of trees with root labelled 1

In this section, we construct bijections between the set of non-decreasing 2-plane trees and the sets of large Schröder numbers and row-increasing tableaux.

4.1. Large Schröder paths

Theorem 4.1. There exists a bijection between the set of non-decreasing 2-plane trees on $n+1$ vertices with roots labelled 1 and the set of large Schröder paths of length $2n$.

Proof. Given a non-decreasing 2-plane tree on $n+1$ vertices, with root labelled 1, we obtain a large Schröder path of length $2n$ by the procedure below: We traverse the tree by preorder traversal and let the root to correspond with the starting point of the large Schröder path.

(i) When a vertex labelled 1 (resp. labelled 1) is encountered as one moves away from the root, draw an up-step $(1,1)$ (resp. horizontal step $(2,0)$).

(ii) As one moves towards the root, if the initial point of the edge is a vertex labelled 1 (resp. labelled 2) then draw a down-step $(1,-1)$ (resp. do not draw any step).

The corresponding structure obtained is a large Schröder path from $(0,0)$ to $(2n,0)$, comprised of the step $(0,1)$, $(1,0)$ and $(2,0)$ on any of the n levels.

Conversely, a non-decreasing 2-plane tree on $n+1$ vertices is obtained from a large Schröder path of length $2n$ as below:

(i) Draw a root of the tree labelled 1. We walk on the lattice path and build up the tree.

(ii) When an up-step is encountered, draw a child (on the right) of the last vertex visited and label it 1.

(iii) For every down-step encountered, move up one step to the vertex labelled 1.

(iv) When a horizontal-step is encountered, draw a child (on the right) of the last vertex visited and label it 2. Move up one step to the parent of the child drawn.

(v) In the tree constructed, for any vertex i labelled 2, let the label of its sibling j on the immediate right be labelled 1. Detach the subtree rooted at j and attach it as a child of i, on the right of the existing children of i (if any).

(v) In the tree obtained, for any vertex labelled i, let the label of its sibling on the immediate right be j. If $j < i$, cut the subtree rooted at vertex j and attach it as a child of vertex i.

The resulting structure is a non-decreasing 2-plane tree on $n+1$ vertices with root labelled 1 and the labels of siblings are non-decreasing from left to right. The bijection is illustrated in Figure 11. □

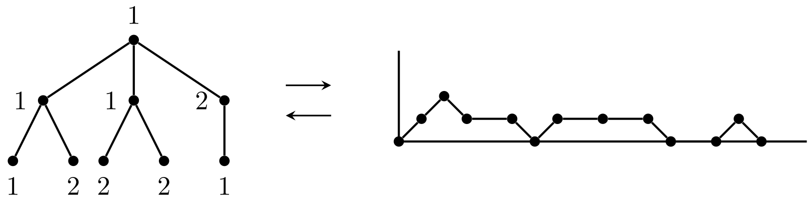


Figure 11: A non-decreasing 2-plane tree on 9 vertices with root labelled 1 and its corresponding large Schröder path of length 16.

4.2. Row-increasing tableaux

The set $\mathsf{RInc}_k(2 \times n)$ of row-increasing tableaux with $2n$ entries and k repeated values was introduced and enumerated in [6]. They are an extension of the increasing tableaux with the entries of the rows being strictly increasing and those of the columns being weakly increasing. As an illustration, in Table 2 we get a row-increasing tableau which is not an increasing tableau.

1	2	3	4
1	2	4	5

Table 2: An example from the set, $\mathsf{RInc}_3(2 \times 4)$.

Theorem 4.2. There exists a bijection between the set of non-decreasing 2-plane trees on $n+1$ vertices with roots labelled 1 such that k vertices are labelled 2 and the set of row-increasing tableaux, $\mathsf{RInc}_k(2 \times n)$.

Proof. The proof of this theorem is similar to that of Theorem 3.4 with the condition that the root in this case is labelled 1 and not 2. Since the children of the root can be labelled either 1 or 2 then it is possible to get two equal values on the same column in the tableau. This satisfies the condition that the values of the columns are weakly increasing. An illustration of this bijection is given in Table 3. □

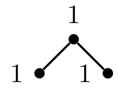
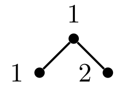
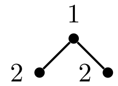
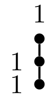
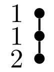
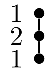
Non-decreasing 2-plane tree	Row-increasing tableau
	1 3 2 4
	1 3 2 3
	1 2 1 2
	1 2 3 4
	1 2 2 3
	1 3 1 4

Table 3: Non-decreasing 2-plane trees on 3 vertices with roots labelled 1 and corresponding row-increasing tableaux when $n = 2$.

5. Conclusion

In this paper, we introduced and enumerated non-decreasing 2-plane trees by number of vertices and root degrees. A study can be carried out to count them by the number of vertices labelled 1 or 2, number of forests and leaves. A k-plane tree was introduced and enumerated by Gu, Prodinger and Wagner in [10]. It would interesting to enumerate their non-decreasing counterparts as a generalization of non-decreasing 2-plane trees introduced in this paper. The non-decreasing 2-noncrossing trees, non-decreasing k-noncrossing trees, non-decreasing binary trees and non-decreasing d-ary trees can also be enumerated and related to various combinatorial structures.

Acknowledgements

The first author wishes to thank German Academic Exchange Service (DAAD) for the financial support towards a PhD study at Moi University. The second author thanks Maseno University for providing a conducive environment to conduct this research.

References

[1] M. Aguiar, W. Moreira, Combinatorics of the free Baxter algebra, arXiv preprint math/0510169 (2005). 1, 3.3

[2] F. Ardila, Algebraic and geometric methods in enumerative combinatorics, Handbook of enumerative combinatorics (2015), 589–678. 1

[3] N. G. de Bruijn, D. E. Knuth, S. O. Rice, The average height of planted plane trees, Graph Theory and Computing, Academic Press, (1972), 15–22. 1

[4] N. G. de Bruijn, B. J. M. Morselt, A note on plane trees, J. Combin. Theory, 2(1) (1967), 27–34. 1

[5] N. Dershowitz, S. Zaks, Enumerations of ordered trees, Discret. Math., 31(1) (1980), 9–28. 1

[6] R. R. X. Du, X. Fan, Y. Zhao, Enumeration on row-increasing tableaux of shape $2 \times n$, arXiv preprint arXiv:1803.01590 (2018). 1, 4.2

[7] I. M. Gessel, Lagrange inversion, J. Combin. Theory, Ser. A 144 (2016), 212-249. 1.1

[8] I. M. Gessel, Schröder numbers, large and small, A talk at CanaDAM (2009). 1

[9] N. S. S. Gu, H. Prodinger, Bijections for 2-plane trees and ternary trees, Europ. J. Combin., 30(4) (2009), 969–985. 2

[10] N. S. S. Gu, H. Prodinger, S. Wagner, Bijections for a class of labelled plane trees, European Journal of Combinatorics, 31(3) (2010), 720–732. 1, 2, 5

[11] Y. W. Kariuki, I. O. Okoth, Bijections of Plane Husimi graphs and certain combinatorial structures, Eur. J. Math. Appl. 3 (2023), 21. 1

[12] K. O. Lumumba, I. O. Okoth, D. M. Kasyoki, Refined enumeration of 2-plane trees, Preprint. 1, 2

[13] W. Mlotkowski, K. A. Penson, The probability measure corresponding to 2-plane trees, arXiv preprint arXiv:1304.6544 (2013). 2

[14] A. P. O. Nyariaro, I. O. Okoth, Bijections for classes of labelled trees, Transactions on Combinatorics, 13(3) (2024), 197–211. 2

[15] I. O. Okoth, Bijections of k-plane trees, Open Journal of Discrete Applied Mathematics, 5(1) (2022), 29–35. 2

[16] I. O. Okoth, On 2-noncrossing increasing trees, Open Journal of Discrete Applied Mathematics, (2021). 2

[17] I. O. Okoth, S. Wagner, Refined enumeration of k-plane trees and k-noncrossing trees, Annals of Combinatorics (2023), 1–33. 2

[18] O. Pechenik, Cyclic sieving of increasing tableaux and small Schroeder paths, J. Combin.Theory, Ser. A, 125(2014), 357–378. 1, 3.4

[19] H. Prodinger, Weighted unary, binary trees, Hex trees, marked ordered trees, and related structures, arXiv preprint arXiv:2106.14782 (2021). 1

[20] H. Prodinger, S. J. Selkirk, S. Wagner, On two subclasses of Motzkin paths and their relation to ternary trees, Algorithmic Combinatorics: Enumerative Combinatorics, Special Functions and Computer Algebra: In Honour of Peter Paule on his 60th Birthday (2020), 297–316. 1

[21] F. Qi, B-N Guo, Some explicit and recursive formulas of the large and little SchrĆ¶der numbers, Arab J. Math. Sci. 23(2) (2017), 141–147. 1

[22] L. W. Shapiro, C. J. Wang, A bijection between 3-Motzkin paths and SchrĆ¶der paths with no peak at odd height, J. Integer Seq., 12(2) (2009), 3. 1

[23] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences (OEIS). Available online at http://oeis.org. 1

[24] R. P. Stanley, Polygon dissections and standard Young tableaux, J. Comb. Theory, Ser. A, 76 (1996), 175–177. 1