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NUMERICAL RANGE OF INNER PRODUCT TYPE INTEGRAL TRANSFORMERS ON HILBERT SPACES

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Abstract. In this paper we compute the algebraic numerical range for inner product type integral transformers and show that the basic properties of the algebraic numerical range holds for this operator.

1. Introduction

Linear operator theory in Hilbert spaces has grown exponentially in the last decade. In this study, our focus is on integral operators, in particular the inner product type integral transformers. The inner product type integral transformation was introduced by Danko, [3] and is defined using the Gel'fand integral. To begin with, let H be a seperable complex Hilbert space, $B(H)$ be the space of all bounded linear operators and (Ω, Σ, μ) be a measure space. Let $A : \Omega \mapsto B(H) : t \mapsto A_t = A(t)$ be an operator-valued function. The operator valued function is said to be weakly*-measurable (weakly*-integrable) if the scalar-valued function $t \mapsto \langle A_t g, h\rangle$ is measurable (integrable) for all $g, h \in H$. If $\langle Af, f\rangle$ is integrable for all $f \in H$, then the mapping $f \mapsto \int_\Omega \langle A_t f, f\rangle d\mu(t)$ represents a quadratic form of the unique bounded operator $\int_\Omega A_t d\mu(t)$ satisfying

$$\left\langle \int_\Omega A_t d\mu(t) f, g \right\rangle = \int_\Omega \langle A_t f, g\rangle \, d\mu(t) \;\; for\ all\ f, g \in H. \tag{1.1}$$

as well as

$$tr\left\{\int_\Omega A_t d\mu(t) Y\right\} = \int_\Omega tr\{A_t Y\} d\mu(t), \forall\, Y \in B(H), Y = f^* \otimes f\ which\ is\ a\ rank\ one\ operator\ and\ f \in D_A,\text{ where } D_A = \{f \in H : \int_\Omega ||A_t f||^2 d\mu(t) < \infty\ and\ (Af)(t) = A_t f\}$$

The unique bounded operator $\int_\Omega A_t d\mu(t)$ is called the Gel'fand integral or weakly*-integral of the operator valued function A_t over Ω.

For every $f \in H$, the function $t \longrightarrow \|A_t f\|$ is also measurable (see [9]) and if additionally $\int_\Omega \|A_t f\|^2 d\mu(t) < \infty$ for all $f \in H$, then there exists weak*-integral $\int_\Omega A_t^* A_t d\mu(t) \in B(H)$ satisfying

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$$\left\langle \int_\Omega A_t^* A_t d\mu(t) f, f \right\rangle = \int_\Omega \langle A_t^* A_t f, f\rangle d\mu(t) = \int_\Omega \|A_t f\|^2 d\mu(t).$$

The family $(A_t)_{t\in\Omega}$ will be called square-integrable $B(H)-valued$ functions. The space of all such functions will be denoted by $L_t^2(\Omega, \mu, B(H))$, that is, the Banach space of all weakly *-measurable functions $A : \Omega \mapsto B(H) : t \mapsto A_t$ such that $\int_\Omega \|A_t f\|^2 d\mu(t) < \infty$ for all $f \in H$ endowed with the norm

$$\|A\|_{L_t^2(\Omega,\mu,B(H))} = \| \int_\Omega A_t^* A_t d\mu(t)\|^{\frac{1}{2}} \ for\ any\ A \in L_t^2(\Omega, \mu, B(H)). \tag{1.2}$$

Let $A, B : \Omega \longrightarrow B(H)$ be weakly *-measurable operator valued (o.v) functions and $\forall\ X \in B(H)$, the function $t \mapsto A_t X B_t$ is also weakly μ^*-measurable. If these functions are Gel'fand integrable for all $X \in B(H)$, then there is a unique bounded operator $\int_\Omega A_t X B_t d\mu(t)$ satisfying Equation (1.1) $\forall X \in B(H)$ with the domain $D_{A,B} = \{f \in H : \int_\Omega ||A_t f||^2 ||B_t f||^2 d\mu(t) < \infty\}$. The linear transformation $X \mapsto \int_\Omega A_t X B_t d\mu(t)$ is called an inner product type (i.p.t.) integral transformer on $B(H)$ and we denote it by

$$T_{\{A,B\}} = \int_\Omega A_t X B_t d\mu(t) = \int_\Omega A_t \otimes B_t d\mu(t). \tag{1.3}$$

We note that when μ is a counting measure on $\mathbb{N}$, then Equation (1.3) becomes

$$T_{\{A,B\}} = \sum_{i=1}^{n} A_t X B_t. \tag{1.4}$$

Transformers of the form (1.4) are called *elementary operators* and have been widely investigated (see [2], [8], [9] and references there in). The inner product type integral operators have been studied by several authors. Danko [3], using the Cauchy-Schwartz inequality showed that $T_{\{A,B\}}$ is bounded and its norm given by

$$\left\|T_{\{A,B\}}\right\| \leq \sqrt{\int_\Omega \|A_t^* A_t\| d\mu(t)} \sqrt{\int_\Omega \|B_t^* B_t\| d\mu(t)}$$

Also, in [3], Danko showed that this operator $\int_\Omega A_t \otimes B_t d\mu(t)$ leaves every unitary invariant norm ideal space $C_{\|.\|}(H)$ invariant.

Dragoljub [1], used the Cauchy-Schwartz and the Aczel-Bellman inequalities to show that if μ is a σ-finite positive measure on Ω and if each of the measurable families (o.v function), $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ consists of commuting normal operators such that $\int_\Omega A^* A d\mu \leq I$ and $\int_\Omega B^* B d\mu \leq I$, then

$$\left|\left|\left|\sqrt{I - \int_\Omega A^* A d\mu} \times \int_\Omega B^* B d\mu\right|\right|\right| \leq \left|\left|\left|X - \int_\Omega A X B d\mu\right|\right|\right|, \text{ for every } X \in C_{|||.|||}(H).$$

The Cauchy-Schwartz operator inequalities and norm inequalities for elementary and inner product type integral transformers have also been investigated, as well as, convergence properties related to these transformers, which depends on the stucture of norm ideals in which they act (see [5]).

In [4], Danko further determined an exact formula of finding the norm of i.p.t. integral transformer $\int_\Omega A_t \otimes B_t$ on the Hilbert-Schmidt class as

$$\| \int_\Omega A_t \otimes B_t d\mu(t) \|_{B(C_2(H))} =$$
$$\lim_{n\longrightarrow\infty} \sqrt[2n]{\int_{\Omega^{2n}} tr \left(\prod_{k=1}^{n} A_{t^*_{n+1-k}} A_{s_{n+1-k}} \right) tr \left(\prod_{k=1}^{n} B_{s_k} B_{t^*_k} \right) \prod_{k=1}^{n} d\mu(s_k) d\mu(t_k)}.$$

whenever $\int_\Omega \|A_t\|_p \|B_t\|_p d\mu(t) < \infty$ for some $p > 0$. It is shown in [6], that the Cauchy-Schwartz inequality and the Landaü inequality in the unitary invariant norm ideals holds for the inner product type integral transformers through the application of the Kortkine identity.
In [7], Danko obtained results on the relationship between the spectra of the inner product type transformers and the unit disc. In particular if $(A_t)_{t\in\Omega}$ and $(B_t)_{t\in\Omega}$ are weakly* measurable families of bounded Hilbert space operators then the transformers $X \longrightarrow \int_\Omega A_t^* X A_t d\mu(t)$ and $X \longrightarrow \int_\Omega B_t^* X B_t d\mu(t)$ on $B(H)$ have their spectra contained in the unit disc and for all bounded operators X,

$$\|\Delta_A X \Delta_B\| \leq \|X - \int_\Omega A_t^* X B_t d\mu(t)\|$$

Where $\Delta_A = s-\lim_{r\longrightarrow 1}(I + \sum_{n=1}^{\infty} r^{2n} \int_\Omega ... \int_\Omega |A_{t1} ... A_{tn}|^2 d\mu^n(t_1, ..., t_n))^{-\frac{1}{2}}$, where r is the spectral radius, with the spectral radius given by

$$r\left(\int_\Omega A^* \otimes B d\mu\right) \leq \sqrt{r(\int_\Omega A^* \otimes A d\mu) r(\int_\Omega B^* \otimes B d\mu)}.$$

The results obtained by Danko and other authors are based on the norm inequalities of the inner product integral transformers. However, the numerical range of the inner product type integral transformers in Hilbert spaces has not been done. In this paper, we shall establish the numerical range of this operator and show that the Toeplitz-Haursdorff property for the usual numerical range holds for this operator.

Properties of inner product type integral transformers have applications in a quantum chemical system by considering the bounded and self-adjoint Hamilitonian operator which helps in estimating the ground state energies of the chemical systems using other subsystems. The study of numerical ranges and numerical radii have applications to areas such as iteration methods, operator theory, Krein space operators, dilation theory, factorization of matrix polynomials, C^*-algebras and unitary similarity.

2. Main results

In this section, we define the algebraic numerical range of the inner product type integral operator $T_{\{A,B\}}$ and show that the basic properties of the numerical range are satisfied. We shall also show that

$$Co\left\{\int_\Omega \lambda_t \beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\}^- \subset V(T_{\{A,B\}}).$$

where Co is the convex hull and $V(T_{\{A,B\}})$, the algebraic numerical range of $T_{\{A,B\}}$.

Definition 2.1. Let $T_{\{A,B\}} = \int_\Omega A_t \otimes B_t d\mu(t)$. The algebraic numerical range of $T_{\{A,B\}}$ is defined as

$$V(T_{\{A,B\}}) = \left\{f(\int_\Omega A_t \otimes B_t d\mu(t)) : f \in P(\Omega)\right\}$$

where $P(\Omega)$ is the set of states in Ω.

Lemma 2.2. *Let $\lambda_t \in W(A_t)$ and $\beta_t \in W(B_t)$,where $W(A_t)$ is the usual numerical range of A_t, then $\lambda_t\beta_t \in W(A_t \otimes B_t)$ and $W(A_t \otimes B_t) \subset V(A_t \otimes B_t)$*

Proof. Let $\lambda \in W(A_t \otimes B_t)$. Then

$$\begin{aligned} \lambda &= \langle A_t \otimes B_t(x \otimes y), (x \otimes y)\rangle, \|x \otimes y\| = \|x\| = \|y\| = 1 \\ &= tr\{y \otimes x(A_t \otimes B_t)x \otimes y\} \\ &= \langle A_t x, x\rangle\langle B_t y, y\rangle \\ &= \lambda_t\beta_t \end{aligned}$$

For the second part, we define a linear functional $g_{x\otimes y}$ by

$$g_{x\otimes y}(X) = tr\{X_{(x\otimes y)}(x \otimes y)\}, with \|x \otimes y\| = \|x\| = \|y\| = 1, \tag{2.1}$$

then

$$\begin{aligned} |g_{x\otimes y}(X)| &\leq \|X_{(x\otimes y)}(x \otimes y)\| \\ &\leq \|x \otimes y\| \times \|X(x \otimes y)\| \\ &\leq \|X(x \otimes y)\| \\ &\leq \|X\| \end{aligned}$$

This implies that $\|g_{x\otimes y}\| = 1$. Since $g_{x\otimes y}(I_H) = tr(x \otimes y)(x \otimes y) = tr(x \otimes x)(y \otimes y) = \langle x, x\rangle\langle y, y\rangle = \|x\|^2\|y\|^2 = 1$, then $\|g_{x\otimes y}\| = g_{x\otimes y}(I_H) = 1$; therefore $g_{x\otimes y}$ is a state on $B(B(H))$.

Then we have,

$$\begin{aligned} g_{x\otimes y}(A_t \otimes B_t) &= tr\{(x \otimes y)(A_t \otimes B_t)(x \otimes y)\} \\ &= tr\{\langle A_t x, x\rangle(y \otimes B_t^* y)\} \\ &= \langle A_t x, x\rangle\langle B_t y, y\rangle \\ &= \lambda_t\beta_t \\ &= \lambda \\ \implies & \lambda \in V(A_t \otimes B_t) \end{aligned}$$

Therefore, $\{\lambda_t\beta_t : \lambda_t \in W(A_t), \beta_t \in W(B_t)\} \subset V(A_t \otimes B_t)$
Hence, $W(A_t \otimes B_t) \subset V(A_t \otimes B_t)$. □

Remark 2.3. From Lemma 2.2, it follows that $\int_\Omega \lambda_t\beta_t d\mu(t) \in W(T_{\{A,B\}})$

Theorem 2.4. *Let $T_{\{A,B\}} = \int_\Omega A_t \otimes B_t d\mu(t)$, and $\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\} \subset V(T_{\{A,B\}})$, then*

$$Co\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\}^- \subset V(T_{\{A,B\}}).$$

Proof. Let $x, y \in H$ such that $\|x\| = \|y\| = 1$. Recall the state as defined in Equation (2.1),

$$\forall F \in B(B(H)) : g_{x\otimes y}(F) = tr\{(F_{(x\otimes y)}(x \otimes y)\},$$

where tr is the trace function, with $\|x \otimes y\| = \|x\| = \|y\| = 1$, then $g_{x\otimes y}$ is a state on $B(B(H))$ as shown in Lemma 2.2 above.

For $T_{\{A,B\}} = \int_\Omega A_t \otimes B_t d\mu(t)$, let $S = A_t \otimes B_t$, using the state as defined in Equation (2.1),

$$\begin{aligned} g_{x\otimes y}(S) &= tr\{S_{x\otimes y}(x\otimes y)\} \\ &= tr\{(A_t \otimes B_t)_{x\otimes y}(x\otimes y)\} \\ &= tr\{(A_t x \otimes B_t y)(x\otimes y)\} \\ &= tr\{(A_t(x\otimes x))(B_t(y\otimes y))\} \\ &= \langle A_t x, x\rangle\langle B_t y, y\rangle \end{aligned}$$

This implies that

$$\begin{aligned} g_{x\otimes y}(T_{\{A,B\}}) &= \int_\Omega \langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) \\ &= \int_\Omega \lambda_t\beta_t d\mu(t). \end{aligned}$$

Thus,

$$\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\} \subset V(T_{\{A,B\}})$$

Since the algebraic numerical range $V(T_{\{A,B\}})$ is compact and convex, then

$$Co\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\}^- \subset V(T_{\{A,B\}}).$$

□

Theorem 2.5. *The algebraic numerical range for inner product type integral transformers satisfies the following properties*

i. $V(T^*_{\{A,B\}}) = \overline{V(T_{\{A,B\}})}$
ii. $V(U^*T_{\{A,B\}}U) = V(T_{\{A,B\}})$
iii. $V(T_{\{A,B\}} + S_{\{A,B\}}) \subseteq V(T_{\{A,B\}}) + V(S_{\{A,B\}})$
iv. $V(\alpha T_{\{A,B\}} + \beta I) = \alpha V(T_{\{A,B\}}) + \beta$

Proof. i. Let $S = A_t \otimes B_t$, then $S^* = B_t^* \otimes A_t^*$, and using the state defined in Equation (2.1),

$$\begin{aligned} g_{x\otimes y}(S^*) &= tr\{S^*_{(x\otimes y)}(x\otimes y)\} \\ &= tr\{(B_t^* \otimes A_t^*)_{(x\otimes y)}(x\otimes y)\} \\ &= tr\{(B_t^* x \otimes A_t^* y)(x\otimes y)\} \\ &= tr\{(B_t^*(x\otimes x))(A_t^*(y\otimes y))\} \\ &= \langle B_t^* x, x\rangle\langle A_t^* y, y\rangle \\ &= \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle \end{aligned}$$

This implies that

$$\begin{aligned} g_{x\otimes y}(T^*_{\{A,B\}}) &= \int_\Omega \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle d\mu(t) \\ &= \int_\Omega \langle y, A_t y\rangle\langle x, B_t x\rangle d\mu(t) \\ &= \int_\Omega \overline{\lambda_t\beta_t} d\mu(t) \end{aligned}$$

Since $\left\{\int_\Omega \overline{\lambda_t\beta_t} d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\} \subset \overline{V(T_{\{A,B\}})}$, then $V(T^*_{\{A,B\}}) = \overline{V(T_{\{A,B\}})}$.

ii. Let $S = U^*A_t \otimes B_tU$, then

$$\begin{aligned} g_{x\otimes y}(S) &= tr\{S_{x\otimes y}(x\otimes y)\} \\ &= tr\{(U^*A_t \otimes B_tU)_{x\otimes y}(x\otimes y)\} \\ &= tr\{(U^*A_t x \otimes B_tU y)(x\otimes y)\} \\ &= tr\{(U^*A_t(x\otimes x))(B_tU(y\otimes y))\} \\ &= \langle U^*A_t x, x\rangle\langle UB_t y, y\rangle \end{aligned}$$

This implies that

$$\begin{aligned} g_{x\otimes y}(T_{\{A,B\}}) &= \int_\Omega \langle U^*A_t x, x\rangle\langle UB_t y, y\rangle d\mu(t) \\ &= \int_\Omega U^*U\langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) \\ &= \int_\Omega \langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) \quad (Since \;\; U^*U = UU^* = I) \\ &= \int_\Omega \lambda_t\beta_t d\mu(t) \end{aligned}$$

It implies that $\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t)\right\} \subset V(T_{\{A,B\}})$.

iii. Let $T_{\{A,B\}} = \int_\Omega A_t \otimes B_t d\mu(t), S_{\{A,B\}} = \int_\Omega P_t \otimes Q_t d\mu(t)$, then

$$\begin{aligned} T_{\{A,B\}} + S_{\{A,B\}} &= \int_\Omega A_t \otimes B_t d\mu(t) + \int_\Omega P_t \otimes Q_t d\mu(t) \\ &= \int_\Omega (A_t \otimes B_t + P_t \otimes Q_t) d\mu(t) \end{aligned}$$

Let $G = A_t \otimes B_t + P_t \otimes Q_t$, then

$$\begin{aligned} g_{x\otimes y}(G) &= tr\{G_{(x\otimes y}(x\otimes y)\} \\ &= tr\{(A_t \otimes B_t + P_t \otimes Q_t)_{x\otimes y}(x\otimes y)\} \\ &= tr\{(A_t \otimes B_t(x\otimes y) + P_t \otimes Q_t(x\otimes y))((x\otimes y)\} \\ &= tr\{((A_t x \otimes B_t y) + (P_t x \otimes Q_t y))(x\otimes y)\} \\ &= tr\{(A_t x \otimes B_t y) + (y\otimes x)(P_t x \otimes Q_t y)((x\otimes y)\} \\ &= tr\{(A_t x \otimes x)(B_t y \otimes y) + (P_t x \otimes x)(Q_t y \otimes y)\} \\ &= \langle A_t x, x\rangle\langle B_t y, y\rangle + \langle P_t x, x\rangle\langle Q_t y, y\rangle \end{aligned}$$

This implies that

$$\begin{aligned} g_{x\otimes y}(T_{\{A,B\}}+S_{\{A,B\}}) &= \int_\Omega \langle A_t x,x\rangle\langle B_t y,y\rangle + \langle P_t x,x\rangle\langle Q_t y,y\rangle d\mu(t) \\ &= \int_\Omega \langle A_t x,x\rangle\langle B_t y,y\rangle d\mu(t) + \int_\Omega \langle P_t x,x\rangle\langle Q_t y,y\rangle d\mu(t) \\ &= \int_\Omega \lambda_t\beta_t d\mu(t) + \int_\Omega \gamma_t\delta_t d\mu(t) \end{aligned}$$

It implies that

$$\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t) + \int_\Omega \gamma_t\delta_t d\mu(t) : \gamma_t \in W(P_t), \delta_t \in W(Q_t)\right\} \subset V(T_{\{A,B\}}) + V(S_{\{A,B\}}).$$

iv. $V(\alpha T_{\{A,B\}}+\beta I) = \alpha V(T_{\{A,B\}})+\beta$

Now, $\alpha T_{\{A,B\}}+\beta I = \alpha\int_\Omega A_t\otimes B_t d\mu(t)+\beta I$

Let $S = \alpha(A_t\otimes B_t)+\beta I$, then

$$\begin{aligned} g_{x\otimes y}(S) &= tr\{S_{(x\otimes y)}(x\otimes y)\} \\ &= tr\{(\alpha(A_t\otimes B_t)+\beta I)_{(x\otimes y)}(x\otimes y)\} \\ &= tr\{(\alpha(A_t\otimes B_t)_{x\otimes y}+\beta I_{x\otimes y})(x\otimes y)\} \\ &= tr\{(\alpha(A_t x\otimes B_t y)+\beta(x\otimes y))(x\otimes y)\} \\ &= tr\{\alpha(A_t x\otimes B_t y)(y\otimes x)+\beta(x\otimes y)(y\otimes x)\} \\ &= tr\{\alpha(A_t x\otimes x)(B_t y\otimes y)+\beta(x\otimes x)(y\otimes y))\} \\ &= \alpha\langle A_t x,x\rangle\langle B_t y,y\rangle+\beta\langle x,x\rangle\langle y,y\rangle \\ &= \alpha\langle A_t x,x\rangle\langle B_t y,y\rangle+\beta \end{aligned}$$

This implies that

$$\begin{aligned} g_{x\otimes y}(\alpha T_{\{A,B\}}+\beta I) &= \int_\Omega \alpha\langle A_t x,x\rangle\langle B_t y,y\rangle d\mu(t)+\beta \\ &= \int_\Omega \alpha\lambda_t\beta_t d\mu(t)+\beta \end{aligned}$$

Hence

$$\alpha\left\{\int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t\in W(A_t), \beta_t\in W(B_t)\right\}+\beta \subset \alpha V(T_{\{A,B\}})+\beta.$$

□

Theorem 2.6. *The algebraic numerical range $V(T_{\{A,B\}})$ is convex.*

Proof. To show that $V(T_{\{A,B\}})$ is convex, we need to show that the intersection of every line with $V(T_{\{A,B\}})$ is connected or empty. Since $T_{\{A,B\}}$ is a bounded linear operator on H, let $\int_\Omega \lambda_t\beta_t d\mu(t)$ and $\int_\Omega \gamma_t\delta_t d\mu(t)$ be distinct points of $W(T_{\{A,B\}})$ with $\int_\Omega \lambda_t\beta_t d\mu(t) = \langle T_{\{A,B\}}x,x\rangle$, $\int_\Omega \gamma_t\delta_t d\mu(t) = \langle T_{\{A,B\}}y,y\rangle$ These values are linearly independent and hence $\langle T_{\{A,B\}}x,x\rangle \neq \langle T_{\{A,B\}}y,y\rangle$, for if we let $x = ay, a \in \mathbb{C}, |a| = 1, \|x\| = 1 = \|y\|$, then $\int_\Omega \lambda_t\beta_t d\mu(t) = \langle T_{\{A,B\}}x,\rangle = \langle T_{\{A,B\}}ay,ay\rangle = \langle aT_{\{A,B\}}y,ay\rangle = \int_\Omega \gamma_t\delta_t d\mu(t)$, which is a contradiction. Similarly, $y \neq ax$ for any

$a \in \mathbb{C}$ so that x and y are linearly independent. Now let $L = span\,\{x, y\}$, a two dimensional subspace of H and let P_L be the projection onto L and $T \in B(H)$. Since $x, y \in L$, $\int_\Omega \lambda_t\beta_t d\mu(t), \int_\Omega \gamma_t\delta_t d\mu(t) \in W(P_L T_{\{A,B\}}|_L) \subseteq W(T_{\{A,B\}}) \subset V(T_{\{A,B\}})$. But since L is two dimensional, $W(P_L T_{\{A,B\}}|_C)$ is convex. Hence, $t\int_\Omega \lambda_t\beta_t d\mu(t) + (1-t)\int_\Omega \gamma_t\delta_t d\mu(t) \in W(P_L T_{\{A,B\}}|_C) \subseteq W(T_{\{A,B\}}) \subset V(T_{\{A,B\}})$ for $0 < t < 1$. Since $\int_\Omega \lambda_t\beta_t d\mu(t), \int_\Omega \gamma_t\delta_t d\mu(t) \in W(T_{\{A,B\}})$ were arbitrary, then $V(T_{\{A,B\}})$ is convex as desired.

□

Theorem 2.7. *The operator* $T_{\{A,B\}} = \int_\Omega A_t \otimes B_t d\mu(t)$ *is self adjoint if and only if* $V(T_{\{A,B\}})$ *is real.*

Proof.

$$\begin{aligned} V(T_{\{A,B\}}) &= \left\{ f\Big(\int_\Omega A_t \otimes B_t d\mu(t)\Big) : f \in P(\Omega) \right\} \\ &= \left\{ \int_\Omega \langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) \right\} \\ &= \left\{ \int_\Omega \lambda_t\beta_t d\mu(t) : \lambda_t \in W(A_t), \beta_t \in W(B_t) \right\} \\ &= \int_\Omega \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle d\mu(t) \\ &= \int_\Omega \langle y, A_t y\rangle\langle x, B_t x\rangle d\mu(t) \\ &= \left\{ \int_\Omega \overline{\lambda_t\beta_t} d\mu(t) : \overline{\lambda_t} \in W(A_t^*), \overline{\beta_t} \in W(B_t^*) \right\} \end{aligned}$$

Hence, we have that $\int_\Omega \overline{\lambda_t\beta_t} d\mu(t) = \int_\Omega \lambda_t\beta_t d\mu(t)$, which implies that $V(T_{\{A,B\}})$ is real.
Conversely,
$f(T_{\{A,B\}}) = \int_\Omega \langle A_t x, x\rangle\langle B_t y, y\rangle d\mu(t) = f(T^*_{\{A,B\}}) = \int_\Omega \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle d\mu(t)$
This implies that,
$f(T_{\{A,B\}}) - f(T^*_{\{A,B\}}) = \langle A_t x, x\rangle\langle B_t y, y\rangle - \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle$ Since A_t and B_t are self adjoint operators, we have $\langle A_t x, x\rangle - \langle A_t^* y, y\rangle = 0$, for $\|x\| = \|y\| = 1$ and $B_t x, x\rangle - \langle B_t^* y, y\rangle$ for $\|x\| = \|y\| = 1$.
Hence $f(T_{\{A,B\}}) - f(T^*_{\{A,B\}}) = \langle A_t x, x\rangle\langle B_t y, y\rangle - \langle A_t^* y, y\rangle\langle B_t^* x, x\rangle = 0. \implies f(T_{\{A,B\}}) - f(T^*_{\{A,B\}}) = 0 \implies f(T_{\{A,B\}} - T^*_{\{A,B\}}) = 0$. Since f is a linear functional on $B(H)$, then $f \neq 0$ and therefore $T_{\{A,B\}} - T^*_{\{A,B\}} = 0$ Which implies that $T_{\{A,B\}} = T^*_{\{A,B\}}$. That is, $T_{\{A,B\}} = \int_\Omega A_t X B_t d\mu(t)$ is self-adjoint. □

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