# CONSTRUCTION OF SOME ECONOMICAL OPTIMUM SECOND ORDER ROTATABLE DESIGNS IN FOUR AND FIVE DIMENSIONS

**BY**

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# DECLARATION

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# DEDICATION

I dedicate this work to my father, Philip Manava Musamusi, who offered unconditional love and support and have always been there for me. I am truly thankful for having you in my life. Thank you so much.

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# ABSTRACT

The technique of fitting a response surface is one widely used especially in the chemical industry to aid in the statistical analysis of experimental work in which the yield of a product depends in some unknown fashion on one or more controllable variables. The world is facing depletion of resources and search for alternative measures is inevitable in all human endeavor. Since the resources are scarce, we need to produce maximally by utilizing design of experiments like in this study. In the current study, optimal economical second order rotatable designs SORDs in four and five dimensions were constructed. The objectives of the study were; to construct economical SORDs in four, and five dimensions, evaluate the alphabetic optimality criteria for the designs, and determine the A- and E- efficiencies for the designs. The sets of points for a sequential rotatable arrangement in four and five dimensions formed rotatable arrangements if their excess functions were zero. All the variables determined were real and positive and this confirmed the existence of a rotatable arrangement. The designs were considered to be SORDs after satisfying both the moment and non-singularity conditions. The moments which formed the elements of the moment matrix were determined by taking the parameter system of interest to be that of a second order model. The moment matrices formed the basis for determination of the optimality criteria for every design considered. The determinant criterion (D-), Average variance criterion (A-), Eigen value criterion (E-) and the trace criterion (T-). Were considered. For each criterion the design with the least value will be optimal to the specific criteria under consideration. The study yielded; 32, 40, 48a, and 48b points SORDs in four dimensions and; 52, 74,100a, and 100b points SORDs in five dimensions respectively. From the Table 2 the design $G_2$ (40 points SORD in four dimensions) is A-, D-, T- and E-optimal. The design $G_6$ (74 points SORD in five dimensions) is A-, D- and-optimal and designs $G_7$ was found to be E-optimal. The analysis of efficiencies facilitated the choice of the most desirable design from the other designs under consideration. In conclusion, the more homogenous the design is, the more optimal it became, and thus the designs obtained provides very essential tools for use in various fields such as in medicine, agriculture and industry. The study recommends evaluation of robustness of missing data for these designs and all other designs to enable researchers make informed decisions whenever missing information is anticipated.

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# ABBREVIATIONS AND ACRONYMS

**CCD** : Central Composite Designs

**DOE** **:** Designs of Experiments

**NED** : Normal Equivalent Deviates.

**RSM** : Response Surface Methodology

**SORD** : Second Order Rotatable Designs.

**TORD** : Third Order Rotatable Designs.

## DEFINITIONS

**D.O.E**: Is a systematic rigorous approach to problem solving that applies principles and techniques at the data collection stage so as to ensure the generation of valid, defensible and supportable conclusions.

**Optimum**: The amount or degree of something that is best or most effective and favorable. It is also the greatest degree attained or attainable under implied or specified conditions.

**R.S.M**: This is a collection of mathematical and statistical techniques useful for analyzing problems in which several independent variables influence a dependent variable and the goal is to optimize the dependent variable.

**Rotatable**: When an orthogonal matrix and a vector are multiplied, the resulting vector having the same magnitude with the vector multiplied with the orthogonal matrix but the two vectors facing different directions.

# CHAPTER ONE

# INTRODUCTION

## 1.0 Introduction

This chapter covers the background of the study, the statement of the problem, the objectives of the study, and the justification of the study.

## 1.1 Background of the Study

The technique of fitting a response surface is one widely used especially in the chemical industry to aid in the statistical analysis of experimental work in which the yield of a product depends in some unknown fashion on one or more controllable variables. Before the details of such an analysis can be carried out, experiments must be performed at predetermined levels of the controllable factors i.e. an experimental design must be selected prior to experimentation. Box and Hunter (1957) suggested designs of a certain type which they called rotatable, as being suitable for such experimentation. Since that time, the work of Carter in 1957 has provided new second order rotatable designs in two factors. However, additional methods were needed which would provide both second and third order designs in three and more factors. Mutiso (1998) in his PhD thesis constructed second order rotatable designs in k-dimensions. The present work represents an alternative method of constructing more economical second order rotatable designs in k-dimensions using the concepts of Draper (1960a). Certain sets of points are used, all of which will be obtained from a basic point set, the basic point set occurs as a result of applying a specified transformation group to a general point in three dimensions. It will be shown that these generated point sets obey all the conditions and the non- singularity conditions for second order rotatable designs.

There is then defined a function called the excess of a point set, which describes the extent to which the outstanding condition is not met, and it is shown that combinations of point sets whose total excess is zero will provide second order rotatable designs or infinite classes of rotatable designs.

### 1.1.1 Transformation Group in Three Dimensions and it's Generated Points Sets

Let W(x, y, z) = (y, z, x). Then $W^2$(x, y, z) = (z, x, y) and $W^3$(x, y, z) = (x, y, z). Thus, W, $W^2$, $W^3$ form a cyclical group of order three. Further, let $R_1$(x, y, z) = (-x, y, z), $R_2$(x, y, z) =(x, -y, z), $R_3$(x, y, z) =(x, y, -z).

The four transformations represented by W, $R_1$, $R_2$ and $R_3$ generate a group of transformations of order 24 with elements.

WJ, WJR1, WJR2, WJR3, WJR2R3, WJR3R1, WJR1R2, WJR1R2R3 (J=1, 2, 3)

It is easily seen that all the 24 elements are distinct while $R_1$ , $R_2$, $R_3$ commute, WJ and $R_I$ do not (i=1,2,3 ;j=1,2,3).

Given a general point (X, Y, Z) in three dimensions, we may apply to it all the transformations of the group (S). In this way we obtain a set of 24 points with co-ordinates.

($\pm$x, $\pm$y, $\pm$z), ($\pm$y, $\pm$z, $\pm$x), ($\pm$z, $\pm$x, $\pm$y). We shall denote this by s(x, y, z)

Note that if (i, m, n) denotes any other point of the set, when operated on by s, will give rise to the same set. Certain special choices of (x, y, z) will coincide in pairs or in triplets or in quadruplets. For example, s (p, q, o) consists of twelve points.

($\pm$p, $\pm$q ,o ) , ($\pm$q, o, $\pm$p ) , ( o, $\pm$p, $\pm$q) ,each occurring twice. We may denote the twelve point set by $\frac{1}{2}$s (p, q, o). This set has excess, EX [$\frac{1}{2}$s (p, q, o)] = 4($p^4$+$q^4$-3$p^2q^2$).

A quantity may be made positive or negative depending on the values of p and q according to Bose and Draper (1959). The sets of points yielding non- zero excesses are suitably combined to give zero excesses where non-zero excess imply values of the excess functions are other than zero according to draper [1960b]. To form rotatable arrangement, we combine several generated points sets and use the excess functions.

### 1.1.2 Optimality Criteria

An optimality criterion is a single number that summarizes how good a design is, and it is maximized or minimized by an optimal design. Two general types of criteria are available: information-based criteria and distance-based criteria. The information-based criteria that are directly available are D- and A- optimality; they are both related to the information matrix $X^{/}X$ for the design. This matrix is important because it is proportional to the inverse of the variance –covariance matrix for the least-squares estimates of the linear parameters of the model. Roughly, a good design should minimize the variance of $(X^{/}X)^{/}$ which is the same as maximizing the information of $X^{/}X$ . The D-, and A- efficiencies are different ways of saying how large $X^{/}X$ and $(X^{/}X)^{/}$ are.

### 1.1.3 Design Efficiency

For experimental designs, efficiency relates to the ability of a design to achieve the objective of the study with minimal expenditure of resources such as time and money. In simple cases, the relative efficiency of designs can be expressed as the ratio of the sample sizes required to achieve a given objective.

### 1.2 Statement of the Problem

The world is facing depletion of resources and search for optimal utilization measures are inevitable in all fields of human endeavor. Since resources are scarce, we need to produce maximally in all spheres by utilizing economical optimum designs in experimentation. The existing second order rotatable designs in four and five dimensions have relatively a higher number of design points, i.e. 44, 48, 52, 56, 64, 68 in four dimensions and 92, 100, 108, 116, 132 & 140 points in five dimensions respectively. The high number of points translates to more resource and time expenditure. This study introduces relatively economical designs in four and five dimensions which will enable experimenters to use minimum resources in their experimental endeavors, thus improving their livelihoods and attaining their countries development goals.

### 1.3 Objectives

The aim of the study was to construct economical optimum second order rotatable designs in four and five dimensions.

The specific objectives were to;

- **i.** Construct some specific and sequential second order rotatable designs in, four and five dimensions.
- **ii.** Evaluate the optimality criteria for the second order rotatable designs in four and five dimensions
- **iii.** Obtain the design's relative efficiencies

## 1.4 Justification

Extensive research in many fields in industry, food processing and agriculture utilizes response surface methodology. As a result of this, economical optimal sequential designs like in this study are developed. These designs enable experimenters to use minimum resources to obtain optimum yield.

# CHAPTER TWO

# LITERATURE REVIEW

## 2.0 Introduction

This section seeks to trace, chronologically, the various streams of thought which have contributed to what we now call Response surface methodology.

## 2.1 Developments in Response Surface Designs

Response surface methodology is defined as a collection of mathematical and statistical techniques useful for analyzing problems in which several independent variables influence a dependent variable, Montgomery (1984). In their review of the literature on Response surface methodology, Hill and Hunter (1966) assert that Response surface methodology was initially developed and described by Box and Wilson (1951). This seems to Mead and Pike (1975) to imply too narrow a view of the subject. Certainly Box and Wilson's paper and the large number of papers by Box and his associates which followed it in the next decade constitutes the single most powerful source of ideas in the investigation of response surfaces but many of the fundamental ideas had been used and discussed much earlier. However, in this early development the design of experiments specifically to investigate response surfaces was not discussed to any extent.

Fechner (1860) transformed proportions to the corresponding normal deviates for data from psychological experiments and fitted a curve to the relationship between the concentrations of a stimulus and the proportions of individuals responding. The word Probit was introduced by Bliss (1935) when he added 5 to $(x-u)/\delta$ called the normal equivalent deviate denoted (N.E.D), where $x=\log_{10}\lambda$ is the dosage or dose Meta meter and $\lambda$ is the dose in a bioassay. Winsor (1932) used the Gompertz curve for a situation where the relative growth rate was believed to decrease exponentially with time and

Wishart (1938, 1939) illustrated orthogonal polynomials for studying growth rates in nutrition studies of pigs. The area where response curves were used and in which for the first time , response surfaces were considered was the agronomic study of the response of crop yield to crop spacing or fertilizer levels where Mitsherlich (1930) found biologically reasonable and asymptotic relationship between plant yield and the supply of an essential growth factor in particular, space, and Yates and Crowther (1941) applied the Mitscherlich response equations in the study of the response of arable crops to several different fertilizers.

In the Box evolution, Box and Wilson (1951) discussed experimental designs whose purpose is to find using the smallest possible number of observations, the point on a response surface at which the optimum is achieved. They introduced the concept of composite designs to generalize the well-known factorial principle of experimental design making it applicable in the response surface methodology. They discussed steepest ascent or descent in the search for the near stationary region around the optimum representing the models using Taylor series expansion and devised the coded level convention. Box (1952) wrote the paper on multifactor designs of First order and Box (1954) explored response surfaces considering general examples. Box and Youle (1955) illustrated the link between the fitted surface and the basic mechanism of the system. Gardiner *et al*. (1956) wrote his doctor of philosophy thesis on some third order rotatable designs. Box and Hunter (1957) suggested designs of a certain type which they called rotatable as being suitable for second order response surfaces experimentation. Gardner *et al.* (1959) gave the moment and the non-singularity conditions for third order response surface designs.

Bose and Draper (1959) introduced a transformation group in three dimensions and its generated point sets and formed second order rotatable designs by combination of several generated points sets. Draper (1960a, 1960b, 1960c) constructed second order rotatable designs in four or more dimensions, third order rotatable designs in three dimensions, and third order rotatable designs in four dimensions respectively. Draper (1961) illustrated construction of specific third order designs in three dimensions. Arap Koske (1984) wrote his doctor of philosophy thesis on fourth order rotatable designs followed in quick succession by Njui (1985) who wrote his doctor philosophy thesis on fifth order rotatable designs with both publishing their works in subsequent years jointly with their supervisor, namely Patel and Arap Koske (1985, 1986, 1987) and Patel and Njui (1988). Arap Koske (1987) gave a fourth order rotatable design in four dimensions.

**2.2 Optimality Criteria for Response Surface Designs**

The literature on optimal design can be traced to Atkinson and Donev (1992) who wrote a book on optimum experimental designs. This was closely followed by Pukelsheim, F. (1993) who wrote a book on optimal design of experiments. Studies on robustness of optimal designs against model variation were done in Huda (1991) using A-efficiency, central composite designs against model variation was examined by Huda and Khan (1993).

The minor technical difficulty that the moment matrix $M_k$ is rank deficient is of no relevance in an age of generalized inverses. For example, Keifers optimality criteria are simply taken to be the mean of order P of the positive Eigen values of $M_k$, Pukelsheim, F. (1993), it is precisely because of these technical differences that the Eigen values decomposition of Kronecker representation moment matrix $M_k$ takes an almost explicit form for third order rotatable designs, Pukelsheim and Draper,(1994). The family of matrix means, $\phi$ (p), with $-\infty \leq p \leq 1$ was introduced by Kiefer (1974) and discussed in detail by Pukelsheim, F. (1993). The detailed discussion by Pukelsheim, F. is purely theoretical and involves much of mathematical jargon**.** There are essentially two ways for the construction of design criteria which incorporate different purposes of the experiment. One approach is the construction of new optimality criteria by averaging several competitive design criteria. Alternatively, one could try to maximize one primary optimality criteria subject to constraints for specific minimum efficiencies of other criteria, Dette and Franke, (2000). Mutiso (1998) constructed optimum designs of order two in three dimensions but the optimality criteria for the construction were not identified. Kosgei (2002) gave the optimality criteria for the specific second order rotatable designs in three dimensions by considering A-, E-, D-, and T- criteria. Goos (2006) discussed in lengthy on IV- optimality. Nyakundi (2016) constructed some third order rotatable designs in three four and five dimensions.

A study of first order rotatable designs assuming the model has correlated errors was initiated by Panda and Das (1994). Das (1997), studied second order rotatable designs assuming that the model has correlated errors. Starting with usual second order rotatable designs, a method of construction of second order rotatable designs, and a method of construction of second order rotatable designs with correlated errors was introduced by Das (1999). The present work gives a method of constructing some second order rotatable designs in in four and five dimensions.

# CHAPTER THREE

# METHODOLOGY

## 3.0 Introduction

This chapter gives a summary on how the specific objectives are achieved. It includes the method of construction of economical second order rotatable designs in four and five dimensions, the methods of obtaining the optimality criteria and efficiencies of the designs.

## 3.1 Construction of Second Order Rotatable Designs in K-Dimensions

The starting point is to investigate the construction of second order specific and sequential rotatable designs in k- dimensions from the existing second order rotatable designs denoted by;

$s_1 = S(a, a, a) + S(c_1, 0,0) + S(c_2, 0, 0)$ [3.1]

$s_2 = S(a_1, a_1, a_1) + S(a_2, a_2, a_2) + S(c, 0,0)$ [3.2]

$s_3 = S(0, f, f) + 2S(c, 0,0)$ [3.3]

$s_4 = S(0, p, p) + S(c_1, 0,0) + S(c_1, 0,0)$ [3.4]

The above sets of points gives 20, 22, 24a and 24b points second order rotatable designs in three dimensions respectively. The designs above exist in Mutiso (1998) PhD thesis.

The sequential second order rotatable designs in four dimensions are obtained by adding a factor to each of the set of points in three dimensions to give;

$G_1 = S(a, a, a, a) + S(c_1, 0,0,0) + S(c_2, 0, 0, 0)$ [3.5]

$G_2 = S(a_1, a_1, a_1, a_1) + S(a_2, a_2, a_2, a_2) + S(c, 0,0,0)$ [3.6]

$G_3 = S(0, f, f, f) + 2S(c, 0,0,0)$ [3.7]

$G_4 = S(0, p, p, p) + S(c_1, 0,0,0) + S(c_1, 0,0,0$ [3.8]

The above sets of points give; 32, 40, 48a and 48b points second order rotatable designs in four dimensions respectively.

The sequential second order rotatable designs in five dimensions are obtained by adding a factor to each of the set of points in four dimensions as below;

$G_5$= S (a, a, a, a, a) +S ($c_1$, 0,0,0,0) +S ($c_2$, 0, 0, 0, 0) [3.9]

$G_6$= S ($a_1$,$a_{1,}$$a_1$, $a_1$, $a_1$) + S ($a_2$,$a_{2,}$$a_2$, $a_2$, $a_2$) +S (c, 0,0,0,0) [3.10]

$G_7$=S (0,f, f, f, f) +2S (c, 0,0,0,0) [3.11]

$G_8$=S (0, p, p, p, p) + S ($c_1$,0,0,0,0) +S ($c_1$,0,0,0,0) [3.12]

The above sets of points give; 56, 74, 100a and 100b points second order rotatable designs in five dimensions respectively.

For each set of combinations, we show that the points form a sequential rotatable arrangement in 4 and 5 dimensions respectively, if all the excess functions are zero. We then estimate one variable such that the other variable obtained after substitution is real and non -negative.

If the moment conditions for second order are achieved, then the sets of points are said to make a rotatable arrangement and subsequently if the non-singularity conditions for second order rotatability are achieved, then the set of points forms a second order rotatable design in 4 and 5 dimensions respectively.

The following are both the moment conditions and non- singularity conditions for second order rotatability

$\sum_{u=1}^{n} x_{iu}^{2} = n\lambda_2$,

$\sum_{u=1}^{n} x_{iu}^{4} = 3\sum_{u=1}^{n} x_{iu}^{2}\, x_{ju}^{2}$, $i \neq j$,

$\sum_{u=1}^{n} x_{iu}^{4}\, x_{ju}^{2} = \sum_{u=1}^{n} x_{iu}^{2}\, x_{ju}^{4}$, $i \neq j$, [3.13]

And the non-singularity conditions given as;

$\frac{\lambda 4}{\lambda_2^2} > \frac{k}{k+2}$ ,  For k=3, 4, 5 [3.14]

**3.2 Optimality Criteria**

A design can do very well in a particular criteria and at the same time do very poorly in another criterion, Kosgei (2002). This is because these criteria play particular roles in improving the parameter estimates whose ultimate goal is to optimize the response (yield). An experimenter who is interested in using a D-optimum second order rotatable design in five dimensions shall do so by using the analysis of D-optimality of all the designs in five dimensions and the least among them is considered as the best for that design. The same case applies to all other criteria.

**3.2.1 D-Criterion**

According to Pukeisheim, F (1993). The determinant criterion is obtained by;

$\emptyset_0(C) = (\det C)^{\frac{1}{s}}$ [3.15]

Where C is the information matrix.

D- Optimum design minimizes the content of the ellipsoidal confidence region for the parameters of the linear model.

The determinant criterion $\emptyset(C)$ differs from the determinant det (C) by taking the $s^{th}$ root where both functions induce the same pre-ordering among information matrices. For comparing different criteria and applying the theory of information functions, the version $\emptyset_0(C) = (\det C)^{\frac{1}{s}}$ is appropriate. Minimizing the determinant of the information matrices is the same as maximizing the determinant of the dispersion matrices because of the formula

$(\text{Det C})^{-1}$ = $(\text{Det C}^{-1})$ [3.16]

According to Rambaei (2014), the determinant criterion is given by;

Det =$[k[\frac{k}{2}]\ \lambda_2^{k}\ \lambda_4[k+2]\ \lambda_4 - k\lambda_2^2]^{\ 1/k+3}$ [3.17]

Where k is the number of dimensions of the design.

### 3.2.2 Average Variance Criterion, A-Criterion

According to Pukeilsheim, F. (1993),

The average variance, A-,$\emptyset_0$(C) =$(\frac{1}{s}\ \text{trace}C^{-1})^{-1}$ (if C is positive definite).

A- Criterion minimizes the sum or average of the variances of the estimates. It is the average of the standardized variances of the optimal estimators for the scalar parameter systems$C_1\beta, C_2\beta \ldots\ C_n\beta$, formed from the columns of n. The average variance criterion

$\emptyset_{-1}$(C) Is given by $\emptyset_0$(C) =$(\frac{1}{s}\ \text{trace}C^{-1})^{-1}$ if C is positive definite.

According to Rambaei (2014), the general formula for obtaining the A- criterion is given by;

$$A - \text{Criterion}= \left[\frac{1}{k+3}\left[\frac{[k+2]\lambda_4}{[k+2]\lambda_4 - k\lambda_2^2} + \frac{1}{k\left[[k+2]\lambda_4 - k\lambda_2^2\right]}\ \frac{k}{\lambda_2}\ \frac{1}{\binom{k}{2}\lambda_4}\right]\right]^{-1} \quad [3.18]$$

### 3.2.3 The Smallest Eigenvalue Criteria, E- Criterion

According to Pukeilsheim, F. (1993),

$\emptyset_{-\infty}$(C) = $\lambda_{min}(C^{-1})$ and, [3.19]

E- Criterion given in (3.19) reduces the variance of each individual parameter estimate. It is the same as minimizing the largest Eigen value of the information matrix.

E- Criterion = $[(1-\gamma)[(k+2)k\lambda_4 - \gamma] - (k\lambda_2)^2]\ (\lambda_2 - \gamma)^k \left[\binom{k}{2}\lambda_4 - \gamma\right] = 0$ [3.20]

### 3.2.4 The Trace Criterion, T-Criterion

The trace criterion, T-,$\emptyset_0(C) = \frac{1}{s}$ trace C [3.21]

The trace criterion is generally used for model discrimination, but in terms of parameter estimation, it has not yet found much use because of the linearity aspect of the T-criterion which makes it susceptible to interpolation. The other matrix means are concave without being linear. The evaluation of the trace criterion is given by

$\emptyset_{-\infty}(C) = \frac{1}{s}$ trace C. [3.22]

Where s is the number of parameters and C is the information matrix.

The generalized Trace or T – criterion as given by Rambaei (2014) is;

T- Criterion $= \frac{1}{K+3}\left[1 + (k+2)k\lambda_4 + k\lambda_2 + \binom{k}{2}\lambda_4\right]$ [3.23]

## 3.3 Relative Efficiencies for SORD in four and five dimensions

In this section, relative efficiencies of the designs discussed earlier for the second order rotatable designs in four and five dimensions respectively are evaluated. The efficiency of particular criteria is arrived at by taking the minimum value of particular optimality criteria of the designs. The design with the minimum value is taken as the most efficient and the other design efficiencies are evaluated relative to the most efficient design.

The relative efficiencies of an arbitrary design $\varepsilon$ with moment matrix M ($\varepsilon$) with respect to an optimal design $\varepsilon^*$ with the moment matrix M ($\varepsilon^*$) are defined as.

### 3.3.1 Relative D-Efficiency

D-Efficiency = $\frac{M\,(\varepsilon)}{M(\varepsilon_D^*)} \times 100$ (3.24)

### 3.3.2 Relative A-Efficiency

A-efficiency = $\frac{\mathrm{tr}(M^{-1}(\varepsilon_A^*))}{tr(M^{-1}(\varepsilon))} \times 100$ (3.25)

### 3.3.3 Relative E-Efficiency

E-Efficiency = $\frac{\mathrm{M}\,(\varepsilon)}{M(\varepsilon_E^*)} \times 100$ (3.26)

### 3.3.4 Relative T-Efficiency

T-Efficiency = $\frac{\mathrm{M}\,(\varepsilon)}{M(\varepsilon_T^*)} \times 100$ (3.27)

# CHAPTER FOUR

# RESULTS AND DISCUSSION

## 4.0 Introduction

In this chapter, the SORDs in four and five dimensions are constructed. Specifically, 32,40, 48a, and 48b SORDs in four dimensions and 52, 74,100a, and 100b in five dimensions are constructed respectively. To assess the designs, their alphabetical optimality criteria are obtained. Finally, the optimality criteria are used to find the relative efficiencies of the designs.

## 4.1 Construction of Economical SORD in Four and Five Dimensions

### 4.1.1 Construction of 32 Points SORD in Four Dimensions

The set of points which form a SORD of 32 points in four dimensions denoted by $G_1$ is given in [3.5]. Substituting $G_1$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 16a^2 + 2c_1^2 + 2c_2^2 = 32\lambda_2 \ , \quad [4.1]$$

$$\sum x_{iu}^4 = 16a^4 + 2c_1^4 + 2c_2^4 = 96\lambda_4 \ , \quad [4.2]$$

$$\sum x_{iu}^2 x_{ju}^2 = 16a^4 \quad [4.3]$$

The excess function obtained from the moment conditions is given by;

$$\sum x_{iu}^2 - 3 \ \sum x_{iu}^2 x_{ju}^2 = 2c_1^4 + 2c_2^4 - 32a^4 = 0, \quad [4.4]$$

Let $c_1^2 = x\, a^2$ , and $c_2^2 = y\, a^2$ . [4.5]

This implies that,

$$x^2 + y^2 = 16 \ . \quad [4.6]$$

Solving equation [4.6] gives;

Let $x = 2$ and $y = 3.464101615$ [4.7]

The value of $x$ that is real and positive is chosen in such a way that the corresponding value of y is real and positive.

Since the values of x and y are real and positive, the set of points form a rotatable arrangement.

Substituting [4.7] to [4.5] gives, $c_1^2 = 2a^2$, and $c_2^2 = 3.464$ [4.8]

Substituting [4.8] to [4.1] and [4.2] gives;

$\lambda_2 = 0.84375\, a^2$, and $\lambda_4 = 0.505208a^4$ [4.9]

Substituting [4.9] to the non-singularity conditions given in [3.14] for k=4, gives,

$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

$0.709 > 0.67$ [4.10]

From [4.10], [3.14] is satisfied, hence $G_1$ forms a SORD in four dimensions.

### 4.1.2 Construction of 40 Points SORD in Four Dimensions

The set of points which form a SORD of 40 points in four dimensions denoted by $G_2$ is given in [3.6]. Substituting $G_2$ to the moment conditions given in [3.13] gives;

$\sum x_{iu}^2 = 2c^2 + 16a_1^2 + 16a_2^2 = 40\lambda_2$ , [4.11]

$\sum x_{iu}^2 = 2c^4 + 16a_1^4 + 16a_2^4 = 120\lambda_4$ [4.12]

$\sum x_{iu}^2 x_{ju}^2 = 16a_1^4 + 16a_2^4$ [4.13]

The excess function obtained from the moment conditions is given by;

$\sum x_{iu}^2 - 3 \sum x_{iu}^2 x_{ju}^2 = c^4 - 16a_1^4 - 16a_2^4 = 0,$ [4.14]

Let $a_1^2 = x\, c^2$ , and $a_2^2 = y\, c^2$ . [4.15]

This implies that,

$x^2 + y^2 = \frac{1}{16}$ . [4.16]

Solving equation [4.16] gives;

$x = 0.05$, and $y = 0.245$ [4.17]

Substituting [4.17] to [4.15], gives

$a_1^2 = 0.05c^2, a_2^2 = 0.245c^2$ [4.18]

Substituting [4.18] to [4.11] and [4.12] gives,

$\lambda_2 = 0.168c^2, \lambda_4 = 0.025c^4$ [4.19]

Substituting [4.19] to the non-singularity conditions given in [3.14] for k=4, gives

Thus; $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

$0.885889077 > 0.6667$ [4.20]

From [4.20], [3.14] is satisfied, hence $G_2$ forms a SORD in four dimensions.

### 4.1.3 Construction of 48a Points SORD in Four Dimensions

The set of points which form a SORD of 48 points in four dimensions denoted by $G_3$ is given in [3.7]. Substituting $G_3$ to the moment conditions given in [3.13] gives;

$\sum x_{iu}^2 = 24f^2 + 2c_1^2 + 2c_2^2 = 48\lambda_2$ , [4.21]

$\sum x_{iu}^2 = 24f^4 + 2c_1^4 + 2c_2^4 = 144\lambda_4$ [4.22]

$\sum x_{iu}^2 x_{ju}^2$=16$f^4$ [4.23]

The excess function obtained from the moment conditions is given by;

$\sum x_{iu}^2 - 3\ \sum x_{iu}^2 x_{ju}^2 = 2c_1^4 + 2c_2^4 - 32f^4 = 0,$ [4.18]

Let $c_1^2 = x\, f^2$ , and $c_2^2 = y\, f^2$ . [4.19]

This implies that,

$x^2 + y^2 = \frac{1}{16}$ . [4.20]

Solving equation [4.20] gives;

$x = 0.05$, and $y = 0.245$ [4.21]

Substituting [4.21], gives

$a_1^2 = 0.05c^2$,$a_2^2 = 0.245c^2$ [4.22]

Substituting [4.22] to [4.15] and [4.16] gives,

$\lambda_2 = 0.168c^2, \lambda_4 = 0.025c^4$ [4.23]

Substituting [4.23] to the non-singularity conditions given in [3.14] for k=4, gives

Thus; $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

$0.885889077 > 0.6667$ [4.24]

From [4.24], [3.14] is satisfied, hence $G_3$ forms a SORD in four dimensions.

### 4.1.4 Construction of 48b points SORD in Four Dimensions

The set of points which form a SORD of 48 points in four dimensions denoted by $G_4$ is given in [3.8]. Substituting $G_4$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 4c^2 + 24p^2 = 48\lambda_2 \qquad [4.25]$$

$$\sum x_{iu}^4 = 4c^4 + 24p^4 = 144\,\lambda_4 \qquad [4.26]$$

$$\sum x_{iu}^2 x_{ju}^2 = 16p^4 \qquad [4.27]$$

The excess function is given by;

$$\sum x_{iu}^4 - 3\sum x_{iu}^2 x_{ju}^2 = 4c^4 - 24p^4 = 0 \qquad [4.28]$$

Solving [4.28] gives,

$$c^4 = 6p^4\ ,\ c^2 = 2.449489743p^2 \qquad [4.29]$$

Substituting [4.29] to [4.25] and [4.26] gives,

$$\lambda_2 = 0.70416666p^2 \text{and}\ \lambda_4 = 0.33333p^4 \qquad [4.30]$$

Substituting [4.30] to the non-singularity conditions given in [3.14] gives,

Thus; $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

$$0.672238647 > 0.6667 \qquad [4.31]$$

From [4.31], [3.14] is satisfied, hence $G_4$ forms a SORD in four dimensions.

### 4.1.5 Construction of 52 Points SORD in five Dimensions

The set of points which form a SORD of 52 points in four dimensions denoted by $G_5$ is given in [3.9]. Substituting $G_5$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 32a^2 + 2c_1^2 + 2c_2^2 = 52\lambda_2 \qquad [4.32]$$

$$\sum x_{iu}^4 = 32a^4 + 2c_1^4 + 2c_2^4 = 156\lambda_4 \qquad [4.33]$$

$$\sum x_{iu}^2 x_{ju}^2 = 32a^4 \qquad [4.34]$$

The excess function is given by,

$$\sum x_{iu}^4 - 3\sum x_{iu}^2 x_{ju}^2 = 2c_1^4 + 2c_2^4 - 64a^4 = 0 \qquad [4.35]$$

Solving [4.35] gives,

$$c_1^2 = 4\,a^2 \text{ and } c_2^2 = 4\,a^2 \qquad [4.36]$$

Since the values of x and y are real and positive, the set of points form a rotatable arrangement.

Substituting [4.36] to [4.32] and [4.33] gives

$$\lambda_2 = 0.923076923\,a^2, \text{ and } \lambda_4 = 0.615384615a^4 \qquad [4.37]$$

Substituting [4.37] to the non-singularity conditions given in [3.14] gives,

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$$

$$0.722222 > 0.714285714 \qquad [4.38]$$

Hence the non-singularity conditions are satisfied.

From [4.38], [3.14] is satisfied, hence $G_5$ forms a SORD in five dimensions.

### 4.1.6 Construction 74 Points SORD in Five Dimensions

The set of points which form a SORD of 74 points in four dimensions denoted by $G_6$ is given in [3.10]. Substituting $G_6$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 2c^2 + 32a_1^2 + 32a_2^2 = 74\lambda_2 \qquad [4.39]$$

$$\sum x_{iu}^2 = 2c^4 + 32a_1^4 + 32a_2^4 = 222\lambda_4 \quad [4.40]$$

$$\sum x_{iu}^2 x_{ju}^2 = 32a_1^4 + 32a_2^4 \quad [4.41]$$

The excess function for this is given by;

$$\sum x_{iu}^2 - 3\sum x_{iu}^2 x_{ju}^2 = 2c^4 - 64a_1^4 - 64a_2^4 = 0 \quad [4.42]$$

Solving [4.42] gives,

$$a_1^2 = 0.05\ c^2 \text{ and } a_2^2 = 0.169558249\ c^2 \quad [4.43]$$

Substituting [4.43] to [4.39] and [4.40] gives,

$$\lambda_2 = 0.121729729\ c^2 \text{ and } \lambda_4 = 0.013513513c^4 \quad [4.44]$$

Substituting [4.44] to the non-singularity conditions given in [3.14] gives,

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$$

$$0.911958276 > 0.714285714 \quad [4.45]$$

Hence the non-singularity conditions are satisfied.

From [4.45], [3.14] is satisfied, hence $G_6$ forms a SORD in five dimensions.

### 4.1.7 Construction of 100a points SORD in Five Dimensions

The set of points which form a SORD of Hundred points in four dimensions denoted by $G_7$ is given in [3.11]. Substituting $G_7$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 4c^2 + 64f^2 = 100\lambda_2 \quad [4.46]$$

$$\sum x_{iu}^4 = 4c^4 + 64f^4 = 300\ \lambda_4 \quad [4.47]$$

$$\sum x_{iu}^2 x_{ju}^2 = 48\ f^4 \quad [4.48]$$

The excess function is given by;

$$\sum x_{iu}^4 - 3\sum x_{iu}^2 x_{ju}^2 = 4c^4 - 80f^4 = 0 \qquad [4.49]$$

Solving [4.49] gives

$$c^4 = 20f^4, c^2 = 4.472135955f^2 \qquad [4.50]$$

Substituting [4.50] to [4.46] and [4.47] gives

$$\lambda_2 = 0.818885438f^2, \lambda_4 = 0.48f^4 \qquad [4.51]$$

Substituting [4.51] to the non-singularity conditions given in [3.14] gives,

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$$

$$0.715805351 > 0.714285714 \qquad [4.52]$$

From [4.52], [3.14] is satisfied, hence $G_7$ forms a SORD in five dimensions.

### 4.1.8 Construction of 100b Points SORD in Five Dimensions

The set of points which form a SORD of Hundred points in four dimensions denoted by $G_8$ is given in [3.12]. Substituting $G_8$ to the moment conditions given in [3.13] gives;

$$\sum x_{iu}^2 = 2c_1^2 + 2c_2^2 + 64p^2 = 100\lambda_2 \qquad [4.53]$$

$$\sum x_{iu}^4 = 2c_1^4 + 2\, c_2^4 + 64p^4 = 300\,\lambda_4 \qquad [4.54]$$

$$\sum x_{iu}^2 x_{ju}^2 = 48f^4 \qquad [4.55]$$

The excess function is given by,

$$\sum x_{iu}^4 - 3\sum x_{iu}^2 x_{ju}^2 = 2c_1^4 + 2\, c_2^4 - 80f^4 = 0 \qquad [4.56]$$

Solving [4.56] gives

$$c_1^2 = 5f^2, c_2^2 = 3.872983546f^2 \quad [4.57]$$

Substituting [4.57] to [4.53] and [4.54] gives,

$$\lambda_2 = 0.81745967f^2, \lambda_4 = 0.48000001f^4 \quad [4.58]$$

Substituting [4.58] to [3.14] gives,

Thus; $\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2}$

$$0.718304465 > 0.714285714 \quad [4.59]$$

From [4.59], [3.14] is satisfied, hence $G_8$ forms a SORD in five dimensions.

## 4.2 Optimality Criteria for the Second Order Rotatable Designs in Four Dimensions

### 4.2.1 D-Optimality criteria

#### 4.2.1.1 D-criterion for 32 Points SORD in Four Dimensions

Substituting the moments given in [4.9] to the generalized D- criterion given in [3.17] gives,

$D_1$= 1.017382475 [4.60]

#### 4.2.1.2 D-criterion for 40 Points SORD in Four Dimensions

Substituting the moments given in [4.19] to the generalized D- criterion given in [3.17] gives,

$D_2$= 0.299064511 [4.61]

#### 4.2.1.3 D-criterion for 48$a$ Points SORD in Four Dimensions

Substituting the moments given in [4.23] to the generalized D- criterion given in [3.17] gives,

$D_3$= 0.957935406 [4.62]

#### 4.2.1.4 D-criterion for 48$b$ Points SORD in Four Dimensions

Substituting the moments given in [4.30] to the generalized D- criterion given in [3.17] gives,

$D_4$= 0.613217973 [4.63]

#### 4.2.1.5 D-Criterion for 52 Points SORD in Five Dimensions

Substituting the moments given in [4.37] to the generalized D- criterion given in [3.17] gives,

$D_5$= 1.687429322 [4.64]

#### 4.2.1.6 D-criterion for 74 Points SORD in Five Dimensions

Substituting the moments given in [4.44] to the generalized D- criterion given in [3.17] gives,

$D_6$= 0.14796891 [4.65]

#### 4.2.1.7 D-criterion for 100$a$ Points SORD in Five Dimensions

Substituting the moments given in [4.51] to the generalized D- criterion given in [3.17] gives,

$D_7$= 1.003415668 [4.66]

#### 4.2.1.8 D-criterion for 100$b$ Points SORD in Five Dimensions

Substituting the moments given in [4.51] to the generalized D- criterion given in [3.17] gives,

$D_8$= 1.009044 [4.67]

### 4.2.2 T- Criterion

#### 4.2.2.1 T- Criterion for 32 Points SORD in Four Dimensions

Substituting the moments given in [4.9] to the generalized T- criterion given in [3.23] gives,

$T_1$= 2.790177143 [4.68]

#### 4.2.2.2 T- Criterion for 40 Points SORD in Four Dimensions

Substituting the moments given in [4.19] to the generalized T- criterion given in [3.23] gives,

$T_2$= 0.792 [4.69]

#### 4.2.2.3 T- Criterion for 48$a$ Points SORD in Four Dimensions

Substituting the moments given in [4.23] to the generalized T- criterion given in [3.23] gives,

$T_3$= 2.226115577 [4.70]

#### 4.2.2.4 T- Criterion for 48$b$ Points SORD in Four Dimensions

Substituting the moments given in [4.30] to the generalized T- criterion given in [3.23] gives,

$T_4$= 2.225335575 [4.71]

#### 4.2.2.5 T- Criterion for 52 Points SORD in five Dimensions

Substituting the moments given in [4.37] to the generalized T- criterion given in [3.23] gives,

$T_5$= 3.307692306 [4.72]

#### 4.2.2.6 T- Criterion for 74 Points SORD in five Dimensions

Substituting the moments given in [4.44] to the generalized T- criterion given in [3.23] gives,

$T_6$ = 0.270332043 [4.73]

#### 4.2.2.7 T- Criterion for 100$a$ Points SORD in five Dimensions

Substituting the moments given in [4.51] to the generalized T- criterion given in [3.23] gives,

$T_7$ = 2.2667934536 [4.74]

#### 4.2.2.8 T- Criterion for 100$b$Points SORD in five Dimensions

Substituting the moments given in [4.58] to the generalized T- criterion given in [3.23] gives,

$T_8$ = 2.667119811 [4.75]

### 4.2.3 A-Criterion

#### 4.2.3.1 A-Criterion for 32 Points in Four Dimensions

Substituting the moments given in [4.9] to the generalized A- criterion given in [3.18] gives,

$A_1$ = 0.267994363 [4.76]

#### 4.2.3.2 A-Criterion for 40 Points SORD in Four Dimensions

Substituting the moments given in [4.19] to the generalized A- criterion given in [3.18] gives,

$$A_2 = 0.031294322 \qquad [4.77]$$

#### 4.2.3.2 A-Criterion for 48$a$ Points SORD in Four Dimensions

Substituting the moments given in [4.23] to the generalized A- criterion given in [3.18] gives,

$$A_3 = 0.\ 322796038 \qquad [4.78]$$

#### 4.2.3.3 A-Criterion for 48$b$ Points SORD in Four Dimensions

Substituting the moments given in [4.30] to the generalized A- criterion given in [3.18] gives,

$$A_4 = 0.046755676 \qquad [4.79]$$

#### 4.2.3.4 A-Criterion for 52 Points in Five Dimensions

Substituting the moments given in [4.37] to the generalized A- criterion given in [3.18] gives,

$$A_5 = 0.087841699 \qquad [4.76]$$

#### 4.2.3.5 A-Criterion for 74 Points SORD in Five Dimensions

Substituting the moments given in [4.44] to the generalized A- criterion given in [3.18] gives,

$$A_6 = 0.00269422 \qquad [4.77]$$

#### 4.2.3.6 A-Criterion for 100$a$ Points SORD in Five Dimensions

Substituting the moments given in [4.51] to the generalized A- criterion given in [3.18] gives,

$A_7$ = 0. 017336505 [4.78]

#### 4.2.3.7 A-Criterion for 100$b$ Points SORD in Five Dimensions

Substituting the moments given in [4.58] to the generalized A- criterion given in [3.18] gives,

$A_8$ = 0.041602646 [4.79]

### 4.2.4 E-Criterion

#### 4.2.4.1 E-Criterion for 32 points SORD in Four Dimensions

Substituting the moments given in [4.9] to the generalized E- criterion given in [3.20] gives,

$\gamma = 3.031248$ or $0.84375$ or $12.37532819$or $0.749663815$ [4.80]

Taking the smallest value gives;

$E_1$= 0.749663815 [4.81]

#### 4.2.4.2 Criterion for 40 Points SORD in Four Dimensions

Substituting the moments given in [4.19] to the generalized E- criterion given in [3.20] gives,

$\gamma = 3.395268711$ or $0.0047312895$ or $0.6$or $0.386$

Taking the smallest value gives;

$E_2$= 0.0047312895 [4.82]

**4.2.4.3 E-Criterion for 48$a$ Points SORD in Four Dimensions**

Substituting the moments given in [4.30] to the generalized E- criterion given in [3.20] gives,

$$\gamma = 10.24839247 \text{ or } 0.06360753 \text{ or } 2.328 \text{or } 0.73570226$$

Taking the smallest value gives;

$E_3$ = 0.06360753 [4.83]

**4.2.4.4 E-Criterion for 48$b$ Points SORD in Four Dimensions**

Substituting the moments given in [4.37] to the generalized E- criterion given in [3.20] gives,

$$\gamma = 3.360594344 \text{ or } -2.36 \text{ or } 1.99 \text{ or } 0.704$$

Taking the smallest value gives;

$E_4$= 0.704 [4.84]

**4.2.4.5 E-Criterion for 52 Points SORD in Five Dimensions**

Substituting the moments given in [4.37] to the generalized E- criterion given in [3.20] gives,

$$\gamma = 22.5274351 \text{ or } 0.010564895 \text{ or } 0.9231 \text{ or } 6.154$$

Taking the smallest value gives;

$E_5$= 0.010564895 [4.85]

**4.2.4.6 E-Criterion for 74 Points SORD in Five Dimensions**

Substituting the moments given in [4.44] to the generalized E- criterion given in [3.20] gives

$\gamma = 1.372604545$ or $0.005773875$ or $0.121729729$ or $0.13513515$

Taking the smallest value gives;

$E_6$= 0.005773875 [4.86]

**4.2.4.7 E-Criterion for 100$a$ Points SORD in Five Dimensions**

Substituting the moments given in [4.51] to the generalized E- criterion given in [3.20] gives

$\gamma = 17.79662857$ or $0.003371425$ or $0.8182$ or $4.8$

Taking the smallest value gives;

$E_7$= 0.003371425 [4.87]

**4.2.4.8 E-Criterion for 100$b$ points SORD in Five Dimensions**

Substituting the moments given in [4.58] to the generalized E- criterion given in [3.20] gives

$\gamma = 17.79471753$ or $0.005282465$ or $0.8182$ or $4.8$

Taking the smallest value gives;

$E_8$= 0.005282465 [4.88]

### 4.2.5 Discussion

**Table 1: Moments and Non-singularity conditions for SORDs in Four and Five dimensions**

| Design | Number of points | Number of factors | $\lambda_2$ | $\lambda_4$ | $\frac{\lambda_4}{\lambda_2^2}$ | $\frac{k}{k+2}$ |
|---|---|---|---|---|---|---|
| $G_1$ | 32 | 4 | 0.84375 | 0.505208 | 0.70900000 | 0.6666667 |
| $G_2$ | 40 | 4 | 0.16800 | 0.025000 | 0.88588907 | 0.6666667 |
| $G_3$ | 48a | 4 | 0.16800 | 0.025000 | 0.88586907 | 0.6666667 |
| $G_4$ | 48b | 4 | 0.70417 | 0.333333 | 0.67223865 | 0.6666667 |
| $G_5$ | 52 | 5 | 0.92308 | 0.615385 | 0.72222222 | 0.7142857 |
| $G_6$ | 74 | 5 | 0.12173 | 0.013514 | 0.91195828 | 0.7142857 |
| $G_7$ | 100a | 5 | 0.81889 | 0.480000 | 0.71580535 | 0.7142857 |
| $G_8$ | 100b | 5 | 0.81746 | 0.480001 | 0.71830447 | 0.7142857 |

Table 1. Above gives a summary on the construction of the SORDs in four and five dimensions. For all the designs under consideration, the sets of points satisfied both the moment conditions and the non-singularity conditions for second order rotatability. These designs have relatively lower number of points compared to those constructed by Mutiso (1998).

**Table 2: Optimality criteria for SORD in four and five dimensions**

| Design | Number of points | Number of factors | D-Criterion | A-Criterion | E-Criterion | T-Criterion |
|---|---|---|---|---|---|---|
| $G_1$ | 32 | 4 | 1.0173824 | 0.26799436 | 0.74966382 | 2.70177143 |
| $G_2$ | 40 | 4 | 0.2990645 | 0.03129432 | 0.00473128 | 0.7920000 |
| $G_3$ | 48a | 4 | 0.9579351 | 0.32279604 | 0.06360753 | 2.2261155 |
| $G_4$ | 48b | 4 | 0.613218 | 0.04675568 | 0.704 | 2.2253355 |
| $G_5$ | 52 | 5 | 1.6874293 | 0.08784169 | 0.01056489 | 3.3076923 |
| $G_6$ | 74 | 5 | 0.1479689 | 0.00269422 | 0.00757738 | 0.270332043 |
| $G_7$ | 100a | 5 | 1.0034157 | 0.01733650 | 0.00337143 | 2.266793454 |
| $G_8$ | 100b | 5 | 1.009044 | 0.04160265 | 0.00528247 | 2.266711981 |

From the Table 2 the design $G_2$ (40 points SORD in four dimensions) is A-, D-, T- and E-optimal. The design $G_6$ (74 points SORD in five dimensions) is A-, D- and T-optimal and designs $G_7$ was found to be E-optimal. There were two pairs of designs with equal number of points, i.e. the 48a, 48b and the 100a & 100b design points. Interestingly, one pair; (100a) points was identified with E-optimality criteria. For an experimenter to choose a better design from this pair, the least optimality value would be used, hence the 100a points design was optimum.

**Table 3: Efficiencies for SORD in four and five dimensions**

| Design | Number of points | Number of factors | D-Criterion | A-Criterion | E-Criterion | T-Criterion |
|---|---|---|---|---|---|---|
| $G_1$ | 32 | 4 | 29.4% | 11.67% | 0.63% | 29.33% |
| $G_2$ | 40 | 4 | 100 % | 100% | 100% | 100% |
| $G_3$ | 48a | 4 | 31.22% | 6.39% | 7.44% | 35.58% |
| $G_4$ | 48b | 4 | 48.76% | 66.93% | 0.67% | 35.59% |
| $G_5$ | 52 | 5 | 8.77% | 3.07% | 31.91% | 8.16% |
| $G_6$ | 74 | 5 | 100% | 100% | 44.49% | 100% |
| $G_7$ | 100a | 5 | 14.75% | 15.54% | 100% | 11.93% |
| $G_8$ | 100b | 5 | 14.66% | 6.48% | 63.82% | 11.92% |

From Table 3, designs $G_1, G_2, G_3\&G_4$ were SORDs in four dimensions while designs $G_5, G_6, G_7\&G_8$ were SORDs in five dimensions. The forty points design, denoted by $G_2$ was found to be the most efficient in all criteria for the designs in four dimensions. For the five dimensions design, the 74 points design denoted by $\mathbf{G_6}$ was the most efficient for all criteria except E-criterion which gave 100 points design denoted by $\mathbf{G_7}$ as the most efficient design.

This technique is important for establishing the most suitable design when faced with the problem of choosing from several competing designs. Designs with higher efficiencies are considered to be more desirable compared with those of relatively low efficiencies.

# CHAPTER FIVE

# CONCLUSIONS AND RECOMMENDATIONS

## 5.0 Introduction

This chapter covers the conclusions and recommendations drawn from the study.

## 5.1 Construction of Economical SORDs in Four and Five Dimensions

In research, experiments must be performed at predetermined levels of controllable factors. Meaning that an experimental design must be selected before the experiment takes place, once an experimenter has chosen a polynomial model of suitable order, the problem arises on how best to choose the settings for the independent variables over which he has control. A particular selection of settings or factor levels at which observations are to be taken is called a design. A design may become inappropriate under special circumstances requiring an increase in factors or levels to make it more desirable. In agriculture for instance, continuous cultivation of crops may exhaust the previously available mineral elements necessitating a sequential appendage of the mineral elements which become deficient in the soil over time. The study will help in the appendage of such elements when the experimenters are faced with scarcity of resources.

The four SORDs in three dimensions initiated by Draper (1960) and developed by Mutiso (1998) were considered. The designs were extended to four dimensions by appending a factor to each that gave; 32, 40, 48a, and 48b points respectively. The designs in four dimensions were further extended to five dimensions to give; 52, 74,100a, and 100b points SORDs respectively. Interestingly, the designs under consideration have relatively a lower number of points as compared to the SORDs in four and five dimensions considered by Mutiso (1998). This gives an economical option for experimenters with scarce resources.

### 5.1.1 Recommendations

There exists a lot of information on design construction. It will be ideal if the evaluated on their robustness for missing data.

## 5.2 Evaluation for the Optimality Criteria

Each of the (A-, D-, E-, and T-) optimality criteria demands a specific statistical property of the best linear unbiased estimator and it amounts to the minimization of a particular Eigen values of C-matrix. The results for evaluation of optimality criteria for the designs under investigation are summarized in the table 2. Evaluation of the particular criteria for the 4 designs in four dimensions were E- optimal except the $G_1$ which was found to be A- optimal. The rest of the SORDs in five dimensions were all E-optimal, except $G_6$ which was found to be A-optimal. Taking the smallest value among the matrix means, identifies the optimality criteria. There is a clear indication that the more homogeneous the design is, the larger the information matrix as evidenced by the values of the information functions. The A-Optimality criteria minimizes the sum or average of the variance of parameter estimates (Atkinson & Donev, 1992). Similarly, the E- Optimum designs reduces the variance of each individual parameter estimates. The Eigen values of the inverse of the information matrix are proportional to the squires of lengths of the axes of the confidence ellipsoid.

### 5.2.1 Applications of the Optimality Criteria

Optimum designs are essential in various fields such as medicine, agriculture and industry, for example, in the manufacture of a certain type of drug, numerous kinds of factors are combined together in different amounts in order to obtain the most effective drug.

### 5.2.2 Recommendations on the optimality Criteria.

This study recommends that the other optimality criteria such as the G-, IV-, U-, DT-, CD- and CDT- Criteria be established for the economical SORDs in this study.

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