OPTIMIZING COMPRESSIVE STRENGTH OF PLINTH CONCRETE MIX USING QUAD-AXIAL WEIGHTED SIMPLEX CENTROID DESIGNS AND SECOND ORDER KRONECKER MODEL

By

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DECLARATION

Declaration by Candidate

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DEDICATION

Am pleased to dedicate this work to my husband Francis Njoroge and my children Fredrick Miring'u, Ednah Wanjiru, Eudias Njeri and Franklin Kiratu. Also to my mother Monicah Njeri and my late father Mr. Stanley Kiratu Ng'ang'a.

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ABSTRACT

In the construction industry, Engineers, Quantity surveyors and other stakeholders work towards obtaining civil structures with desired compressive strength at minimum costs. In Kenya and many parts of the world, many cases of collapsing buildings causing fatal damage are reported from time to time. A study by the National Construction Authority of Kenya () attributed the causes of these collapses, among others to the compromised concrete mixes. Researches done on optimizing concrete mixes have dealt with three variables, Cement, Sand (fine aggregate) and Ballast (coarse aggregate), while keeping water constant. This study was geared to find a procedure for the optimal strength of M25 class of concrete to mitigate the collapsing of buildings. This included water a major contribution of strength as one of the variables for finding the optimal mix for the said class. To actualize the study, an experiment was conducted in the concrete laboratory at the Jomo Kenyatta University of Agriculture and Technology (JKUAT). The main objective was to obtain a statistical model using second order Kronecker model and a quad-axial weighted simplex centroid designs, satisfying the D- G- and I-optimality tests performed in order to locate the optimum values of the design. The specific objectives for the study were to construct an optimal inscribed tetrahedral weighted simplex centroid design. To evaluate its D-, G- and I- optimality conditions using the $\mathcal{H}$-Invariant matrices with two weighted designs namely; equally and unequally weighted simplex centroid axial design (EWSCAD) and (UWSCAD). To perform a concrete mixture experiment using the design and to fit a second-degree Kronecker model for the experiment. Finally, to obtain the optimal mix for the experiment and to evaluate its optimality conditions. The study applied Response Surface Methodology (RSM). The results revealed that the centroid obtained the best D- and G-optimal values. The UWSCAD was D-efficient while EWSCAD was G-efficient. I-optimality of the two designs occurred at similar design points. The concrete model obtained the same optimality conditions as the adopted design. The second-degree Kronecker model fitted showed that the adjusted R-squared was 0.9951. The variance inflation factors (V.I.F) for the squared portions and the interactions were 3.3344 and 5.0891 respectively, hence no serious multi-collinearity problem. The descriptive statistics showed the distribution of the experiment outcomes while the contours and the response surfaces showed the effect on compressive strength due to interactions of two components. The response trace plot revealed the optimal point for the ratio water: cement: sand: ballast as 0.52:1:1.4:2.8 for the optimal compressive strength of $27.63 N/mm^2$for the M25 class. In conclusion, the model obtained was appropriate in estimating the optimal ratios since it occurred in the class of interest. The study therefore recommends that the procedure used for this study be applied in search of the concrete mixing ratio for the construction of the plinth of M25 class, for the four components which may differ due to the source of variables.

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ABBREVIATIONS AND ACRONYMS

ANN	Artificial Neural Networks.
ANOVA	Analysis Of Variance
BS	British Standard
CAC	Coarse Aggregate Content
CCD	Central Composite Design
COST	Concrete Optimization Software Tool
CMD	Concrete Mix Design
EWSCAD	Equally Weighted Simplex Centroid Axial Design
FAC	Fine Aggregate Content
FD	Factorial designs
GA	Genetic Algorithm
GDP	Gross Domestic Product.
GL(s)	General Linear Group
HPC	High performance concretes
IFRC	International Federation of Red Cross and Red Crescent societies.
IS	Indian Standard
JKUAT	Jomo Kenyatta University of Agriculture and Technology
MDG	Millennium Development Goals
NND	None Negative Definite
RSM	Response Surface Methodology
SDG	Sustainable Development goals

TAC	Total Aggregate Content
UHPC	Ultra high performance concrete
UWSCAD	Unequally Weighted Simplex Centroid Axial Design
VIF	Variance Inflation Factor

CHAPTER ONE

INTRODUCTION

This chapter laid the foundation for the study by explaining some of the important concepts that were used in the study. It outlined the following; the background, the statement of the problem, the justification, the purpose, the general and specific objectives, the scope, the significance, delimitations, limitations and assumptions of the study. The chapter two covered the related studies, chapter three the methodology, chapter four the results and discussions and in chapter five, the conclusions and recommendations.

1.1 BACKGROUND OF THE STUDY

The Sustainable Development Goals (SDG) of 2015, which replaced the Millennium Development Goals (MDG) of 2000, targeted to reduce by half the one billion people living in slums in the world by provision of decent housing. The Vision 2030 of the Kenyan government launched in 2002, aimed at providing decent housing to the citizens especially those living in the slums and one of the 'Big Four Agenda' for the Kenyan government for the period 2017-2022 was to provide affordable housing for citizens. All these goals confirm the need for research on affordable concrete mix. This study reviewed literature and incidences that are currently affecting construction industry. The study looked into depth some relevant studies on concrete development and the methods used. The history of the problem that has prompted the study was outlined from several reports, hence accelerating the desire to model the optimal compressive strength of plinth of affordable houses.

1.1.1 Response Surface Methodology

(Montgomery, 1997) defined Response Surface Methodology as a collection of Mathematical and Statistical techniques that are useful for the modeling and analysis of problems in which a response of interest is influenced by several variables and the objective is to optimize this response.

Given two variables $x_1 and\ x_2$ the process yield function is given by,

$$y = f(x_1, x_2) + \varepsilon \tag{1.1}$$

Where $\varepsilon \sim N(0, \sigma^2)$ represents the noise or error observed in the response y. Errors are assumed to be identically and independently distributed with a constant variance. The Response surface methodology then, is a collection of experimental strategies, mathematical methods and statistical inferences, which enables an experimenter to make efficient empirical exploration of the system of interest according to Box and Wilson (1951). Response Surface Methodology is useful for developing, improving and optimizing processes and therefore it is the performance measure of a given process. The response surface is defined by the expected value denoted by

$$\eta = f(x_1, x_2) \tag{1.2}$$

1.1.2 Mixture experiments

An experiment is a process or a study that results in the collection of data. Usually statistical experiments are conducted in situations which the researcher can manipulate the conditions of the experiment and can control the factors that are irrelevant to the stated research objectives, in order to minimize errors. A mixture experiment is a process that involves mixing of proportions of two or more components to make different compositions of the final product, (Cornell, 2002)

In mixture problems, the purpose of an experiment is to model the blending surface with some form of mathematical equation called a regression function, to find the most appropriate mixture with respect to a well-defined response variable and an optimality criterion. The assumptions of regression models also hold for the mixture problems, including that the response surface is continuous over the region under study.

Mixture experiments that relate to the component proportions of a mixture were discussed first by (Quenouille, 1953) and later by (Scheffe, 1958), (Scheffé, 1963)The conditions of a mixture experiment with q components according to (Scheffe, 1958)are:

$$\sum_{i=1}^{q} x_i = 1, \quad 0 \leq x_i \leq 1, \; i = 1, 2, \dots, q \tag{1.3}$$

where q is the number of components in the experiments.

A mixture experiment then, is a special type of a response surface experiment in which the factors are the ingredients or components of the mixture. The response is a function of the proportions of each ingredient. The proportional amounts of each ingredient are measured by either weight, volume, mole or ratio and any other method as described by Myers et.al. (2009). Mixture experiments are useful in the study of quality of products such as paint, alloys, glass, animal feeds, concrete and many others, which depends mainly on the combinations of the components that maximizes the quality.

Cornell (1990) described a mixture amount experiment as one that is performed at two or more levels of the total amount. The response in this case is deemed to depend on the proportions of the given ingredients and at the same time the amount of the blend. These kinds of mixtures increases production when the amount levels are increased and the vice versa. That is, response changes by varying the amounts of ingredients applied. For example, if a certain fertilizer is applied in greater amounts up to some specified level, the more the production of the yield in terms of size and weight, and beyond the specified level, other effects would set in. The designs for fitting mixture amount models called the mixture-amount designs were developed by (Piepel & Cornell, 1987). Mixtures in some constituent proportions were discussed in this study.

1.1.3 An experimental design

An experimental design is the process of planning a study to meet specified objectives. A good design is effective if it enables one to obtain sufficient data to fit an interpolating model that provides unbiased predictions with sufficient precision. It is efficient if it enables one to obtain the most precise estimates for a given budget on the number of runs Cornell (1990). There are many and diverse experimental designs depending on the desire of the experimenter and the expected outcomes. Some designs relevant to this study are discussed here below.

1.1.4 Simplex Centroid Design

(Scheffé, 1963) introduced the q- component Simplex Centroid Designs as the designs where

$2^q - 1$ points are located at the vertices, the edges, on the faces and at the Centre of the $q - 1$ dimension simplex. The q-component Simplex Centroid Design involves;

$2^q - 1 = q + \binom{q}{2} + \cdots + \binom{q}{r} + \cdots + 1$ distinct design points in total. There are q pure components $(1,0,0, \dots ,0)$ which occur at the vertices, the $\binom{q}{2}$ permutations of binary mixtures $(\frac{1}{2}, \frac{1}{2}, 0,0, \dots ,0)$, the $\binom{q}{3}$ permutations of ternary mixture $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0,0 \dots ,0)$, the $\binom{q}{4}$ permutation of quaternary mixture $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0,0, \dots ,0)$, up to mixtures involving $(\frac{1}{q}, \frac{1}{q}, \frac{1}{q}, \dots , \frac{1}{q})$of equal proportion of all q components or q-nary mixtures.

1.1.5 An Axial Simplex Centroid Design

An Axial design is a design that consists mainly of complete or q component blends where most of the points are located inside the simplex. Axial designs are recommended to be used when component effects are to be measured, and in screening experiments. The simplest form of the axial design is one whose points are equidistant from the Centroid $(\frac{1}{q}, \frac{1}{q}, \frac{1}{q}, \dots, \frac{1}{q})$, and has q axes Cornell (2002). While (Sinha, Das, Mandal, & Pal , 2014)described axial designs as the designs with interior points $x_i = 0, x_j = \frac{1}{1-q}$ $, \forall\, j \neq i$ and $x_i = 1, x_j = 0, \forall\, j \neq i,$ which contains the points of the form $[\frac{1+(q-1)\Delta}{q}, \frac{1-\Delta}{q}, \dots, \frac{1-\Delta}{q}]$ and its permutations $\frac{-1}{q-1} < \Delta < 1$.

1.1.6 Kronecker models

(Draper, Heiligers, & Pukelsheim, 1998) proposed K-models or Kronecker models. The K-models are alternative representations of the Scheffe mixture models, so referred to as the S- models. The K-models involves the Kronecker square multiplication $t \otimes t$, which results to a $q^2 \times 1$ vector of squares and cross products of the components in lexicographic order of subscripts given as;

$$t \otimes t = (t_1 t_1, \dots, t_1 t_q, t_2 t_1, \dots, t_2 t_q, \dots, t_q t_1, \dots, t_q t_q) \tag{1.4}$$

The polynomial function related to the K-model is

$$E[Y_{ij}] = \sum_{i=1}^{q} \sum_{j=1}^{q} t_i t_j \theta_{ij} = (t \otimes t\,)'\theta \tag{1.5}$$

In the view of (Prescott, Dean, Draper, & Lewis, 2002)K-models are better models than the Scheffe models since they have increased symmetry due to the repletion of cross product terms that results to larger moment matrices, compact notations and homogeneous model functions. The model is also less susceptible to ill conditioning and the associated problem of unstable models with highly correlated parameters and large standard error.

1.1.7 Optimization

(Castro , Silva, Tirapegui , Borsato, & Bona, 2003) defined optimization as the choice of the best alternative starting from a specified set of possibilities, in development of a formula that entails optimization. This is the process of determining the optimum levels of the components or the key ingredients of a given mixture. These components are the independent variables and response is the dependent variable to be optimized.

(Wanida, Chomtee , & Borkowski, 2019), defined a mixture experiment as a special case of the response surface design that aims to optimize the product by combining several components of a mixture. They outlined that the optimal designs are constructed by specifying the model; this is done by choosing an optimality criterion and using an algorithm to select n design points from a finite set of candidate points. They proposed a Genetic Algorithm (GA) to create a mixture design with a robustness property. Effectiveness of such an algorithm depended on the choice of design region, encoding scheme, evaluation function and genetic operators.

In this study, optimization entailed developing a design, to model the key components (independent variables) that contributed to the best response of the concrete mix. This study's expectation was to optimize the concrete components namely; Water, Cement, Fine aggregate (Sand) and Coarse aggregate (Ballast), which resulted to a concrete mix with optimized compression strength for an affordable plinth of low cost houses.

1.2 STATEMENT OF THE PROBLEM

Kenya has a long history of collapsing buildings. A study by the National Construction Authority (2019), determined that construction industry is one of the drivers of economic growth and a major pillar of the Vision 2030. They outlined the following as some of the causes of building failures; erroneous building designs, poor workmanship, use of substandard materials, non-compliance with building standards and regulations, poor mix ratios, lack of supervision by professionals and many others. They were optimistic that their report would be a useful source of information to the National government, County governments and relevant stakeholders in addressing the problem of collapsing buildings.

Many researchers using different methods have tackled the problem of optimization of concrete mixtures. (Alabi, Olanitori , & Afolayan, 2012) used a linear programming technique. Their research highlighted the properties of pit-sand produced in Akure South West Nigeria, commonly used as fine aggregate and the properties of resulting concrete. They researched into the characteristics and behavior of locally available aggregates, which were expected to improve the knowledge of Structural and Civil Engineers who made use of these aggregates and hence the concretes.

(Comput , 1999) used the neural networks, his paper aimed at demonstrating the possibilities of adapting Artificial Neural Networks (ANN) to predict the compressive strength of high-performance concrete. The study concluded that, a strength model based on ANN is more accurate than a model based on regression analysis and that, it is convenient and easy to use ANN models for numerical experiments to review the effects of the proportions of each variable on the concrete mix.

In their study, (Okere, Onwuka, Onwuka, & Arimanwa, 2013)used a mathematical model based on simplex method formulated for optimization of concrete cube strength. The model aimed at providing all possible mix ratios that could yield the desired concrete cube strength.

This study, in pursuit of mitigating the menace of collapsing of concrete structures, employed the second-order Kronecker model using the Simplex methods to optimize the slope of the response surface of a weighted simplex centroid design. This was for deriving a procedure for obtaining an optimal concrete mix for the plinth with an optimal compressive strength of the desired class of concrete meant for low cost houses.

1.3 JUSTIFICATION OF THE STUDY

The study was prompted by the frequent cases reported on collapsing of buildings. This called for an in-depth investigation on the causes of collapsed concrete structures. One of the major findings was that most of the collapsed buildings caused fatal injuries on human life as enumerated.

IFRC (International Federation of Red Cross and Red Crescent societies) organization on 23 January 2006, reported with the headline "Buried alive as building collapses in Kenya". The Daily Nation of Kenya reported on 9 June 2012 about a collapsed building in Mlolongo Estate in Athi-River, where some people died and many others severely injured. The cause of the collapse was said to be due to rusting of metal bars since the foundation was flooded with water.

On Monday May 2 2016, Kenya Daily Nation wrote on the "History of collapsed buildings in Kenya", about a building that had collapsed in Huruma Estate, Nairobi on April 29 2016.

The BBC news on May 6 2016 reported on "Demolition of unfit buildings" in Kenya where 42 people were killed and that these unfit buildings were homes for more than 600 people.

A live video by the Kenya Citizen Television in August 2 2016 showed a five-storrey building collapsing in the Kariobangi South Estate in Nairobi.

On 6 December 2019 in Tassia Estate Nairobi Kenya, a building collapsed causing fatalities, this being a report by the A4 architects.

New York Times conducted a research and reported on June 13 2017 that about 58% of residential houses were unfit for habitation in Kenya.

All these reports confirmed that our study was relevant to what is happening to the construction industry.

1.4 PURPOSE OF THE STUDY

The purpose of this research is to contribute to the knowledge of securing good and strong buildings. Some of the reasons outlined in the study by National Construction Authority of Kenya (NCAK, 2019)on the causes of buildings failure in Kenya included; poor workmanship, use of substandard materials, noncompliance with building standards and regulations for instance safety requirements, materials mix ratios, lack of building approvals and others. Inadequate structural design, overloading of the structure and inadequate maintenance.

Other causes outlined by the British Broadcasting corporation (BBC) news reports on collapsing of buildings were:

a) Foundations were weak which resulted from poor soils that were not fit to hold high-rise buildings.

b) Building materials are not strong enough. Some contractors fail to follow directives by the civil engineers and reduce the amount of materials, especially cement which is the most important ingredient in the strength of concrete.

c) Workers (builders) make mistakes. The supervisors and contractors sometimes overlook rules of construction.

d) Load is heavier than expected – In Nigeria a church building collapsed killing hundreds of people on 12th September2014 reason being that the people were heavier than the building could hold.

e) Lack of testing the concrete strength. It is one of the construction requirements that samples of concrete should be tested for strength in a concrete laboratory.

The Guardian paper and CNN on 23rd January 2006 reported that a building in Nairobi's Central Business District (CBD) collapsed and that the owners had not obtained the occupancy permit. Others general reports showed that the construction industry was the second most corrupt sector in the country then.

Section 32 of Building Code 1968 of Kenya specifies that materials should be; of suitable nature and quality for the purpose, which they are used, adequately prepared and mixed according to the British standards, applied in the proper manner. Section 34, requires that materials need to be tested to ensure they meet a certain threshold.

This study sought to correct some of the mistakes analyzed above by following the construction guidelines. Several concrete cubes were constructed and tested for their respective compressive strength according to the adopted statistical design.

1.5 OBJECTIVES OF THE STUDY

1.5.1 General objective

This study aimed at optimizing the compressive strength of plinth concrete mix for low-cost houses using second-degree Kronecker model for quad-axial weighted simplex centroid design, which meets the D-, G- and I- optimality criteria.

1.5.2 Specific Objectives

The specific objectives of the study were to:

i) Construct an inscribed tetrahedral Weighted Simplex Centroid Design (WSCD).
ii) Evaluate the D-, G- and I- optimality criteria of the design using two weighted simplex centroid designs.
iii) Perform a concrete mixture experiment and fit a second-degree Kronecker model.
iv) Find the optimal mix and to evaluate the D-, G-, and I-optimality criteria for the experiment model.

1.6 SCOPE OF THE STUDY

The study restricted itself to a four-component experiment was conducted in the area of concrete mixes, these were Water, Cement, Fine aggregate (Sand) and the Coarse aggregate (Ballast). These components were represented by $t_i,\ i = 1,2,3,4$ in this study.

The experimental domain τ for the mixture components used was as indicated by Draper and (Pukelsheim, 1993) was given by equation (1.6).

$$\tau = \{t \in [0,1]^m : 1'_m t = 1\} \qquad (1.6)$$

They too proposed the Kronecker polynomial regression model, consisting of squares and cross products of the components of t in Lexicographic order of the subscripts. They called it the K-model expressed as equation (1.5), and proposed the set of weighted centroid designs for a four-ingredient second-degree model as given in equation (1.7).

$$C = \{\alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4 : (\alpha_1, \alpha_2, \alpha_3, \alpha_4)' \in \tau\} \quad (1.7)$$

The information matrix that was used was obtained by utilizing, the H-invariant matrix as in the equation (1.8).

$$D_c = HCH' \quad (1.8)$$

The matrix D_c is an improved information matrix obtained from the information matrix $C = C_k(M)$ and $H,$ the slope or the derivative matrix of the design matrix. These were applied in order to obtain the response surface with an optimal slope.

The D-, G- and I optimal values for Equally Weighted Simplex Centroid Axial Design (EWSCAD) and Unequally Weighted Simplex Centroid Axial Design (UWSCAD) were then evaluated to assess suitability of the design and the concrete model.

The data for this study was obtained from an experiment that was carried out at the Jomo Kenyatta University of Agriculture and Technology (JKUAT) in the months of September and October 2019.

The study used R-Gui (4.0.0) for analysis of data and in the computation of the D-, G- and I-optimality criteria values and construction of the response surfaces. Design expert 12 and Excel helped to develop the normality plot and the response trace plot respectively.

1.7 SIGNIFICANCE OF THE STUDY

Housing is one of the parameters that determine the Gross Domestic Product (GDP) of a country. According to the Sustainable Development Goals (SDG) 2015, the target was to reduce by half the one billion people living in slums in the world by providing decent housing. Shelter is one of the basic needs but due to the levels of poverty in

Africa, many people have not afforded decent housing. Hence, the need for a research on affordable slab mix for decent and durable houses. The foundation of a house determines its durability and safety for its occupants.

In the year 2017, the Jubilee government in Kenya launched the ‘Big Four Agenda’; Affordable decent housing being one of them. The government was determined to eradicate poverty by making decent housing accessible to all.

The plinth is the lower slab at the base of a column according to the Oxford English Dictionary. Middle income earners in Africa, make a slab of Water, Cement, Sand and Ballast, then the perpendicular walls projected on the slab may be made of any other materials such as stone blocks, panels, iron sheets, timber and others, which minimizes the cost of building houses.

According to (Day, 1995), the Chemical Admixtures could solve intractable technical problems and substantial cost; though also have a potential to create technical problems if improperly used. We opted not to use admixtures in this study, since nonprofessionals understand better the traditional concrete mix that only includes the above-mentioned ingredients. Application of admixtures requires more water to enable the workability of concrete, but has a possibility of reducing strength.

In the past, some buildings constructed in Kenya were found with cracks others eventually collapsing and causing fatalities. Partly, this was because of inadequate concrete mixes therefore, the need to obtain optimal ratios relevant to the local construction ingredients. (Kioko, 2014)in his paper suggested that to reduce the incidences of collapsing buildings that the national society of engineers and other government agencies avail a code of practice matching the local materials.

1.8 DELIMITATIONS

The study narrowed down to a simple concrete mix without reinforcement, since the focus of the study was on low cost houses. The two designs considered for comparison were the equally and unequally weighted simplex centroid axial designs. These were used to assess the better design in terms of the optimality criteria and in location of the optimal response.

1.9 LIMITATIONS

Concrete strength depends on the source of materials especially the fine and coarse aggregates, which are prone to pollution and weather changes. Cement majorly depends on the raw materials and the processes for its production. This means that the previous optimal mix 1:2:4 (cement: sand: ballast) of strength class M20 by the British standards, may differ from mix to mix whose materials are acquired from different geographical locations. The standard compressive strengths for the many ratios used were obtained very long ago as indicated in Britain standards (BS 5328).

To perform the concrete experiment took a long time due to failure of accessing a government concrete testing laboratory. Private laboratories were too expensive and would not have been possible to hire the facility due to lack of funds. The results of the first experiment conducted in 2016 were compromised due inaccuracies in measurements by the technicians, hence discarded.

The second experiment conducted in 2019, was expensive in terms of time, purchase of materials and paying for the laboratory services.

1.10 ASSUMPTIONS OF THE STUDY

It was assumed that; the instruments used in the experiment, the compressor and others were assumed to be calibrated, the measuring of ingredients was accurate, since the researcher was directly involved and spillage had been accounted for. The Engineers standard ratios of 1:2:4 and 1:1.5:3 taken in this study as the control, with a water/cement ratio taken approximately as 0.5, and whose average compressive strengths are 20 N/mm^2 and 25 N/mm^2 respectively, were assumed to be accurate.

Another assumption is that the materials used were standard. The experts advice that concrete materials are affected by weathering reagents, sources and manufacturing processes and therefore may not compare accurately one on one with the materials in our experiment. Therefore the more reason of using regression to determine these optimal conditions.

Subsequently, after looking at the background of this study, the following chapter outlines the literature related to this study. Chapter 3 discussed the methods used for the study, whereas chapter four displayed the results. The study ended with chapter five deliberating on discussions of the results, conclusions and the recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.0 INTRODUCTION

This chapter reviewed literature related to response surfaces methodology, Kronecker models, slope mixture and invariant designs, optimality criterion of mixture experiments, decision variables in mixtures and concrete mix designs.

2.1 RESPONSE SURFACE METHODOLOGY

(Wang & Fang, 2010) said that medicine is composed of several components, and that the quality of medicine depends on the proportions of the components. They studied the component proportions of medicine SIBELIUM capsule. They used the design points of symmetric-simplex design and design points generated by XVERT algorithm. For the two quality characteristics under consideration, the two-optimization methods produced similar results.

(Koske, Kinyanjui, Mutiso, & Cherutich, 2009) indicated that the goal of an experimenter is to obtain a design that gives maximum information. They investigated mixture experiments on second-degree Kronecker model and showed that a parameter subspace can improve a design.

The main design problem for this study was to obtain a design with maximum information for the maximal parameter subsystem $K'\theta$, subject to the side conditions. The maximum information obtained was used to evaluate the D-, G- and I-optimality criteria of weighted simplex centroid designs, which follows the Kiefer-Wolfowitz equivalence theorem as given in subsection 3.2.4.

A second degree Kronecker model suggested by (Draper & Pukelsheim, Mixture models based on homogeneous polynomials, 1998) given in (3.2), $\mathrm{Y_t}$ are the observed responses under the experimental conditions $t \in T$, is taken to be a scalar random variable and an unknown parameter$\Theta = (\theta_{11}, \theta_{22}, \dots, \theta_{mm})' \in R^m$. The moment matrix given by (3.18) for the second-degree Kronecker-model has all entries homogeneous in degree four, and reflects the statistical properties of a design τ. (Kinyanjui, 2007), showed that second-degree mixture experiments for maximal parameter subsystem with $m \geq 2$ ingredients, unique D-and A-optimal weighted centroid designs for $K'\theta$ exist and in the same study, E-optimal WSCD mixture experiment with two ingredients too was obtained. This research focuses on deriving D- , G-and I-optimal weighted simplex centroid designs for four ingredients.

One of the most important concerns of the experimenter is to learn more about the subsystems of interest. This allows the designer to evaluate the performance of a design relative to the subsystems of interest only. The parameter system of the mixture experiments contains many repeated terms making it rank deficient hence not all the parameters are efficiently estimated. The parameters in the subsystem of interest have similar properties to those of the full parameter system. K the maximal coefficient matrix for M, in this study is a $\binom{m+1}{2} \times 16$ matrix.

2.2 SLOPE DESIGNS AND INVARIANCE

A study by (Korir , 2019) showed that the set of weighted centroid constitutes a minimal class designs for the Kiefer ordering design and that any design that is not a weighted centroid design could be improved upon by convex combination of appropriate elementary designs. At the same time, exchangeable moment matrices were constructed which are symmetrical, balanced, and invariant that had homogeneous entries, these were considered as good properties for an optimal design.

Another study by (Kleins, 2002) analyzed a quadratic subspace of block matrices, which are invariant under the action of a group $\mathcal{H}$ arising from the design of mixture experiments. He found that there are two sets of novel results: one, finding an orthogonal basis of the quadratic subspace and a multiplication table for the matrix blocks allowing efficient handling of $\mathcal{H}$-invariant symmetric matrices. Two, he presented a spectral analysis of $\mathcal{H}$-invariant symmetric matrices. The results were used to calculate optimal designs of mixture experiments analytically as well as numerically.

While (Wambua A, et al., 2017) investigated some optimal slope mixture designs in second-degree Kronecker model for mixture experiments. The study was restricted to weighted centroid designs for three components, where H-invariant symmetric matrices containing the information matrices were used to obtain optimal designs for mixture experiments analytically.

2.3 OPTIMALITY CRITERION

The research by (Cheruiyot, Koske, & Mutiso, 2017) furthered on the study by (Cornell, 2002) on optimizing the eradication of pests using the Third order weighted simplex models. They used the I-optimality equivalence theorem, which showed that the uniformly weighted designs performed better than the other weighted centroid design. The results also showed that the pure blend (1,0,0,0) and the centroid $\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)$ yielded more optimal results and recommended that Vendex alone be used in eradication of mites or a combination of the four namely; Vendex, Omite, Kelthane and Dibrom in equal proportions.

A study done by (Huda & AL-Shiha , 1999) extended the concepts of D-, E- and A-optimality to consider designs for estimating the slopes of a response surface. Optimal designs under the D-optimality criteria were obtained for second-order models over spherical regions.

Another scholar (Klein, 2004) investigated optimal designs in the second-degree Kronecker model for mixture experiments. He presented three results as characterization of feasible weighted centroid designs for a maximum parameter system, derivations of D-, A-, and E-optimal weighted centroid designs and finally, numerically $\emptyset_p$-optimal weighted centroid designs. The main tool throughout the problem was the quadratic subspace of invariant symmetric matrices containing the information matrices.

In application of the weighted simplex centroid design to obtain V-optimal allocations of the observations, (Shuanzhe & Heinz , 1995) showed that optimality over the entire simplex was obtained using the equivalence theorem. Utami et.al. (2013), showed that the performance of D-optimal designs in terms of the I-optimality criteria strongly depends on which of the D-optimal designs are replicated. They concluded that I-optimal designs are more superior than D-optimal designs, as the latter focus on precise model estimation than precise predictions, and noting that I-optimality minimizes the average variance of prediction.

In their study (Gaylor & Sweeny , 1965) on the optimum allocation in a region of interest, which does not necessarily correspond to the region available for experimentation $0 \leq x \leq 1$. They found that the allocation of experimental data points which minimizes the average variance of the predicted values occurring according to the density function in the region of prediction is derived as $\hat{y} = \alpha + \beta z$. The errors of this relation were assumed to be uncorrelated and of a common varianceσ_y^2.

(Wanida, Chomtee , & Borkowski, 2019), developed a new G-optimality criterion using the concept of weighted optimality criteria, where they aimed to minimize the weighted average of maximum scaled prediction variance in the design region over a set of reduced models. Having used the genetic algorithm (GA), they found out that it had model-robust properties and performed over other generated designs by PROC OPTEX algorithms.

2.4 PROPERTIES OF CONCRETE INGREDIENTS

A research conducted by National Construction Authority of Kenya (2019) led to the conclusion that materials should be of suitable nature and quality for the purpose they are used.

The Engineering services Pvt. Ltd explained in the Durocrete Mix Design Manual (2013) that concrete is a composition of cement, sand (fine aggregate), ballast (coarse aggregate) and water, which are the basic raw materials. Concrete can have other additives such as silica, fluorspar and others, where some are useful for the hardening of concrete.

Testing of concrete for slabs is one of the requirements to ensure the right quality. There are two major test methods. Compressive strength is the test designed to obtain the concrete strength without reinforcement maximum load per area before failing. The other is tensile strength which involves concrete and steel, the test is performed in such a way that the concrete is pulled apart until the tension reaches a breaking point, usually this is done for reinforced concrete.

2.4.1 Cement

The strength and grade of cement are denoted by the same value. For instance, Grade 32.5 of cement gives a minimum strength of $32.5N/mm^2$. Higher grade of cement demands a higher water-to-cement ratio and the vice versa. Cement curves gives indication of the water content required to achieving a given strength. Users of cement have observed that overstayed cement loses its strength and may not be useful for construction purposes. Curing before testing of concrete done in either 1, 3, 7, 28 or 56 days depends on the user of the concrete. Waiting for 28 days or more produces better results since concrete hardens with time. It has been noted that Ordinary Portland cement requires less curing period for higher strengths measured at 42.5 N/mm^2 unlike Pozzolana (blended) cements whose strength are at 32.5 N/mm^2.

Cement has initial setting and final setting times. Initial setting time is the time taken for the cement paste to lose its plasticity. Minimum initial setting time by IS456-2000 standard is 30 minutes. Nevertheless, current cements give initial setting time greater than or equal to 60 minutes. The beginning of hardening of cement implies final setting of cement. Maximum time permitted by IS8112:1989 Indian Standard is 600 minutes. Currently final setting of concrete is between 3 to 5 hours.

This study has chosen a concrete of strength 32.5 N/mm^2 since it is affordable and is sufficient for plinth construction. The curing time was set at 28 days in order to obtain better and dependable results.

2.4.2 Fine Aggregate (Sand)

According to Day (1995), fine aggregate goes through grading given by sieve analysis. This is passing of sand through a set of standard sieves. The fineness of sand found by sieve analysis governs the proportion of sand in concrete. The overall fineness given by a factor called fine modulus varies from two to four. Another characteristic of sand is specific gravity given by

$$Specific\ gravity = \frac{Solid\ Density\ of\ Sand\ Particles}{Water\ Density}$$

The greater the specific gravity the heavier the density of the particles of sand and the vice versa. It is important to note that high Silt content affects the workability of concrete and results to a higher water content ratio hence lower concrete strength.

2.4.3 Coarse Aggregate

The width of the coarse aggregate measured in diameter varies in approximate sizes of 40mm, 25mm, 20mm, 12.5mm and 10mm according to the Durocrete Mix Design Manual (2013).

After sieving, 90% of the aggregate should pass. The size of the aggregate affects the workability and strength of concrete. Coarse aggregate demands water and fine aggregate content in order to achieve a cohesive mix. A smaller maximum size of coarse aggregate requires a greater fine aggregate content, while it is noted that particles of 40mm coarse aggregate requires less water (where less water means a lower water content ratio) than 20mm for the same workability. A higher size of aggregate reduces cement consumption. For grades up to M35, it is advisable to use a greater maximum size of aggregate since mortar failure is predominant. A lower water content leads to higher strength of mortar, which leads to higher concrete strength. For grades M40 and above, bond failure is predominant; therefore, it was recommended that a lower maximum size aggregate be used to prevent bond failure.

Grading of coarse aggregate helps us to obtain cohesiveness and density of concrete. All in (that is, all sizes of coarse aggregates), fills the voids left by larger coarse aggregate particles since they are filled by smaller coarse aggregate particles, leading to minimum cement-sand-water paste to be used to fill the voids. The purpose is to improve compactness and minimizes segregation of concrete, the more the compacted the concrete is, the higher the strength.

The rounded shaped coarse aggregates have a lower water demand. Lower mortar paste requirement leads to higher concrete strength this is best for M35 grades and lower though expensive. Angular/irregular shaped coarse aggregates are best for M40 grades and higher. Angular aggregate is preferred with more surface area, increases in water demand and increases the tendency to segregation.

Strength of coarse aggregate is indicated by crushing strength of rock, aggregate crushing value, aggregate impact value and the aggregate abrasion value (100N/mm^2 in Maharashtra), hence aggregates rarely fail in strength. Aggregates can absorb water up to 2% by weight when in bone-dry state and in some cases as high as 5%. Aggregate absorption is used for applying a correction factor.

2.5 DECISION VARIABLES IN MIX DESIGN

The decision variables in a mix design are Water-to-Cement Ratio, Cement Content, Relative Proportion of Fine, Coarse Aggregates and Use of Admixtures.

Water-to-Cement Ratio is the single most important factor that governs the strength and durability of concrete. Abram's law states that the higher the water-to-cement ratio, the lower the strength of concrete and the vice versa. As a rule of thumb, every 1% increase in water causes a 5% decrease in concrete strength. This is to say that every litre of water added reduces concrete strength by 2 to 3 N/mm^2 and increases workability by 25mm.

Cement Content is a core ingredient in concrete. For durability considerations, cement must not be reduced to less than 300kg/m^3. Higher cement content is required for severe weather conditions due to cracking. Higher cement content may not result to higher concrete strength, as shown in recent findings in (Anand , 2013). It showed that in the same water-to-cement ratio, a leaner mix gives a higher strength. A lower water-to-cement ratio leads to lower workability, which means that if a lower water-to-cement ratio is to be achieved without disturbing the workability, cement content should be increased. The cement content can be worked-out as follows;

$$Cement\ content[kg/\,m^3] = \frac{Water\ required\ to\ achieve\ workability[lit\ /\ m^3]}{Water -\ to - cement\ ratio}$$

Relative Proportion of Fine & Coarse Aggregates requires gradation, the Coarse aggregate (Ballast) should be retained on a standard IS 4.75mm sieve while Fine aggregate (Sand) should pass through a standard IS 4.75mm sieve. The proportion depends on the Fineness of sand. That is to say the finer the sand, the less coarse aggregate is required and vice-versa. It also depends on the size and shape of coarse aggregate in that the greater the size, the less fine aggregate is required. For cement content, the leaner the mix, the less of cement and more of fine aggregate is required.

The experiment for this study did not incorporate the use of admixtures; it is recommended that in the future it could be incorporated as a fifth component in a compressive strength study. Admixtures such as plasticizers and super-plasticizers, retarders, accelerators, entraining agents, shrinkage-compensating admixtures and waterproofing admixtures can change the properties of concrete making it a little expensive but workable.

On Concrete Mix Design Methods, the basic objective of concrete mix design is to find the most economical proportions (optimized) to achieve strength, cohesion, workability and durability (Materials and Tests units , 2019). Concrete mix design (CMD) is the process of taking trials with certain proportions. Scientific methods have been used to develop these proportions. No mix design directly gives the exact proportions that will most economically achieve the desired result. These methods only serve as a basis to start and achieve the result with the fewest possible trials.

Other factors that affect properties of concrete include quality and quantity of cement, water and aggregates, batching transportation, placing compaction and curing.

The basic steps of mix designs are; finding the target mean strength, determining the curve of cement based on its strength, determining the water-to-cement ratio, determining the cement content and determining the fine and coarse aggregate proportions.

2.6 CONCRETE MIX DESIGNS

As suggested by (Day, 1995), there are hundreds of systems of concrete mix designs just as there are hundreds of cures for the common cold. In the case of concrete mix design, there is evidence that nearly all systems end by suggesting adjustment of a trial mix by eye and most commercial concrete results from the continued ad hoc modification of existing mixes without any application of the former mix design.

Dewar in (Day, 1995) foreword stated that Concrete was fast moving from the stage of an art to that of a science, in that there was a blend of theory and experience and the development of expert systems aided by computer. It therefore meant that, just like medicine is improved through formulation, so is concrete formulation supposedly every time for better results.

The study by (Ahmad & Alghamdi, 2014) used a systematic statistical approach to obtain optimum proportioning of concrete mixtures using the data obtained through a statistically planned experimental program. They said that the utility of the proposed approach for optimizing the design of concrete mixture was illustrated considering a typical case in which trial mixtures were considered according to a full factorial experiment design involving three factors and their three levels (3^3). Experimental data were utilized to carry out analysis of variance (ANOVA) and hence developed a polynomial regression model for compressive strength in terms of the three design factors. The statistical model developed was used to show how optimization of concrete mixtures could be carried out with different possible options.

On optimizing concrete mixture (Marcia, 2003), used statistical methods. They investigated the feasibility of using statistical experimental design of analysis to optimize concrete mixture proportions. The research developed an internet based software program for optimization. Two experimental designs were investigated, classical and factorial based central composite design in laboratory experiment. In each case, six component materials were used and mixtures optimized for four performance criteria. They used a system Concrete Optimization Software Tool (COST), which employed six interactive procedures starting with material selection, working through the batches, testing and analyzing the test results. The recommendation was that the mixture proportions should achieve the desired performance levels.

By use of analytical and numerical methods (Shakhmenko & Birsh , 1998) obtained an aggregate mix design that optimized the cost of raw materials, quality of aggregate packing, water and cement.

On the other hand, (Ahmad & Alghamdi, 2014) wrote a paper on laboratory trial procedure for optimum design on concrete mixes using locally available ingredients. The optimization procedure was formulated to find the minimum cost of concrete mix, by trying different combinations of coarse aggregate to total aggregate ratio, and total aggregate ratio to cement ratio within their optimum ranges keeping water to cement ratio constant. One of his finding was that, the optimum coarse aggregate to total aggregate ratio was 0.62, while total aggregate to cement ratio was 4.88 respectively, since this combination attained the required 28-day minimum strength requirement of $35N/mm^2$.

A comparative study of concrete design by (Shah & Shah, 2014), by adding various types of Admixtures, correlation between rheological parameters and compressive strength were used, instead of water-cement ratio versus compressive strength relationship. They determined the water-cement ratio and the aggregate volume to paste volume ratio from the rheological behavior. They were able to estimate parameters like compressive strength and economical costing at the design stage for a given strength in addition to concrete ingredients.

Some engineers (Sohail, et al., 2018), studied Advancements in Concrete Mix Designs for High performance concretes (HPC) and Ultra High performance concretes (UHPC), from 1970 to 2016. They reviewed methods adopted to produce HPC and UHPC. They highlighted the earlier techniques used to obtain cementitious materials with high strength and durability. They found that high compressive strength was achieved with denser mixtures, that is, particle-packing density was a major attribute in achievement of low porosity, flow ability, durability and reduced defects in concrete.

The scholars (Marcia , Eric , & Kenneth , 1997) worked on a six-concrete component design using Scheffe model involving a set of constraints. It was found that many components and several properties of interest, trial and error method could easily miss the optimal conditions, resulting to higher costs to producers.

This study generated a regression model, emanating from a suggested weighted simplex centroid design subjected to $\mathcal{H}$-invariance slope matrices, aimed to achieve a stationary point. Also to evaluate D-,G- and I-optimality criteria for the design and model that obtained an optimal mix for concrete class M25. While other authors had their own interests as shown above, this study sought to address the challenge of collapsing of buildings caused by poor construction standards.

CHAPTER THREE

METHODOLOGY

3.0 INTRODUCTION

This chapter dealt with methodologies that aimed at constructing a weighted simplex centroid design for reaching the optimal concrete mix for an affordable plinth. These included methodologies used; to construct the desired design, to evaluate the D-, G- and I- optimal conditions of the constructed design, for designing a concrete experiment and for evaluating the optimality of the obtained experiment model.

3.1 CONSTRUCTION OF AN INSCRIBED TETRAHEDRAL WSCD

The measured response in the general mixture problem, is assumed to depend only on the proportions of the ingredients present in the mixture and not on the amount of the mixture as stipulated by (Cornell, 2002). The mixture ingredients $t_1, t_2, \dots, t_q$ are such that $t_i \geq 0$ and that $\sum_{i=1}^{q} t_i = 1$. Thus, the experimental region is given by the probability simplex,

$$T_q = \left\{ t = \left(t_1, \dots, t_q\right)' \in [0,1]^q : \sum_{i=1}^{q} t_i = 1 \right\}, t \in T_q \tag{3.1}$$

In this study, the second order polynomial in t was used to model the expected response $E(Y_t)$ as was suggested by Draper and (Pukelsheim, 1993) in the second degree Kronecker model

$$E(Y_t) = f(t)'\theta = (t \otimes t)'\theta = \sum_{i=1}^{q} \theta_{ii} t_i^2 + \sum_{i,j=1,i<j}^{q} (\theta_{ij} + \theta_{ji}) t_i t_j \tag{3.2}$$

With the unknown parameter vector $\Theta = (\theta_{11}, \theta_{22}, \dots, \theta_{mm})' \in R^{m^2}$ and the regression function $f(t) = (t \otimes t)$, all observations t_i from an experiment are assumed to be of equal unknown variance and are uncorrelated.

For the full system of interest second-degree Kronecker model for four ingredients is given as,

$$E(Y) = \theta_{11}t_1^2 + \theta_{22}t_2^2 + \theta_{33}t_3^2 + \theta_{44}t_4^2 + \theta_{12}t_1t_2 + \theta_{13}t_1t_3 + \theta_{14}t_1t_4 + \theta_{21}t_2t_1 + \theta_{23}t_2t_3 + \theta_{24}t_2t_4 + \theta_{31}t_3t_1 + \theta_{32}t_3t_2 + \theta_{34}t_3t_4 + \theta_{41}t_4t_1 + \theta_{42}t_4t_2 + \theta_{43}t_4t_3 \, . \quad (3.3)$$

This study aimed at obtaining the optimal slope designs using the $\mathcal{H}$-Invariant matrices, the $\phi_p$optimal values for the parameter subsystem of interest, and evaluating the optimality conditions of the D-, G- and I-criteria. The parameter subsystem of interest with fewer terms than the full parameter system of interest was obtained by merging the similar interactions as given by equation (3.4). The subsystem reduces the bulky computations that results when using the equation (3.3).

$$E(Y) = \theta_{11}t_1^2 + \theta_{22}t_2^2 + \theta_{33}t_3^2 + \theta_{44}t_4^2 + \theta_{12}t_1t_2 + \theta_{13}t_1t_3 + \theta_{14}t_1t_4 + \theta_{23}t_2t_3 + \theta_{24}t_2t_4 + \theta_{34}t_3t_4 \quad (3.4)$$

3.1.1 The Tetrahedral Axial Design

Unlike many other mixtures, the Concrete mixture must have some proportion of each of the ingredients. This study focused on four ingredients namely; Water, Cement, Sand (Fine aggregate) and Ballast (Coarse aggregate). Therefore, the coded design that was selected was commensurate with the fact that there is no pure blends, binary blends and tertiary blends for the four component concrete mixture. The kind of simplex design generated was a Simplex Centroid inscribed in another Simplex Centroid design, normally called the axial design. It is referred to as axial since all the corresponding points in each face, vertex and line of this design are equidistant from the Simplex Centroid. The assumption in this study was to let the vertices of the simplex to be a distance h from the main vertices of the original Simplex, which is a regular tetrahedron of vertices (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1). This value h also denoted by$\triangle$, should have maximum value given by $\frac{q-1}{q}$, according to (Cornell, 2002).

The four vertex points that make design η_1 and whose moment matrix $M(\eta_1)$ were generated as;

$$\eta_1 = \left(1-\mathrm{h}, \frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}\right), \left(\frac{\mathrm{h}}{3}, 1-\mathrm{h}, \frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}\right), \left(\frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}, 1-\mathrm{h}, \frac{\mathrm{h}}{3}\right), \left(\frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}, \frac{\mathrm{h}}{3}, 1-\mathrm{h}\right)$$

The six edge midpoints form the design η_2 whose moment matrix is $M(\eta_2)$were given as;

$$\eta_2 = \left(\frac{3-2h}{6}, \frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \right), \left(\frac{3-2h}{6}, \frac{h}{3}, \frac{3-2h}{6}, \frac{h}{3}, \right), \left(\frac{h}{3}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{h}{3}\right),$$

$$\left(\frac{h}{3}, \frac{3-2h}{6}, \frac{h}{3}, \frac{3-2h}{6}\right), \left(\frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{3-2h}{6}\right), \left(\frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}\right)$$

The four points on faces of the Tetrahedron forming the design η_3, whose moment matrix is $M(\eta_3)$ are the mixture blends generated as;

$$\eta_3 = \left(\frac{3-h}{9}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{h}{3}\right), \left(\frac{3-h}{9}, \frac{3-h}{9}, \frac{h}{3}, \frac{3-h}{9}, \right), \left(\frac{3-h}{9}, \frac{h}{3}, \frac{3-h}{9}, \frac{3-h}{9}\right), \left(\frac{h}{3}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{3-h}{9}\right)$$

The single point which forms the design η_4 is the centre of the simplex design referred to as the Centroid $\eta_4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, its moment matrix is $M(\eta_4)$.

3.1.2 Determining the Weights of the design

Equal weights means that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{4}$. Outlined below are two methods of obtaining the unequal weights.

Firstly, the fourth order moments of the axial design were obtained using the set of equations (3.5).

$\mu_4 = \frac{1}{2^q-1}\sum_{j=1}^{15} t_{1j}^4.$

$\mu_{31} = \frac{1}{2^q-1}\sum_{j=1}^{15} t_{1j}^3 t_{2j}.$

$\mu_{22} = \frac{1}{2^q-1}\sum_{j=1}^{15} t_{1j}^2\, t_{2j}^2.$ (3.5)

$\mu_{211} = \frac{1}{2^q-1}\sum_{j=1}^{15} t_{1j}^2\, t_{2j} t_{3j}.$

$\mu_{1111} = \frac{1}{2^q-1}\sum_{j=1}^{15} t_{1j}\, t_{2j} t_{3j} t_{4j}.$

Where $j = 1,2, \dots ,15$ and t_1, t_2, t_3 and t_4 were given as

$t_1 = \left(1-h, \frac{h}{3}, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \frac{h}{3}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{h}{3}, \frac{1}{4}\right).$

$t_2 = (\frac{h}{3}, 1-h, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{h}{3}, \frac{3-2h}{6}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{h}{3}, \frac{3-h}{9}, \frac{1}{4}).$

$t_3 = (\frac{h}{3}, \frac{h}{3}, 1-h, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{3-h}{9}, \frac{h}{3}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{1}{4}).$ (3.6)

$t_4 = (\frac{h}{3}, \frac{h}{3}, \frac{h}{3}, 1-h, \frac{h}{3}, \frac{h}{3}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{3-2h}{6}, \frac{h}{3}, \frac{h}{3}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{3-h}{9}, \frac{1}{4}).$

The value h used in the sets of equations (3.6) was arbitrarily chosen subject to a maximum of $\frac{q-1}{q}$according to Cornell (1990) forming the design matrix X as shown below

$X = [t_1, t_2, t_3, t_4]$, t_1, t_2, t_3 and t_4 are as shown in equation (3.6) above.

In order to calculate the weights of this design it was necessary to find the lower order moments, which were expressed as sums of the fourth order moments shown in (3.7), according to Draper and (Pukelsheim, 1993).

$$\mu_{111} = \int t_1 t_2 t_3\,(t_1 + t_2 + t_3 + t_4)dt = 3\mu_{211} + \mu_{1111}$$

$$\mu_{21} = \int t_1^2 t_2\,(t_1 + t_2 + t_3 + t_4)dt = \mu_{31} + \mu_{22} + 2\mu_{211}$$

$$\mu_3 = \int t_1^3\,(t_1 + t_2 + t_3 + t_4)dt = \mu_4 + 3\mu_{31} \qquad (3.7)$$

$$\mu_2 = \int t_1^2(t_1 + t_2 + t_3 + t_4)dt = \mu_4 + 3\mu_{31} + 3\mu_{22} + 6\mu_{211}$$

$$\mu_{11} = \int t_1 t_2\,(t_1 + t_2 + t_3 + t_4)dt = 2\mu_{31} + 2\mu_{22} + 10\mu_{211} + 2\mu_{1111}$$

$$\mu_1 = \int t_1\,(t_1 + t_2 + t_3 + t_4)dt = \mu_4 + 12\mu_{31} + 9\mu_{22} + 36\mu_{211} + 6\mu_{1111}$$

Subject to the Simplex restriction

$$4\mu_4 + 48\mu_{31} + 36\mu_{22} + 144\mu_{211} + 24\mu_{1111} = 1 \qquad (3.8)$$

The elementary designs η_1, η_2, η_3 *and* η_4 were used to generate the weighted simplex centroid design η Such that equation (3.9), improves upon a given exchangeable design $\bar{\tau}$ according to the Loewner ordering of having $M(\eta) \geq M(\bar{\tau})$.

$$\eta = \alpha_1\eta_1 + \alpha_2\eta_2 + \alpha_3\eta_3 + \alpha_4\eta_4 \qquad (3.9)$$

The weights $\alpha_1, \alpha_2, \alpha_3$ *and* $\alpha_4 \geq 0$ in the set of equations (3.10) are such that $\sum_{i=1}^{m} \alpha_i = 1$.

$$\mu_{1111} = \frac{1}{4}\alpha_1\eta_{14} + \frac{1}{6}\alpha_2\eta_{24} + \frac{1}{4}\alpha_3\eta_{34} + \alpha_4\eta_{44}$$

$$\mu_{111} = \frac{1}{4}\alpha_1\eta_{13} + \frac{1}{6}\alpha_2\eta_{23} + \frac{1}{4}\alpha_3\eta_{33} + \alpha_4\eta_{43} \qquad (3.10)$$

$$\mu_{11} = \frac{1}{4}\alpha_1\eta_{12} + \frac{1}{6}\alpha_2\eta_{22} + \frac{1}{4}\alpha_3\eta_{32} + \alpha_4\eta_{42}$$

$\mu_1 = \frac{1}{4}\alpha_1\eta_{11} + \frac{1}{6}\alpha_2\eta_{21} + \frac{1}{4}\alpha_3\eta_{31} + \alpha_4\eta_{41}$

But

$$\mu_1 = \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_4 = 0.25 \quad (3.11)$$

Since

$$\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3 \quad (3.12)$$

The notations shown in (3.10) above namely, $\eta_{14}, \eta_{24}, \eta_{34}$ *and* η_{44} refer to all design points in the elementary designs η_1, η_2, η_3 *and* η_4 and for $\eta_{13}, \eta_{23}, \eta_{33}$ *and* η_{43}, $\eta_{12}, \eta_{22}, \eta_{32}$ *and* η_{42} $\eta_{11}, \eta_{21}, \eta_{31}$ *and* η_{41} the first three, two and one design points as they correspond with the moments $\mu_{1111}, \mu_{111}, \mu_{11}$ *and* μ_1 respectively.

The equations (3.10) to (3.12) with the actual values for μ_1, μ_2, μ_3 *and* μ_4 were solved simultaneously in chapter 4.

Secondly, the α_i would also be found generally using the equation

$$\alpha_i = \frac{\binom{q}{i}}{2^q - 1},\ i = 1,2, \dots, q \quad (3.13)$$

where q is the number of the components in the experiment. This confirmed that the method used to generate $\alpha's$ in the equations (3.10) to (3.12) was appropriate.

3.1.3 Obtaining the Information Matrices of the WSCADs

The information matrix for the four component weighted simplex centroid axial designs (WSCADs) namely; Equally Weighted Simplex Centroid Axial Design (EWSCAD) and Unequally Weighted Simplex Centroid Axial Design (UWSCAD) were obtained.

The amount of information that the design τ contains on the parameter system $K'\theta$ is contained in the information matrix$C_k M(\tau)$.

3.1.3.1 The coefficient matrix

An experimenter may find it expensive and unnecessary to work with the full model θ, and therefore may wish to study s out of the $k, s \leq k$ components. This was achieved by integrating and averaging similar outcomes, hence the linear parameter subsystem of interest $k'\theta$ for some $k \times s$ matrix K. K is referred to as the coefficient matrix of the parameter subsystem $K'\theta$, which is estimable when there exists an unbiased linear estimator for θ. This can only happen if there is some matrix L, the left inverse of **K**, which satisfies the equation.

$$LX = K \quad . \tag{3.14}$$

(Draper & Pukelsheim, 1998) proposed a representation involving the Kronecker square$t \otimes t$, the $m^2 \times 1$ vector consisting of the squares and cross products of the components of t in lexicographic order. For a four-component mixture experiment, the regression function for the full model is as given by (3.3).

By merging similar component replicates, reduces the full system matrix to a subsystem of interest coefficient matrix given by:

$$(K'\theta)' = [\theta_{11}, \theta_{22}, \theta_{33}, \theta_{44}, \frac{\theta_{12}+\theta_{21}}{2}, \frac{\theta_{13}+\theta_{31}}{2}, \frac{\theta_{14}+\theta_{41}}{2}, \frac{\theta_{23}+\theta_{32}}{2}, \frac{\theta_{24}+\theta_{42}}{2}, \frac{\theta_{34}+\theta_{43}}{2}] \tag{3.15}$$

The subsystem of interest equation (3.4) has the coefficients of the variables t_i shown by (3.15). The left inverse of K is given as

$$L = (K'K)^{-1}K' \tag{3.16}$$

3.1.3.2 Moment Matrix

(Pukelsheim, 1993) stated from the General Equivalence Theorem, that if $M \in \mathcal{M}$ is a competing moment matrix that is feasible for $K'\theta$ with information matrix $C = C_K(M)$ Then M is $\phi -$optimal for $K'\theta$ in $\mathcal{M}$ if and only if there exists an NND(s) matrix D that solves the polarity equation

$$\emptyset(M)\emptyset^{\infty}(N) = trace\ MN = 1 \tag{3.17}$$

Note, the elements of the leading diagonal of the matrix CD, are the weights of the design. And also there exists a generalized inverse G of M, such that matrix $N = GKCDCK'G'$ that satisfies the *normality inequality*

$trace\ AN \leq 1$, for all $A \in \mathcal{M}$

It is noted that for optimality, equality is obtained in the normality inequality if M replaces A.

(Pukelsheim, 1993) defined the admissibility of a moment matrix as a moment matrix $M \in \mathcal{M}$ is called admissible in $\mathcal{M}$ when every competing moment matrix $A \in \mathcal{M}$ with $A \geq M$ is actually equal to M. A further explanation to this is that, the weakest requirement for a moment matrix M to be worthy of consideration is that M should be maximal in the Loewner ordering, implying that M cannot be improved by another moment matrix A such that

$$A \geq M \Leftrightarrow A - M \geq 0 \Leftrightarrow A - M \in NND(K).$$

The moment matrix at each design point was obtained by the summation of Kronecker product shown in (3.18).

$$M(\eta_i) = \int (t \otimes t)(t \otimes t)' d\tau. \tag{3.18}$$

The overall moment matrix which is $q^2 \times q^2$ matrix shown by equation (3.19), where q is the number of the components in the mix.

$$M(\eta) = \alpha_1 M(\eta_1) + \alpha_2 M(\eta_2) + \alpha_3 M(\eta_3) + \alpha_4 M(\eta_4) \tag{3.19}$$

The moment matrix $M(\eta)$ becomes

$$\begin{bmatrix}
\mu_4 & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} \\
\mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} \\
\mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} \\
\mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} \\
\mu_{22} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_4 & \mu_{31} & \mu_{31} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} \\
\mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} \\
\mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} \\
\mu_{211} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} \\
\mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{211} & \mu_{31} & \mu_{31} & \mu_4 & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{22} \\
\mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} \\
\mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} \\
\mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{1111} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} \\
\mu_{22} & \mu_{211} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{22} & \mu_{211} & \mu_{31} & \mu_{211} & \mu_{211} & \mu_{22} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_{31} & \mu_4
\end{bmatrix}$$

The values for $\mu_4, \mu_{31}, \mu_{211}$ *and* μ_{1111} also known as the fourth order moments were calculated in chapter 4.

3.1.3.3 The improved information matrix

The great scholar (Pukelsheim, 1993) gave the definition of an information matrix as a design ξ with the moment matrix M. The information matrix for $K'\theta$ with $k \times s$ coefficient matrix K of full column S, is defined to be $C_k(M)$ where the mapping C_k from the cone NND (k) into the space sym(S) is given by

$$C_k(A) = \min_{L \in R^{s\times k}; LK = I_s} LAL', \text{ for all } A \in NND(k) \tag{3.20}$$

If A is equal to the competing moment matrix M then the information matrix becomes

$$C_k(M) = \min_{L \in R^{s\times k}; LK = I_s} LML', \quad \text{for all } M \in NND(k) \tag{3.21}$$

Where, the minimum was according to Loewner ordering over all the left inverses L of K.

An information matrix $C_k(M) \in C$ is called admissible in C when every competing information matrix $C_k(A) \in C$ with $C_k(A) \geq C_k(M)$ is actually equals to $C_k(M)$.

The improved information matrices are the slope matrices, obtained by utilizing the equation$D_c = HCH'$.

Where D_c is the improved information matrix of the slope obtained from the $C = C_k(M)$ and H, the derivative of the elements of the design matrix $M(\tau)$ that is $H = \frac{dM(\tau)}{d\tau}$, where $M(\tau)$ is given by (3.22).

$$M(\tau) = t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 \tag{3.22}$$

The general derivative matrix H was given as

$$H=\begin{bmatrix} 2t_1 & 0 & 0 & 0 & t_2 & t_3 & t_4 & 0 & 0 & 0 \\ 0 & 2t_2 & 0 & 0 & t_1 & 0 & 0 & t_3 & t_4 & 0 \\ 0 & 0 & 2t_3 & 0 & 0 & t_1 & 0 & t_2 & 0 & t_4 \\ 0 & 0 & 0 & 2t_4 & 0 & 0 & t_1 & 0 & t_2 & t_3 \end{bmatrix} \qquad (3.23)$$

It was noted by (Pukelsheim, 1993) that for an arbitrary subset $\mathcal{H}$ of $s \times s$ matrices, defines a symmetric $s \times s$ matrix C to be $\mathcal{H}$ invariant if $D_c = HCH'$, for all $H \in \mathcal{H}$.

The set of all $\mathcal{H}$ invariant symmetric $s \times s$ matrices are expressed as $sym(s\mathcal{H})$. The following equation expresses the three different examples of $sym(s\mathcal{H})$.

$$sym(s\mathcal{H}) = \begin{cases} \{\Delta_z : Z \in \mathbb{R}^s\}, & for\ \mathcal{H} = sign(s) \\ \{\alpha I_s + \beta 1_s 1'_s : \alpha, \beta \in \mathbb{R}\}, & for\ \mathcal{H} = perm(s). \\ \{\alpha I_s : \alpha \in \mathbb{R}\}, & for\ \mathcal{H} = orth(s) \end{cases} \qquad (3.24)$$

The first group $sign(s)$ implies that $\mathcal{H}$ invariant matrices are diagonal matrices. The second group $perm(s)$implies that the matrices are completely symmetric, that is they have identical on-diagonal entries and identical off-diagonals entries. The third group $orth(s)$ shows that the matrices are multiples of the Identity matrix under the full orthogonal group.

The gradient techniques help in computation of stationary points at the steepest ascent or descent. In this research, we obtained slope matrices at different points of the design in order to get improved D- slope optimal values.

3.2 EVALUATING D-G- I- OPTIMALITY OF THE DESIGN USING TWO WSCDS

Alphabets were used to name the designs optimality criteria. Smith (1918) was the first author of optimality criteria, namely the G or the global optimality. (Kiefer &

Wolfowitz , 1959) ,(Pukelsheim, 1993) were other authors who developed many other optimality criteria like the known D-A-E-T criterion.

The optimality tests performed in order to locate the optimal values of a design according to each criterion were compared in order to obtain the design with the best characteristics. D-, G- and I-Optimal values for two different designs namely the, EWSCAD and UWSCAD were compared in this study.

3.2.1 D-Optimality

D-optimality criterion developed by Kiefer in 1958 is among the most popular optimality criteria since it is simple to compute. The D-optimality focuses on estimation of model parameters through the admirable attributes of the moment matrix, which is defined as

$$M = \frac{X\prime X}{N} \tag{3.25}$$

$X'X$ is the information matrix, and N, the total number of trials.

The D criterion is the commonly used optimality criterion, which seeks designs that maximize the determinant of the information matrix. A design D^* is said to be optimal if it belongs to the design space Ω such that the determinant of the information matrix (3.26) is maximum.

$$M(D^*) = maxD \in \Omega|M(D)|. \tag{3.26}$$

The normality assumptions, suggest that $|X'X|$ is inversely proportional to the square of the volume of the confidence region for the regression coefficients. Therefore, the larger the$|X'X|$, the better the estimation of the model parameters. The aim of D-optimality criterion is essentially parameter estimation. The determinant of $(X'X)^{-1}$

provides a measure of the overall uncertainty of the parameter estimates, and a design that minimizes this determinant is D-optimal. The D-optimality criterion is equivalent to minimizing the volume of confidence regions for finding the actual parameters.

For the comparison of different criteria and for applying the theory of information functions, the version $\phi_0(C) = (detC)^{\frac{1}{s}}$ is appropriate for the maximization of the determinant of the information matrices, which is the same as minimizing the determinant of the dispersion matrices given by (3.27).

$$(det\ C)^{-1} = \det(C)^{-1} \tag{3.27}$$

(Pukelsheim, 1993) found $\det(HCH') = det(H)^2 detC$ to be a pleasing property on-singular $s \times s$ Matrix H. If the parameter sub-system $K'\theta$ is re-parameterized according to $H'K'\theta$, forms a special case of iterated information matrices that provide the identities.

$$C_{KH}(A) = C_H(C_K(A)) = (H'(C_K(A))^{-1}H)^{-1} = H^{-1}C_K(A)H'^{-1}$$

$$detC_{KH}(A) = \frac{detC_K(A)}{detH^2} \tag{3.28}$$

The $\det C_K(A)$ is proportional to $C_{KH}(A)$. The determinant induced ordering in invariance under re-parametrization causes the determinant criterion to be the only criterion for which the function induced ordering has this invariance property.

Another invariant property pertains to the determinant function itself that is, the criterion is invariant under re-parametrization with matrices $H$and fulfils $\det H = \pm 1$, so we have $detC_{KH}(A) = detC_K(A)$. By definition, an information function ϕ on

NND(s) is called $\mathcal{H}$ invariant when $\mathcal{H}$is a subgroup of the general linear group GL(s) and all transformations $H \in \mathcal{H}$ fulfill

$$\phi(C) = \phi(HCH'), for\ all\ C \in NND(s).$$

The following lemma has a proof towards the same.

Let $\mathcal{H}$ be a subgroup of GL(s) and let ϕ be an information function on NND(s) (Invariance), the determinant criterion ϕ_0 is $\mathcal{H}$ invariant if and only if $\mathcal{H}$ is a subgroup of the unimodular matrices

$$\mathcal{H} \subseteq unim(S).$$

The proof followed as shown

Invariance implies $\theta_0(H'H) = 1$ for every $H \in \mathcal{H}$. This is the same as $|\det H| = 1$. Conversely for $H \in unim(s)$ we have invariance $\phi_0(H'CH) = ((detH^2)\det H)^{\frac{1}{2}} = \phi_0 C$

(Pukelsheim, 1993) gave these as outlined in pgs. 136,343,344.

3.2.2 G- Optimality

The G optimality criterion also known as the global optimality criterion is a parameter prediction criterion introduced by Smith in 1918. It is a design that minimizes the worst case expected error in prediction of a parameter. (Rady, Abd EL-Monsef, & Seyam, 2009) provided the G optimality criterion definition as $MinMaxVar(\widehat{y_x})$ which is equivalent to;

$MinMax\big(d(x\xi)\big) = MinMaxf^T(x)M^{-1}(\xi)f(x)$ for a full parameter system and

$$MinMax\left(d^*(x\xi)\right) = MinMaxf^T(x)C^{-1}(\xi)f(x) \tag{3.29}$$

for the subsystem of interest.

To get best G-optimal value we find the $f^T(x)C^{-1}(\xi)f(x)$ of every design point, the inverse of the improved information matrix D_C^{-1} was used instead of the inverse of the information matrix to obtain the G-slope optimal values.

Thomas and Stephen (2013) defined G-Optimal design as the design which minimizes the maximum variance of the estimated response function over the given design region.

3.2.3 I-Optimality

The $I -$ criterion also known as the IV-criterion, denotes integration over the candidate space. I-optimal designs minimize average or integrated prediction variance over the experimental region χ given by

$$Average\ variance = \frac{\int_\chi\ f^T(x)M(\xi)^{-1}f(x)}{\int_\chi\ dx} \tag{3.30}$$

To calculate this average variance, we exploited the fact that, when calculating the trace of a matrix product, we cyclically permutated the matrices as shown in 3.31.

$$trace[f^T(x)M(\xi)^{-1}f(x)] = trace[M(\xi)^{-1}f(x)f^T(x)]$$

Thus (3.31)

$$\int_\chi\ f^T(x)M(\xi)^{-1}f(x)dx = tr[M(\xi)^{-1}\int_\chi\ f(x)f^T(x)dx]$$

Since for any given design ξ, the information matrix $M(\xi)$ is constant as far as the integration is concerned, we can therefore rewrite the formula for the average prediction variance as was stated by (Goos, Jones , & Syafitri, 2013).

$$Average\ variance = \frac{1}{\int_{\chi} dx} trace[M(\xi)^{-1} \int_{\chi} f(x) f^{T}(x) dx]$$

In 2014, the same researchers Goos and Syafitri came up with a more convenient way of finding the average variance as given in 3.32.

$$Average\ variance = \frac{1}{\int_{\chi} dx} trace[M(\xi)^{-1} B] \tag{3.32}$$

where B is obtained as

$$B = \int_{s_{q-1}} X_1^{p1}, X_2^{p2}, \dots, X_q^{pq}\, dx_1, dx_2, \dots, dx_p = \frac{\prod_{i=1}^{q} \Gamma(p_i+1)}{\Gamma(q+\sum_{i=1}^{q} p_i)} = \frac{\prod_{i=1}^{q} p_i!}{(\sum_{i=1}^{q} p_i+q-1)!} \tag{3.33}$$

$$L_{ij} = K(\frac{\prod_{i=1}^{q} p_i!}{(\sum_{i=1}^{q} p_i+q-1)!}) \tag{3.34}$$

Average variance=$tr[M^{-1}L]$ and $L = \Gamma(q)B$. L is the moment matrix since the elements are moments of a uniform distribution on the experimental region S_{q-1}, and M the information matrix of the full model.

$\Gamma(q) = K = (q-1)!$ in addition, $M = X'\Lambda X$, where $X = [f(t_1), \dots, f(t_p)]$is the $p \times p$ model matrix corresponding to p points of the simplex centroid design. $\Lambda = \text{diag}(r_1 I_{11}, \dots, r_q I_{qq})$, $r_1, \dots, r_q$ are the weights of the different design points. M is the information matrix under the *I-Optimality*, for the full Kronecker model and average variance is given by

$$Average\ Variance = \frac{1}{\int_t \ dt} tr[M^{-1} \int_t \ f(t) f'(t) dt]. \tag{3.35}$$

The focus of this study was the subsystem of interest, hence the average variance was obtained by $Ave.variance = tr[C^{-1}L]$.

The information matrix C of the subsystem of interest was obtained by the equation (3.21).

3.2.4 D-, G- and I- optimality Equivalence Theorems

Theorem by Kiefer and Wolfowitz established the equivalence of G and D optimal designs in the limiting case that the number of experiments at a particular setting of independent variables can take on non-integer values.

Theorem 7.20 of optimal design of Experiments by (Pukelsheim, 1993), considers a matrix mean ϕ_p with parameter p such that $p \in (-\infty, 1]$. If M is a moment matrix that is feasible for $K'\theta$ with information Matrix $C = C_k(M)$, then M is ϕ_P- optimal for $K'\theta$ in $M(\Xi)$ if and only if there exists a generalized inverse G of M which satisfies the normality inequality

$$x'GKC^{p+1}K'G'x \leq traceC^p, for\ all\ x \in \chi \tag{3.36}$$

The equality in the normality inequality implies optimality, if any support points x_i of design $\xi \in \Xi$ replaces x and is $\phi_p$optimal for $K'\theta$ in Ξ . D-optimality is a matrix mean ϕ_p such that $p = 0$. The bound for Global optimality is given in the Lemma by (Pukelsheim, 1993) pg. 211, that every moment matrix $M \in M(\Xi)$ satisfies $d(M) \geq K$, or $g(M) = \frac{1}{k}$ where k is the size of the square matrix C or $k = trace\ MM^{-1}$.

The proof to the Lemma was that, given M is a non-singular matrix and belongs to the design

$\xi \in \Xi$ then the bounds become

$$traceMM^{-1} \leq d(M)\,. \tag{3.37}$$

The upper bound $\frac{1}{k}$ for the minimum information g and the lower bound k for maximum variance d are the optimal values.

For I-optimality (Rady, El-Monsef & Seyam, 2009) showed that a continuous design with information matrix M for the full model is I-optimal if and only if

$$f'(t)M^{-1}LM^{-1}f(t) \leq tr(M^{-1}L) \tag{3.38}$$

And for each design point in the experimental region of the subsystem of interest

$$f'(t)C^{-1}LC^{-1}f(t) \leq tr(C^{-1}L) \tag{3.39}$$

The general equivalence theorem provides a methodology to check the optimality of a given continuous design, for any convex and differentiable design optimality criterion.

3.2.5 Efficiencies for D-, G- and I- optimal designs

Let $K'\theta$ be a parameter subsystem with the coefficient matrix K of full column rank s. $\mathcal{M}$ is a set of competing moment matrices that intersects the feasibility cone $\mathcal{A}(K)$. Given an information function ϕ on NND(s) the general design problem is to

Maximize $\quad \phi \circ C_k(M)$

Subject to $\quad M \in \mathcal{M}$

Now the ϕ −efficiency of a design $\xi \in \Xi$ is defined by

$$\phi - eff(\xi) = \frac{\phi C_k(M(\xi))}{V(\phi)} \tag{3.40}$$

This efficiency is a number between zero and one usually expressed as a percentage. It gives the extent to which the design ξ exhausts the maximum information $V(\phi)$ for $K'\theta$ in $\mathcal{M}$.

Where $V(\phi)$ is the optimal value of the problem and is defined by

$$V(\phi) = \max_{M \in \mathcal{M}} \phi(C_K(M)) \tag{3.41}$$

A moment matrix $M \in \mathcal{M}$ is said to be formally $\phi -$ optimal for $K'\theta$ in $\mathcal{M}$ when $\phi(C_K(M))$ attains the optimal value (ϕ) . (Pukelsheim, 1993), pg. 131,132).

Given the designs ξ and ξ^*, which correspond to the UWSCAD and EWSCAD respectively, then the D-efficiency is given by

$$D_{eff}(\tau) = \left\{\frac{|M(\xi)|}{|M(\xi^*)|}\right\}^{\frac{1}{s}} \times 100 \tag{3.42}$$

where $M(\xi)$ and $M(\xi^*)$ are equivalent to the improved information matrices $D_{ciu} and\ D_{cie}$ respectively and s is the number of the parameters being estimated, as given in (4.18) and (4.19). If the D-efficiency is greater than one, then the UWSCAD is better than EWSCAD.

For G-optimality, $MinMax\big(d^*(x\xi)\big) = MinMax\ f'(x)C^{-1}(\xi)f(x)$is the G-optimal value for the EWSCAD and $MinMax\big(d^{**}(x\xi)\big) = MinMax\ f'(x)C^{*-1}(\xi)f(x)$ is the G-optimal value for the UWSCAD. The G-efficiency is given by;

$$G_{eff}(\tau) = \frac{MinMax(d^{**}(x\xi))}{MinMax(d^*(x\xi))} \times 100 \tag{3.43}$$

As in the case of D-efficiency, if G-efficiency is greater than one, then the UWSCAD is better than EWSCAD.

Now turning to I-optimality, if P_1 is the average variance of prediction of design ξ and P_2 is the average variance of prediction of a second design ξ^*, then the I-efficiency of the former design compared to the latter is computed as $I - optimality = \frac{P_2}{P_1}$, in this case I-efficiency is given by

$$I_{eff}(\tau) = \frac{tr[C^{*-1}L^*(\xi^*)]}{tr[C^{-1}L(\xi)]} \times 100 \qquad (3.44)$$

I-efficiency larger than one indicates that ξ^* is better than ξ in terms of the average prediction variance.

3.3 FITTING A SECOND DEGREE KRONECKER MODEL TO THE EXPERIMENT

Cornell (1990) said the objectives of the analysis of mixture data are; one, to fit a proposed model for describing the shape of the response surface over the simplex factor space. Two, to determine the roles played by the individual components. The same analysis can achieve the two objectives at once.

3.3.1 The mix design process

The following five steps were outlined by (Teychenne , Franklin , & Erntroy , 1988) in determining the framework of the desired design.

First was to obtain the statistical margin. The experiment was planned according to the simplex design to produce 15 outcomes. These being less than 20, the standard deviation (S) adopted was arbitrarily chosen to fit the characteristic strengths in M20 class. To find the standard deviation for the larger experiment of 45 outcomes, the minimum for more than 20 results line B of figure 3 of (Teychenne , Franklin , & Erntroy , 1988) pp.12 was used.

$$M = K * S \quad (3.45)$$

K is the critical value Z at a given significant level $\alpha\%$ and S is a predetermined standard deviation according to figure 3 borrowed from (Teychenne , Franklin , & Erntroy , 1988) pg. 12.

Secondly, the mean target strength was obtained as shown in (3.46).

$$f_m = f_c + M \quad (3.46)$$

Where f_c is the specified characteristic strength and M is the value defined by (3.45).

Thirdly, (3.47) gave the cement content. The free water content was determined using table 3.1. It was arrived at, by choosing an average slump height of 30-60mm and an average maximum aggregate size of 20mm, for crushed aggregates.

$$Cement\ Content(C) = \frac{free\ water\ content}{freewater{:}cement\ ratio} \quad (3.47)$$

Table 3. 1: Approximate free-water contents (kg/m^3)

Slump	Workability levels	0-10	10-30	30-60	60-180
Max. size Aggregate(mm)	Type of Aggregate				
10	Uncrushed	150	180	205	225
	Crushed	180	205	230	250
20	Uncrushed	135	160	180	195
	Crushed	170	190	210	225
40	Uncrushed	115	140	160	175
	Crushed	155	175	190	205

The free water content ratio 210 Kg/m^2in table 3.1 was obtained from table 3 of (Teychenne , Franklin , & Erntroy , 1988).

Table 3. 2: Approximate compressive strength (N/mm^2) for Coarse aggregate

Cement Strength class	Type of Coarse aggregate	Compressive strengths (N/mm^2) Age(days)			
		3	7	28	91
32.5	Uncrushed	15	23	36	44
	Crushed	20	29	43	51
42.5	Uncrushed	22	30	42	49
	Crushed	27	36	49	56
52.5	Uncrushed	29	37	48	54
	Crushed	34	43	55	61

In table 3.2 the value 43 (in bold), was the interpolated strength obtained for a mix of free-water cement ratio of 0.5. The concrete age and the aggregates used were specified. The required compressive strength of cement strength class 32.5N/mm^2 for this study was interpolated from table 2, the strength value obtained was plotted on figure 4 all of (Teychenne , Franklin , & Erntroy , 1988). A curve was drawn from the given point parallel to the given curves intercepted the horizontal line equal to the target mean strength, hence determining the free water cement ratio for the experiment.

Fourthly, the total aggregate content or density for saturated surface dry aggregates was obtained by wet concrete density (D) less Cement content (C) shown in (3.47) above, less the free water content (W), all in Kg/m^3.

$$TAC = D - C - W \tag{3.48}$$

The relative density of concrete was assumed to be $2.7kg/m^3$ for crushed aggregates and $2.6kg/m^3$ for the uncrushed.

Fifthly, the fine and coarse aggregate contents were defined by the equation 3.49.

$$FAC = TAC * Proportion\ of\ fine\ aggregates$$

$$CAC = TAC - FAC \tag{3.49}$$

The Coarse Aggregate Content (CAC) are given in diameter of the sizes of 10mm, 20mm and 40mm. The proportions depend on the shape and usage of the concrete. The general rule is that for concrete ratio 1:2:4 (class 20 or M20), a combination of 10mm and 20mm should be used, while for 1:1.5:3(class 25or M25), a combination of 10mm,20mm and 40mm should be used. The fine aggregate content was determined from figure 6 (Teychenne , Franklin , & Erntroy , 1988)pg. 15) using an average of 20mm of the maximum aggregate size and the free water/cement ratio.

These five processes were fitted in the mix design process form Table A shown in Appendix I which was originally formulated by (Teychenne , Franklin , & Erntroy , 1988)

3.3.2 Model Validity

The proposed Kronecker model for the subsystem of interest as shown by (3.15) is,

$$E(Y) = \theta_{11}t_1^2 + \theta_{22}t_2^2 + \theta_{33}t_3^2 + \theta_{44}t_4^2 + \theta_{12}t_1t_2 + \theta_{13}t_1t_3 + \theta_{14}t_1t_4 + \theta_{23}t_2t_3 + \theta_{24}t_2t_4 + \theta_{34}t_3t_4.$$

The following three main steps were used to analyze second order response surface model given by the general equation (3.15). First, the Normal Probability plot was constructed which was used to check for the outliers using the "Design Expert" statistical software Programme. Secondly the adequacy of the model obtained was tested which included obtaining the Student-t test, Analysis of Variance and finding the Coefficient of Variation. Finally, descriptive statistics, the response surfaces and the contour plots of the predicted responses were constructed using the R software program. The result of the most optimal combination from the experiment was obtained by use of a response trace plot and a recommendation given.

3.3.3 The Analysis of Variance

The Analysis of variance for the second order Kronecker model fitted was obtained. The regression sum of squares (SSR), Error sum of squares (SSE) and Total sum of squares (SST) were obtained as shown in (3.50).

$$SSR = \sum_{w=1}^{N}(\hat{y}_w - \bar{y})^2$$

$$SSE = \sum_{w=1}^{N}(y_w - \hat{y}_w)^2 \tag{3.50}$$

$$SST = \sum_{w=1}^{N}(y_w - \bar{y})^2, w = 1,2,\dots,15$$

3.3.3.1 Testing usefulness of terms in the Kronecker Polynomial

When Kronecker polynomials are used to model the response surface and to provide measures of the blending characteristics of the components, many times the model includes all the terms given in $E(Y) = \sum_{i=1}^{q}\theta_{ii}X_{ii}^2 + \sum_{i<j}^{q}\sum_{i=2}^{q}\theta_{ij}X_iX_j + \cdots$, depending on the number of components being blended and the degree of the polynomial. To choose the model to use, tests of hypotheses were performed on the $\theta's$, the coefficients of the Kronecker model (3.4) also denoted by $\theta_{ij},\ i,j = 1,2,3,4$ for

each group of parameters, which are either squares of the components or cross products of two components. So we tested the null hypothesis that

$H_0: \theta_{11} = \theta_{22} = \theta_{33} = \theta_{44}, \ \ \theta_{ij} = 0, \ i < j, i = 1,2,3,4.$

Versus

$$H_1: At\ least\ one\ of\ the\ equality\ is\ false. \tag{3.51}$$

The F calculated ratio is given by

$$F = \frac{SSR/(P-1)}{SSE/(N-P)} = \frac{MSR}{MSE} \tag{3.52}$$

where $P - 1\ and\ N - P$ are the numerator and denominator degrees of freedom respectively.

This ratio was compared to the table value $F_{(P-1,N-P,\alpha)}$, H_0 is rejected at $\alpha -$ level of significance, if the F calculated value exceeds the F critical or the table value. When we reject H_0 implies that the compressive strength does depend on the mix components. The summary is provided in the ANOVA table 3.3.

Table 3. 3: Analysis of Variance

Sources of variations	Degrees of freedom	Sum of Squares	MSS	F
Regression	$P - 1$	$SSR = \sum_{w=1}^{N}(\hat{y}_w - \bar{y})^2$	$\frac{SSR}{P-1} = MSR$	$\frac{MSR}{MSE}$
Residual	$N - P$	$SSE = \sum_{w=1}^{N}(y_w - \hat{y}_w)^2$	$\frac{SSE}{N-P} = MSE$	
Total	$N - 1$	$SST = \sum_{w=1}^{N}(y_w - \bar{y})^2$		

3.3.4 Coefficient of Variation

The coefficient of determination R^2 was used to determine how the estimated model fits the data obtained. R^2 lies in the range [0, 1], it measures the variation of y to $\bar{y}$ that is explained by the regression model. The closer the R^2 is to one, the better the model and the vice versa. However R^2 is not stable since addition of a variable to the model changes its value. Therefore the adjusted coefficient of determination R_A^2 is used instead. The formulae for the two are given as:

$$R^2 = \frac{SSR}{SST} \text{ and } R_A^2 = 1 - \frac{MSE}{MST} \tag{3.53}$$

3.3.5 Testing the Adequacy of the Parameters

The model validation provided important examination of the fitted model whether it offered an adequate approximation of the true response surface. At the same time it assured that none of the least squares regression assumptions were violated, therefore residual analysis was performed. R statistical software was used to obtain the Student-t test, on every term of the regression model, as shown in table 4.13. The parameter or an interaction with the smallest error at any given $\alpha - level$ of significance was approved to be better than others.

3.4 OPTIMAL MIX AND EVALUATING D-G-I-OPTIMALITY OF THE EXPERIMENT

Finding optimal settings is the goal of every experimenter. These were obtained by constructing several plots and graphs in order to justify the findings.

3.4.1 Descriptive Statistics Plots

The descriptive statistics plots that were used to describe the distribution for the experimental results were, scatter diagrams, boxplots and histogram, for each concrete ingredient versus compressive strength, as shown in figures 4.2, 4.3, 4.4 and 4.5.

3.4.2 Contours and the Quadratic Response Surface

(Montgomery, 1997) stated that a stationary point of a response surface may be obtained by use of Canonical analysis, where the regression model is transformed to a new co-ordinate system. The most straightforward way is to examine a contour plot of the fitted model. The R statistical package was used to construct contours and three dimensions quadratic response surfaces. These were used to study the effect or contribution of the combinations of ingredients on the response variable.

3.4.3 Response Trace Plot

Cornell (2002) defined the response trace as a plot of the estimated values using the fitted model along the component directions. Where components proportions are restricted by lower bounds and upper bounds of the form, $0 < L_i \leq x_i \leq U_i < 1$, a response trace can be used to determine the effect of each component on the response variable and to determine the optimal point for the experiment.

A reference mixture or blend is used other than the centroid of the simplex to work out an alternative direction which is an imaginary line projected from the reference mixture to the vertex $x_i = 1$. The rest of the co-ordinates are worked out by letting the proportions of q components at the reference mixture be $S = (s_1, s_2, \dots, s_q)$, such that $s_1 + s_2 + \cdots + s_q = 1$. Then changing each proportion of component i at s_i by Δ_i, where Δ_i can be positive or negative depending on the position of the proportion in reference to the reference mixture proportion s_i, as shown by equation (3.54).

$$x_i = s_i + \Delta_i \tag{3.54}$$

The remaining $q - 1$ components resulting from (3.54), were denoted by (3.55).

$$x_j = s_j - \frac{\Delta_i s_j}{1-s_i}, \; j = 1,2, \dots, q. \;\; j \neq i \tag{3.55}$$

The ratio of the proportions of components $j \; and \; k$ given by (3.56) must be the same as the ratio of the original components.

$$\frac{x_j}{x_k} = \frac{s_j(1-s_i-\Delta_i)}{s_k(1-s_i-\Delta_i)} = \frac{s_j}{s_k} \tag{3.56}$$

The y hat (estimated strength) values were obtained by substituting the values of x_i and x_j given in equations (3.54) and (3.55) above, in the regression equation (4.29).

3.4.4 Evaluating the D-, G- and I-Optimality of the Experiment

The research sought to find out if the optimal settings obtained corresponded with the optimality region of the tetrahedral simplex centroid design adopted. The D- and G-optimality criterion were evaluated and compared in two ways; between the two weighted designs and between the criterions, while I-optimality were evaluated and compared for the two weighted simplex designs.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 INTRODUCTION

This chapter aimed at reaching the optimal settings as discussed in chapter 3. Two weighted designs namely the Equally Weighted Simplex Centroid Axial Designs (EWSCAD) and Unequally Weighted Simplex Centroid Axial Designs (UWSCAD) were used. The slope information matrices were used to obtain D-, G-optimal values for the inscribed tetrahedral design. I-optimal values, the efficiencies and equivalence theorems for the two designs were evaluated and compared. Finally, a regression model was constructed from the experiment results and optimal settings of the concrete mixture were obtained using the response trace plot comparing with the current optimal strength range of the concrete strength class M25. The D-, G- and I-optimal values for the concrete model were compared with those of the tetrahedral weighted simplex centroid design adopted.

4.1 CONSTRUCTING AN INSCRIBED TETRAHEDRAL WSCD

The weighted simplex centroid design with a weight vector, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_q) \in \mathrm{T}_q$ where the weights satisfy the equation $\sum_{i=1}^{q} \alpha_i = 1$ is given by $\eta = \sum_{i=1}^{q} \alpha_i \eta_i$.

4.1.1 The Tetrahedral Axial Design

As discussed in section 3.1.1, this study adopted a tetrahedral design such that the vertices of the simplex were chosen a distance h from the main vertices of the original simplex which is a tetrahedron (1,0,0,0), (0,1,0,0), (0,0,1,0) and (0,0,0,1). This is equivalent to choosing $1 - h$ from the center of the axes.

The value h was arbitrarily selected for this study as 0.3 such that $1 - h \le \frac{q-1}{q}$, Cornell (1990) pg. 69, therefore the fifteen point axial simplex design became;

$$X' = \frac{1}{10}\begin{bmatrix} 7 & 1 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 1 & 2.5 \\ 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 1 & 4 & 3 & 3 & 1 & 3 & 2.5 \\ 1 & 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 4 & 3 & 1 & 3 & 3 & 2.5 \\ 1 & 1 & 1 & 7 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 3 & 3 & 3 & 2.5 \end{bmatrix} \tag{4.1}$$

The larger design η given in (4.1) is divided into sub designs $\eta_1, \eta_2, \eta_3\ and\ \eta_4$ as shown in (4.2).

$$\eta_1 = \begin{Bmatrix} 0.7 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.7 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.7 \end{Bmatrix}$$

$$\eta_2 = \begin{Bmatrix} 0.4 & 0.4 & 0.1 & 0.1 \\ 0.4 & 0.1 & 0.4 & 0.1 \\ 0.4 & 0.1 & 0.1 & 0.4 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.4 & 0.4 & 0.1 \end{Bmatrix} \tag{4.2}$$

$$\eta_3 = \begin{Bmatrix} 0.3 & 0.3 & 0.3 & 0.1 \\ 0.3 & 0.3 & 0.1 & 0.3 \\ 0.3 & 0.1 & 0.3 & 0.3 \\ 0.1 & 0.3 & 0.3 & 0.3 \end{Bmatrix}$$

$$\eta_4 = \{0.25 \quad 0.25 \quad 0.25 \quad 0.25\}$$

4.1.2 Determining the weights of the axial design

The values of the fourth order moments obtained from equation (3.5) are

$$\mu_4 = \frac{1}{2^4 - 1}\sum_{j=1}^{15} t_{1j}^4 = 0.023054$$

$$\mu_{31} = \frac{1}{2^4-1}\sum_{j=1}^{15} t_{1j}^3 t_{2j} = 0.006507$$

$$\mu_{22} = \tfrac{1}{2^4-1}\textstyle\sum_{j=1}^{15} t_{1j}^2 t_{2j}^2 = 0.004267 \qquad (4.3)$$

$$\mu_{211} = \frac{1}{2^4-1}\sum_{j=1}^{15} t_{1j}^2 t_{2j} t_{3j} = 0.002767$$

$$\mu_{1111} = \tfrac{1}{2^4-1}\textstyle\sum_{j=1}^{15} t_1 t_2 t_3 t_4 = 0.001807$$

Where $j = 1,2,\dots,15$ and t_1, t_2, t_3 and t_4 are given in (4.4).

$$t_1 = (\tfrac{7}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{4}{10},\tfrac{4}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{1}{10},\tfrac{1}{4})$$

$$t_2 = (\tfrac{1}{10},\tfrac{7}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{1}{10},\tfrac{3}{10},\tfrac{1}{4}) \qquad (4.4)$$

$$t_3 = (\tfrac{1}{10},\tfrac{1}{10},\tfrac{7}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{4}{10},\tfrac{3}{10},\tfrac{1}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{1}{4})$$

$$t_4 = (\tfrac{1}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{7}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{4}{10},\tfrac{4}{10},\tfrac{4}{10},\tfrac{1}{10},\tfrac{1}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{3}{10},\tfrac{1}{4})$$

The values of the lower order moments were calculated from the fourth order moments as shown in the set of equations (3.7) as, μ_{1111} =0.001807, $\mu_{111} = 0.010108, \mu_{11} = 0.052832, and\ \mu_1 = 0.25$ as shown in equation (4.3).

The elementary designs $\eta_1, \eta_2, \eta_3\ and\ \eta_4$ were used to generate the weighted Centroid design η with points as shown in (3.9). Using the moments, $\mu_{1111}, \mu_{111}, \mu_{11}\ and\ \mu_1$ the weights were determined by solving simultaneously the four equations in (4.5) and as were illustrated in (3.10).

$$0.001807 = \frac{7}{10000}\alpha_1 + \frac{16}{10000}\alpha_2 + \frac{27}{10000}\alpha_3 + \frac{1}{256}\alpha_4$$

$$0.010108 = \frac{11}{2000}\alpha_1 + \frac{1}{100}\alpha_2 + \frac{27}{2000}\alpha_3 + \frac{1}{64}\alpha_4 \qquad (4.5)$$

$$0.052832 = \frac{1}{25}\alpha_1 + \frac{11}{200}\alpha_2 + \frac{3}{50}\alpha_3 + \frac{1}{16}\alpha_4$$

$$0.25 = \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2 + \frac{1}{4}\alpha_3 + \frac{1}{4}\alpha_4$$

Letting

$\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3$

The values of alpha obtained were:

$\alpha_1 = \frac{4}{15}, \alpha_2 = \frac{6}{15}, \alpha_3 = \frac{4}{15}$ *and* $\alpha_4 = \frac{1}{15}$

The general equation $\alpha_i = \frac{\binom{q}{i}}{2^q - 1}, (i = 1,2,3,4) and\ q = 4$, confirmed the workings in (4.5).

4.1.3 Obtaining the information matrices for the Weighted Simplex Centroid Designs

In order to evaluate and compare optimality two designs namely, EWSCAD and UWSCAD were used. The amount of information which the design τ contains on the parameter subsystem $K'\theta$, is contained in the information matrix $C_k M(\tau) = LML'$. First, we obtain the coefficient matrix and its left inverse and then the moment matrices of EWSCAD and UWSCAD.

4.1.3.1 The coefficient matrix

The coefficient matrix K is the matrix that transforms the full parameter system to the subsystem of interest. The full system and the subsystem of interest are as stated in equations (3.3) and (3.4) respectively. Equation (4.6) give the full system coefficient matrix, while (3.15) gave the subsystem of interest.

$$\theta' = [\theta_{11}, \theta_{12}, \theta_{13}, \theta_{14}, \theta_{21}, \theta_{22}, \theta_{23}, \theta_{24}, \theta_{31}, \theta_{32}, \theta_{33}, \theta_{34}, \theta_{41}, \theta_{42}, \theta_{43}, \theta_{44}] \tag{4.6}$$

Equation (4.7) give the transpose of coefficient matrix K.

$$K' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{4.7}$$

The left inverse of a coefficient matrix determined using R-software is the inverse that helps to obtain the greatest determinant. The left inverse of K is $L = (K'K)^{-1}K'$.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.8}$$

4.1.3.2 Moment Matrices

The moment matrix obtained from the summation of the Kronecker product equation (4.9) is equivalent to the moment matrix equation in (3.19).

$$M(\eta_i) = \int (t \otimes t)(t \otimes t)' d\tau \qquad (4.9)$$

The moment matrices for η_1, η_2, η_3 and η_4 designs were worked out as shown in (4.10)

$M(\eta_1) =$

$$\frac{1}{10000}\begin{bmatrix}
601 & 88 & 88 & 88 & 88 & 25 & 16 & 16 & 88 & 16 & 25 & 16 & 88 & 16 & 16 & 25 \\
88 & 25 & 16 & 16 & 25 & 88 & 16 & 16 & 16 & 16 & 16 & 7 & 16 & 16 & 7 & 16 \\
88 & 16 & 25 & 16 & 16 & 16 & 16 & 7 & 25 & 16 & 88 & 16 & 16 & 7 & 16 & 16 \\
88 & 16 & 16 & 25 & 16 & 16 & 7 & 16 & 16 & 7 & 16 & 16 & 25 & 16 & 16 & 88 \\
88 & 25 & 16 & 16 & 25 & 88 & 16 & 16 & 16 & 16 & 16 & 7 & 16 & 16 & 7 & 16 \\
25 & 88 & 16 & 16 & 88 & 601 & 88 & 88 & 16 & 88 & 25 & 16 & 16 & 88 & 16 & 25 \\
16 & 16 & 16 & 7 & 16 & 88 & 25 & 16 & 16 & 25 & 88 & 16 & 7 & 16 & 16 & 16 \\
16 & 16 & 7 & 16 & 16 & 88 & 16 & 25 & 7 & 16 & 16 & 16 & 16 & 25 & 16 & 88 \\
88 & 16 & 25 & 16 & 16 & 16 & 16 & 7 & 25 & 16 & 88 & 16 & 16 & 7 & 16 & 16 \\
16 & 16 & 16 & 7 & 16 & 88 & 25 & 16 & 16 & 25 & 88 & 16 & 7 & 16 & 16 & 16 \\
25 & 16 & 88 & 16 & 16 & 25 & 88 & 16 & 88 & 88 & 601 & 88 & 16 & 16 & 88 & 25 \\
16 & 7 & 16 & 16 & 7 & 16 & 16 & 16 & 16 & 16 & 88 & 25 & 16 & 16 & 25 & 88 \\
88 & 16 & 16 & 25 & 16 & 16 & 7 & 16 & 16 & 7 & 16 & 16 & 25 & 16 & 16 & 88 \\
16 & 16 & 7 & 16 & 16 & 88 & 16 & 25 & 7 & 16 & 16 & 16 & 16 & 25 & 16 & 88 \\
16 & 7 & 16 & 16 & 7 & 16 & 16 & 16 & 16 & 16 & 88 & 25 & 16 & 16 & 25 & 88 \\
25 & 16 & 16 & 88 & 16 & 25 & 16 & 88 & 16 & 16 & 25 & 88 & 88 & 88 & 88 & 601
\end{bmatrix}$$

$M(\eta_2) =$

$$\frac{1}{10000}\begin{bmatrix}
129 & 66 & 66 & 66 & 66 & 54 & 28 & 28 & 66 & 28 & 54 & 28 & 66 & 28 & 28 & 54 \\
66 & 54 & 28 & 28 & 54 & 66 & 28 & 28 & 28 & 28 & 28 & 16 & 28 & 28 & 16 & 28 \\
66 & 28 & 54 & 28 & 28 & 28 & 28 & 16 & 54 & 28 & 66 & 28 & 28 & 16 & 28 & 28 \\
66 & 28 & 28 & 54 & 28 & 28 & 16 & 28 & 28 & 16 & 28 & 28 & 54 & 28 & 28 & 66 \\
66 & 54 & 28 & 28 & 54 & 66 & 28 & 28 & 28 & 28 & 28 & 16 & 28 & 28 & 16 & 28 \\
54 & 66 & 28 & 28 & 66 & 129 & 66 & 66 & 28 & 66 & 54 & 28 & 28 & 66 & 28 & 54 \\
28 & 28 & 28 & 16 & 28 & 66 & 54 & 28 & 28 & 54 & 66 & 28 & 16 & 28 & 28 & 28 \\
28 & 28 & 16 & 28 & 28 & 66 & 28 & 54 & 16 & 28 & 28 & 28 & 28 & 54 & 28 & 66 \\
66 & 28 & 54 & 28 & 28 & 28 & 28 & 16 & 54 & 28 & 66 & 28 & 28 & 16 & 28 & 28 \\
28 & 28 & 28 & 16 & 28 & 28 & 54 & 28 & 28 & 54 & 66 & 28 & 16 & 28 & 28 & 28 \\
54 & 28 & 66 & 28 & 28 & 54 & 66 & 28 & 66 & 66 & 129 & 66 & 28 & 28 & 66 & 54 \\
28 & 16 & 28 & 28 & 16 & 28 & 28 & 28 & 28 & 28 & 66 & 54 & 28 & 28 & 54 & 66 \\
66 & 28 & 28 & 54 & 28 & 28 & 16 & 28 & 28 & 16 & 28 & 28 & 54 & 28 & 28 & 66 \\
28 & 28 & 16 & 28 & 28 & 66 & 28 & 54 & 16 & 28 & 28 & 28 & 28 & 54 & 28 & 66 \\
28 & 16 & 28 & 28 & 16 & 28 & 28 & 28 & 28 & 28 & 66 & 54 & 28 & 28 & 54 & 66 \\
54 & 28 & 28 & 66 & 28 & 54 & 28 & 66 & 28 & 28 & 54 & 66 & 66 & 66 & 66 & 129
\end{bmatrix}$$

$$M(\eta_3) = \frac{1}{10000}\begin{bmatrix} 61 & 48 & 48 & 48 & 48 & 45 & 36 & 36 & 48 & 36 & 45 & 36 & 48 & 36 & 36 & 45 \\ 48 & 45 & 36 & 36 & 45 & 48 & 36 & 36 & 36 & 36 & 36 & 27 & 36 & 36 & 27 & 36 \\ 48 & 36 & 45 & 36 & 36 & 36 & 36 & 27 & 45 & 36 & 48 & 36 & 36 & 27 & 36 & 36 \\ 48 & 36 & 36 & 45 & 36 & 36 & 27 & 45 & 45 & 27 & 36 & 36 & 45 & 36 & 36 & 48 \\ 48 & 45 & 36 & 36 & 45 & 48 & 36 & 36 & 36 & 36 & 36 & 27 & 36 & 36 & 27 & 36 \\ 45 & 48 & 36 & 36 & 48 & 61 & 48 & 48 & 36 & 48 & 45 & 36 & 36 & 48 & 36 & 45 \\ 36 & 36 & 36 & 27 & 36 & 48 & 45 & 36 & 36 & 45 & 48 & 36 & 27 & 36 & 36 & 36 \\ 36 & 36 & 27 & 45 & 36 & 48 & 36 & 45 & 27 & 36 & 36 & 36 & 36 & 45 & 36 & 48 \\ 48 & 36 & 45 & 45 & 36 & 36 & 36 & 27 & 45 & 36 & 48 & 36 & 36 & 27 & 36 & 36 \\ 36 & 36 & 36 & 27 & 36 & 48 & 45 & 36 & 36 & 45 & 48 & 36 & 27 & 36 & 36 & 36 \\ 45 & 36 & 48 & 36 & 36 & 45 & 48 & 36 & 48 & 48 & 61 & 48 & 36 & 36 & 48 & 45 \\ 36 & 27 & 36 & 36 & 27 & 36 & 36 & 36 & 36 & 36 & 48 & 45 & 36 & 36 & 45 & 48 \\ 48 & 36 & 36 & 45 & 36 & 36 & 27 & 36 & 36 & 27 & 36 & 36 & 45 & 36 & 36 & 48 \\ 36 & 36 & 27 & 36 & 36 & 48 & 36 & 45 & 27 & 36 & 36 & 36 & 36 & 45 & 36 & 48 \\ 36 & 27 & 36 & 36 & 27 & 36 & 36 & 36 & 36 & 36 & 48 & 45 & 36 & 36 & 45 & 48 \\ 45 & 36 & 36 & 48 & 36 & 45 & 36 & 48 & 36 & 36 & 45 & 48 & 48 & 48 & 48 & 61 \end{bmatrix} \quad (4.10)$$

$$M(\eta_4) = \frac{1}{10000}\begin{bmatrix} 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \\ 39 & 39 & 39 & 39 & 39 & 39 & 39 & 3.9 & 39 & 39 & 39 & 39 & 39 & 39 & 39 & 39 \end{bmatrix}$$

The Equally weighted simplex centroid design is the one such that all the α_i are qual. The moment matrices for the designs were worked using the R- statistical program using the formula;

$$M_{ne} = \frac{1}{4}M(\eta_1) + \frac{1}{4}M(\eta_2) + \frac{1}{4}M(\eta_3) + \frac{1}{4}M(\eta_4) \quad (4.11)$$

$$M_{\eta e} = \frac{1}{10000}\begin{bmatrix} 207 & 60 & 60 & 60 & 60 & 41 & 30 & 30 & 60 & 30 & 41 & 30 & 60 & 30 & 30 & 41 \\ 60 & 41 & 30 & 30 & 41 & 60 & 30 & 30 & 30 & 30 & 30 & 22 & 30 & 30 & 22 & 30 \\ 60 & 30 & 41 & 30 & 30 & 30 & 30 & 22 & 41 & 30 & 60 & 30 & 30 & 22 & 30 & 30 \\ 60 & 30 & 30 & 41 & 30 & 30 & 22 & 30 & 30 & 22 & 30 & 30 & 41 & 30 & 30 & 60 \\ 60 & 41 & 30 & 30 & 41 & 60 & 30 & 30 & 30 & 30 & 30 & 22 & 30 & 30 & 22 & 30 \\ 41 & 60 & 30 & 30 & 60 & 207 & 60 & 60 & 30 & 60 & 41 & 30 & 30 & 60 & 30 & 41 \\ 30 & 30 & 30 & 22 & 30 & 60 & 41 & 30 & 30 & 41 & 60 & 30 & 22 & 30 & 30 & 30 \\ 30 & 30 & 22 & 30 & 30 & 60 & 30 & 41 & 22 & 30 & 30 & 30 & 30 & 41 & 30 & 60 \\ 60 & 30 & 41 & 30 & 30 & 30 & 30 & 22 & 41 & 30 & 60 & 30 & 30 & 22 & 30 & 30 \\ 30 & 30 & 30 & 22 & 30 & 60 & 41 & 30 & 30 & 41 & 60 & 30 & 22 & 30 & 30 & 30 \\ 41 & 30 & 60 & 30 & 30 & 41 & 60 & 30 & 60 & 60 & 207 & 60 & 30 & 30 & 60 & 41 \\ 30 & 22 & 30 & 30 & 22 & 30 & 30 & 30 & 30 & 30 & 60 & 41 & 30 & 30 & 41 & 60 \\ 60 & 30 & 30 & 41 & 30 & 30 & 22 & 30 & 30 & 22 & 30 & 30 & 41 & 30 & 30 & 60 \\ 30 & 30 & 22 & 30 & 30 & 60 & 30 & 41 & 22 & 30 & 30 & 30 & 30 & 41 & 30 & 60 \\ 30 & 22 & 30 & 30 & 22 & 30 & 30 & 30 & 30 & 30 & 60 & 41 & 30 & 30 & 41 & 60 \\ 41 & 30 & 30 & 60 & 30 & 41 & 30 & 60 & 30 & 30 & 41 & 60 & 60 & 60 & 60 & 207 \end{bmatrix} \tag{4.12}$$

The moment matrix for the unequally weighted simplex centroid design ($M_{\eta u}$) was worked out using R program given by (4.14).

$$M_{\eta u} = \frac{4}{15}M(\eta_1) + \frac{6}{15}M(\eta_2) + \frac{4}{15}M(\eta_3) + \frac{1}{15}M(\eta_4) \tag{4.13}$$

$$M_{\eta u} = \frac{1}{10000}\begin{bmatrix} 231 & 65 & 65 & 65 & 65 & 43 & 28 & 28 & 65 & 28 & 43 & 28 & 65 & 28 & 28 & 43 \\ 65 & 43 & 28 & 28 & 43 & 65 & 28 & 28 & 28 & 28 & 28 & 18 & 28 & 28 & 18 & 28 \\ 65 & 28 & 43 & 28 & 28 & 28 & 28 & 18 & 43 & 28 & 65 & 28 & 28 & 18 & 28 & 28 \\ 65 & 28 & 28 & 43 & 28 & 28 & 18 & 28 & 28 & 18 & 28 & 28 & 43 & 28 & 28 & 65 \\ 65 & 43 & 28 & 28 & 43 & 65 & 28 & 28 & 28 & 28 & 28 & 18 & 28 & 28 & 18 & 28 \\ 43 & 65 & 28 & 28 & 65 & 231 & 65 & 65 & 28 & 65 & 43 & 28 & 28 & 65 & 28 & 43 \\ 28 & 28 & 28 & 18 & 28 & 65 & 43 & 28 & 28 & 43 & 65 & 28 & 18 & 28 & 28 & 28 \\ 28 & 28 & 18 & 28 & 28 & 65 & 28 & 43 & 18 & 28 & 28 & 28 & 28 & 43 & 28 & 65 \\ 65 & 28 & 43 & 28 & 28 & 28 & 28 & 18 & 43 & 28 & 65 & 28 & 28 & 18 & 28 & 28 \\ 28 & 28 & 28 & 18 & 28 & 65 & 43 & 28 & 28 & 43 & 65 & 28 & 18 & 28 & 28 & 28 \\ 43 & 28 & 65 & 28 & 28 & 43 & 65 & 28 & 65 & 65 & 231 & 65 & 28 & 28 & 65 & 43 \\ 28 & 18 & 28 & 28 & 18 & 28 & 28 & 28 & 28 & 28 & 65 & 43 & 28 & 28 & 43 & 65 \\ 65 & 28 & 28 & 43 & 28 & 28 & 18 & 28 & 28 & 18 & 28 & 28 & 43 & 28 & 28 & 65 \\ 28 & 28 & 18 & 28 & 28 & 65 & 28 & 43 & 18 & 28 & 28 & 28 & 28 & 43 & 28 & 65 \\ 28 & 18 & 28 & 28 & 18 & 28 & 28 & 28 & 28 & 28 & 65 & 43 & 28 & 28 & 43 & 65 \\ 43 & 28 & 28 & 65 & 28 & 43 & 28 & 65 & 28 & 28 & 43 & 65 & 65 & 65 & 65 & 231 \end{bmatrix} \tag{4.14}$$

4.1.3.3 The Improved Information Matrices

The information matrix $C_e = LM_{ne}L'$ for the weighted centroid design with equal weights was given by the matrix (4.15), while (4.16) denote the Information matrix $C_u = LM_{nu}L'$ for the unequally weighted simplex centroid axial design.

$$C_e = \frac{1}{10000}\begin{bmatrix} 207 & 41 & 41 & 41 & 120 & 120 & 120 & 60 & 60 & 60 \\ 41 & 207 & 41 & 41 & 120 & 60 & 60 & 120 & 120 & 60 \\ 41 & 41 & 207 & 41 & 60 & 120 & 60 & 120 & 60 & 120 \\ 41 & 41 & 41 & 207 & 60 & 60 & 120 & 60 & 120 & 120 \\ 120 & 120 & 60 & 60 & 164 & 120 & 120 & 120 & 120 & 88 \\ 120 & 60 & 120 & 60 & 120 & 164 & 120 & 120 & 88 & 120 \\ 120 & 60 & 60 & 120 & 120 & 120 & 164 & 88 & 120 & 120 \\ 60 & 120 & 120 & 60 & 120 & 120 & 88 & 164 & 120 & 120 \\ 60 & 120 & 60 & 120 & 120 & 88 & 120 & 120 & 164 & 120 \\ 60 & 60 & 120 & 120 & 88 & 120 & 120 & 120 & 120 & 164 \end{bmatrix} \quad (4.15)$$

$$C_u = \frac{1}{10000}\begin{bmatrix} 231 & 43 & 43 & 43 & 130 & 130 & 130 & 56 & 56 & 56 \\ 43 & 231 & 43 & 43 & 130 & 56 & 56 & 130 & 130 & 56 \\ 43 & 43 & 231 & 43 & 56 & 130 & 56 & 130 & 56 & 130 \\ 43 & 43 & 43 & 231 & 56 & 56 & 130 & 56 & 130 & 130 \\ 130 & 130 & 56 & 56 & 172 & 112 & 112 & 112 & 112 & 72 \\ 130 & 56 & 130 & 56 & 112 & 172 & 112 & 112 & 72 & 112 \\ 130 & 56 & 56 & 130 & 112 & 112 & 172 & 72 & 112 & 112 \\ 56 & 130 & 130 & 56 & 112 & 112 & 72 & 172 & 112 & 112 \\ 56 & 130 & 56 & 130 & 112 & 72 & 112 & 112 & 172 & 112 \\ 56 & 56 & 130 & 130 & 72 & 112 & 112 & 112 & 112 & 172 \end{bmatrix} \quad (4.16)$$

The improved information matrix D_c (slope information matrix) was derived from the information matrix C such that $D_c = HCH'$. H is the invariant matrix of derivatives of the elements of the equation (3.22), shown by equation (3.23).

The H-invariant matrices for the design points $\eta_1=\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}$, $\eta_2=\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}$, $\eta_3=\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$ and $\eta_4=\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ were given by (4.17).

$$H_1 = \frac{1}{10}\begin{bmatrix} 14 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 7 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 7 & 0 & 1 & 1 \end{bmatrix}$$

$$H_2 = \frac{1}{10}\begin{bmatrix} 8 & 0 & 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 4 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 4 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 4 & 0 & 4 & 1 \end{bmatrix} \quad (4.17)$$

$$H_3 = \frac{1}{10}\begin{bmatrix} 6 & 0 & 0 & 0 & 3 & 1 & 3 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 3 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 7 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 3 & 0 & 3 & 3 \end{bmatrix}$$

$$H_4 = \frac{1}{10}\begin{bmatrix} 5 & 0 & 0 & 0 & 2.5 & 2.5 & 2.5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 2.5 & 0 & 0 & 2.5 & 2.5 & 0 \\ 0 & 0 & 5 & 0 & 0 & 2.5 & 0 & 2.5 & 0 & 2.5 \\ 0 & 0 & 0 & 5 & 0 & 0 & 2.5 & 0 & 25 & 2.5 \end{bmatrix}$$

The gradient techniques help in computation of stationary points of the steepest ascent or descent, which is the most optimal point of a response surface.

This research aimed at obtaining slope information matrices at different points of the design in order to get the D-optimal values.

4.2 EVALUATING D-G- I- OPTIMALITY OF THE DESIGN USING TWO WSCDs

The optimal values of the D-, G- and I- were obtained for the tetrahedral axial design, as well as the corresponding efficiencies and the equivalence theorems.

4.2.1 D- slope optimal Values, Efficiencies and Equivalence theorems

The D criterion is the most commonly used optimality criterion that seeks designs that maximize the determinant of the information matrix. The aim of D-optimality is essentially a parameter estimation. For the comparison of different criteria and for applying the theory of information functions, the version $\emptyset_0(C) = (detC)^{\frac{1}{s}}$ was appropriate for the maximization of the determinant of the information matrices. This is the same as minimizing the determinant of the dispersion matrices given by the formula$(detC)^{-1} = det(C)^{-1}$.

Now the D-optimality criterion for each design point for the two designs was given by $(\det(D_c))^{\frac{1}{s}}$ as derived in (4.18) and (4.19) respectively. The efficiencies were worked out as given in (3.37).

$$D_{c1e} = H_1 C_e H'_1 = \frac{1}{1000000}\begin{bmatrix} 51864 & 18552 & 18552 & 18552 \\ 18552 & 17112 & 11880 & 11880 \\ 18552 & 11880 & 17112 & 11880 \\ 18552 & 11880 & 17112 & 17112 \end{bmatrix}$$

$$D_{s1e} = (\det(D_{c1e}))^{\frac{1}{10}} = 0.176753$$

$$D_{c2e} = H_2 C_e H'_2 = \frac{1}{1000000}\begin{bmatrix} 29880 & 17184 & 14136 & 14136 \\ 17184 & 29880 & 14136 & 14136 \\ 14136 & 14136 & 16320 & 11304 \\ 14136 & 14136 & 11304 & 16320 \end{bmatrix}$$

$$D_{s2e} = (\det(D_{c2e}))^{\frac{1}{10}} = 0.1779787 \qquad (4.18)$$

$$D_{c3e} = H_3 C_e H'_3 = \frac{1}{1000000}\begin{bmatrix} 24248 & 14760 & 14760 & 12888 \\ 14760 & 24248 & 14760 & 12888 \\ 14760 & 14760 & 24248 & 12888 \\ 12888 & 12888 & 12888 & 16056 \end{bmatrix}$$

$$D_{s3e} = (\det(D_{c3e}))^{\frac{1}{10}} = 0.1785106$$

$$D_{c4e} = H_4 C_e H'_4 = \frac{1}{100000}\begin{bmatrix} 2175 & 1365 & 1365 & 1365 \\ 1365 & 2175 & 1365 & 1365 \\ 1365 & 1365 & 2175 & 1365 \\ 1365 & 1365 & 1365 & 2175 \end{bmatrix}$$

$$D_{s4e} = (\det(D_{c4e}))^{\frac{1}{10}} = 0.1787608$$

The D-optimal values for the four designs are $D_{s1e} = 0.176753$, $D_{s2e} = 0.1779787$, $D_{s3e} = 0.1785106$ and $D_{s4e} = 0.1787608$.

This showed that η_4 obtained the highest D-optimal value followed by η_3, then η_2 and η_1 having the least value.

The improved (slope) information matrices for the weighted centroid design with unequal weights and the corresponding Determinant criterion values were obtained in (4.19).

$$D_{c1u} = H_1 C_u H'_1 = \frac{1}{1000000}\begin{bmatrix} 57384 & 19360 & 19360 & 19360 \\ 19360 & 17736 & 11056 & 11056 \\ 19360 & 11056 & 17736 & 11056 \\ 19360 & 11056 & 11056 & 17736 \end{bmatrix}$$

$$D_{s1u} = (\det(D_{c1u}))^{\frac{1}{10}} = 0.186846$$

$$D_{c2u} = H_2 C_u H'_2 = \frac{1}{1000000}\begin{bmatrix} 32376 & 17776 & 13876 & 13876 \\ 17776 & 32376 & 13876 & 13876 \\ 13876 & 13876 & 16656 & 10336 \\ 13876 & 13876 & 10336 & 16656 \end{bmatrix}$$

$$D_{s2u} = (\det(D_{c2u}))^{\frac{1}{10}} = 0.1875484 \tag{4.19}$$

$$D_{c3u} = H_3 C_u H'_3 = \frac{1}{1000000}\begin{bmatrix} 25864 & 14704 & 14704 & 12320 \\ 14704 & 25864 & 14704 & 12320 \\ 14704 & 14704 & 25864 & 12320 \\ 12320 & 12320 & 12320 & 16296 \end{bmatrix}$$

$$D_{s3u} = (\det(D_{c3u}))^{\frac{1}{10}} = 0.1881068$$

$$D_{c4u} = H_4 C_u H'_4 = \frac{1}{100000}\begin{bmatrix} 2295 & 1330 & 1330 & 1330 \\ 1330 & 2295 & 1330 & 1330 \\ 1330 & 1330 & 2295 & 1330 \\ 1330 & 1330 & 1330 & 2295 \end{bmatrix}$$

$$D_{s4u} = (\det(D_{c4u}))^{\frac{1}{10}} = 0.1884468$$

The four D-optimal values obtained for the four design points are $D_{s1u} = 0.186846$,

$D_{s2u} = 0.187484$, $D_{s3u} = 0.1881068$ and $D_{s4u} = 0.1884468$.

Similarly, the design η_4 had the largest value and hence the most optimal, while η_1 had the lowest, as was the case for the equally weighted design.

Table 4. 1: D-optimal Slope values and Efficiencies

Design	Blends	D-Optimal values EWSCAD	D-Optimal values UWSCAD	Efficiency $D_{eff} = \left\{\frac{\lvert D_{siu}\rvert}{\lvert D_{sie}\rvert}\right\}^{\frac{1}{10}}$
η_1	$\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}$	0.176753	0.186846	1.0571
η_2	$\frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}$	0.1779787	0.1875484	1.0534
η_4	$\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}$	0.1785106	0.1881068	1.0538
η_4	$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	0.1787608	0.1884468	1.0542

The centroid in the two designs was the most D-optimal point. Comparing the efficiencies, UWSCAD proved to be a better design with η_1 obtaining the highest efficiency.

Optimality was obtained if the normality inequality shown in (3.36) is satisfied. The D criterion is the matrix mean ϕ_p where $p = 0$. The matrix C_e^p which is the information matrix of the EWSCAD is a 10×10 matrix of ones and $trace\ C_e^p = 10$. The matrix C_e^{p+1} is the same as C_e is as shown in (4.15). The generalized inverse G of the moment matrix of this design was indicated as Mnp in R-codes in Appendix II, the D Equivalence theorem at each point of this design were given in table 4.2.

Table 4.2: D- Equivalence Theorem for EWSCAD

DESIGN	$f'(t)GKC_e^{p+1}K'G'f(t)$		$trace\ C_e^p$	Remark
η_1	15.3782	$>$	10	Out of range
η_2	16.9803	$>$	10	Out of range
η_3	5.4968	$<$	10	Within range
η_4	1.8289	$<$	10	Within range

Equivalently for UWSCAD, C_u^p has its trace equal to 10 and in addition$C_u^{p+1} = C_u$, is as given in (4.16). The generalized inverse of the moment matrix of the unequally weighted centroid design Mnn is as indicated in R-codes in Appendix II. Table 4.3 shows the values for the equivalence theorem of this design.

Table 4.3: D- Equivalence Theorem for UWSCAD

BLEND	$f'(t)GKC_u^{p+1}K'G'f(t)$		$trace\ C_u^p$	Remark
$\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}$	14.3118	>	10	Out of range
$\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}$	11.6188	>	10	Out of range
$\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$	4.7961	<	10	Within range
$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	2.3319	<	10	Within range

For both designs the blends $\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$ and $\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},$ satisfied the equivalence theorem.

4.2.2 G-slope optimality Values, Efficiencies and Equivalence theorems

The equation (3.37) was utilized to work out the G-optimal values at each point of the designs. We obtained the maximum of $f^T(x)C^{-1}(\xi)f(x)$ for every design point, and then obtained the minimum of the maximum variances. The improved information matrices given in (4.18) and (4.19) as $D_{ciw}^{-1}\ i = 1,2,3,4\ and\ w = e,u$ were utilised instead of C_i^{-1}.

The G-optimal values for the EWSCAD were obtained by equation (4.20).

$$MinMax\big(d(x\xi)\big) = MinMaxf'(x)D_{cie}^{-1}f(x) \qquad (4.20)$$

where *f(x)* are the different design points. The set of equations (4.21) gave the optimal values at each point.

$$\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c1e}^{-1} \left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right) = 12.6846$$

$$\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c1e}^{-1} \left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right) = 56.0556$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right)' D_{c1e}^{-1} \left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right) = 56.0556$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right)' D_{c1e}^{-1} \left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right) = 56.0556$$

$$\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c2e}^{-1} \left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right) = 10.4983$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right)' D_{c2e}^{-1} \left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right) = 17.0446$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right)' D_{c2e}^{-1} \left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right) = 17.0446$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right)' D_{c2e}^{-1} \left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right) = 17.0446 \tag{4.21}$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right)' D_{c2e}^{-1} \left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right) = 22.1439$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right)' D_{c2e}^{-1} \left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right) = 17.0446$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right)' D_{c3e}^{-1} \left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right) = 6.9945$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right)' D_{c3e}^{-1} \left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right) = 8.4233$$

$$\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right)' D_{c3e}^{-1} \left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right) = 8.4233$$

$$\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right)' D_{c3e}^{-1} \left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right) = 8.4233$$

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)' D_{c4e}^{-1}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) = 3.9872$$

The G-optimal values for the UWSCAD were given by the equation (4.22).

$$MinMax\big(d(x\xi)\big) = MinMax f'(x) D_{ciu}^{-1} f(x) \tag{4.22}$$

where *f(x)* are the different design points. The optimal values at each design point were calculated as shown in the set of equations (4.23).

$$\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c1u}^{-1}\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right) = 11.2853$$

$$\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c1u}^{-1}\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right) = 46.6038$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right)' D_{c1u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right) = 46.6038$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right)' D_{c1u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right) = 46.6038$$

$$\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right)' D_{c2u}^{-1}\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right) = 8.9116$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right)' D_{c2u}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right) = 14.8380$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right)' D_{c2u}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right) = 14.8380$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right)' D_{c2u}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right) = 14.8380 \tag{4.23}$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right)' D_{c2u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right) = 20.8184$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right)' D_{c2u}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right) = 14.8380$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right)' D_{c3u}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right) = 6.1411$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right)' D_{c3u}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right) = 7.9170$$

$$\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right)' D_{c3u}^{-1}\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right) = 7.9170$$

$$\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right)' D_{c3u}^{-1}\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right) = 7.9170$$

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)' D_{c4u}^{-1}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) = 3.9777.$$

Table 4.4: G-optimal values and Efficiency

Design	Blends	G slope-optimal values EWSCAD	G slope-optimal values UWSCAD	Efficiencies $G_{ef} = \frac{f'(t)D_{ciu}^{-1}f(t)}{f'(t)D_{cie}^{-1}f(t)}$
η_1	$\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}$	56.0556	46.6038	0.8314
η_2	$\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}$	22.1439	20.8184	0.9401
η_3	$\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$	8.4233	7.9170	0.9399
η_4	$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	3.9872	3.9777	0.9976

The table shows that in both cases the Centroid obtained the lowest G-optimality values, hence the best G- optimal points. Using equation (3.43) the efficiencies calculated at the given design points, showed that the EWSCAD design was more efficient, with the centroid being the most efficient.

The G-optimality equivalence theorem explained in section 3.2.4 showed that equivalence was obtained when the equation (3.37) was satisfied. The design chosen produced a moment matrix that is singular the information matrix was used for this study since it gives the best information for the subsystem of interest.

The equations (4.15) and (4.16) gave the information matrices of EWSCAD and UWSCAD. Their inverses that were run in the R program showed that $traceC_eC_e^{-1} = 10$ and $traceC_uC_u^{-1} = 10$, $k = 10$ is the size of the information matrices C_e and C_u. This showed that the two designs satisfied the G- Equivalence theorem.

4.2.3 I-Optimality Values, Efficiencies and Equivalence theorems

The I-optimality values were worked out as shown in (3.39).

Working out the integrals of the 10×10 matrix were given by the set of equations (4.24). The components are represented by $i, j, k = 1,2,3,4$, while $n\ and\ m$ represent the rows and columns of the moment matrix L, the constant $K = \Gamma(4) = 6$. The matrix L is a symmetric matrix along the main diagonal.

$$M_{nn} = K \int t_i^4 dt_i = \frac{3!4!}{(4+4-1)!} = \frac{1}{35},\ i = n = 1,2,3,4, n = m.$$

$$M_{nm} = K \int t_i^2 t_j^2 dt_i dt_j = \frac{1}{210},\ (i \neq j), (n, m) = 1,2,3,4, n \neq m. \qquad (4.24)$$

$$M_{nn} = K \int t_i^2 t_j^2 dt_i dt_j = \frac{1}{210},\ (i \neq j),\ n = 5,6,7,8,9,10,\ n = m.$$

$$M_{nm} = K \int t_i^3 t_j dt_i dt_j = \frac{1}{140}, i \neq j.$$

$(n,m) = (1,5),(1,6),(1,7),(2,5),(2,8),(2,9),(3,6),(3,8),(3,10),(4,7),(4,9),(4,10), n \neq m$

$M_{nm} = K\int t_i^2 t_2 t_3 dt_i dt_j dt_k = \frac{1}{420},\ i \neq j \neq k.$

$(n,m) = (1,8),(1,9),(1,10),(2,6),(2,7),(2,10),(3,5),(3,7),(3,9),(4,5),(4,6),(4,8),(5,6),$

$(5,7),(5,8),(5,9),(6,7),(6,8),(6,10),(7,9),(7,10),(8,9),(8,10),(9,10), n \neq m.$

$$M_{nm} = K\int t_1 t_2 t_3 t_4 dt_1 dt_2 dt_3 dt_4 = \frac{1}{840}, (n,m) = (5,10),(6,9),(7,8), n \neq m.$$

The summary of above workings are given in the moment and information matrices (4.25) and (4.26) respectively.

$$B = \frac{1}{840}\begin{bmatrix} 24 & 4 & 4 & 4 & 6 & 6 & 6 & 2 & 2 & 2 \\ 4 & 24 & 4 & 4 & 6 & 2 & 2 & 6 & 6 & 2 \\ 4 & 4 & 24 & 4 & 2 & 6 & 2 & 6 & 2 & 6 \\ 4 & 4 & 4 & 24 & 2 & 2 & 6 & 2 & 6 & 6 \\ 6 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 \\ 6 & 2 & 6 & 2 & 2 & 4 & 2 & 2 & 1 & 2 \\ 6 & 2 & 2 & 6 & 2 & 2 & 4 & 1 & 2 & 2 \\ 2 & 6 & 6 & 2 & 2 & 2 & 1 & 4 & 2 & 2 \\ 2 & 6 & 2 & 6 & 2 & 1 & 2 & 2 & 4 & 2 \\ 2 & 2 & 6 & 6 & 1 & 2 & 2 & 2 & 2 & 4 \end{bmatrix} \tag{4.25}$$

The matrix $L = \Gamma(q) \times B = 6B$

$$L = \frac{1}{140}\begin{bmatrix} 24 & 4 & 4 & 4 & 6 & 6 & 6 & 2 & 2 & 2 \\ 4 & 24 & 4 & 4 & 6 & 2 & 2 & 6 & 6 & 2 \\ 4 & 4 & 24 & 4 & 2 & 6 & 2 & 6 & 2 & 6 \\ 4 & 4 & 4 & 24 & 2 & 2 & 6 & 2 & 6 & 6 \\ 6 & 6 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 1 \\ 6 & 2 & 6 & 2 & 2 & 4 & 2 & 2 & 1 & 2 \\ 6 & 2 & 2 & 6 & 2 & 2 & 4 & 1 & 2 & 2 \\ 2 & 6 & 6 & 2 & 2 & 2 & 1 & 4 & 2 & 2 \\ 2 & 6 & 2 & 6 & 2 & 1 & 2 & 2 & 4 & 2 \\ 2 & 2 & 6 & 6 & 1 & 2 & 2 & 2 & 2 & 4 \end{bmatrix} \tag{4.26}$$

For optimality LHS of the I-optimality equivalence inequality of each design point should be less or equal to $trace[C_e^{-1}L] = 73.9209$ and $trace[C_u^{-1}L] = 64.9382$ for

EWSCAD and UWSCAD respectively. The left hand side of the equivalence theorem values given by

$f'(t)C_e^{-1}LC_e^{-1}f(t)$ and $f'(t)C_u^{-1}LC_u^{-1}f(t)$ for both designs were given in tables 4.5and 4.6 respectively.

Table 4.5: I-optimality Equivalence Theorem for EWSCAD.

BLEND	$f'(t)C_e^{-1}LC_e^{-1}f(t)$		$trace[C_e^{-1}L]$	Optimality
$\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}$	316.7287	>	73.9209	Not optimal
$\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}$	48.9044	<	73.9209	Optimal
$\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$	12.5650	<	73.9209	Optimal
$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	5.3604	<	73.9209	Optimal

Table 4.6: I-optimality Equivalence Theorem for UWSCAD.

BLENDS	$f'(t)C_u^{-1}LC_u^{-1}f(t)$		$trace[C_u^{-1}L]$	Optimality
$\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}$	292.0415	>	64.9382	Not optimal
$\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}$	36.3542	<	64.9382	Optimal
$\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}$	9.6759	<	64.9382	Optimal
$\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$	3.6996	<	64.9382	Optimal

The design points $\eta_2, \eta_3 and \eta_4$, for the two designs attained the optimality according to the equivalence theorem. The I-efficiency given by the equation (3.44) was utilized to compare the efficiency of the two designs namely the EWSCAD and UWSCAD as shown below

$I_{eff} = \frac{64.9382}{73.9209} \times 100 = 87.85\%$. This implied that the EWSCAD was a more I-efficient design than UWSCAD.

4.3 FITTING A SECOND DEGREE KRONECKER MODEL FOR THE EXPERIMENT

The design adopted was a regular Tetrahedron inscribed in another regular Tetrahedron as described in section 3.1.1 and shown in (4.1).

$$X' = \frac{1}{10}\begin{bmatrix} 7 & 1 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 1 & 3 & 3 & 3 & 1 & 2.5 \\ 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 1 & 4 & 3 & 3 & 1 & 3 & 2.5 \\ 1 & 1 & 7 & 1 & 1 & 4 & 1 & 1 & 4 & 4 & 3 & 1 & 3 & 3 & 2.5 \\ 1 & 1 & 1 & 7 & 1 & 1 & 4 & 4 & 4 & 1 & 1 & 3 & 3 & 3 & 2.5 \end{bmatrix}$$

The mix for the low cost houses chosen depended on the process of trial mix shown in section 3.3.1. The trial minimum mix table 4.7 and trial maximum mix table 4.8 showed that ratios required for the experiment were 0.55:1:1.49:2.9 and 0.52:1:1.35:2.75. These are very close to the traditional M25 concrete class ratio 1:1.5:3 for cement, sand (fine aggregate), and ballast (coarse aggregate). The minimum and maximum water cement ratio for these mixes were selected from the lower values of the table 2, figure 4 values 0.66 and 0.61 respectively of (Teychenne , Franklin , & Erntroy , 1988), and the arbitrarily chosen values for water content were 0.55 and 0.52 respectively. The recommended water content values for these classes should be a minimum of 0.5 and a maximum of 0.88. The less the water used the higher strength of concrete. This means that due to selection of lower water cement ratio the results obtained changed the minimum strength of class M25. The experts recommend the usage of class M25 for a stronger plinth.

The water demanded depended on the required workability of concrete. The use of admixtures would help to reduce the usage of cement and hence increase the workability of concrete. The common person may not access the admixtures easily; therefore, the experimenter opted to use the primary ingredients namely; Water, Cement, Sand (fine aggregate) and Ballast (coarse aggregate).

4.3.1 The Transformation Design

The experimental points were obtained by the use of the plan shown in table 4.7 and 4.8 extracted from the work of (Teychenne , Franklin , & Erntroy , 1988) pg. 11. It was modified as table A in appendix II. The general custom in the United Kingdom was to specify concrete by a system of proportions for example 1:2:4 for cement: fine aggregate: coarse aggregate, by mass or volume. These systems are said to provide simplicity of expression but are not quite convenient when discussing the effect of mix parameters on the characteristics of concrete.

The characteristic strength target class was M20 which range from $20\,N/mm^2$ to $24.99N/mm^2$. Then due to adjustments done in the tables 4.7 and 4.8 for the mix process the experiment feasible region matched class M25 which ranges from $25N/mm^2$ to $29.99N/mm^2$, which is the recommended class for a strong plinth. A standard deviation of $3N/mm^2$ was chosen arbitrarily and an error of 5% in order to fix the maximum targeted strength at $24.92N/mm^2$. A one tail normal curve was used with $\mu = 20, \sigma = 3\ and\ Z_c = 1.64$, such that

$$Z_c = \frac{x-\mu}{\sigma} \tag{4.27}$$

The upper targeted strength is $x = 24.92 N/mm^2$.The two characteristic strengths $20 N/mm^2$ and $24.92 N/mm^2$ formed the minimum (standard) and the maximum values for the target strength that were used in the transformation of ratios of the components used in the experiment as shown in the tables below. The standard deviation of 4N/mm^2 used in the tables 4.7 and 4.8 below was obtained from figure 3 of (Teychenne , Franklin , & Erntroy , 1988)pg. 12, reason being that the experiment was planned to use 45 testing cubes which were more than 20 according to the line B.

An interpolation for the values corresponding to cement strength 32.5N/mm^2 from the already given values 42.5 N/mm^2 and 52.5 N/mm^2 were presented in table 3.2, so as to use the most affordable cement in the market. The free water cement ratio 0.66 and 0.61 were respectively obtained by plotting the target mean strengths 26.6 N/mm^2 and 31.58 N/mm^2 (from tables 4.7 and 4.8) on the vertical axis against the free-water/cement ratio axis of figure 4 of (Teychenne , Franklin , & Erntroy , 1988) on page12 with 0.5 free water cement ratio starting line. The concrete density 2270 kg/m^3 was found in figure 5 by plotting the relative density of aggregate 2.5 kg/m^3 against the free water content 210 kg/m^3

The 60% passing sieve was used in the experiment. The proportion of the fine aggregate out of the total aggregates was obtained from figure 6 of (Teychenne , Franklin , & Erntroy , 1988) on page 15 by finding the intersection of the free-water/cement ratio for the 30-60mm slump height graph and the 60% passing sieve curve. The balance left from the total aggregate content is the coarse aggregate content. Since all the measurements were in the same units, cement as a standard measure of strength was used to obtain the ratios W: C: S: B.

Table 4.7: Processing of the ingredient ratios using the minimum required strength

Stage		Item	Values and calculations			
1	1.1	Characteristic strength (specified)	{ 20 N/mm^2 at 28 days}			
		Proportion Defective	{ 5 %}			
	1.2	Standard deviation	{ 4 N/mm^2} (see figure 3 Teychenne) (K= 1.65) { 1.65 * 4 = 6.6 N/mm^2}			
	1.3	Margin (see (3.45))	20 + 6.6 = 26.6 N/mm^2			
			32.5 N/mm^2 .			
	1.4	Target mean Strength(see(3.46))				
	1.5	Cement strength (see table 3.2)				
	1.6	Aggregate type(Coarse/fine)	(crushed / uncrushed)			
	1.7	Free water/cement ratio	0.66(see table3.2 and figure 4 of Teychenne)			
	1.8	Maximum free water/cement ratio.	0.55 (specified) Use lower value 0.55.			
2	2.1	Slump	30-60 mm (specified)			
	2.2	Maximum aggregate size	40 mm (specified)			
	2.3	Average aggregate size	20 mm (see table 3.1)			
	2.4	Free water content	210 Kg/m^3 (see table 3.1)			
3	3.1	Cement content(see(3.47))	$210 \div 0.55 = 382$ kg/m^3			
4	4.1	Relative density of aggregate	2.5 Assumed/known			
	4.2	Concrete density	2270 Kg/m^3 (See figure 5 Teychenne)			
	4.3	Total agg. content(see(3.48))	2270 - 382 - 210 = 1678 Kg/m^3			
5	5.1	Grading of fine aggregate	60 % passing sieve			
	5.2	Proportion of fine aggregate	34 % (see figure 6 in Teychenne)			
	5.3	Fine aggregate content(see(3.49))	$0.34 \times 1678 = 570$ kg/m^3			
	5.4	Coarse aggregate content	$1678 - 570 = 1108$ kg/m^3			
6			Water (kg)	Cement (kg)	Fine Aggregate (kg)	Coarse Aggregate(20mm) (kg)
		Per m^3	210	382	570	1108
		Ratios	**0.55**	**1**	**1.49**	**2.9**

Table 4.8: Processing of the ingredient ratios using the maximum required strength

Stage		Item	Values and calculations
1	1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8	Characteristic strength (specified) Proportion Defective Standard deviation Margin (see (3.45)) Target mean Strength(see(3.46)) Cement strength (see table 3.2) Aggregate type(Coarse/fine) Free water/cement ratio Maximum free water/cement ratio.	{ 24.92 N/mm^2 at 28 days} { 5 %} { 4 N/mm^2} (see figure 3 in Teychenne) (K= 1.65) { 1.65 * 4 = 6.6 N/mm^2} 24.92 + 6.6 = 31.58 N/mm^2 32.5 N/mm^2 (crushed / uncrushed) 0.61. (see table 3.2 and figure 4 of Teychenne) 0.52(specified). Use lower value 0.52.
2	2.1 2.2 2.3 2.4	Slump Maximum aggregate size Average aggregate size Free water content	30-60 mm (specified) 40 mm (specified) 20 mm (see table 3.1) 210 Kg/m^3 (see table 3.1)
3	3.1	Cement content(see(3.47))	$210 \div 0.52 = 404$ kg/m^3
4	4.1 4.2 4.3	Relative density of aggregate Concrete density Total aggregate content(see(3.48))	2.5 Assumed/known 2270 Kg/m^3 (See figure 5 in Teychenne) 2270 - 404 - 210 = 1656 Kg/m^3
5	5.1 5.2 5.3 5.4	Grading of fine aggregate Proportion of fine aggregate Fine aggregate content (see(3.49)) Coarse aggregate content	60 % passing sieve 33 % (see figure 6 in Teychenne) $0.33 \times 1656 = 546$ kg/m^3 $1656 - 546 = 1110$ kg/m^3
6			

	Water (kg)	Cement (kg)	Fine Aggregate (kg)	Coarse Aggregate (20mm) (kg)
Per m^3	210	404	546	1110
Ratios	**0.52**	**1**	**1.35**	**2.75**

The transformation from the design points to ratios for every experimental point was worked out using the following formula

$$R = P(M - S) + S \tag{4.28}$$

Where R, P, M and S represent ratio, proportion, maximum and standard (minimum).

The ratios for concrete ingredients are measured against the cement content, therefore this experiment used the same format, for example 1:2:4 means for every one unit of cement two and four units of fine and coarse aggregates respectively are used. The quantities per m^3 for cement (as the standard measure of strength) as shown in tables 4.7 and 4.8 above were subjected to a 25% increase to cater for evaporation and spillage for all the ingredients. Each of these values were divided by 296.3 due to the fact that one meter cube consists of 296.3 cubes of length 15cm. The average of the two values was used to transform the entire design from ratios to masses required to construct one cube at each design point. These workings were shown on tables 4.9 and 4.10 respectively.

Table 4.9: Cement component as a measure of transformation

	Masses		Ratios		Add 25%		Divide by 296.3		Average
	Min	Max	Min	Max	Min	Max	Min	Max	
Water	210	210	0.55	0.52					
Cement	382	404	1	1	477.5	505	1.6115	1.7044	1.6579
Sand	570	546	1.49	1.35					
Ballast	1108	1110	2.9	2.75					

Table 4.10: Experiment Transformation Plan

DESIGN					RATIO= Prop(max-min)+min				MASS (kg) for 1 CUBE= RATIO*1.6579			
Point	**t1**	**t2**	**t3**	**t4**	**W**	**C**	**S**	**B**	**W**	**C**	**S**	**B**
1	0.7	0.1	0.1	0.1	0.541	1	1.364	2.765	0.896924	1.6579	2.261376	4.584094
2	0.1	0.7	0.1	0.1	0.523	1	1.364	2.765	0.867082	1.6579	2.261376	4.584094
3	0.1	0.1	0.7	0.1	0.523	1	1.448	2.765	0.867082	1.6579	2.400639	4.584094
4	0.1	0.1	0.1	0.7	0.523	1	1.364	2.855	0.867082	1.6579	2.261376	4.733305
5	0.4	0.4	0.1	0.1	0.532	1	1.364	2.765	0.882003	1.6579	2.261376	4.584094
6	0.4	0.1	0.4	0.1	0.532	1	1.406	2.765	0.882003	1.6579	2.331007	4.584094
7	0.4	0.1	0.1	0.4	0.532	1	1.364	2.81	0.882003	1.6579	2.261376	4.658699
8	0.1	0.4	0.1	0.4	0.523	1	1.364	2.81	0.867082	1.6579	2.261376	4.658699
9	0.1	0.1	0.4	0.4	0.523	1	1.406	2.81	0.867082	1.6579	2.331007	4.658699
10	0.1	0.4	0.4	0.1	0.523	1	1.406	2.765	0.867082	1.6579	2.331007	4.584094
11	0.3	0.3	0.3	0.1	0.529	1	1.392	2.765	0.877029	1.6579	2.307797	4.584094
12	0.3	0.3	0.1	0.3	0.529	1	1.364	2.795	0.877029	1.6579	2.261376	4.633831
13	0.3	0.1	0.3	0.3	0.529	1	1.392	2.795	0.877029	1.6579	2.307797	4.633831
14	0.1	0.3	0.3	0.3	0.523	1	1.392	2.795	0.867082	1.6579	2.307797	4.633831
15	0.25	0.25	0.25	0.25	0.5275	1	1.385	2.7875	0.874542	1.6579	2.296192	4.621396

Three cubes per design point, were molded and cured for 28 days and then crushed. Three responses were measured in this experiment namely, slump height (mm) for each wet mix immediately after the mixing, masses (kg) and strength (N/mm^2) for each cube after 28 days of curing. This study sought to find the point that gives optimal compressive strength using the statistical methods for the M25 class of concrete.

4.3.2 The Experiment results

The ratio that gives optimal strength for the construction of affordable plinth is the core for this study. The study tried to compare with the traditional Engineering ratio 1:1.5:3 for the class M25. The table 4.11 shows the testing results per the design point.

Table 4.11: Experiment Results for 28 day testing Compressive Strength (N/mm2)

POINT	DESIGN				CUBE 1 y_{k1}	CUBE 2 y_{k2}	CUBE 3 y_{k3}	AV.ST. $\bar{y}_k$
1	0.7	0.1	0.1	0.1	24.647	24.452	23.296	24.132
2	0.1	0.7	0.1	0.1	27.058	33.502	28.752	29.771
3	0.1	0.1	0.7	0.1	30.941	30.834	32.752	31.509
4	0.1	0.1	0.1	0.7	30.445	21.668	31.915	28.009
5	0.4	0.4	0.1	0.1	26.052	24.653	27.108	25.938
6	0.4	0.1	0.4	0.1	27.828	27.026	26.963	27.272
7	0.4	0.1	0.1	0.4	27.593	28.04	28.128	27.920
8	0.1	0.4	0.1	0.4	26.529	27.363	27.5	27.131
9	0.1	0.1	0.4	0.4	28.373	28.227	26.74	27.780
10	0.1	0.4	0.4	0.1	25.486	24.737	28.567	26.263
11	0.3	0.3	0.3	0.1	25.73	24.758	24.513	25.000
12	0.3	0.3	0.1	0.3	23.323	24.609	22.158	23.363
13	0.3	0.1	0.3	0.3	29.374	30.016	23.412	27.601
14	0.1	0.3	0.3	0.3	32.405	33.317	22.963	29.562
15	0.25	0.25	0.25	0.25	26.44	27.684	27.508	27.211

$k = 1,2,3,\dots,15.$

4.3.3 Estimated model

Equation (4.29) gave the estimated second-degree Kronecker model obtained through R statistical software from the experiment results.

$$E(\hat{Y}) = 23.41t_1^2 + 37.95t_2^2 + 37.31t_3^2 + 28.39t_4^2 + 34.93t_1t_2 + 53.46t_1t_3 + 60.53t_1t_4 + 38.83t_2t_3 + 48.34t_2t_4 + 59.27t_3t_4. \tag{4.29}$$

The regression coefficients are all positive. This showed that the individual components and their interactions were of paramount importance in determination of concrete compressive strength.

4.3.4 Model Validity

This model was subjected to a validation process. The normality probability plot was done to assess any outliers. Analysis of variance was also carried out to further examine the model.

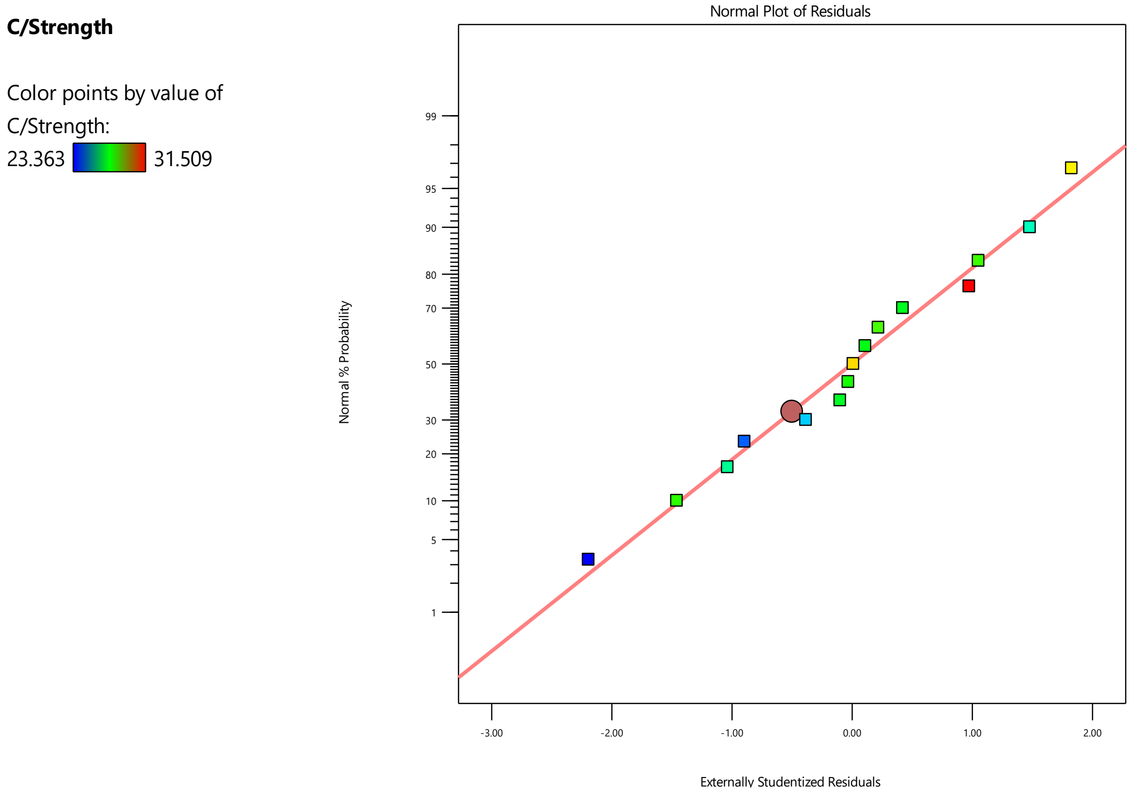

Figure 4. 1: Normality Probability Plot of Residuals

A normality probability plot is a graph that assesses whether or not a given data set is nearly normally distributed. The normality probability plot of residuals figure 4.1 was plotted using "Design Expert" version 12 statistical package. It is clear from the plot that the compressive strengths from the experiment were highly normally distributed as most of the points fell on or very near the normality line. According to the assumptions of the regression model, the error terms e_i must be normally distributed having a zero mean and constant variance σ^2. The error terms e_i were obtained as

$$e_i = y_i - \hat{y}_i \tag{4.30}$$

where y_i , were the observed values and $\hat{y}_i$, the fitted or estimated values.

4.3.5 The Analysis of Variance

According to (Marquardt & Snee, 1974) on testing the models with no constant term, the sum of squares and the associated degrees of freedom follow directly. That is there is no loss of degrees of freedom since the null hypothesis model ($\theta_0 = 0$) has no parameters estimated from the data. The Analysis of variance for the fitted Kronecker model shown on table 4.13, where the calculated values for SST, SSR and SSE as shown in (3.50) were 389.905, 136.703 and 253.102 respectively, and the F calculated value 2.10 was obtained as indicated in the equations (3.52).

The effect of the components and their interactions was shown in the table 4.12. All the P values were less than 5% and all the F calculated values were greater than the F-critical values at 5% significant level. Therefore, every component and their interactions were very significant in the expected response.

Table 4.12: ANOVA for effect of the concrete components on compressive strength

TERM	DF	SS	MSS	F	F critical	$\Pr(> F)$
t_{11}	1	3612.3	3612.3	986.340	6.61	6.145e-07***
t_{22}	1	2951.5	2951.5	805.906	6.61	1.016e-06***
t_{33}	1	2399.9	2399.9	655.292	6.61	1.699e-06***
t_{44}	1	1597.0	1597.0	436.074	6.61	4.664e-06***
t_{12}	1	199.9	199.9	54.592	6.61	0.0007142***
t_{13}	1	145.5	145.5	39.724	6.61	0.0014795**
t_{14}	1	106.2	106.2	29.008	6.61	0.0029762**
t_{23}	1	63.5	63.5	17.343	6.61	0.0087854**
t_{24}	1	48.4	48.4	13.208	6.61	0.0149924*
t_{34}	1	44.2	44.2	12.061	6.61	0.0177947*
Residuals	5	18.3	3.66			

Table 4.13: Summarized Analysis of Variance for the concrete experiment

Sources of variations	Degrees of freedom	Sum of Squares	MSS	F
Regression	9	136.703	15.189	2.10
Residual	35	253.102	7.231	
Total	44	389.905		

The statistical hypothesis that was tested using the ANOVA table 4.13 above was as given by hypothesis stated in (3.51).

The F calculated value is 2.10 and $F_{(0.1,9,35)} = 1.79$, hence we reject the hypothesis and conclude that the model estimates were different from zero and therefore significant. The compressive strength of concrete depends on the different combinations of the components.

4.3.6 Variation Measurement

The coefficients of determination R^2 and the R_A^2 where the latter is the adjusted coefficient of determination were calculated as shown in equations (3.53). The values of these coefficients were 0.9984 and 0.9951 respectively. The adjusted R-squared adjusts the statistic based on the number of the independent variables in the model. Therefore, the model explains 99.51% of the variability in the response variable. A good model should allow at most 10% error, therefore the model proved to be very good and hence the results.

4.3.7 Testing the significance of the parameters

The student t test was carried out to test the significance of each parameter in the model as shown in the table 4.14 below.

Table 4. 14: The T-test

Term	Coeff.Est.	Std.Error	t-value	$Pr(> \|t\|)$	VIF
t_{11}	23.411	5.942	3.940	0.01096*	3.3344
t_{22}	37.953	5.942	6.387	0.00139**	3.3344
t_{33}	37.308	5.942	6.278	0.00151**	3.3344
t_{44}	28.388	5.942	4.777	0.00498**	3.3344
t_{12}	34.927	17.066	2.047	0.09606	5.0891
t_{13}	53.461	17.066	3.133	0.02588*	5.0891
t_{14}	60.528	17.066	3.547	0.01644*	5.0891
t_{23}	38.833	17.066	2.275	0.07195	5.0891
t_{24}	48.345	17.066	2.833	0.03655*	5.0891
t_{34}	59.267	17.066	3.473	0.01779*	5.0891

From the table 4.14, all the parameters in the model except the interactions t_{12} and t_{23} were significant at 5%, but all the parameters would be significant by allowing a larger error of 10%. This implied that the effect of the two interactions on strength was not significant at 5%. These two had a p-value of 0.09606 and 0.07195 respectively. The blends with the higher value of one component (at the vertices), had a standard error of 5.942 and the interactions 17.066, implying that there was less errors for the vertices mixes compared to the other design points. The Variance Inflation Factors (VIF) were 3.3344 and 5.0891 for squared portions and interactions respectively. The rule of thumb is that if VIF is less than 10, there is no serious multi-collinearity, but at the same time if it goes beyond 4, then it calls for further investigations. In this experiment, though the interactions posed a high VIF, it is impossible to do away with any ingredient since all the four must be present in order to make the concrete.

4.4 OPTIMAL MIX AND EVALUATING D-G-I-OPTIMALITY OF THE EXPERIMENT

In this section, descriptive statistics were used to show the distribution of the experimental data. The contribution of each component to compressive strength was displayed in the individual scatter diagrams and the individual box plots. The overall box plot and the histogram showed the general distribution of strength across the whole design. The response surface diagrams showed contributions of interactions on strength, while the response trace plot helped determine the optimal value for the M25 class of concrete.

4.4.1 Descriptive Statistics Plots

The four set of scatter diagrams in Figure 4.2 shows how each ingredient at the given ratios. contributed to the compressive strength. It showed that cement and sand at proportion 0.7 contributed to high compressive strength, while ballast at the same level was average within the class and for water at the same level caused reduction of strength. This confirms the rule of thumb that more water reduces strength though it is valuable for concrete workability.

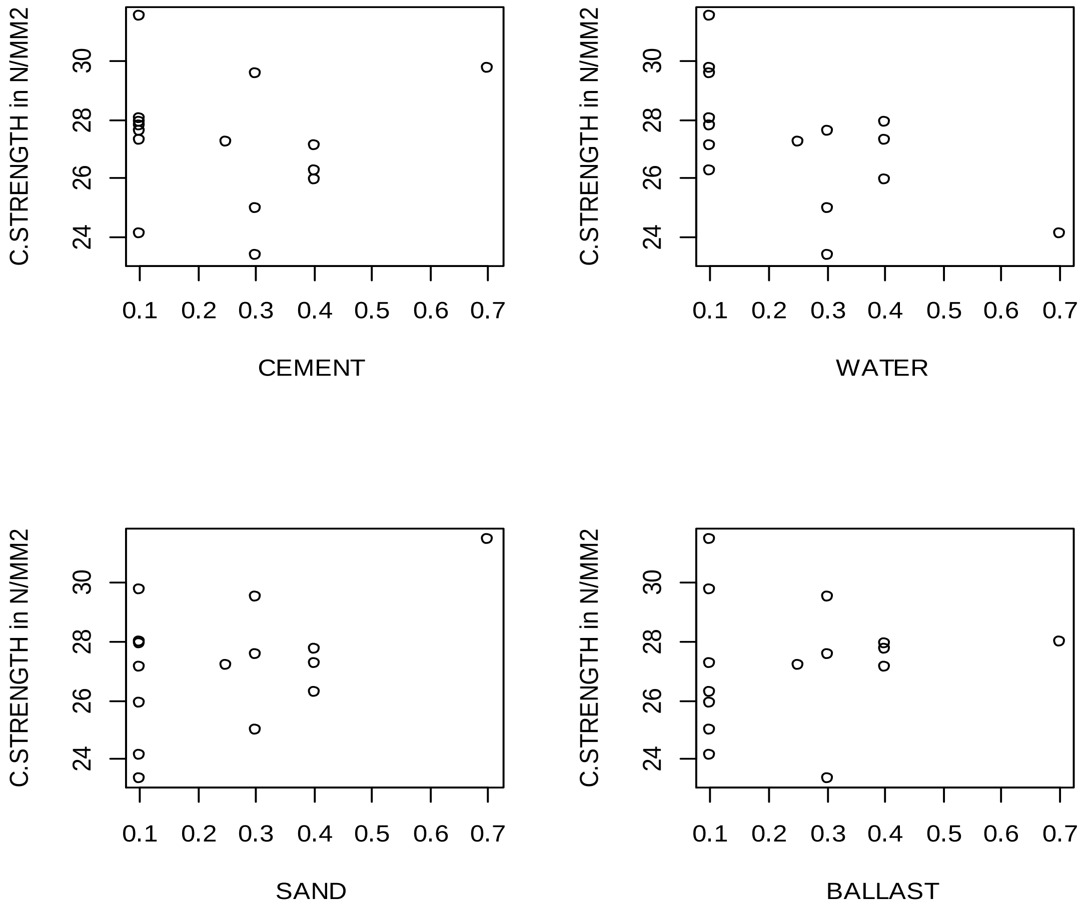

Figure 4. 2: Individual component scatter plots

Figure 4.3 below shows a boxplot. It indicates that the median of the data is approximately $27.5N/mm^2$. It also shows that the reading $31.509N/mm^2$ is an outlier.

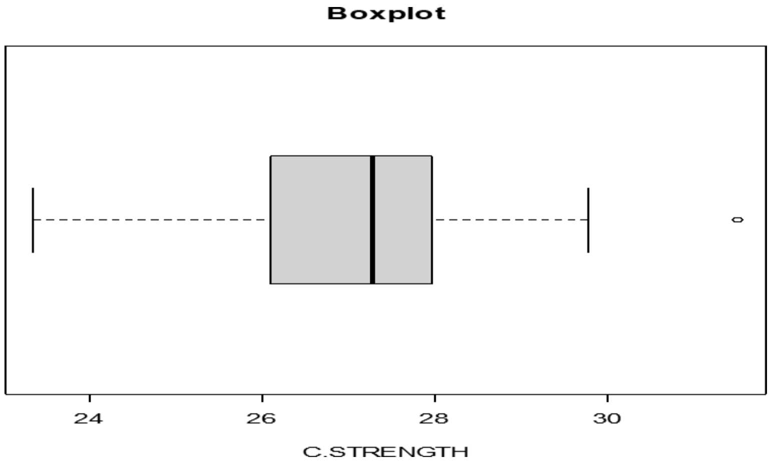

Figure 4. 3: Overall Boxplot

The boxplots in figure 4.4 correspond with the scatter plots in figure 4.2. They define the effect of each component on compressive strength at different design points.

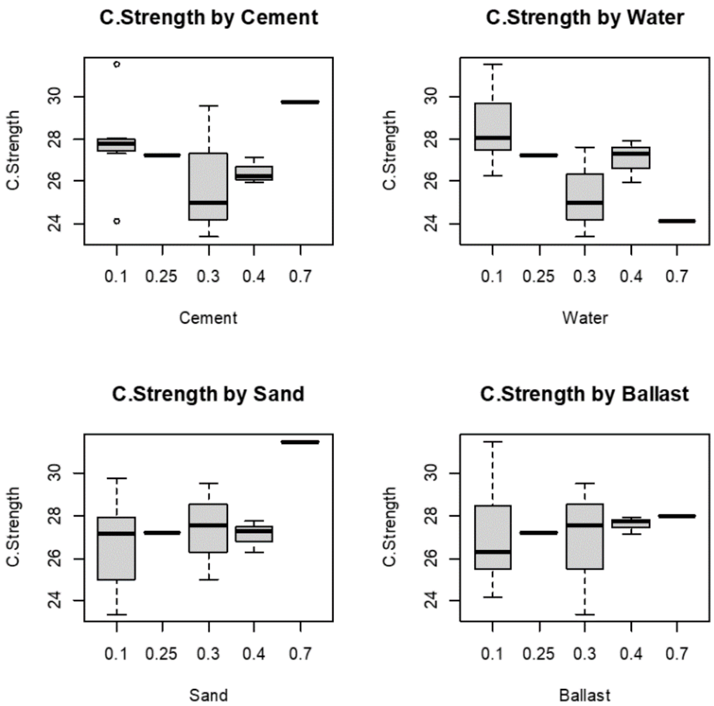

Figure 4. 4: Boxplots for individual component

The figure 4.5 is a histogram, which shows that seven out of fifteen design points give an average strength of between 26 and28 N/mm^2.

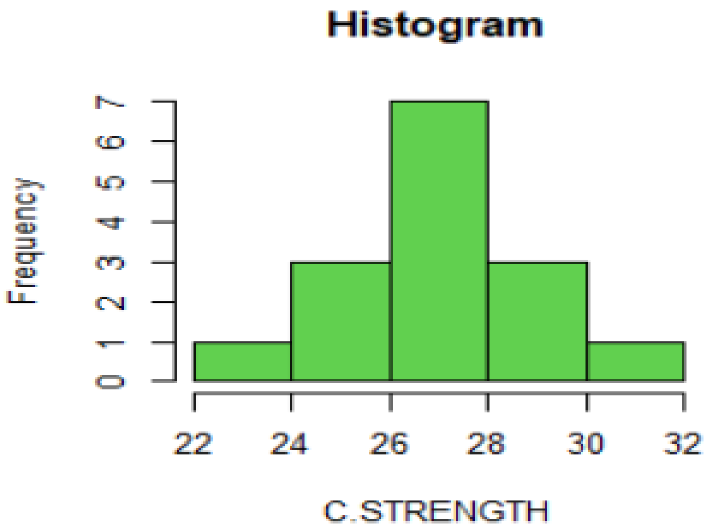

Figure 4. 5: Overall Histogram

4.4.2 Contours and the Quadratic Response surfaces

The images, contours and response surfaces shown in figures 4.6, 4.7, 4.8, 4.9, 4.10 and 4.11 show the effect on compressive strength as a result of interaction of two concrete ingredients.

Figure 4.6 shows how strength was affected due to water and cement interaction. At ratio of 0.4 of cement there is a steep decent as water was increased. This implied that as water was increased with a constant cement ratio, the strength was decreased.

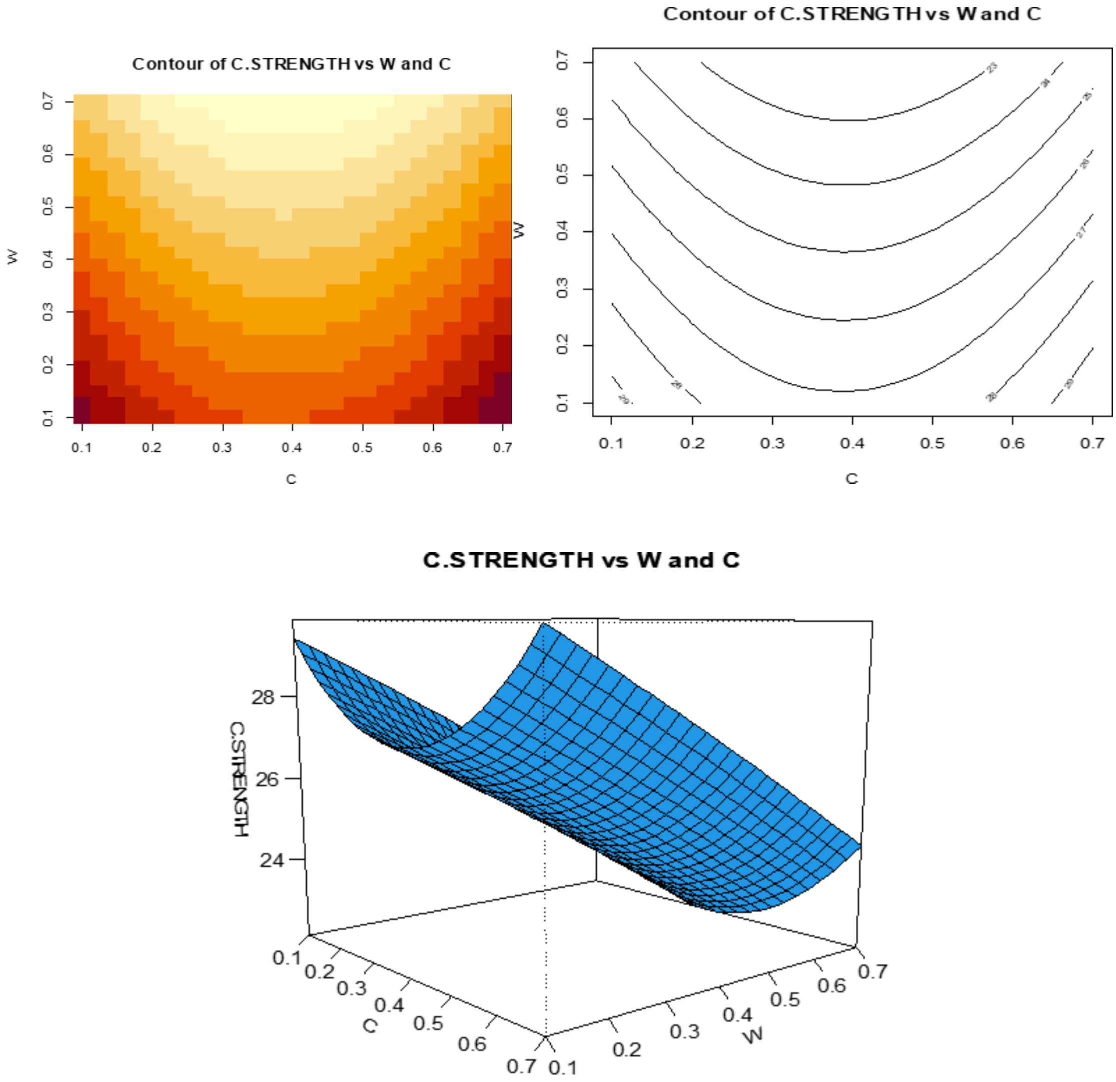

Figure 4. 6: Response Surface for effect of Water and Cement on Compressive Strength

Figure 4.7 shows a great decent of strength from high levels of water and low levels of sand to high levels of sand and low levels of water. This implied that strength of concrete increased at the inverse relationship of water and sand.

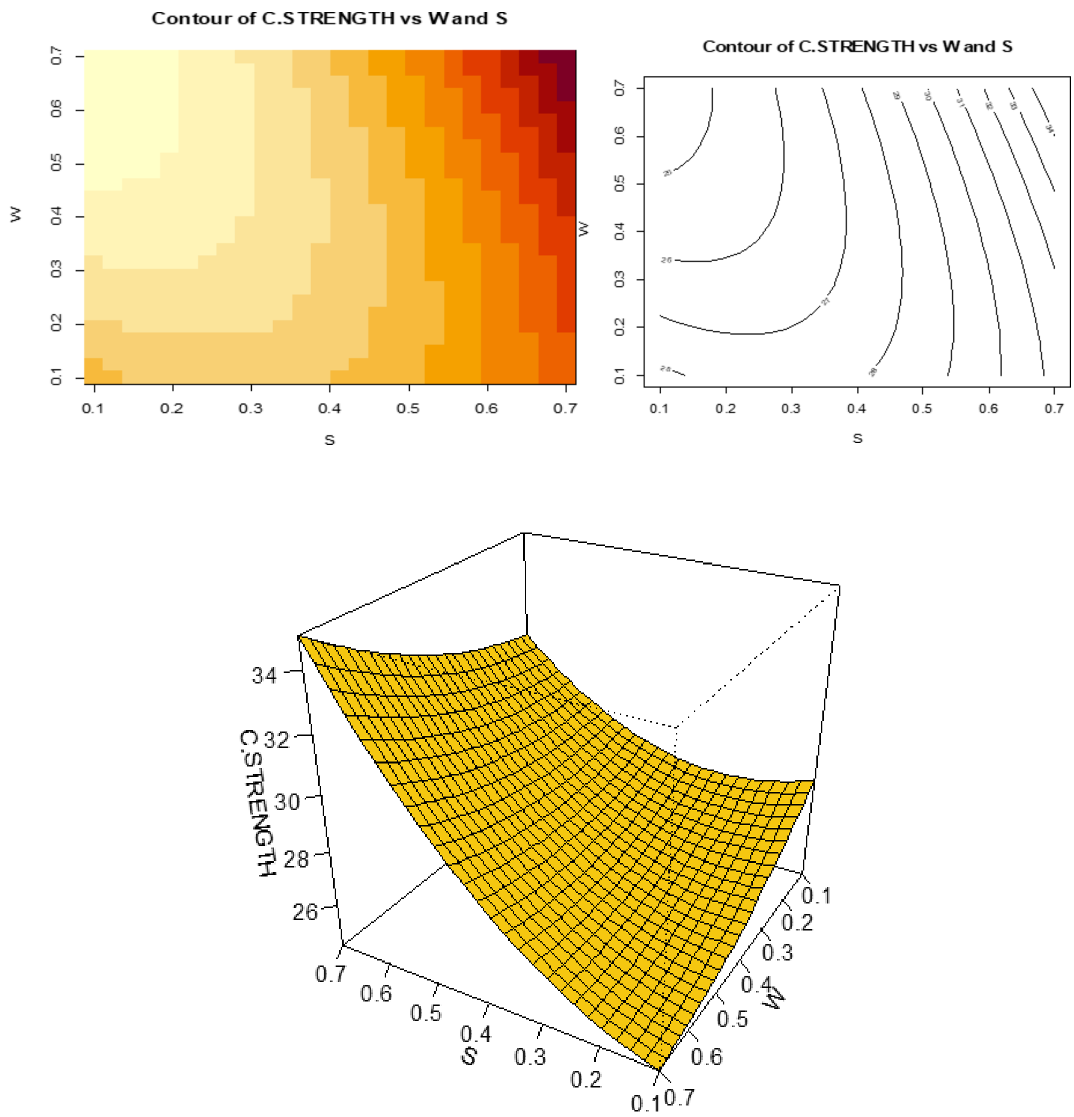

Figure 4. 7: Response Surface for effect of Water and Sand on Compressive Strength

Figure 4.8 shows a gentler decent of strength from high levels of water and low levels of ballast to high levels of ballast and low levels of water, compared to figure 4.7. This implied that strength of concrete increased at a gentler slope of the inverse relationship of water and ballast.

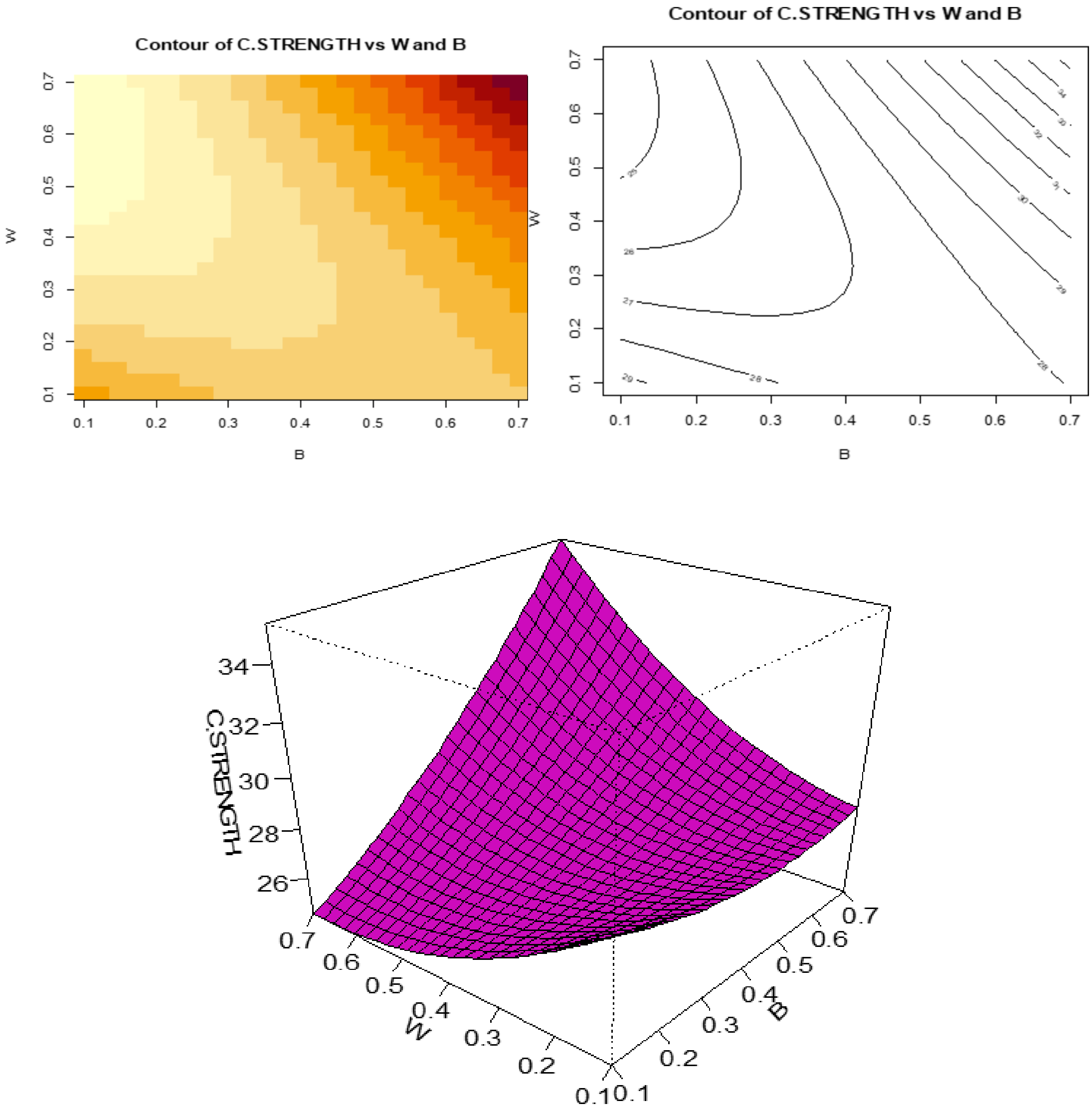

Figure 4. 8: Response Surface for effect of Water and Ballast on Compressive Strength

Figure 4.9 shows an increase in strength of concrete as there was a gentle increase in cement and sand components. This meant that almost direct proportionality of cement and sand caused an increase in strength.

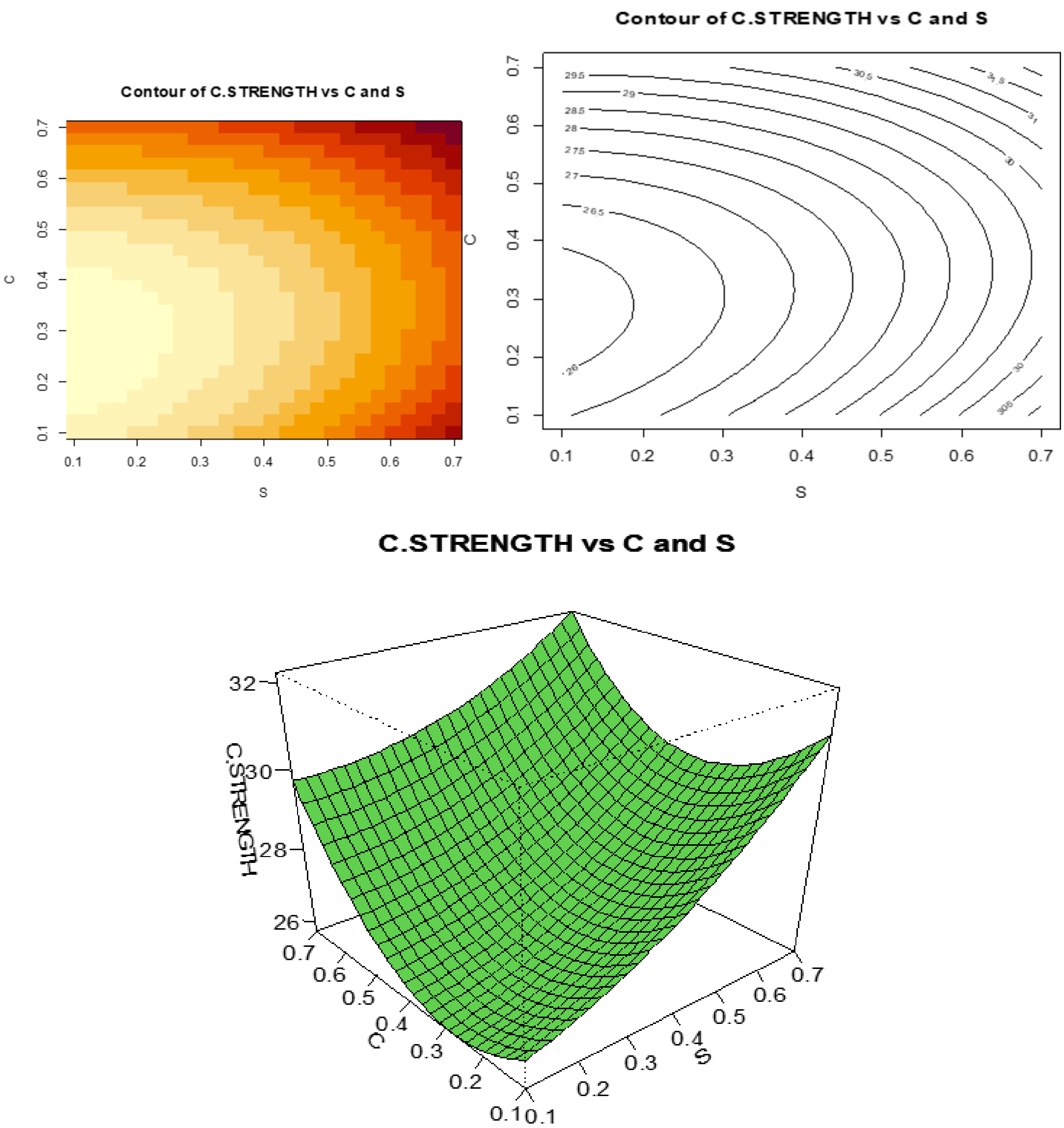

Figure 4. 9: Response Surface for effect of Cement and Sand on Compressive Strength

Figure 4.10 shows a steep increase in concrete strength as a result of a very gentle decreasing slope of cement and ballast.

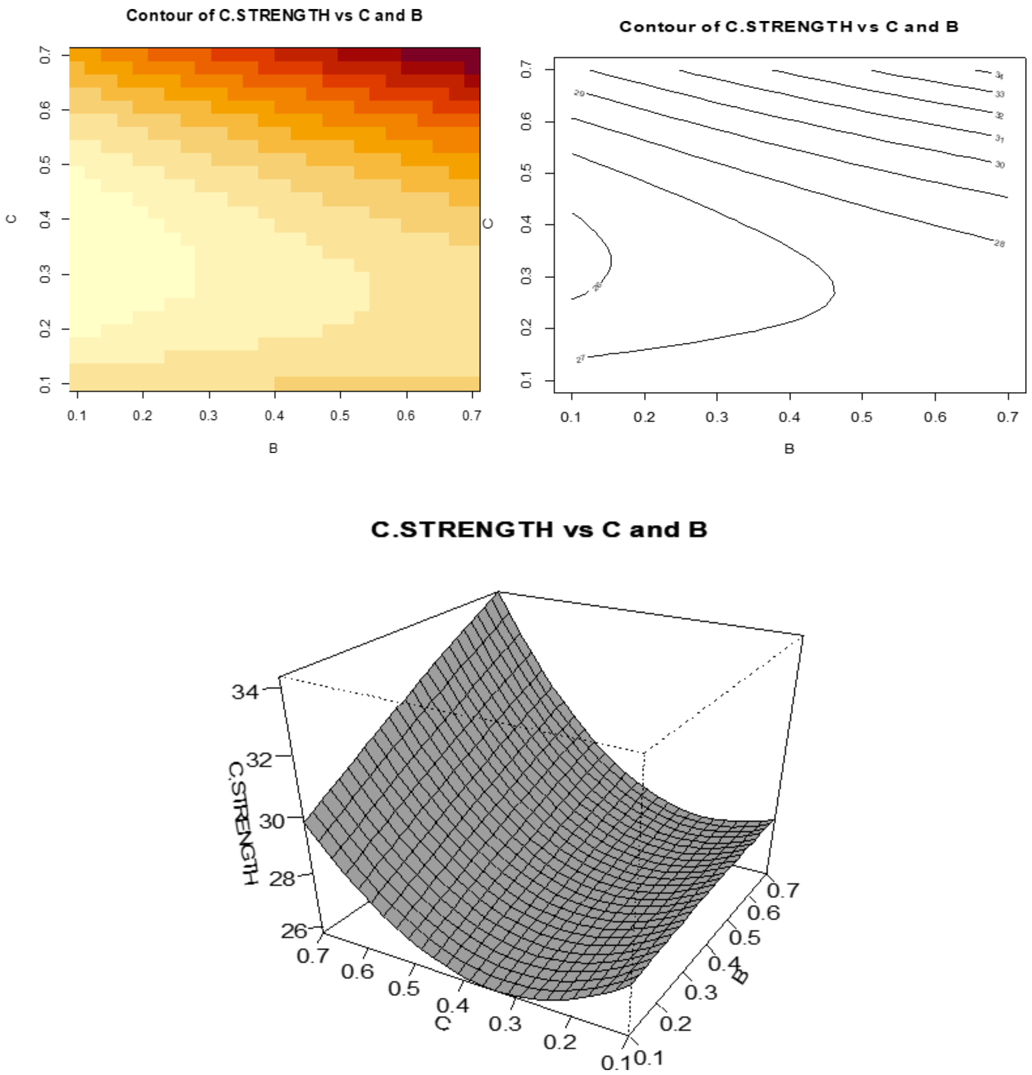

Figure 4. 10: Response Surface for effect of Cement and Ballast on Compressive Strength

Figure 4.11 shows a steep increase in strength of concrete due to an almost direct proportion variation of sand and ballast.This means that as sand and ballast were increased, the strength of concrete increased while keeping water and cement constant.

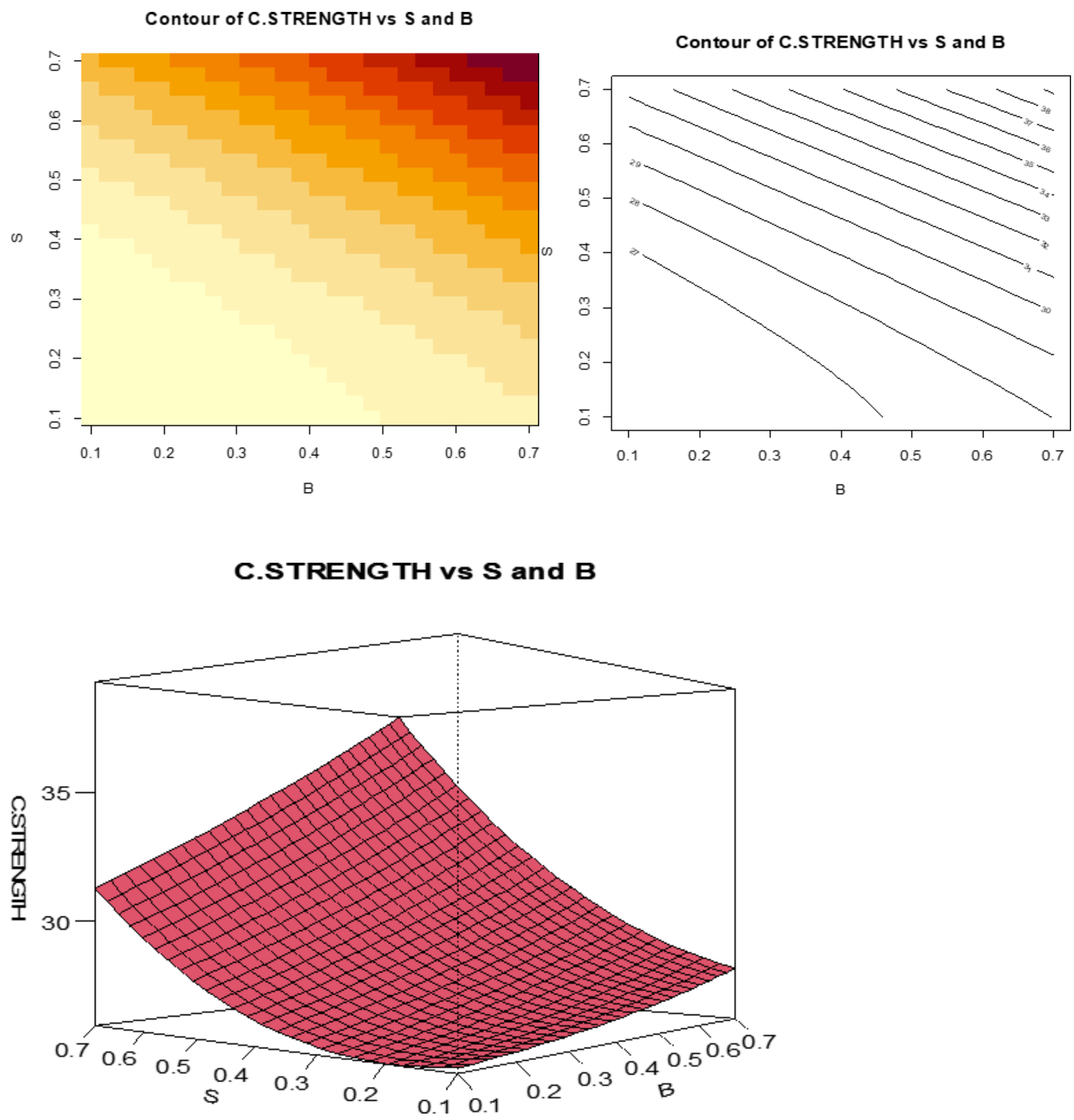

Figure 4. 11: Response Surface for effect of Sand and Ballast on Compressive Strength

4.4.3 The Response Surface Plot

The overall centroid (reference blend) chosen for this experiment results was blend 3, whose co-ordinates are (0.1,0.1,0.7,0.1) since it had the highest strength value. The increamental changes were arbitrarily chosen as $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = 0.03$. The Yhat values in the table 4.15 below were worked out using the model (4.29) and equations (3.54) and (3.55).

Table 4. 15: Response Trace Component Directions

COMPONENT 1 DIRECTION

X11	X21	X31	X41	Yhat1
0.04	0.107	0.747	0.107	31.966
0.07	0.103	0.723	0.103	31.584
0.1	0.1	0.7	0.1	31.227
0.13	0.097	0.677	0.097	30.875
0.16	0.093	0.653	0.093	30.514
0.19	0.09	0.63	0.09	30.176
0.22	0.087	0.607	0.087	29.845
0.25	0.083	0.583	0.083	29.504
0.28	0.08	0.56	0.08	29.185
0.31	0.077	0.537	0.077	28.874
0.34	0.073	0.513	0.073	28.553
0.37	0.07	0.49	0.07	28.254
0.4	0.067	0.467	0.067	27.963
0.43	0.063	0.443	0.063	27.662
0.46	0.06	0.42	0.06	27.383

COMPONENT 2 DIRECTION

X21	X22	X23	X24	Yhat2
0.107	0.04	0.747	0.107	32.639
0.103	0.07	0.723	0.103	31.895
0.1	0.1	0.7	0.1	31.227
0.097	0.13	0.677	0.097	30.616
0.093	0.16	0.653	0.093	30.046
0.09	0.19	0.63	0.09	29.552
0.087	0.22	0.607	0.087	29.115
0.083	0.25	0.583	0.083	28.72
0.08	0.28	0.56	0.08	28.399
0.077	0.31	0.537	0.077	28.136
0.073	0.34	0.513	0.073	27.914
0.07	0.37	0.49	0.07	27.767
0.067	0.4	0.467	0.067	27.678
0.063	0.43	0.443	0.063	27.631
0.06	0.46	0.42	0.06	27.657

COMPONENT 3 DIRECTION

X31	X32	X33	X34	Yhat3
0.12	0.12	0.64	0.12	30.285
0.11	0.11	0.67	0.11	30.744
0.1	0.1	0.7	0.1	31.227
0.09	0.09	0.73	0.09	31.732
0.08	0.08	0.76	0.08	32.26
0.07	0.07	0.79	0.07	32.811
0.06	0.06	0.82	0.06	33.385
0.05	0.05	0.85	0.05	33.982
0.04	0.04	0.88	0.04	34.601
0.03	0.03	0.91	0.03	35.244
0.02	0.02	0.94	0.02	35.91
0.01	0.01	0.97	0.01	36.598
0	0	1	0	37.31
-0.01	-0.01	1.03	-0.01	38.044
-0.02	-0.02	1.06	-0.02	38.802

COMPONENT 4 DIRECTION

X41	X42	X43	X44	Yhat4
0.107	0.107	0.747	0.04	31.532
0.103	0.103	0.723	0.07	31.37
0.1	0.1	0.7	0.1	31.227
0.097	0.097	0.677	0.13	31.087
0.093	0.093	0.653	0.16	30.935
0.09	0.09	0.63	0.19	30.802
0.087	0.087	0.607	0.22	30.673
0.083	0.083	0.583	0.25	30.531
0.08	0.08	0.56	0.28	30.409
0.077	0.077	0.537	0.31	30.291
0.073	0.073	0.513	0.34	30.159
0.07	0.07	0.49	0.37	30.047
0.067	0.067	0.467	0.4	29.939
0.063	0.063	0.443	0.43	29.818
0.06	0.06	0.42	0.46	29.717

The values plotted on the response plot figure 4.12 were summarised in the table 4.16.

Table 4. 16: Response Trace plot co-ordinates

X1	X2	X3	X4	Yhat1	Yhat2	Yhat3	Yhat4
0.04	0.04	0.64	0.04	31.966	32.639	30.285	31.532
0.07	0.07	0.67	0.07	31.584	31.895	30.744	31.370
0.1	0.1	0.7	0.1	31.227	31.227	31.227	31.227
0.13	0.13	0.73	0.13	30.875	30.616	31.732	31.087
0.16	0.16	0.76	0.16	30.514	30.046	32.260	30.935
0.19	0.19	0.79	0.19	30.176	29.552	32.811	30.802
0.22	0.22	0.82	0.22	29.845	29.115	33.385	30.673
0.25	0.25	0.85	0.25	29.504	28.720	33.982	30.531
0.28	0.28	0.88	0.28	29.185	28.399	34.601	30.409
0.31	0.31	0.91	0.31	28.874	28.136	35.244	30.291
0.34	0.34	0.94	0.34	28.553	27.914	35.910	30.159
0.37	0.37	0.97	0.37	28.254	27.767	36.598	30.047
0.4	0.4	1	0.4	27.963	27.678	37.310	29.939
0.43	0.43	1.03	0.43	27.662	27.631	38.044	29.818
0.46	0.46	1.06	0.46	27.383	27.657	38.802	29.717

Response Trace Plot (Cox)

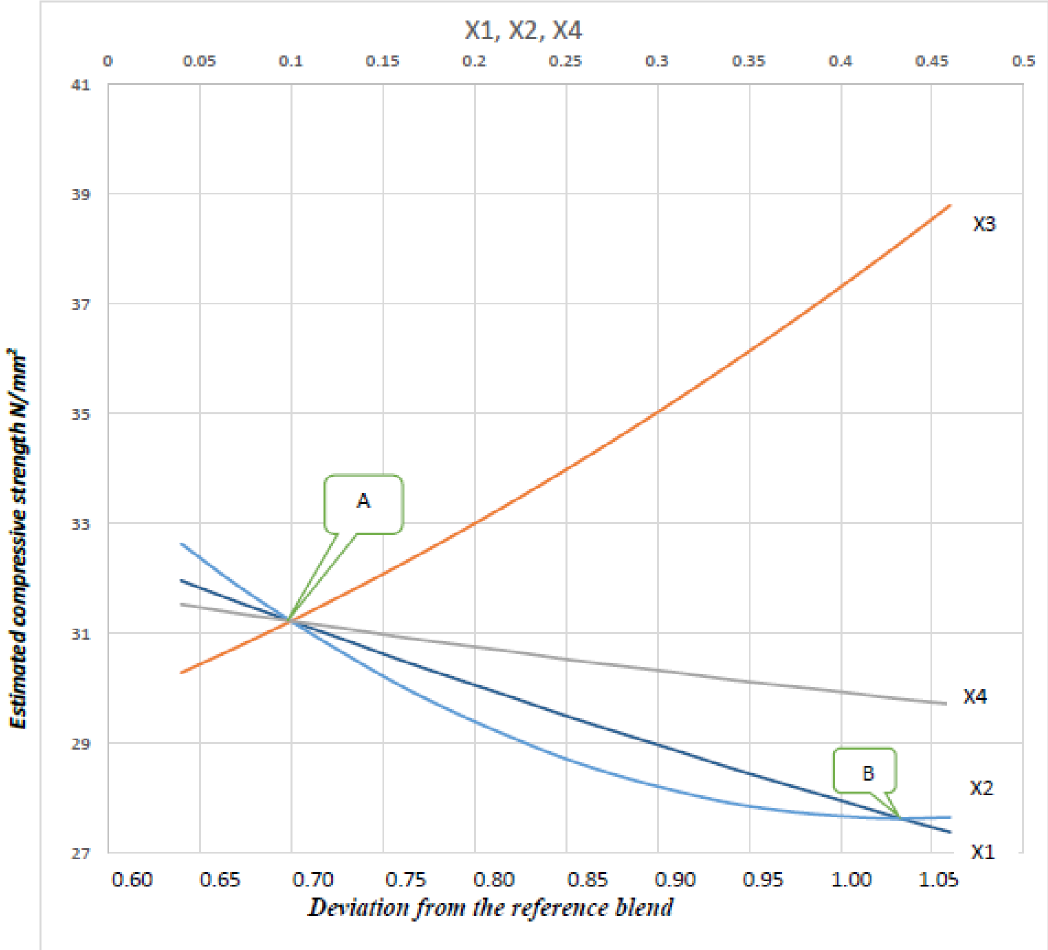

Figure 4. 12: Response Trace Plot

In the figure 4.12 the point labeled A is the reference blend, the point that had the highest compressive strength from the experimental results. The point labeled B, the intersection of component directions X1 and X2 was the lowest point of the X2- ray in the response trace diagram. It is clear that concrete strength is very sensitive to each of its components. At the reference blend sand had the highest value, deviation from this point to the left showed a decrease in strength and to the right increase. The other three components behaved in a contrary manner such that, increasing water and ballast showed great decrease in strength while cement adopted a parabolic nature so that at point B was the lowest and hence to start increasing. The intersection of the water and cement the greatest determinants of strength meant that this point should be the lowest for the M25 class of this experiment's compressive strength, with an approximate strength of $27.63N/mm^2$.

The approximate co-ordinate setting being (x_1, x_2, x_3, x_4)= (0.063, 0.43, 0.443, 0.063), whose original ratio was $(s_1, s_2, s_3, s_4) = 0.1, 0.3, 0.3, 0.3)$ which translated to (0.523:1: 1.392: 2.795).

Researchers (Okoloekwe & Okafor, 2007)on their paper developed a computer generated concrete mix design chart, which presented mix proportions of various grades ranging from $10N/mm^2$ to $50N/mm^2$. This was meant to avoid the tedious mix design procedures. These researchers used already developed ratios while this study developed a ratio.

Another study on concrete mixture maximization was by (Marcia , Eric , & Kenneth , 1997) who optimized six-component concrete mixture subject to several performance constraints. The experiment was performed to test the technique for proportioning of high performance concrete mixtures. By use of simplex methods on a 39 design point plan (including replicates), they found that using a quadratic model was adequate. Their study included displaying results on a response trace plot, which interestingly showed the same effect for water, sand and ballast from the reference blend as for this study. The effect of increasing cement for their study was superimposed with that of sand, while for this study took a parabolic nature with a minimum value at $27.63N/mm^2$.

4.4.4 Evaluating the D-, G- and I- Optimality for the experiment model

The D-, G- and I-optimality characteristics of the concrete experiment model equation (4.29) were evaluated under this section.

4.4.4.1 The H-invariant matrices for the Concrete model

Differentiating the model with respect to each variable$t_1, t_{2,} t_3$ and t_4, we obtain the H matrices given by (4.31)

$$H=\begin{bmatrix} 46.82t_1 & 0 & 0 & 0 & 34.93t_2 & 53.46t_3 & 60.53t_4 & 0 & 0 & 0 \\ 0 & 75.90t_2 & 0 & 0 & 34.93t_1 & 0 & 0 & 38.83t_3 & 48.34t_4 & 0 \\ 0 & 0 & 74.62t_3 & 0 & 0 & 53.46t_1 & 0 & 38.83t_2 & 0 & 59.27t_4 \\ 0 & 0 & 0 & 56.78t_4 & 0 & 0 & 60.53t_1 & 0 & 48.34t_2 & 59.27t_3 \end{bmatrix}$$

(4.31)

The H-invariant matrices for the design points $(\eta_1, \eta_2, \eta_3, \eta_4) = \{\left(\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}\right), \left(\frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}\right), \left(\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}\right), \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)\}$ for the concrete model are given by the set of equations (4.32).

$H_{1c} =$

$$\frac{1}{1000}\begin{bmatrix} 32774 & 0 & 0 & 0 & 3493 & 5346 & 6053 & 0 & 0 & 0 \\ 0 & 7590 & 0 & 0 & 24451 & 0 & 0 & 3883 & 4834 & 0 \\ 0 & 0 & 7462 & 0 & 0 & 37422 & 0 & 3883 & 0 & 5927 \\ 0 & 0 & 0 & 5678 & 0 & 0 & 42371 & 0 & 4834 & 5927 \end{bmatrix}$$

(4.32)

$H_{2c} =$

$$\frac{1}{1000}\begin{bmatrix} 18728 & 0 & 0 & 0 & 13972 & 5346 & 6053 & 0 & 0 & 0 \\ 0 & 30360 & 0 & 0 & 13972 & 0 & 0 & 3883 & 4834 & 0 \\ 0 & 0 & 7462 & 0 & 0 & 21384 & 0 & 15532 & 0 & 5927 \\ 0 & 0 & 0 & 5678 & 0 & 0 & 24212 & 0 & 19336 & 5927 \end{bmatrix}$$

$H_{3c} =$

$$\frac{1}{1000}\begin{bmatrix} 14046 & 0 & 0 & 0 & 10479 & 16038 & 6053 & 0 & 0 & 0 \\ 0 & 22770 & 0 & 0 & 10479 & 0 & 0 & 11649 & 4834 & 0 \\ 0 & 0 & 22386 & 0 & 0 & 16038 & 0 & 11649 & 0 & 5927 \\ 0 & 0 & 0 & 5678 & 0 & 0 & 18159 & 0 & 14502 & 17781 \end{bmatrix}$$

$H_{4c} =$

$$\frac{1}{1000}\begin{bmatrix} 11705 & 0 & 0 & 0 & 8733 & 13365 & 15133 & 0 & 0 & 0 \\ 0 & 18975 & 0 & 0 & 8733 & 0 & 0 & 9708 & 12085 & 0 \\ 0 & 0 & 18655 & 0 & 0 & 13365 & 0 & 9708 & 0 & 14818 \\ 0 & 0 & 0 & 14195 & 0 & 0 & 15133 & 0 & 12085 & 14818 \end{bmatrix}$$

4.4.4.2 The D-optimal Values Efficiencies and Equivalence theorems for the Experiment

The improved information matrices for the weighted centroid design with equal weights and the corresponding Determinant criterion values for the concrete model are given by (4.33) below

$$DCEt1 = H_{1c}C_{ce}H'_{1c} = \frac{1}{100000}\begin{bmatrix} 51864 & 18552 & 18552 & 18552 \\ 19333 & 17112 & 11880 & 11880 \\ 18552 & 11880 & 17112 & 11880 \\ 18552 & 11880 & 11880 & 17112 \end{bmatrix}$$

$$DEt1 = (\det(DCEt1))^{\frac{1}{10}} = 0.176753$$

$$DCEt2 = H_{2c}C_{ce}H'_{2c} = \frac{1}{100000}\begin{bmatrix} 29880 & 17184 & 14136 & 14136 \\ 17184 & 29880 & 14136 & 14136 \\ 14136 & 14136 & 16320 & 11304 \\ 14136 & 14136 & 11304 & 16320 \end{bmatrix}$$

$$DEt2 = (\det(DCEt2))^{\frac{1}{10}} = 0.1779787 \qquad (4.33)$$

$$DCEt3 = H_{3c}C_{ce}H'_{3c} = \frac{1}{100000}\begin{bmatrix} 24248 & 14760 & 14760 & 12888 \\ 14760 & 24248 & 14760 & 12888 \\ 14760 & 14760 & 24248 & 12888 \\ 12888 & 12888 & 12888 & 16056 \end{bmatrix}$$

$$DEt3 = (\det(DCEt3))^{\frac{1}{10}} = 0.178106$$

$$DCEt4 = H_{4c}C_{ce}H'_{4c} = \frac{1}{10000}\begin{bmatrix} 2175 & 1365 & 1365 & 1365 \\ 1365 & 2175 & 1365 & 1365 \\ 1365 & 1365 & 2175 & 1365 \\ 1365 & 1365 & 1365 & 2175 \end{bmatrix}$$

$$DEt4 = (\det(DCEt4))^{\frac{1}{10}} = 0.1787608$$

The improved information matrices for the weighted centroid design with unequal weights and the corresponding Determinant criterion values for the concrete model are given by (4.34) below

$$DCUt1 = H_{1c}C_{cu}H'_{1c} = \frac{1}{100000}\begin{bmatrix} 57384 & 19360 & 19360 & 19360 \\ 19360 & 17736 & 11056 & 11056 \\ 19360 & 11056 & 17736 & 11056 \\ 19360 & 11056 & 11056 & 17736 \end{bmatrix}$$

$$DUt1 = (\det(DCUt1))^{\frac{1}{10}} = 0.186846.$$

$$DCUt2 = H_{2c}C_{cu}H'_{2c} = \frac{1}{100000}\begin{bmatrix} 32376 & 17776 & 13876 & 13876 \\ 17776 & 32376 & 13876 & 13876 \\ 13876 & 13876 & 16656 & 10336 \\ 13876 & 13876 & 10336 & 16656 \end{bmatrix}$$

$$DUt2 = (\det(DCUt2))^{\frac{1}{10}} = 0.1875484. \qquad (4.34)$$

$$DCUt3 = H_{3c}C_{cu}H'_{3c} = \frac{1}{100000}\begin{bmatrix} 25864 & 14704 & 14704 & 12320 \\ 14704 & 25864 & 14704 & 12320 \\ 14704 & 14704 & 25864 & 12320 \\ 12320 & 12320 & 12320 & 16296 \end{bmatrix}$$

$$DUt3 = (\det(DCUt3))^{\frac{1}{10}} = 0.1881068.$$

$$DCUt4 = H_{4c}C_{cu}H'_{4c} = \frac{1}{10000}\begin{bmatrix} 2295 & 1330 & 1330 & 1330 \\ 1330 & 2295 & 1330 & 1330 \\ 1330 & 1330 & 2295 & 1330 \\ 1330 & 1330 & 1330 & 2295 \end{bmatrix}$$

$DUt4 = (\det(DCUt4))^{\frac{1}{10}} = 0.1884468.$

Table 4. 17: D-Optimality values and Efficiency for the Concrete model

Blends	D-Optimal values EWSCAD	D-Optimal values UWSCAD	Efficiency $D_{eff} = \left\{\frac{\lvert DUi \rvert}{\lvert DEi \rvert}\right\}^{\frac{1}{10}}$
$\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}$	0.176753	0.186846	1.0571
$\frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}$	0.1779787	0.1875484	1.0538
$\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}$	0.178106	0.1881068	1.0538
$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	0.1787608	0.1884468	1.0542

The design point $\eta_4 = \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$ was more optimal than the other design points in the both designs. While the UWSCAD was a more efficient design than EWSCAD, with η_1 having the highest efficiency and $\eta_2 and \eta_3$ having the same efficiency.

The table 4.16 and table 4.17, showed that the concrete model satisfied the D equivalence theorem for both designs.

Table 4. 18: D- Equivalence Theorem for the Concrete model EWSCAD design

DESIGN	$f'(t)GKC_{ce}^{p+1}K'G'f(t)$		$trace\ C_{ce}^{p}$	Remarks
η_1	15.3782	>	10	Out of range
η_2	16.9803	>	10	Out of range
η_3	5.4968	<	10	Within range
η_4	1.8289	<	10	Within range

Table 4. 19: D- Optimal Equivalence Theorem for the Concrete model UWSCAD design

DESIGN	$f'(t)GKC_{cu}^{p+1}K'G'f(t)$		$trace\ C_{cu}^{p}$	Remarks
η_1	14.3118	>	10	Out of range
η_2	11.6188	>	10	Out of range
η_3	4.7961	<	10	Within range
η_4	2.3319	<	10	Within range

The design points $\eta_3\ and\ \eta_4$ for the two designs satisfied the D-equivalence theorem.

4.4.4.3 G-slope optimal Values Efficiencies and Equivalence Theorems for the Experiment

The inverse of the improved information matrices of the EWSCAD for the concrete model at different points of the design are $D_{ct1e}^{-1}, D_{ct2e}^{-1}, D_{ct3e}^{-1}\ and D_{ct4e}^{-1}$, hence the G-optimal values shown in equations (4.35) were obtained as given by (3.29).

$$\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)' D_{ct1e}^{-1}\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right) = 12.6846.$$

$$\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right)' D_{ct1e}^{-1}\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right) = 56.0556.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right)' D_{ct1e}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right) = 56.0556.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right)' D_{ct1e}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right) = 56.0556.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right)' D_{ct1e}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right) = 10.4983.$$

$$\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right)' D_{ct2e}^{-1}\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right) = 17.0446.$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right)' D_{ct2e}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right) = 17.0446.$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right)' D_{ct2e}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right) = 17.0446. \quad (4.35)$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right)' D_{ct2e}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right) = 17.0446.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right)' D_{ct2e}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right) = 22.1439.$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right)' D_{ct2e}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right) = 17.0446.$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right)' D_{ct3e}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right) = 6.9945.$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right)' D_{ct3e}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right) = 8.4233.$$

$$\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right)' D_{ct3e}^{-1}\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right) = 8.4233.$$

$$\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right)' D_{ct3e}^{-1}\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right) = 8.4233.$$

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)' D_{ct4e}^{-1}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) = 3.9872.$$

The inverse of the improved information matrices of the UWSCAD at different points of the design are $D_{cc1u}^{-1}, D_{cc2u}^{-1}, D_{cc3u}^{-1}\ and D_{cc4u}^{-1}$. Hence the G-optimal values were obtained as given by (3.29).

At each point the optimal values are given by set of equations (4.36) shown below

$$\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right)' D_{cc1u}^{-1}\left(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10}\right) = 11.2853.$$

$$\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right)' D_{cc1u}^{-1}\left(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10}\right) = 46.6038.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right)' D_{cc1u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10}\right) = 46.6038.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right)' D_{cc1u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10}\right) = 46.6038.$$

$$\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right)' D_{cc2u}^{-1}\left(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10}\right) =8.9116.$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right)' D_{cc2u}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10}\right) = 14.8380.$$

$$\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right)' D_{cc2u}^{-1}\left(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10}\right) = 14.8380.$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right)' D_{cc2u}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10}\right) = 14.8380.$$

$$\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right)' D_{cc2u}^{-1}\left(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10}\right) = 20.8184. \tag{4.36}$$

$$\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right)' D_{cc2u}^{-1}\left(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10}\right) = 14.8380.$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right)' D_{cc3u}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10}\right) = 6.1411.$$

$$\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right)' D_{cc3u}^{-1}\left(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10}\right) = 7.9170.$$

$$\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right)' D_{cc3u}^{-1}\left(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10}\right) = 7.9170.$$

$$\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right)' D_{cc3u}^{-1}\left(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10}\right) = 7.9170.$$

$$\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)' D_{cc4u}^{-1}\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) =3.9777$$

Table 4. 20: G-Optimality values and Efficiency for the Concrete model

Blends	G-Optimal values EWSCAD	G-Optimal values UWSCAD	Efficiency $G_{eff} = \frac{f'(t)D_{ccciu}^{-1}f(t)}{f'(t)D_{ccie}^{-1}f(t)}$
$\frac{7}{10}, \frac{1}{10}, \frac{1}{10}, \frac{1}{10}$	56.0556	46.6038	0.8314
$\frac{4}{10}, \frac{4}{10}, \frac{1}{10}, \frac{1}{10}$	22.1439	20.8184	0.9401
$\frac{3}{10}, \frac{3}{10}, \frac{3}{10}, \frac{1}{10}$	8.4233	7.9170	0.9399
$\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}$	3.9872	3.9777	0.9976

The design point η_4 for both designs was the most G-optimal, optimality reduced outwards. The same design point was the most efficient among the points and EWSCAD was the better design.

The Concrete model satisfied the G- equivalence theorem since $C_{ce}C_{ce}^{-1} = C_{cu}C_{cu}^{-1} = 10$, which is the size of the information matrix.

4.4.4.4 I-Optimality Values Efficiencies and Equivalence Theorems for the Experiment

The 10×10 moment matrix B for the subsystem of interest, which is an equivalent of the moment matrix of the full model given by (3.19) was also represented by the integral matrix (4.25) is

B

$$
= \begin{bmatrix}
t_1^4 & t_1^2t_2^2 & t_1^2t_3^2 & t_1^2t_4^2 & t_1^3t_2 & t_1^3t_3 & t_1^3t_4 & t_1^2t_2t_3 & t_1^2t_2t_4 & t_1^2t_3t_4 \\
t_1^2t_2^2 & t_2^4 & t_2^2t_3^2 & t_2^2t_4^2 & t_2^3t_1 & t_2^2t_1t_3 & t_2^2t_1t_4 & t_2^3t_3 & t_2^3t_4 & t_2^2t_3t_4 \\
t_1^2t_3^2 & t_2^2t_3^2 & t_3^4 & t_3^2t_4^2 & t_3^2t_1t_2 & t_3^3t_1 & t_3^2t_1t_4 & t_3^3t_2 & t_3^2t_2t_4 & t_3^3t_4 \\
t_1^2t_4^2 & t_2^2t_4^2 & t_3^2t_4^2 & t_4^4 & t_4^2t_1t_2 & t_4^2t_1t_3 & t_4^3t_1 & t_4^2t_2t_3 & t_4^3t_2 & t_4^3t_3 \\
t_1^3t_2 & t_2^3t_1 & t_3^2t_1t_2 & t_4^2t_1t_2 & t_1^2t_2^2 & t_1^2t_2t_3 & t_1^2t_2t_4 & t_2^2t_1t_3 & t_2^2t_1t_4 & t_1t_2t_3t_4 \\
t_1^3t_3 & t_2^2t_1t_3 & t_3^3t_1 & t_4^2t_1t_3 & t_1^2t_2t_3 & t_1^2t_3^2 & t_1^2t_3t_4 & t_3^2t_1t_2 & t_1t_2t_3t_4 & t_3^2t_1t_4 \\
t_1^3t_4 & t_2^2t_1t_4 & t_3^2t_1t_4 & t_4^3t_1 & t_1^2t_2t_4 & t_1^2t_3t_4 & t_1^2t_4^2 & t_1t_2t_3t_4 & t_4^2t_1t_2 & t_4^2t_1t_3 \\
t_1^2t_2t_3 & t_2^3t_3 & t_3^3t_2 & t_4^2t_2t_3 & t_2^2t_1t_3 & t_3^2t_1t_2 & t_1t_2t_3t_4 & t_2^2t_3^2 & t_2^2t_3t_4 & t_3^2t_2t_4 \\
t_1^2t_2t_4 & t_2^3t_4 & t_3^2t_2t_4 & t_4^3t_2 & t_2^2t_1t_4 & t_1t_2t_3t_4 & t_4^2t_1t_2 & t_2^2t_3t_4 & t_2^2t_4^2 & t_4^2t_2t_3 \\
t_1^2t_3t_4 & t_2^2t_3t_4 & t_3^3t_4 & t_4^3t_3 & t_1t_2t_3t_4 & t_3^2t_1t_4 & t_4^2t_1t_3 & t_3^2t_2t_4 & t_4^2t_2t_3 & t_3^2t_4^2
\end{bmatrix}
$$

The L_2 information matrix of the concrete experiment Kronecker model below was obtained by the use of the product of integral matrix shown in (4.25) and the regression coefficients 10×10 matrix for the regression model (4.29).

$$
L_2 = \begin{bmatrix}
15.658 & 4.231 & 4.159 & 3.165 & 5.841 & 8.939 & 10.121 & 2.164 & 2.694 & 3.304 \\
4.231 & 41.147 & 6.742 & 5.130 & 9.469 & 4.830 & 5.469 & 10.526 & 13.104 & 5.355 \\
4.159 & 6.742 & 39.772 & 5.044 & 3.103 & 14.427 & 5.377 & 10.348 & 4.294 & 15.795 \\
3.165 & 5.130 & 5.044 & 23.028 & 2.361 & 3.614 & 12.275 & 2.625 & 9.803 & 12.019 \\
5.841 & 9.469 & 3.103 & 2.361 & 5.810 & 4.460 & 5.034 & 3.229 & 4.020 & 2.465 \\
8.939 & 4.830 & 14.427 & 3.614 & 4.460 & 13.609 & 7.705 & 4.943 & 3.076 & 7.544 \\
10.121 & 5.469 & 5.377 & 12.275 & 5.034 & 7.705 & 17.447 & 2.798 & 6.967 & 8.542 \\
2.164 & 10.526 & 10.348 & 2.625 & 3.229 & 4.943 & 2.798 & 7.180 & 4.469 & 5.480 \\
2.694 & 13.104 & 4.294 & 9.803 & 4.020 & 3.076 & 6.967 & 4.469 & 11.127 & 6.822 \\
3.304 & 5.355 & 15.795 & 12.019 & 2.465 & 7.544 & 8.542 & 5.480 & 6.822 & 16.728
\end{bmatrix}
$$

(4.37)

To obtain the I-Optimality values for the Concrete model of EWSCAD, the matrix $C_e^{-1}L_2$ was obtained from matrices (4.15) and (4.24) in order to get the RHS value of the equivalence theorem.

$$
\begin{bmatrix}
740.35 & 386.99 & 203.52 & 107.95 & 109.90 & 17.642 & -39.45 & 204.12 & 255.92 & 271.83 \\
710.19 & 4090.38 & 1117.90 & 767.11 & 867.70 & 913.90 & 1012.37 & 944.58 & 1101.53 & 978.84 \\
289.31 & 634.88 & 3244.16 & 370.73 & 296.22 & 726.81 & 529.08 & 617.71 & 446.76 & 783.75 \\
-39.84 & 52.28 & 24.28 & 1139.65 & 40.88 & 95.63 & -44.06 & 71.95 & 51.17 & -5.35 \\
-1222.6 & -2344.04 & -19.45 & -519.16 & -164.25 & -1107.08 & -1570.7 & -765.23 & -1217.99 & 122.69 \\
390.29 & 864.82 & -357.02 & -379.26 & 181.22 & 1886.89 & -453.14 & -34.99 & 354.02 & -748.68 \\
1034.44 & 728.49 & 29.82 & 673.66 & 382.19 & 191.86 & 3236.62 & 312.24 & 207.43 & -115.1 \\
-248.58 & -1989.78 & -2579.99 & -509.43 & -655.16 & -1377.44 & 26.25 & -126.62 & -1101.15 & -1699.13 \\
-147.76 & -510.28 & 153.51 & -692.34 & -273.36 & 291.7 & -928.16 & -251.06 & 1058.09 & -98.72 \\
-609.20 & 106.91 & 113.44 & 250.06 & -98.72 & -615.16 & -723.93 & -106.65 & -170.74 & 2372.25
\end{bmatrix}
$$

The average variance for the Concrete Kronecker model with equal weights is given by

$A.V = tr[C_e^{-1}L_2] = 17{,}477.51.$

The design is said to be I-optimal if and only if

$f'(t)C_e^{-1}L_2C_e^{-1}f(t) \leq 17{,}477.51.$

Table 4. 21: I-Optimality values for η_1e for the Concrete model

$\eta_1 design\ points$	$(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10})$
LHS	19,021.4	77,562.46	59,897.66	26,686.15
Comment:	All not optimal			

Table 4.22: I-Optimality values for η_{2e} for the Concrete model

$\eta_2 design\ \ points$	$(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10})$	$(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10})$
LHS	8,070.3	12,563.3	12,509.7	12,103.6	14,231.8	10,250.9
Comment:	All optimal					

Table 4. 23: I-Optimality values for η_3e for the Concrete model

$\eta_3 design\ points$	$(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10})$	$(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10})$	$(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10})$	$(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10})$
LHS	1,922.27	2,140.67	3,085.06	2,426.19
Comment:	All optimal			

Table 4. 24: I-Optimality values for η_4e for the Concrete model

$\eta_4\ design\ points$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
LHS	819.4624
Comment:	Optimal.

The I-Optimality values for the Concrete model of UWSCAD were obtained in the same way as for the EWSCAD. The matrix $C_u^{-1}L_2$ was obtained from matrices (4.16) and (4.25) in order to get the RHS value of the equivalence theorem.

$$\begin{bmatrix} 732.18 & 442.61 & 248.64 & 95.58 & 123.99 & 41.82 & -30.63 & 188.26 & 213.11 & 206.87 \\ 615.27 & 3889.48 & 1047.52 & 672.30 & 797.96 & 771.85 & 827.48 & 885.26 & 1015.96 & 825.01 \\ 256.46 & 273.64 & 3121.01 & 334.24 & 260.72 & 687.81 & 415.39 & 605.64 & 388.32 & 742.33 \\ -31.62 & 164.91 & 103.14 & 1132.30 & 37.46 & 65.67 & -17.34 & 79.45 & 95.60 & 40.54 \\ -889.19 & -2021.14 & -106.53 & -384.93 & -183.35 & -731.27 & -1013.83 & -589.28 & -887.73 & -101.41 \\ 183.73 & 516.68 & -597.91 & -202.07 & 120.21 & 1182.02 & -222.59 & -70.35 & 154.52 & -497.72 \\ 645.51 & 569.25 & 185.11 & 370.07 & 299.41 & 244.59 & 2186.44 & 217.73 & 206.83 & 52.60 \\ -276.86 & -1852.20 & -2412.19 & -431.15 & -545.03 & -1032.21 & -233.69 & -249.23 & -835.69 & -1230.35 \\ -141.08 & -830.14 & 48.84 & -600.51 & -257.32 & 48.87 & -611.16 & -244.15 & 582.12 & -508.03 \\ -312.16 & 152.68 & -264.04 & 63.85 & -56.26 & -378.34 & -367.92 & -76.73 & -56.13 & 1509.16 \end{bmatrix}$$

The average variance for the Concrete Kronecker model with unequal weights is given by

$A.V = tr[C_u^{-1}L_2]$ =13,902.12

The design is said to be I-optimal if and only if

$f'(t)C_u^{-1}L_2C_u^{-1}f(t)$ ≤13,902.12.

Table 4. 25: I-Optimality values for η_1u for the Concrete model

$\eta_1 design\ points$	$(\frac{7}{10},\frac{1}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{7}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{7}{10},\frac{1}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{1}{10},\frac{7}{10})$
LHS	17,196.14	69,009.86	54,036.29	24,253.16
Comment:	All not optimal			

Table 4.26: I-Optimality values for η_2u for the Concrete model

$\eta_2 design\ \ points$	$(\frac{4}{10},\frac{4}{10},\frac{1}{10},\frac{1}{10})$	$(\frac{4}{10},\frac{1}{10},\frac{4}{10},\frac{1}{10})$	$(\frac{4}{10},\frac{1}{10},\frac{1}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{4}{10},\frac{1}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{1}{10},\frac{4}{10},\frac{4}{10})$	$(\frac{1}{10},\frac{4}{10},\frac{4}{10},\frac{1}{10})$
LHS	4,991.74	7,034.65	6,246.03	7,214.64	7,954.72	6,803.42
Comment:	All optimal					

Table 4.27: I-Optimality values for η_3u for the Concrete model

$\eta_3 design\ points$	$(\frac{3}{10},\frac{3}{10},\frac{3}{10},\frac{1}{10})$	$(\frac{3}{10},\frac{3}{10},\frac{1}{10},\frac{3}{10})$	$(\frac{3}{10},\frac{1}{10},\frac{3}{10},\frac{3}{10})$	$(\frac{1}{10},\frac{3}{10},\frac{3}{10},\frac{3}{10})$
LHS	1,397.19	1,513.79	2,076.86	1,738.49
Comment:	All optimal			

Table 4. 28:I-Optimality values for η_4ufor the Concrete model

$\eta_4 design\ points$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
LHS	595.557
Comment:	Optimal.

The concrete model has shown that optimality was obtained in the $\eta_2, \eta_3 and\ \eta_4$ design points for both EWSCAD and UWSCAD.

The efficiency of the two designs on the concrete experiment was worked out using the equation (3.44). $I_{eff} = \frac{13,902.12}{17,477.51} = 0.7954$.

This showed that the EWSCAD was a more efficient than UWSCAD just as it was in the adopted tetrahedral axial design.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.0 INTRODUCTION

This chapter summarizes the major research findings, giving the conclusions and recommendations. It also gives suggestions for further research. The study aimed at obtaining a quad-axial weighted simplex centroid design using second order Kronecker model to optimize the plinth concrete mix components for low cost houses. This was guided by four objectives namely; to construct an inscribed tetrahedral Weighted Simplex Centroid Design (WSCD). Secondly, to Evaluate the D-, G- and I- optimality criteria of the design using two weighted simplex centroid designs. Thirdly to fit a second-degree Kronecker model to a mixture experiment and finally to find the optimal mix hence evaluating the D-, G-, and I-optimality criteria of the experiment.

5.1 SUMMARY

The first objective of the study was to construct an inscribed tetrahedral weighted centroid design described in section 3.1.1 and obtained as (4.1).

Secondly, evaluating the D- and G- optimal slope values as well as the I-optimal values, for both Unequally Weighted Simplex Centroid Axial Design (UWSCAD) and Equally Weighted Simplex Centroid Axial Design (EWSCAD). The slope D- and G- optimal values were obtained using the $\mathcal{H}$-Invariant matrices. The most D- optimal point was the centroid and optimality decreased towards the vertices. The unequally weighted design (UWSCAD) proved to be more D-efficient than the EWSCAD, while the design points $\eta_3 and\ \eta_4$ satisfied the D-equivalence theorem for both designs. The most G-optimal point was the centroid since it had the lowest value. Optimality became worse towards the vertices. The equally weighted design (EWSCAD) was more G-efficient than the UWSCAD, and the two designs satisfied the G- equivalence theorem. The two designs obtained I-optimality at the design points η_2, η_3 and η_4. The efficiency ratio of UWSCAD to EWSCAD was 0.8785 therefore; EWSCAD was a more I-efficient design than UWSCAD.

Thirdly was to fit an optimal statistical second-degree Kronecker model to the experimental data. The variability in the response variable explained by the concrete model (4.27) was 0.9951. This indicated that the variation in concrete strength was largely explained by components in the model, and only 0.49% was explained by other factors. The analysis of variance (ANOVA) in both the effects of concrete components table 4.12, and the summary table 4.13 showed that the calculated F values were greater than the F-critical values at each point. This showed that the individual components were very significant in the response variable. In testing the adequacy of the model, Variance Inflation Factors (V.I.F) were calculated to test multi-collinearity in the model. The values for the squared portions and the interactions were 3.3344 and 5.0891 respectively. The rule of thumb is; if VIF is more than 10, there is a serious multi collinearity problem. If it is greater than four, then a further investigation should be done. Since the kind of experiment performed was one where all the components must be present to give the desired response, there was no need of further adequacy tests.

Fourthly, was to find the optimal mix and to evaluate the D-, G-, and I-optimality criteria of the experiment model. From figure 4.12 (the response trace diagram), the component directions X_1, X_2, X_3 and X_4 showed that as we increase component 1 (water), the compressive strength dropped rapidly on a straight line. As component 2 (cement) was increased, compressive strength decreased hyperbolically to a minimum indicated by point B. There after the increase of the same component showed an increase in compressive strength. Component 3 (sand) had a direct proportionality with compressive strength. Increasing component 4 (ballast) in relation to other components dropped the compressive strength gradually on a straight line. Therefore, the design point that gave minimum compressive strength for the said class was (0.1, 0.3, 0.3, 0.3), which translated to the ratio read from table 4.10 as (0.523:1:1.392:2.795) whose corresponding compressive strength of $27.63N/mm^2$. This therefore meant that the right amounts of water, sand and ballast required for one unit of cement for the class of concrete under study are 0.52, 1.4 and 2.8 respectively.

The concrete model obtained the D-optimality as the design constructed in (4.1). The whole model satisfied the D-and the G- equivalence theorems. It obtained the G-optimality and G-efficiency the same way as in the design (4.1) except for the η_4 design point. Both the adopted design and the experimental design points obtained the same I-optimality conditions, where $\eta_2, \eta_3 \; and \; \eta_4$ satisfied the I-optimality equivalence theorem. In testing the efficiency of the designs, EWSCAD Proved to be better than UWSCAD.

5.2 CONCLUSION

The design that was adopted shown by (4.1) proved to be an effective design. The design attained satisfactory optimality conditions of the D-, G- and I- criterion.

In evaluating for the optimality conditions, the notable differences between the adopted design and the concrete model were that all the design points for the concrete model in the two weighted designs satisfied the D-, G- and I-optimality conditions like the adopted design. By working out efficiencies, EWSCAD was a more G- and I- efficient design while UWSCAD was a more D-efficient design. Therefore, the adopted design and the Concrete model agreed largely with very small differences. It was concluded that the model was dependable to generate trustworthy results.

A second-degree Kronecker model was adopted because its quadratic nature helped obtain the optimal point at the turning point. Secondly, the Kronecker multiplication helped to increase symmetry due to the repletion of cross product terms that results to larger moment matrices. The model is also less susceptible to ill conditioning and the associated problem of unstable models with highly correlated parameters and large standard error. (Prescott, Dean, Draper, & Lewis, 2002)

According to the summary given above, the second-degree Kronecker model that was developed showed that the proportion of variation in the dependent variable explained by the model was very high (R_A^2=0.9951). The outcome from the analysis of variance (ANOVA) showed that the calculated F values were greater than the F-critical values hence; the individual components were very significant in the response variable. The variance inflation factors (V.I.F) for the squared portions and the interactions were 3.3344 and 5.0891 respectively implying that there was no serious problem of multi-collinearity hence no need of further adequacy tests.

The response surface methods (RSM) applied are dependable, as it is evident from the results that the optimal condition (minimum point) was attained as $27.63 N/mm^2$ for the M25 class of concrete. This point gave the ratio 0.52:1:1.4:2.8 for water: cement: sand: ballast. Higher strengths for this class may be achieved by higher ratios, which would address the causes of collapsing of buildings build by contractors trying to save on cost of components.

5.3 RECOMENDATIONS

From the findings of this study, the following recommendations were given

That the process that was used to realize the results be applied by the stakeholders in the building industry as it proved authentic.

That the ratio 0.52: 1:1.4:2.8 for Water: Cement Sand Ballast obtained by the study be used for low cost plinth, whose optimal (minimum) strength of $27.63N/mm^2$ for the M25 class of concrete was achieved.

That there be collaboration of Statisticians, Civil Engineers and other construction stakeholders, in order to produce valuable and sustainable structures to safeguard lives.

That the governments support such researches so as to achieve their desired goals for providing decent housing for their subjects.

5.4 FURTHER RESEARCH

Many researchers have used different statistical methods to optimize concrete components. This means that there is room for future researchers to apply other statistical methods to verify or challenge these findings. Other RSM models may be applied to leverage on these findings.

The use of admixtures in this kind of experiment is an open door for further research.

Comparing components from different places would also be a valuable research as well as testing for tensile strength.

Compressive strength is not the only response from a concrete experiment; future researchers may optimize other outcomes.

The design adopted for this study was an inscribed tetrahedron within another. A design with original vertices (1,0,0,0),(0,1,0,0),(0,0,1,0) and (0,0,0,1) is an open window of research.

Furthering the works of other statisticians or engineers using statistical methods is another identifiable study gap.

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Appendix I

Table A: The Mix Process Form

Stage		Item	Values and calculations
1	1.1	Characteristic strength (specified) Proportion Defective	{__________N/MM^2 at__________days} {____________%}
	1.2	Standard deviation (see figure in Appendix B)	{______________N/MM^2}
	1.3	Margin (see (3.42))	
	1.4	Target mean Strength(see(3.43))	(K=____) {______*______= ______N/MM^2}
	1.5	Cement strength (see working)	___________+ ___________N/MM^2
	1.6	Aggregate type(Coarse/fine)	___________.
	1.7	Free water/cement ratio (table)	(____________/____________)
	1.8	Maximum free water/cement ratio.	__________. __________Use lower value_______.
2	2.1	Slump	______________Mm
	2.2	Maximum aggregate size	______________Mm
	2.3	Average aggregate size	_____________ Mm
	2.4	Free water content	______________Kg/m^3
3	3.1	Cement content(see(3.44))	________÷________=________kg/m^3
4	4.1	Relative density of aggregate	__________Assumed
	4.2	Concrete density(see figure)	___________Kg/m^3
	4.3	Total aggregate content(see(3.45))	_____ _ _____ __ _____= _____**Kg/m^3**
5	5.1	Grading of fine aggregate	_________%
	5.2	Proportion of fine aggregate	_________%
	5.3	Fine aggregate content(see(3.46))	________×________=________kg/m^3
	5.4	Coarse aggregate content	________–________=________kg/m^3
6		Water (kg or litres) / Cement (kg) / Fine Aggregate (kg) / Coarse Aggregate mm Per m^3 ___________ __________ ____________ ____________. Ratios ___________ __________ ____________ __________.	

Appendix II

The R-Codes

```
#### Coefficient Matrix, Transpose And Left Inverse
K=matrix(k, nrow=16, ncol=10, byrow=T)
TK=t(K)
L=solve(TK%*%K)%*%TK
###### Moment Matrices Per Designs
t1=rbind(7/10,1/10,1/10,1/10)
t2=rbind(1/10,7/10,1/10,1/10)
t3=rbind(1/10,1/10,7/10,1/10)
t4=rbind(1/10,1/10,1/10,7/10)
t1kt1=kronecker(t1,t1)
ta=t1kt1%*%t(t1kt1)
mn4=round(n4,4)
## Moment And Information Matrices For Ewscad
Mne=round(((1/4)*mn1+(1/4)*mn2+(1/4)*mn3+(1/4)*mn4),4)
Mnp=ginv(Mne)
Ce=L%*%Mne%*%t(L)
Cee=solve(Ce)
Pe=Ce^0
Put1=sum(diag(Pe))
pe1=Ce^1
####### Moment And Information Matrices For Uwscad
Mnu=round(((4/15)*mn1+(6/15)*mn2+(4/15)*mn3+(1/15)*mn4),4)
Mnn=ginv(Mnu)
Cu=L%*%Mnu%*%t(L)
Cuu=solve(Cu)
Pu=Cu^0
Put=sum(diag(Pu))
pu1=Cu^1
##### Determinant Criterion for each design EWSCAD
```

```
######7/10,1/10,1/10,1/10
H1=matrix(h1, nrow=4, ncol=10, byrow=T)
Dc1e=H1%*%Ce%*%t(H1)
(det(Dc1e))^0.1
#########EFFICIENCY OF THE D-OPTIMAL DESIGN
EF1=(((det(Dc1))^0.1)/((det(Dc1e))^0.1))*100
EF2=(((det(Dc2))^0.1)/((det(Dc2e))^0.1))*100
EF3=(((det(Dc3))^0.1)/((det(Dc3e))^0.1))*100
EF4=(((det(Dc4))^0.1)/((det(Dc4e))^0.1))*100
########### D EQUIVALENCE THEOREM FOR EWSCAD with Ce
DET12=t(t1kt1)%*%Mnp%*%K%*%Ce%*%t(K)%*%t(Mnp)%*%t1kt1
DET152=t(t15kt15)%*%Mnp%*%K%*%Ce%*%t(K)%*%t(Mnp)%*%t15kt15
########## D EQUIVALENCE THEOREM FOR UWSCAD with Cu
DET1a=t(t1kt1)%*%Mnn%*%K%*%Cu%*%t(K)%*%t(Mnn)%*%t1kt1
##### G-OPTIMALITY FOR EWSCAD WITH IMPROVED INFORMATION MATRIX with Ce
G2A=t(t1)%*%solve(Dc1e)%*%t1
##### G-OPTIMALITY FOR UWSCAD WITH IMPROVED INFORMATION MATRIX with Cu
G1A=t(t1)%*%solve(Dc1)%*%t1
####### G OPTIMALITY TRACE MM-1
MMne=Mne%*%Mnp
MMne1=sum(diag(MMne))
MMnu=Mnu%*%Mnn
MMnu1=sum(diag(MMnu))
TRu=sum(diag(Cu%*%Cuu))
TRu
TRe=sum(diag(Ce%*%Cee))
#########EFFICIENCY OF THE G-OPTIMAL DESIGN
GEF1=G1D/G2D
GEF2=G1I/G2I
GEF3=G1L/G2L
GEF4=G1P/G2P
```

```
####  I-OPTIMALITY for EWSCAD with Ce
La=matrix(la, nrow=10, ncol=10, byrow=T)
Lb=round(La/140,4)
CLe=Cee%*%Lb
AVe=sum(diag(CLe))
f11=c(49,1,1,1,7,7,7,1,1,1)
F11=matrix(f11, nrow=1, ncol=10, byrow=T)
F11a=F11/100
F111r=F11a%*%Cee%*%Lb%*%Cee%*%t(F11a)
##### I-OPTIMALITY for design with UWSCAD with Cu
La=matrix(la, nrow=10, ncol=10, byrow=T)
Lb=round(La/140,4)
CLu=Cuu%*%Lb
AVu=sum(diag(CLu))
f11=c(49,1,1,1,7,7,7,1,1,1)
F11=matrix(f11, nrow=1, ncol=10, byrow=T)
F11a=F11/100
F111n=F11a%*%Cuu%*%Lb%*%Cuu%*%t(F11a)
####### I-EFFICIENCY (DESIGN)
IEF=AVu/AVe
###########   CONCRETE EXPERIMENT
##########The Concrete experiment K,L,M and C matrices
KC=matrix(kc, nrow=16, ncol=10, byrow=T)
TKC=t(KC)
LC=solve(TKC%*%KC)%*%TKC
############ EWSCAD
Cce=LC%*%Mne%*%t(LC)
Ccee=solve(Cce)
###########  UWSCAD
Ccu=LC%*%Mnu%*%t(LC)
Ccuu=solve(Ccu)
#####################  PART 1  PART 1
```

```
##### DETERMINANT criterion for CONCRETE MODEL design EWSCAD with Ce
######7/10,1/10,1/10,1/10
H1C=matrix(h1c, nrow=4, ncol=10, byrow=T)
Dcc1e=H1C%*%Ce%*%t(H1C)
###D EQUIVALENCE THEOREM FOR EWSCAD (Ce)
DT12b=t(t1kt1)%*%Mnp%*%KC%*%Ce%*%t(KC)%*%t(Mnp)%*%t1kt1
###D EQUIVALENCE THEOREM FOR UWSCAD for Concrete model(Cu) PART 1
DT1ab=t(t1kt1)%*%Mnn%*%KC%*%Cu%*%t(KC)%*%t(Mnn)%*%t1kt1
###CONCRETE D-EFFICIENCY (part 1)
CDEF1=(((det(Dcc1u))^0.1)/((det(Dcc1e))^0.1))*100
### CONCRETE G-EFFICIENCY
###C1GEF1=G12/G22
TRa=sum(diag(Ccu%*%Ccuu))
TRa
TRb=sum(diag(Cce%*%Ccee))
TRb
####### I-OPTIMAL for the concrete model MOMENT MATRIX
R1=c(15.658,4.231,4.159,3.165,5.841,8.939,10.121,2.164,2.694,3.304)
############ I-Optimal Equivalence Theorem EWSCAD with (Cee)
f11=c(49,1,1,1,7,7,7,1,1,1)
F11=matrix(f11, nrow=1, ncol=10, byrow=T)
F11a=F11/100
F111eq=F11a%*%Cee%*%CL2%*%Cee%*%t(F11a)
#### TRACE AV PREDICTION UWSCAD############
CL5=Cuu%*%CL2
CL6=sum(diag(CL5))
#############I-Optimal Equivalence Theorem UWSCAD with Cu
f11=c(49,1,1,1,7,7,7,1,1,1)
F11=matrix(f11, nrow=1, ncol=10, byrow=T)
F11a=F11/100
F111u=F11a%*%Cuu%*%CL2%*%Cuu%*%t(F11a)
###########   CONCRETE   PART 2    Cce and Ccu      #############
```

```
##### DETERMINANT CRITERION for EWSCAD  with Cce
###### 7/10, 1/10,1/10,1/10
H1C=matrix(h1c, nrow=4, ncol=10, byrow=T)
##### DETERMINANT CRITERION for UWSCAD with Ccu
######7/10,1/10,1/10,1/10
H1C=matrix(h1c, nrow=4, ncol=10, byrow=T)
######### 3/10,3/10,3/10,1/10
H3C=matrix(h3c, nrow=4, ncol=10, byrow=T)
Dct3u=H3C%*%Ccu%*%t(H3C)
(det(Dct3u))^0.1
#####1/4,1/4,1/4,1/4,1/4
H4C=matrix(h4c, nrow=4, ncol=10, byrow=T)
Dct4u=H4C%*%Ccu%*%t(H4C)
(det(Dct4u))^0.1
###CONCRETE D-EFFICIENCY (part 2)
CDEF12=(((det(Dct1u))^0.1)/((det(Dct1e))^0.1))*100
########### D EQUIVALENCE THEOREM FOR EWSCAD (Cce)
DT12=t(t1kt1)%*%Mnp%*%KC%*%Cce%*%t(KC)%*%t(Mnp)%*%t1kt1
########## D EQUIVALENCE THEOREM FOR UWSCAD for Concrete model(Ccu)
DT1a=t(t1kt1)%*%Mnn%*%KC%*%Ccu%*%t(KC)%*%t(Mnn)%*%t1kt1
#####CONCRETE G-OPTIMALITY FOR EWSCAD  with Cce
G21t=t(t1)%*%solve(Dct1e)%*%t1
######  CONCRETE G-EQUIVALENCE THEOREM
TRa
TRb
######   CONCRETE EFFICIENCY OF THE G-OPTIMAL DESIGN
CGEF1=G12t/G22t
CGEF1
CGEF2=G19t/G29t
CGEF2
CGEF3=G113t/G213t
```

```
CGEF3
CGEF4=G115t/G215t
CGEF4
########### I-OPTIMAL Equivalence Theorem EWSCAD with Ccee
CL4t=sum(diag(Ccee%*%CL2))
CL4t
f11t=c(49,1,1,1,7,7,7,1,1,1)
F11t=matrix(f11t, nrow=1, ncol=10, byrow=T)
F11at=F11t/100
F11eqt=F11at%*%Ccee%*%CL2%*%Ccee%*%t(F11at)
#####   TRACE AV PREDICTION UWSCAD  with Ccuu
CL5t=Ccuu%*%CL2
CL5t
CL6t=sum(diag(CL5t))
CL6t
###### I-Optimal Equivalence Theorem UWSCAD with Ccuu
f11=c(49,1,1,1,7,7,7,1,1,1)
F11=matrix(f11, nrow=1, ncol=10, byrow=T)
F11a=F11/100
F11tu=F11a%*%Ccuu%*%CL2%*%Ccuu%*%t(F11a)
####CONCRETE I-EFFICIENCY(2)
Ieff2=CL6t/CL4t
Ieff2
```

Appendix III: Project Pictures

Handling Sand Sieves

Sieving ballast

Drying components

Dried sand

Dried ballast

Weighing components

Compacting the mixture

Finishing cubes

Labelling *Demolding*

Demolded cubes *Cubes ready for curing*

Cubes after curing *Measuring compression strength*

Appendix IV: Publications

American Journal of Theoretical and Applied Statistics
2021; 10(1): 32-37
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I-Optimal Axial Designs for Four Ingredient Concrete Experiment

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Abstract: Stakeholders in the construction industry work towards obtaining optimal concrete mixes with an aim of producing structures with the best compressive strength. In many instances, Kenya has witnessed collapse of buildings leading to death and huge financial loses, which has been associated largely to poor concrete mixes. This paper aims at evaluating the I-optimal designs for a concrete mixture experiment for both Equally Weighted Simplex Centroid Axial Design and Unequally Weighted Simplex Centroid Axial Design, based on the second-degree Kronecker model. Optimality tests are performed to locate the optimum values of a design. In various studies, I-optimality has been shown to be among the best criteria in obtaining the most optimal outcomes. In this study, Response Surface Methodology is applied in evaluating I-optimal designs, which are known to minimize average or integrated prediction variance over the experimental region. I-optimality equivalence conditions for the inscribed tetrahedral design and for the concrete experiment model are identical with the boundary points, mid-face points and

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D- and G- Optimal Axial Slope Designs for Four Ingredient Mixture

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Abstract This paper aims at investigating and comparing the D- and G-optimal criteria for non-pure blends slope designs. The study used a parameter subsystem of interest based on the second-degree Kronecker model to obtain the H-invariant information matrices for both Equally Weighted Simplex Centroid Axial Design and Un-equally Weighted Simplex Centroid Axial Design. The D- and G- optimal values worked out revealed that the centroid achieved the best D- and G-optimality values and that the best D-efficient and G-efficient design points were η_1 with 105.71% and η_4 99.76% respectively. The latter design was more D-efficient while former design was more G-efficient.

Keywords: *axial design, subsystem, H-invariant, centroid, D-optimal, G-optimal, efficiency*

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