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Numerical Solutions of the Burgers' System in Two Dimensions under Varied Initial and Boundary Conditions

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Abstract

In this paper, we generate varied sets of exact initial and Dirichlet boundary conditions for the 2-D Burgers' equations from general analytical solutions via Hopf-Cole transformation and separation of variables. These conditions are then used for the numerical solutions of this equation using finite difference methods (FDMs) and in particular the Crank-Nicolson (C-N) and the explicit schemes. The effects of the variation in the Reynolds number are investigated and the accuracy of these schemes is determined by the $\mathbf{L}^1$ error. The results of the explicit scheme are found to compare well with those of the C-N scheme for a wide range of parameter values. The variation in the values of the Reynolds number does not adversely affect the numerical solutions.

Keywords: Hopf-Cole transformation, finite difference methods (FDMs), analytic solution, Crank-Nicolson (C-N) scheme, explicit scheme

1 Introduction

The Burgers' equation was named after the great Physicist Johannes Martinus Burgers' (1895-1981). This is an important non-linear parabolic partial differential equation (PDE) widely used to model several physical flow phenomena in fluid dynamics teaching and in engineering such as turbulence, boundary layer behaviour, shock wave formation, and mass transport, Pandey[8]. In general, this equation is suited to modelling fluid flows because it incorporates directly the interaction between the non-linear convection processes and the diffusive viscous processes, Fletcher[4]. Consequently, it is one of the principle model equations used to test the accuracy of new numerical methods or computational algorithms, Kanti[6]. The 2-D coupled non-linear Burgers' equations are a special form of incompressible Navier-Stokes equations without the pressure term and the continuity equation, Vineet[10].

It is widely known that non-linear PDEs do not have precise analytic solutions, Taghizadeh[9]. The first attempt to solve the Burgers' equation analytically was done by Bateman[2], who derived the steady-state solution for the one-dimensional equation, which was used by Burgers'[3] to model turbulence, Mohammad[7]. Due to its wide range of applicability, several researchers, both scientists and engineers, have been interested in studying the properties of the Burgers' equation using various numerical techniques. They have successfully used it to develop new computational algorithms and to test the existing ones, Kanti[6]. In most of these cases, researchers have used varying initial and boundary conditions but the most commonly used are credited to Hopf-Cole transformation and used it to generate initial and boundary conditions. Vineet[10] used two different sets of initial and boundary conditions to test the accuracy of the C-N scheme. Newton's method was used to linearize the non-linear algebraic system of equations after which Gauss elimination with partial pivoting was used to solve the resultant linear system. Bahadir[1] also used the same sets of conditions to test the accuracy of his scheme, the fully implicit finite difference scheme. Hongqing[5] and Young[11] used similar conditions to test their discrete Adomian decomposition method and the Eulerian-Lagrangian method of fundamental solutions respectively. Mohammad[7] developed a semi-implicit finite difference approach to solve the equations using an additional set of exact solutions.

In this paper, we generate three sets of varied initial and boundary conditions from general analytic solutions via Hopf-Cole transformation and separation of variables. These conditions are used to find numerical solutions of the 2-D Burgers' system using the C-N and the explicit schemes. The accuracy in terms of convergence, consistency, and stability of these schemes is determined by $\mathbf{L}^1$ error. The Reynolds number is varied to determine its effect on the solution.

2 Mathematical Formulation

The 2-D Burgers' model is given by;

$$u_t + uu_x + vu_y = \frac{1}{Re}(u_{xx} + u_{yy}) \tag{2.1}$$

$$v_t + uv_x + vv_y = \frac{1}{Re}(v_{xx} + v_{yy}) \tag{2.2}$$

subject to the initial conditions

$$\left.\begin{array}{l} u(x,y,0) = \varphi_1(x,y) \\ v(x,y,0) = \varphi_2(x,y) \end{array}\right\} (x,y) \in \Omega \tag{2.3}$$

and Dirichlet boundary conditions

$$\left.\begin{array}{l} u(x,y,t) = \zeta(x,y,t) \\ v(x,y,t) = \xi(x,y,t) \end{array}\right\} (x,y) \in \partial\Omega, \quad t > 0 \tag{2.4}$$

where $\Omega = \{(x,y) : a \leq x \leq b, a \leq y \leq b\}$ is the computational domain which in this study is taken to be a square domain, because of its convenience for finite difference methods (FDMs), and $\partial\Omega$ is its boundary; $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined; φ_1, φ_2, ζ, and ξ are known functions; u_t is the unsteady term; uu_x is the non-linear convection term; Re is the Reynolds number, and $\frac{1}{Re}(u_{xx} + u_{yy})$ is the diffusion term.

We find analytic solutions to the equations (2.1) and (2.2) via Hopf-Cole transformation in order to derive varied sets of initial and boundary conditions (2.3) and (2.4) respectively. The process of transformation is given by the following steps;

1. Linearization of the Burgers' equations by relating a function, $\phi(x,y,t)$, to $u(x,y,t)$ and $v(x,y,t)$ in the following way;

$$u = \frac{-2}{Re}\frac{\phi_x}{\phi} \tag{2.5}$$

$$v = \frac{-2}{Re}\frac{\phi_y}{\phi} \tag{2.6}$$

For simplicity in calculations, let

$$u = f_1(\phi) \tag{2.7}$$

$$v = f_2(\phi) \tag{2.8}$$

2. The derivatives of u and v with respect to t, x, and y are found and substituted back into the equations (2.1) and (2.2) to obtain;

$$f_1'(\phi)\phi_t + f_1(\phi)f_1'(\phi)\phi_x + f_2(\phi)f_1'(\phi)\phi_y$$
$$= \frac{1}{Re}(f_1''(\phi)\phi_x^2 + f_1'(\phi)\phi_{xx} + f_1''(\phi)\phi_y^2 + f_1'(\phi)\phi_{yy}) \tag{2.9}$$

$$
\begin{aligned}
& f_2'(\phi)\phi_t + f_1(\phi)f_2'(\phi)\phi_x + f_2(\phi)f_2'(\phi)\phi_y \\
= & \frac{1}{Re}(f_2''(\phi)\phi_x^2 + f_2'(\phi)\phi_{xx} + f_2''(\phi)\phi_y^2 + f_2'(\phi)\phi_{yy}) \qquad (2.10)
\end{aligned}
$$

Taking any of the above equations (2.9) and (2.10), the same solution is arrived at and therefore there is no need for repetition. We assume that ϕ is bounded and therefore it implies that $f_1'(\phi)$ and $f_2'(\phi)$ are all non-zero functions. Thus considering the first equation (2.9), and dividing through by $f_1'(\phi)$ results in;

$$
\begin{aligned}
\phi_t + f_1(\phi)\phi_x + f_2(\phi)\phi_y \quad = \quad & \frac{1}{Re}(\frac{f_1''(\phi)\phi_x^2}{f_1'(\phi)} + \phi_{xx} + \\
& \frac{f_1''(\phi)\phi_y^2}{f_1'(\phi)} + \phi_{yy}) \qquad (2.11)
\end{aligned}
$$

But from expressions (2.5) to (2.8), we determine derivatives with respect to ϕ and substitute into (2.11) to obtain;

$$
\phi_t = \frac{1}{Re}(\phi_{xx} + \phi_{yy}). \qquad (2.12)
$$

3. Equation (2.12) is linear and can be solved by separation of variables after which the solution ϕ is transformed back to the original solutions of u and v using (2.5) and (2.6) respectively.

We seek a general solution of the form;

$$
\phi(x, y, t) = a + bx + cy + dxy + X(x)Y(y)T(t) \qquad (2.13)
$$

which is the sum of the bilinear solution $a + bx + cy + dxy$ and the separable solution $X(x)Y(y)T(t)$. The bilinear solution is denoted by $\phi_1(x, y)$, and the separable solution by $\phi_2(x, y, t)$. The bilinear solution is added as a stabilizer while the separable solution is obtained from the transformed equation and can be written as

$$
\phi_2(x, y, t) = X(x)Y(y)T(t) = W(x, y)T(t) \qquad (2.14)
$$

Note that the first separation is done between space and time followed by space and space for convenience. On substitution of the expression (2.14) into equation (2.12) we obtain

$$
WT' = \frac{1}{Re}(W_{xx}''T + W_{yy}''T) \qquad (2.15)
$$

For simplicity, equation (2.15) can also be written as

$$
Re(WT') = (\Delta W)T \qquad (2.16)
$$

where Δ is the Laplacian operator. Finally, rearrangement gives

$$Re\frac{T'}{T} = \frac{\Delta W}{W} = -\alpha^2 \tag{2.17}$$

Where α^2 is a separation constant and the negative sign is used because a decaying function of time is anticipated. Thus the separated equations are

$$T' + \frac{\alpha^2 T}{Re} = 0 \tag{2.18}$$

$$\Delta W + \alpha^2 W = 0 \tag{2.19}$$

Solving equation (2.18) yields

$$T(t) = Ae^{\frac{-\alpha^2 t}{Re}} \tag{2.20}$$

Consequently, equation (2.19) is solved but it is at this stage that the function $W(x, y)$ is separated into $X(x)Y(y)$ that is space and space to arrive at;

$$X''Y + XY'' + \alpha^2 XY = 0 \tag{2.21}$$

Or

$$\frac{X''}{X} = -\frac{Y''}{Y} - \alpha^2 = -\beta^2 \tag{2.22}$$

where β^2 is a separation constant. From the expression (2.22) two equations are obtained of the form

$$X'' + \beta^2 X = 0 \tag{2.23}$$

$$Y'' + (\alpha^2 - \beta^2)Y = 0 \tag{2.24}$$

The general solutions of equations (2.23) and (2.24) are given by

$$X(x) = B\sin(\beta x) + C\cos(\beta x) \tag{2.25}$$

$$Y(y) = D\sin(\gamma y) + E\cos(\gamma y) \tag{2.26}$$

where $\gamma = (\alpha^2 - \beta^2)$. Substituting the solutions $\phi_1(x, y, t)$ and $\phi_2(x, y, t)$ into the general solution (2.13) yields;

$$\phi(x, y, t) = a + bx + cy + dxy + (B\sin\beta x + C\cos\beta x)(D\sin\gamma y + E\cos\gamma y)e^{\frac{-\alpha^2 t}{Re}} \tag{2.27}$$

At this point we transform the solution $\phi(x, y, t)$ to the original solutions $u(x, y, t)$ and $v(x, y, t)$ as stated earlier to obtain;

$$u(x,y,t) = \frac{-2[b + dy + \beta(B\cos\beta x - C\sin\beta x)(D\sin\gamma y + E\cos\gamma y)Ae^{\frac{-\alpha^2 t}{Re}}]}{Re[a+bx+cy+dxy+(B\sin\beta x+C\cos\beta x)(D\sin\gamma y+E\cos\gamma y)Ae^{\frac{-\alpha^2 t}{Re}}]} \quad (2.28)$$

$$v(x,y,t) = \frac{-2[c + dx + \gamma(B\sin\beta x + C\cos\beta x)(D\cos\gamma y - E\sin\gamma y)Ae^{\frac{-\alpha^2 t}{Re}}]}{Re[a+bx+cy+dxy+(B\sin\beta x+C\cos\beta x)(D\sin\gamma y+E\cos\gamma y)Ae^{\frac{-\alpha^2 t}{Re}}]} \quad (2.29)$$

Equations (2.28) and (2.29) are the general analytic solutions to the 2-D Burgers' system. We now choose three sets of parameters a, b, c, d, A, B, C, D, α, β, and γ to arrive at three sets of exact solutions from which we shall derive varied sets of initial and boundary conditions for numerical computation. Note that the parameters are chosen carefully to ensure that the solutions are not trivial.

The discretization of the Burgers equations is done by the explicit and the C-N schemes. For the explicit scheme, we discretize in time by the forward Euler scheme and in space by the second order central difference scheme. For the C-N, it is the trapezoidal rule in time and second order central difference scheme in space. This results in linear and non-linear algebraic systems of equations which are solved by a direct method and Newton's method respectively. The direct method used in this paper is the LU decomposition which is also used for the C-N after linearization of the non-linear systems of algebraic equations by the Newton's method. The explicit and C-N schemes are given mathematically by the following recurrence relations.
For the explicit scheme we have

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{k} = -u_{i,j}^{n}\frac{(u_{i+1,j}^{n} - u_{i-1,j}^{n})}{2h} - v_{i,j}^{n}\frac{(u_{i,j+1}^{n} - u_{i,j-1}^{n})}{2h} + \frac{(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n})}{\mathrm{Re}h^2} + \frac{(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})}{\mathrm{Re}h^2} \quad (2.30)$$

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{k} = -u_{i,j}^{n}\frac{(v_{i+1,j}^{n} - v_{i-1,j}^{n})}{2h} - v_{i,j}^{n}\frac{(v_{i,j+1}^{n} - v_{i,j-1}^{n})}{2h} + \frac{(v_{i+1,j}^{n} - 2v_{i,j}^{n} + v_{i-1,j}^{n})}{\mathrm{Re}h^2} + \frac{(v_{i,j+1}^{n} - 2v_{i,j}^{n} + v_{i,j-1}^{n})}{\mathrm{Re}h^2} \quad (2.31)$$

For the C-N scheme we have

$$\begin{aligned}\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{k} &= -\frac{1}{2}[u_{i,j}^{n+1}(\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h})+u_{i,j}^{n}(\frac{u_{i+1,j}^{n}-u_{i-1,j}^{n}}{2h})] \\ &\quad -\frac{1}{2}[v_{i,j}^{n+1}(\frac{u_{i,j+1}^{n+1}-u_{i,j-1}^{n+1}}{2h})+v_{i,j}^{n}(\frac{u_{i,j+1}^{n}-u_{i,j-1}^{n}}{2h})] \\ &\quad +\frac{1}{Re}[\frac{1}{2}\{(\frac{u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1}}{h^2})+(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2})\} \\ &\quad +\frac{1}{2}\{(\frac{u_{i,j+1}^{n+1}-2u_{i,j}^{n+1}+u_{i,j-1}^{n+1}}{h^2})+(\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2})\}] \end{aligned} \tag{2.32}$$

$$\begin{aligned}\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{k} &= -\frac{1}{2}[u_{i,j}^{n+1}(\frac{v_{i+1,j}^{n+1}-v_{i-1,j}^{n+1}}{2h})+u_{i,j}^{n}(\frac{v_{i+1,j}^{n}-v_{i-1,j}^{n}}{2h})] \\ &\quad -\frac{1}{2}[v_{i,j}^{n+1}(\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2h})+v_{i,j}^{n}(\frac{v_{i,j+1}^{n}-v_{i,j-1}^{n}}{2h})] \\ &\quad +\frac{1}{Re}[\frac{1}{2}\{(\frac{v_{i+1,j}^{n+1}-2v_{i,j}^{n+1}+v_{i-1,j}^{n+1}}{h^2})+(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{h^2})\} \\ &\quad +\frac{1}{2}\{(\frac{v_{i,j+1}^{n+1}-2v_{i,j}^{n+1}+v_{i,j-1}^{n+1}}{h^2})+(\frac{v_{i,j+1}^{n}-2v_{i,j}^{n}+v_{i,j-1}^{n}}{h^2})\}] \end{aligned} \tag{2.33}$$

where $h = \Delta x = \Delta y$, $k = \Delta t$, and $h^2 = \Delta x^2 = \Delta y^2$ due to the square computational domain.

3 Numerical Results by C-N and the Explicit Schemes

1. For the first set of parameter values given by; $a = 100$, $b = 0$, $c = 0$, $d = 1$, $A = 1$, $B = 1$, $C = 1$, $D = 1$, $E = 0$, $\beta = \pi$, $\gamma = \pi$
   the exact solutions is given by;

$$u(x,y,t) = \frac{-2y - 2\pi e^{\frac{-2\pi^2 t}{Re}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))}{Re(100 + xy + e^{\frac{-2\pi^2 t}{Re}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))} \tag{3.1}$$

$$v(x,y,t) = \frac{-2x - 2\pi e^{\frac{-2\pi^2 t}{Re}}((\cos(\pi x) + \sin(\pi x))\cos(\pi y))}{Re(100 + xy + e^{\frac{-2\pi^2 t}{Re}}((\cos(\pi x) - \sin(\pi x))\sin(\pi y))} \tag{3.2}$$

2. For the second set of parameter values given by; $a = 0$, $b = 5$, $c = 10$, $d = 0$, $A = 1$, $B = 0$, $C = 1$, $D = 0$, $E = 1$, $\beta = 0$, $\gamma = 2\pi$, the exact

solutions are given by;

$$u(x,y,t) = \frac{-10}{Re}(5x + 10y + e^{\frac{-4\pi^2 t}{Re}} \cos(2\pi y)) \tag{3.3}$$

$$v(x,y,t) = \frac{-20 + 4\pi e^{\frac{-4\pi^2 t}{Re}} \sin(2\pi y)}{Re(5x + 10y + e^{\frac{-4\pi^2 t}{Re}} \cos(2\pi y))} \tag{3.4}$$

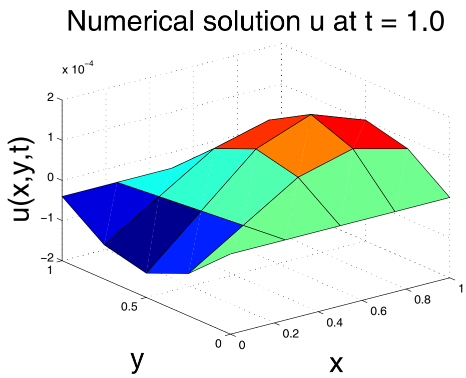
3. For the third set of parameter values given by; $a = 10$, $b = 50$, $c = 0$, $d = 0$, $A = 1$, $B = 0$, $C = 1$, $D = 1$, $E = 0$, $\beta = 2\pi$, $\gamma = 2\pi$, the exact solutions are given by;

$$u(x,y,t) = \frac{-100 + 4\pi e^{\frac{-8\pi^2 t}{Re}} \sin(2\pi x)\sin(2\pi y)}{Re(10 + 50x + e^{\frac{-8\pi^2 t}{Re}} \cos(2\pi x)\sin(2\pi y))} \tag{3.5}$$

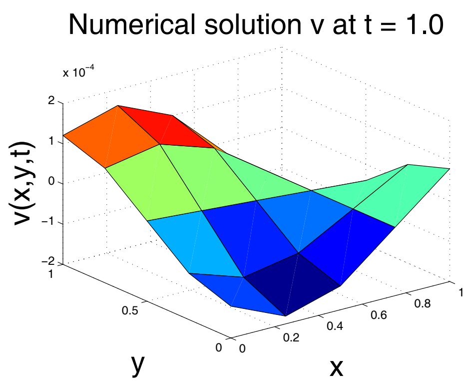
$$v(x,y,t) = \frac{-4\pi e^{\frac{-8\pi^2 t}{Re}} \cos(2\pi x)\cos(2\pi y)}{Re(10 + 50x + e^{\frac{-8\pi^2 t}{Re}} \cos(2\pi x)\sin(2\pi y))} \tag{3.6}$$

From the above sets of exact conditions, three sets of initial and boundary conditions are derived to obtain the varying numerical solutions. We vary the Reynolds number and the grid size and find the effect on the numerical solutions and the stability of the explicit scheme. We provide graphical solutions for the explicit scheme since they are as accurate as those of the C-N scheme and no difference can be noticed by way of sight.

(i) The first set of initial and boundary conditions with Re = 500 and 4×4 grid by the explicit scheme yields;



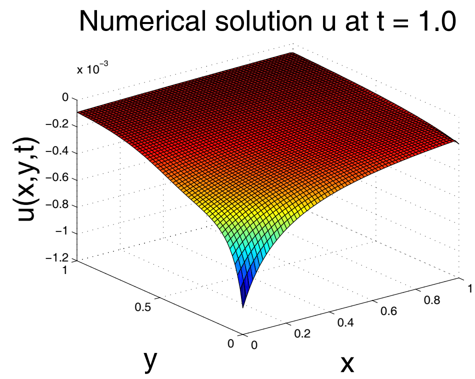
(a)



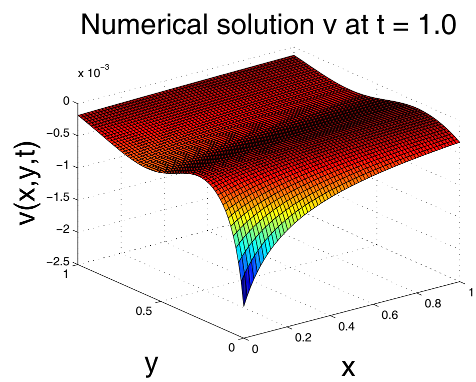
(b)

Figure 1: Numerical solutions for u and v with dt = 0.001 and t = 1.0 seconds

(ii) The second set of initial and boundary conditions with Re = 10,000 and 64×64 grid by the explicit scheme yields;



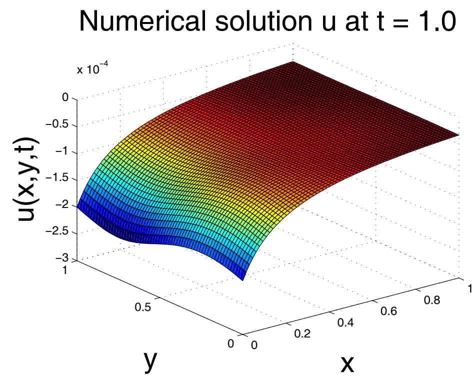
(a)



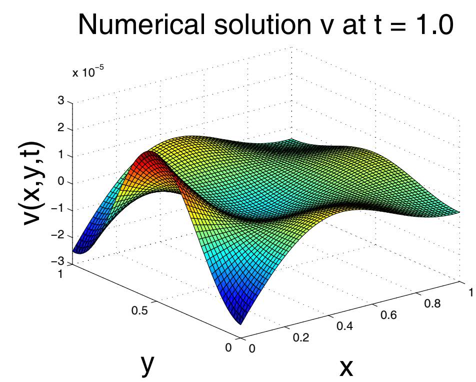
(b)

Figure 2: Numerical solutions for u and v with dt = 0.001 and t = 1.0 seconds

(iii) The third and last set of initial and boundary conditions with Re = 50,000 and 64×64 grid by the explicit scheme yields;



(a)



(b)

Figure 3: Numerical solutions for u and v with dt = 0.001 and t = 1.0 seconds

We now turn our attention to the $\mathbf{L}^1$ error analysis. We determine this error for the first set of initial and boundary conditions as follows;

(i) For the explicit scheme, the first set of solutions in u and v respectively yields;

Table 1: Order of Convergence for solution u and v at Re = 4000, t = 1 sec, dt = 0.001

No. of Cells	$\mathbf{L}^1$ error in u	Order	No. of Cells	$\mathbf{L}^1$ error in v	Order
(4,4)	1.372671e-09		(4,4)	7.73968e-10	
(8,8)	4.589355e-10	1.580623	(8,8)	3.559509e-10	1.12060
(16,16)	1.270045e-10	1.853412	(16,16)	1.109287e-10	1.68205
(32,32)	3.295523e-11	1.946300	(32,32)	3.024347e-11	1.87494
(64,64)	8.277211e-12	1.993291	(64,64)	7.734937e-12	1.96716
(128,128)	1.999203e-12	2.049720	(128,128)	1.876935e-12	2.04301

(ii) For the C-N scheme, the first set of solutions in u and v respectively yields;

Table 2: Order of Convergence for solution u and v at Re = 4000, t = 1 sec, dt = 0.001

No. of Cells	$\mathbf{L}^1$ error in u	Order	No. of Cells	$\mathbf{L}^1$ error in v	Order
(4,4)	1.372733e-09		(4,4)	7.740016e-10	
(8,8)	4.590219e-10	1.58042	(8,8)	3.560174e-10	1.12039
(16,16)	1.271022e-10	1.85257	(16,16)	1.110144e-10	1.68120
(32,32)	3.305757e-11	1.94294	(32,32)	3.033822e-11	1.87154
(64,64)	8.381246e-12	1.97974	(64,64)	7.833241e-12	1.95346

4 Conclusion

From tables 1 and 2, it is clearly noticed that the explicit and C-N schemes are accurate and compare well with each other and are of second order convergence in space. The explicit scheme is stable for small time stepping and high Reynolds number. Furthermore since the L^1 error approaches zero as the mesh is refined, consistency is achieved in these schemes. Variation in the Reynolds number does not affect the numerical solutions thus justifies the balance between the non-linear convection terms and the diffusion terms in the Burgers' equation.

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