CONSTRUCTION OF MODIFIED OPTIMAL SECOND ORDER ROTATABLE DESIGNS

BY

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DECLARATIONS

Declaration by the student

I declare that this thesis is my original work and has not been presented for any academic award in any institution and shall not be reproduced in part or full, or in any format without prior written permission from the author and/or Moi University.

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DEDICATION

To my mother Teresa Kemunto, my brothers Charles and Douglas and my future family.

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ABSTRACT

Optimal rotatable designs are experimental designs that are applicable in agricultural, industrial and pharmaceutical processes to provide optimum responses subject to various input variables. Most researchers in these fields seek alternative experimental measures to improve productivity by optimizing the scarce resources available in order to cut on the cost of experimentation. The purpose of the study was to construct optimal modified Second Order Rotatable Designs (SORD) with reduced number of designs points from the existing SORD. The objectives were: Construction of SORD with reduced number of designs points from existing SORD; construction of Modified Second Order Slope Rotatable Designs (MSOSRD) with reduced designs points from existing modified slope rotatable designs, evaluation of Average (A-), Determinant (D-), Eigenvalue (E-) and Trace (T-) optimality criteria and illustrate the application of the constructed reduced designs points with hypothetical example. Construction of reduced SORD was done by taking a fraction of a suitable set of points for existing design points while keeping the other set of points constant and subjecting them to rotatability conditions. The MSOSRD with reduced designs points were obtained by taking fractions of suitable factorial combination obtained from $2^{t(v)}$ fractional factorial designs. The parameter system of interest considered linear, pure and mixed quadratic factors to determine the moment matrix used for the evaluation of the alphabetic optimality criteria (A-, D-, E- and T-). A practical hypothetical example of sixteen design points was used in the analysis of response surface design using Mintab version 17.All the SORD and MSOSRD considered in this study were reducible and rotatable. From the evaluation of optimality criteria in three dimensions; the design with fourteen (14) points was E- optimal design with an eigenvalue of 0.004492. In four dimensions, the design with twenty four (24) design points was A-optimal with an optimal trace value of 0.001437 while in five dimensions, the forty two (42) design points was also A- optimal with an optimal trace value of 0.000357. Designs in three, four and five dimensions with reduced number of design points were constructed and their optimality criteria evaluated. The reduced number of designs points imply fewer experimental runs therefore minimizes the cost of experimentation. The study recommends utilization of optimal reduced designs for cost effectiveness in designing of experiments for production processes in various sectors of the economy. Further study can be done in higher order rotatable designs if the second order is established to be inadequate.

LIST OF ABBREVIATIONS

RSM: Response Surface Methodology.

SORD: Second Order Rotatable Design.

TORD: Third Order Rotatable Design.

CCD: Central Composite Design

SRCCD: Slope Rotatable Central Composite Design

SOSRCCD: Second Order Slope Rotatable Central Composite Design

MSOSRD: Modified Second Order Slope Rotatable Design

D-: Determinant criterion

E-: Eigen value criterion

A-: Average variance criterion

T-: Trace criterion

C-: Criterion

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CHAPTER ONE
INTRODUCTION

1.1 Introduction

This chapter covers the background of the study, the statement of the problem, objectives of the study and the significance of the study.

1.2 Background of the study

Experimentation plays an important role in any scientific research. It involves the allocation of treatments to experimental units, and then estimation of one or more responses. A well planned and designed experiment is an efficient method of exploring practical problems about the world. It is part of scientific method which requires observing and gathering information about how processes and systems work. In an experiment, some input transform into an output that has one or more observable response. Therefore, useful results and conclusions can be drawn from experiments. In many applications of Response Surface Methodology (RSM) such as in agriculture, industry and pharmacy most of the responses or yields about products and processes are majorly derived from experiments. In these economic hard times, the world is facing scarcity of resources which has prompted researchers to come up with robust methods on the utilization of scarce resources available for optimal production. Therefore it is important for researchers to carefully plan and design experiments before conducting the actual experiment. Among the basic few considerations in planning and designing of experiments are; the assessment of the resources available, time and cost of conducting experiments and the prior knowledge of the experimental procedures. Box and Hunter (1957) referred these types of experimental designs as rotatable designs and suggested that they can be utilized in experimentations. In such designs, the experimenters can use the optimality criteria to determine the adequacy of a proposed experimental design prior to running it. If several alternative designs are proposed, the optimality properties can be compared to aid in the choice of the best design. The most common empirical model used for approximation of the true model over the experimental region is a polynomial.

In some applications of response surface methodology, according to Box and Draper (1959), experimenters are usually interested at estimating either the absolute response or the parameters of a model providing the relationship between the response and the factors.Researchers may also need to determine rates of change in the yield or a response for a given unit change in input variables .In such cases, slope rotatable designs are of great interest. In most frequent cases, estimation of slope occurs in practical situations. For example, there are circumstances in which the experimenter wants to estimate the rate of reaction in a chemical experiment or the rate of change in the yield of a crop for various fertilizer doses, Victorbabu (2005).

Different authors have greatly contributed in the construction of many second; third, fourth and fifth order rotatable designs where by some designs have been applied in experimentation. This study considered second order rotatable designs. Draper and Herzberg (1968) reveals that some of these design points are of theoretical interests and the chance of them being utilized in an experimental investigation is currently small due to the number of points and levels involved. Therefore, reduction of these design points are needed to optimize some experimental constraints such as scarcity of inputs, cost of production and the little time available to carry out experiments. Therefore, further advancement is required in developing experimental designs with fewer experimental runs especially in developing countries where the cost of production and living is high. The aim of this study was to construct optimal modified second order rotatable designs by utilizing the existing second order rotatable designs constructed by Draper (1960b) and Victorbabu (2005).

1.3 Basic definitions

1.3.1 Second Order Design

A Second order design also referred to as a polynomial of degree two is a design for fitting a second-degree model used to approximate the response surface with a parabolic curvature that comprises the linear, quadratic and cross product (interaction) terms.

1.3.2 Rotatable Design

A design is said to be rotatable if the variance of the response estimate is a function only of the distance of the points from the design center or is a design whose prediction variance is constant at all points that are equidistant from the design center.

1.3.3 Optimal Design

An optimal design is a class of experimental design that is optimal with respect to a certain optimality criterion. In the design of experiments for estimating statistical models, optimal designs allow parameters to be estimated without bias and with minimum variance.

1.4 Statement of the Problem

In any experimental setting, the primary objective of the application of optimal rotatable designs is to optimally combine the available inputs to obtain maximum yields. However, some of the available rotatable designs in literature generally have a substantially large number of design points. These design points may not be desirable to experimenters who are constrained by resources, time and the cost involved in carrying out the experiment. This study therefore considered existing rotatable designs with the aim of reducing the number of design points in order to minimize the cost of experimentation when carrying out the actual experiment.

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1.5 Objectives of the study

1.5.1 General Objective

The general objective for this study was to construct optimal modified second order rotatable designs.

1.5.2 Specific Objectives

In this study the specific objectives were to;

1. Construct second order rotatable designs with reduced number of points from existing second order rotatable designs in k-dimensions.
2. Construct modified slope rotatable central composite designs with reduced number of designs points from existing second order slope rotatable central composite designs.
3. Evaluate A-, D-, E-, and T- optimality criteria for the reduced rotatable designs constructed.
4. Illustrate the application of the constructed reduced designs points with hypothetical example.

1.6 Significance of the study

The utilization of these reduced designs can be used in order to optimize production in agriculture and industrial processes .Since the reduced experimental runs could require fewer resources and little time to conduct and obtain optimal responses, proper utilization of such experimental designs would enable developing countries realize sustainable development goals.

CHAPTER TWO
LITERATURE REVIEW

2.1 Introduction

In this section the study traces the streams of thought that led to this study in response surface methodology.

2.2 Response Surface Methodology and Construction of Second Order Rotatable Designs

Response Surface Methodology (RSM) is a collection of mathematical and statistical techniques useful for analyzing experiments where the yield is believed to be influenced by one or more controllable factors. Experimenters are required to make choices of the experimental designs before the actual experiments to avoid incurring more experimental costs. Therefore, an experimental design must be selected prior to experimentation. Box and Hunter (1957) suggested that these types of experimental designs are suitable for such experimentations and called them rotatable designs. In accordance with Montgomery *et al*., (2009), there are several experimental designs some of which can be applied in food or chemical companies to test ingredients, prepare and reformulate a new food product or even to optimize the conditions leading to an optimal process and perhaps more important in estimating rate of change of a given response. Designs for fitting first-degree models are called first-order designs and those for fitting second-degree models are referred to as second-order designs. Some of these designs are; full factorial design, fractional factorial designs, saturated designs; central composite designs, slope rotatable designs etc, Myers *et al*., (2009).

In this context, Box and Hunter (1957) introduced rotatable designs in order to explore the response surfaces. They developed second order rotatable design through geometrical configurations. This was closely followed by the work of Carter (1957) who constructed some second order rotatable designs in two dimensions.

Bose and Draper (1959) gave a method of constructing second order rotatable designs in three dimensions. There was need to have a general method for construction of second order rotatable designs in four or more dimensions and Draper (1960a) provided the method, where he constructed second order rotatable designs in three dimensions and gave the conditions for existence of second order rotatable designs in k-dimensions. Herzberg (1967) came up with an alternative method of constructing second order rotatable designs in k-dimensions. When comparing her method with Draper's method, Herzberg's method gave designs with very large number of points but there were no conditions to be satisfied like the case in Draper's method.

Gardiner *et al*., (1959) gave both the moments and the non-singularity conditions for third order rotatability. Their work was followed by Patel and Arap Koske (1985) who also gave the moments and the non-singularity conditions for fourth order rotatability. Njui and Patel (1988) gave the moments and non-singularity conditions for fifth order rotatability.

Since then, different authors have constructed several second, third and fourth order rotatable designs in different dimensions. Draper (1960b) constructed some third order rotatable designs in three dimensions. Huda (1982a, 1982b) gave an alternative method of constructing some third order rotatable designs. Arap koske and Patel (1986) constructed a fourth order rotatable design in three dimensions and thereafter Arap Koske (1987) used some hints from Draper and Herzberg (1985) to construct a fourth order rotatable design in four dimensions. Mutiso (1998) constructed specific and sequential optimal rotatable designs in three, four and five dimensions. Mutai (2012) used the method of Huda (1982b) to construct third order rotatable designs in k-dimensions under balanced incomplete block designs. Kosgei (2013) constructed a five level modified third order rotatable design using balanced incomplete block design. Otieno *et al*., (2016) gave cost effectiveness analysis of optimal malaria control strategies in Kenya. Draper and Herzberg (1968) in their paper on further research on second order rotatablility suggested that an experimenter can take fractions of the existing points set and carefully combine them with other point sets to form reduced second order rotatable designs.

There was need to estimate the slope of the response where in many applications of response surface methodology estimation of rate of change was of great interest. This was made possible by Atkinson (1970) who used the least squares estimation of the coefficients in a first order polynomial model to estimate the slope of a response surface. Das *et al*., (1999) pioneered the construction of modified rotatable designs. Hader and Park (1978) introduced slope rotatability for central composite designs on analogous lines to Box and Hunter (1957) central composite rotatable designs. Victorbabu (2002a, 2002b) studied second order slope rotatable designs (SOSRD) and constructed SOSRD using different methods. Victorbabu (2005) introduced and constructed modified slope rotatable central composite designs for $2 \leq v \leq 17$ number of factors. Victorbabu (2006) constructed modified SORD using BIBD. Victorbabu (2007) suggested a review on SOSRD. It is evident that a lot of work has been done in construction of second, third and fourth order designs. However, these designs have relatively large number of points and may not be desirable to experimenters with scarce resources such as experimental materials, money and little time required to carry out the experiment. Therefore, experimenters and researches who are interested in carrying out cost effective experimental tests would prefer to choose designs with minimal points in their experimental investigations. The current study therefore focused on construction of optimal modified second order rotatable designs with reduced design points from the existing second order rotatable designs constructed by Draper (1960b) and Victorbabu (2005).

2.3 Optimality Criteria

In many experimental investigations, accordance to Montgomery (2009) and Myers *et al.,* (2009) there are several experimental designs that can be applied in food or chemical companies to test ingredients and/or to prepare or reformulate a new food product or even to optimize the conditions leading to an optimal process. In such designs, one may need experiments with optimal settings on the design of interest. Finding an optimal experimental design is considered one of the most important aspects in the context of the experimental design. Before experimentation, the experimenter needs to decide on which design is suitable for his or her experiment. This is achieved by analyzing the optimality criteria of the designs. An optimality criterion is a criterion which summarizes how good a design is, and it is maximized or minimized by an optimal design. There are several alphabetic optimality criteria that are used to evaluate optimality of designs. These criteria are classified into three categories i.e. parameter estimation criteria, model discrimination criteria and others. Those for parameter estimation include; Determinant, D- Criterion, Average variance, A- Criterion and Eigen value, E- Criterion. Those for model discrimination include the C- Criterion and Trace, T- Criterion. In the analysis of the designs, all the criteria are evaluated with respect to a particular design and the one with the least value is taken as the optimality criterion of that design, Kosgei (2002).

Optimal designs are experimental designs that are generated based on a particular optimality criterion and are generally optimal only for a specific statistical model. The work of optimal experimental designs extends back to Smith (1918) who was one of the first authors to state a criterion and obtain optimal designs for regression problems. Many years later, Kiefer (1959) developed useful computational procedures for finding optimum designs in regression problems of statistical inference. There are many optimality criteria, these criteria are sometimes called alphabetical optimality criteria. In this study, alphabetic optimality criteria (A-, D-, E- and T-) were considered.

The D- optimality, considered as the most important and popular design criterion in experimental applications, was introduced by Wald (1943) who put the emphasis on the quality of the parameter estimates. The D- optimality criterion also known as the determinant criterion is essentially a parameter estimation criterion. This was called later, D- optimality by Kiefer and Wolfowitz (1959*).* The D- optimality is the most studied criterion which is widely seen in the literature by Kiefer (1959), Fedorov (1972), Silvey (1980), Pázman (1986), Pukelsheim (1993) and Mandal (2000). Mandal (2000) considered the construction of D- optimal designs in a variety of examples which is used in maximizing the determinant of the moment matrix, or equivalently, minimizing the determinant of the inverse of the moment matrix.

The A- optimality criterion was introduced by Chernoff (1953) which involves the use of Fisher's information matrix. An algebraic approach for constructing A- optimal design under generalized linear models was presented by Yang (2008).The A- optimality is used in minimizing the average variance of the parameter estimates.

The E- optimality was introduced by Ehrenfeld (1955), but the Computations of E- optimal polynomial regression design was introduced by Heiligers (1996). A method for computing E- optimal designs for a broad class of two parameter models was presented by Dette and Haines (1994). The procedure that was employed here builds on finding the design which maximizes the minimum eigenvalue of the moment matrix or equivalently minimize the maximum eigenvalue of the moment matrix. E- optimality minimizes the maximum variance of all possible normalized linear combinations of parameter estimates.

The T- optimal design is an optimality criterion used in discriminating between two or more models. Atkinson and Fedorov (1975a, 1975b) introduced experimental designs for discriminating between two models and also between several models. There are two choices for defining T–optimality criterion according to the number of models under discrimination. One of the choices is by discriminating between two models and discriminating between several models.

According to Pukelsheim (2006), real optimality criteria are functions with properties that measure largeness of information matrices. These properties includes; positive homogeneity, superadditive, non-negative, non-constant and upper semicontinuity. Such criteria are called information functions. The most prominent information functions are matrix means; $\phi_p, P \in [-\infty;1]$. The matrix means comprise the classical optimality *D*-, *A*-, *E*- and *T*- .Mutiso (1998) constructed designs of order two but the optimality criteria for the constructions were not evaluated. Kosgei (2002) gave the optimality criteria for the specific second order rotatable designs in three dimensions constructed by Mutiso (1998). Kosgei *et al.*, (2006) gave optimality of second order rotatable designs in three dimensions. Rambaei (2014) considered second order rotatability and developed general formulae for their optimality criteria. Mutai et al. (2012) gave optimal designs for mixture of experiments and their applications in agricultural research. Koech (2013) gave E- optimal designs for second degree kronecker model mixture experiments .Kiplagat *et al.*, (2015) gave designs with optimal values for second degree kronecker model mixture of experiments with four or more ingredients. Otieno *et al.,* (2016) carried out all possible combinations of cost-effectiveness analysis of optimal malaria control strategies in Kenya. Rajyalakshmi and Victorbabu (2016) suggested an empirical study of second order rotatable designs under tri-diagonal correlated structure of errors using incomplete block designs. From the existing literature, more optimal designs with reduced number of design points are needed for experimenters who would prefer designs with fewer experimental runs for their investigation. This study therefore focused on the construction of modified optimal second order rotatable designs with reduced number of designs points.

CHAPTER THREE
METHODOLOGY

3.1 Introduction

In this chapter, the methods of construction of optimal second order rotatable designs with reduced designs points and for evaluation of their optimality criteria were presented.

3.2 Method of Construction of SORDs with reduced number of design points in k- dimensions

3.2.1 Second order model

The second order model for fitting a response surface design in k- dimensions is given by;

$$Y_u = \beta_0 + \sum_{i=1}^{k} \beta_i x_{iu} + \sum_{i=1}^{k} \beta_{ii} x_{iu}^2 + \sum_{i=1}^{k} \sum_{i<j}^{k} \beta_{ij} x_{iu} x_{ju} + e_u \ . \qquad (3.1)$$

Where x_{iu} denotes the level of the ith factor $(i = 1,2,\dots,k)$ in the uth run $(u = 1,2,\dots,N)$ of the experiment, $e_u{}'s$ are uncorrelated random errors with mean zero and variance δ^2.

3.2.2 Transformation group in three dimensions and its generated points sets

Let (x, y, z) be a general point in three dimensions, according to Bose and Draper (1959), applying a transformation group to the point gave a set of 24 elements with coordinates $(\pm x, \pm y, \pm z)$, $(\pm y, \pm z, \pm x)$ and $(\pm z, \pm x, \pm y)$ denoted by $s(x, y, z)$ which consists of eight points. In other cases, some special choices of $s(x, y, z)$ may coincide in pairs or in triplets or in quadruplets for example, $\mathrm{s(p, q, 0)}$which consists of twelve points presented in coordinate form as;$(\pm p, \pm q, 0)$, $(\pm p, 0, \pm q)$ and $(0, \pm p, \pm q)$ or $\mathrm{s}(c_i, 0, 0)$ as $(\pm c_1, 0,0), (0, \pm c_1, 0)$ and $(0,0, \pm c_1)$ which consists six points .Let the excess of these sets of points be denoted by Ex such that $Ex[s(x, y, z] = \sum_{u=1}^{N} x_{iu}^4-3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2$. The twelve point set may be denoted by S $\mathrm{(p, q, 0)}$ with an excess of $Ex[\ \mathrm{(p, q, 0)}]$ = 4 $(\mathrm{p}^4 + \mathrm{q}^4 - 3\mathrm{p}^2\mathrm{p}^2)$.

3.2.3 Conditions for second order rotatable designs

A second-order response surface design is said to be a second order rotatable design if the design points satisfy the following rotatable arrangement given by Box and Hunter (1957);

a) Moment conditions

i. $\sum_{u=1}^{N} x_{iu}^2 = N\lambda_2$, $i = 1,2,\ldots,k$,

ii. $\sum_{u=1}^{N} x_{iu}^4 = cN\lambda_4$, i = 1,2,…,k, (3.2)

iii. $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = N\lambda_4$, $i \neq j = 1,2,\ldots,k$.

For all $i \neq j = 1,2,3$ and all other sums and products and powers up to and including order four are zero. The excess is given as;

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2, \ i \neq j = 1, 2\ldots k. \quad (3.3)$$

b) Non-singularity conditions

$$\frac{\lambda_4}{\lambda_2^2} > \frac{k}{k+2} . \quad (3.4)$$

Consider the existing design points for designs constructed by Draper (1960). Let D_{ij} denote existing designs points where, $i = 3,4,5$ is the number of dimensions and $j = 1,2,3$ is the set points in the i^{th} dimension such that in the following designs are given as;

(i) Three Dimensions

D_{31}= s(a, a, a)+s $(c_1, 0, 0)$ + s $(c_2, 0, 0)$ (twenty points);

D_{32}= s(a_1, a_1, a_1) + s (a_2, a_2, a_2) + s (c, 0 ,0); (twenty two points); (3.5)

D_{33}= s(f, f, 0) + s$(c_1, 0, 0)$ + s$(c_2, 0, 0)$ (twenty four)

The construction of reduced second order rotatable designs in three dimensions based on Draper and Herzberg (1968) suggested method was done by taking half fraction of $s(a, a, a)$ factorial points in D_{31} while keeping the other axial points set constant i.e.

$s(c_1, 0,0)$ and $s(c_2, 0,0)$ respectively. D_{32} was reduced by taking half fraction of; $s(a_1, a_1, a_1)$ and $s(a_2, a_2, a_2)$ factorial points set respectively while keeping $s(c, 0,0)$ axial points constant. D_{33} was reduced by taking half fraction of the $s(f, f, 0)$ experimental runs while keeping the other axial runs of $s(c_1, 0,0)$ and $s(c_2, 0,0)$ constant.

(ii) Four dimensions

D_{41}= s(a, a, a, a)+s (c_1, 0, 0,0) + s (c_2, 0, 0,0) (thirty two points); (3.6)

D_{42}= s(a_1,a_1,a_1, a_1) + s (a_2,a_2,a_2, a_2) + s (c, 0,0,0) (forty points).

The reduced design points were obtained by taking half fraction of $s(a, a, a, a)$ factorial points in D_{41} while keeping the other axial points set constant i.e. $s(c_1, 0,0,0)$ and $s(c_2, 0,0,0)$. D_{42} was also reduced by taking half fraction of; $s(a_1, a_1, a_1, a_1)$ and $s(a_2, a_2, a_2, a_2)$ factorial point's sets respectively while keeping $s(c, 0,0,0)$ axial points constant.

(iii) Five dimensions

D_{51}= s(a, a, a, a, a)+s (c_1, 0, 0,0,0) + s (c_2, 0, 0,0,0) (fifty two points); (3.7)

D_{52}= s(a_1,a_1,a_1, $a_{1,}$, a_1) + s (a_2,a_2,a_2, a_2, a_2)) + s (c, 0,0,0,0) (seventy four points).

D_{51} was reduced by taking half fraction of $s(a, a, a, a, a)$ factorial points while keeping the other axial points set constant i.e. $s(c_1, 0,0,0,0)$ and $s(c_2, 0,0,0,0)$. D_{52} was also reduced by taking half fraction of; $s(a_1, a_1, a_1, a_1, a_1)$ and $s(a_2, a_2, a_2, a_2, a_2)$ factorial points sets respectively while keeping $s(c, 0,0,0,0)$ axial points set constant.

The reduced points sets from (3.5), (3.6) and (3.7) in three, four and five dimensions respectively were subjected to the rotatability conditions given in (3.2) and (3.3) to test if they were rotatable.

3.3 Method of Construction of modified slope rotatable designs from existing designs

Consider the two existing modified slope rotatable central composite designs points constructed by Victorbabu (2005) in four and five factors given as;

i. Four factors

$$\text{v} = 4, \text{t(v)} = 4, n_a = 2, a^2 = 4, n_0 = 32 \text{ and } N = 64, \quad (3.8)$$

ii. Five factors

$$\text{v} = 5, \text{t(v)} = 4, n_a = 2, a^2 = 4, n_0 = 28 \text{ and } N = 64.$$

The reduced modified designs were obtained by taking half fraction of $2^{t(v)}$ factorial combinations of the existing modified slope rotatable central composite design presented as; $(\pm a, a, a, \dots, a)$, $(a, \pm a, \dots a)$,…, $(a, a, \dots, \pm a)$ while keeping the axial and the central points constant given in the form; $(\pm a, 0, 0, \dots, 0)$, $(0, \pm a, 0, \dots, 0)$, $(0, 0, \dots \pm a)$ and $(0, 0, 0, \dots, 0)$ respectively. The combination of the reduced ($2^{t(v)-1}$) factorial points together with the axial and central points were then subjected to modified slope rotatability conditions to test if they were rotatable.

3.3.1 Conditions for Modified slope Rotatable Designs.

Victorbabu (2005, 2006) gave the conditions for modified slope second order rotatability as;

a) Moment conditions

i. $\sum_{u=1}^{N} x_{iu}^2 = 2^{t(v)-1} + 2n_a a^2 = N\lambda_2$,

ii. $\sum_{u=1}^{N} x_{iu}^4 = 2^{t(v)-1} + 2n_a a^4 = cN\lambda_4$, (3.9)

iii. $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 2^{t(v)-1} = N\lambda_4$.

iv. $\text{N} = 2^{t(v)-1} + 2vn_a + n_0$

Where $c = 5$, and N is the total number of design points.

The application of some restriction that indicate some relationship among $\sum_{u=1}^{N} x_{iu}^2$, $\sum_{u=1}^{N} x_{iu}^4$ and $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$ solves the unknown levels in equations (3.9). Precisely, $\sum_{u=1}^{N} x_{iu}^4$ equated to $5\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$ solves the unknown values of n_a and a^4 .

b) Non-singularity conditions

$$(c+v-1)\lambda_4 > v\lambda_2^2, \tag{3.10}$$

$$\lambda_4[v(5-c)-(c-3)^2]+\lambda_2^2[v(c-5)+4]=0.$$

Where c, λ_2 and λ_4 are constants.

c) Modified condition

Is given as;

$$\lambda_2^2 = \lambda_4. \tag{3.11}$$

The utilization of the modified condition in (3.11) gives the unknown values of N and n_0.

d) The variances and covariances

The variances and covariances of the estimated parameters for the reduced designs points were given as;

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\delta^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$V(\hat{b}_i) = \frac{\delta^2}{N\sqrt{\lambda_4}},$$

$$V(\hat{b}_{ij}) = \frac{\delta^2}{N\lambda_4}, \tag{3.12}$$

$$V(\hat{b}_{ii}) = \frac{\delta^2}{(c-1)-N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\delta^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right],$$

$$Cov(\hat{b}_0,\hat{b}_{ii}) = \frac{-\lambda_2\delta^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj}) = \frac{(\lambda_2^2-\lambda_4)\delta^2}{(c-1)-N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]}$$ and other covariances are zero.

According to Victorbabu (2007), for a second order model, the partial derivative of the response with respect to the independent variables is given by;

$\frac{\delta\hat{Y}}{\delta x_i} = \hat{b}_i + 2\hat{b}_{ii}x_{iu} + \sum_{i\neq j} b_{ij}x_{ju}$.

The variance of the estimated response becomes;

$V(\frac{\delta\hat{Y}}{\delta x_i}) = V(\hat{b}_i) + 4x_{iu}^2 V(\hat{b}_{ii}) + \sum_{i\neq j} x_{ij}^2 V(\hat{b}_{ij})$.

Upon simplification this gives

$$V\left(\frac{\delta\hat{Y}}{\delta x_i}\right) = \left[\frac{\sqrt{\lambda_4}}{N\lambda_4} + \frac{d^2}{N\lambda_4}\right]\delta^2. \quad (3.13)$$

The two examples used provided a basis for further generation of other reduced designs for 4 ≤ v ≤10 factors. Table 3.1 below shows the list of existing designs and their respective components that were considered in this study.

Table 3.1 A list of Modified SRCCD for 4≤ v ≤10

No. of Factors(v)	t(v)	n_a	a^2	n_0	$N(Unreduced)$	$V(\hat{b}_i)\delta^{-2}$	$V\left(\frac{\delta\hat{Y}}{\delta x_i}\right)\delta^{-2}$
4	4	2	4	32	64	0.0313	0.0313+0.0625d^2
5	4	2	4	28	64	0.0313	0.0313+0.0625d^2
6	5	1	8	28	72	0.0208	0.0208+0.0313d^2
7	6	2	8	52	144	0.0104	0.0104+0.0156d^2
8	6	2	8	48	144	0.0104	0.0104+0.0156d^2
9	7	1	1	54	200	0.0063	0.0063+0.0078d^2
10	7	1	1	52	200	0.0063	0.0063+0.0078d^2

3.4 Method of Evaluation of Optimality Criteria for the Reduced Designs

Illustration on method of evaluation of particular optimality criteria was presented.

3.4.1 Design Matrix

A design matrix for second order rotatability is given by;

$$X'=\begin{bmatrix} x_{o1}\ x_{11} \dots x_{k1}\ x_{11}^2 \dots x_{k1}^2 \\ x_{o2}\ x_{12} \dots x_{k2}\ x_{12}^2 \dots x_{k2}^2 \\ \cdot \\ \cdot \\ \cdot \\ x_{on}\ x_{1n} \dots x_{kn}\ x_{1n}^2 \dots x_{kn}^2 \end{bmatrix}_{[n\times k]} \tag{3.14}$$

Then the transpose of X is X' such that $N^{-1}(X'X)$ is the moment matrix of N points in k-dimensions.

3.4.2 The Moment Matrix for Second Order Rotatability

The moment matrix for a second order model given in (3.1) is given by;

$$M = \frac{1}{N}(X'X), \tag{3.15}$$

Where the design matrix X is given in (3.14) and N is the number of design points.

Therefore, using (3.14) in (3.15), the following were obtained;

$$M = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \tag{3.16}$$

Where;

$$A_1 = \begin{bmatrix} 1 & \lambda_2 & \lambda_2 & \lambda_2 \dots \lambda_2 \\ \lambda_2 & 3\lambda_4 & \lambda_4 & \lambda_4 \dots \lambda_4 \\ \lambda_2 & \lambda_4 & 3\lambda_4 & \dots \lambda_4 \\ & & \cdot & \\ & \cdot & & \cdot \\ & \cdot & & \cdot \\ \lambda_2 & \lambda_4 & \lambda_4 & \lambda_4 \dots 3\lambda_4 \end{bmatrix}_{[k+1]\times[k+1]}, \tag{3.17}$$

$$A_2 = \begin{bmatrix} \lambda_2 & 0 & 0 \dots 0 \\ 0 & \lambda_2 & 0 \dots 0 \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ 0 & 0 & 0 \dots \lambda_2 \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]}, \tag{3.18}$$

and

$$A_3 = \begin{bmatrix} \lambda_4 \; 0 \; 0 \ldots 0 \\ 0 \; \lambda_4 \; 0 \ldots 0 \\ 0 \; 0 \; \lambda_4 \ldots 0 \\ . \\ . \\ . \\ 0 \; 0 \; 0 \ldots \lambda_4 \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]} . \quad (3.19)$$

From (3.16);

$$M^{-1} = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} \quad (3.20)$$

Where,

$$A_1^{-1} = \frac{1}{D}\begin{bmatrix} a & b & b & \ldots & b \\ b & c & d & \ldots & d \\ b & d & c & \ldots & d \\ & & \vdots & & \\ b & d & d & \ldots & c \end{bmatrix}_{[k+1]\times[k+1]} \quad (3.21)$$

In which; a= $2(k+2)\lambda_4^2$, b= $-2\lambda_2\lambda_4$, d= λ_2^2-λ_4 and c = $(k+1)\lambda_4$-$(k-1)\lambda_2^2$, D= $2[(k+2)\lambda_4^2$ - $k\lambda_2^2\lambda_4]$ and k is the number of factors;

$$A_2^{-1} = \frac{1}{\lambda_2}\begin{bmatrix} 1 \; 0 \; 0 \ldots 0 \\ 0 \; 1 \; 0 \ldots 0 \\ 0 \; 0 \; 1 \ldots 0 \\ . \\ . \\ . \\ 0 \; 0 \; 0 \ldots 1 \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]} , \quad (3.22)$$

and

$$A_3^{-1} = \frac{1}{\lambda_4}\begin{bmatrix} 1 \; 0 \; 0 \ldots 0 \\ 0 \; 1 \; 0 \ldots 0 \\ 0 \; 0 \; 1 \ldots 0 \\ . \\ . \\ . \\ 0 \; 0 \; 0 \ldots 1 \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]} . \quad (3.23)$$

Now let;

$$M^* = \begin{bmatrix} A_1^* \; 0 \; 0 \\ 0 \; A_2^* \; 0 \\ 0 \; 0 \; A_3^* \end{bmatrix} \qquad (3.24)$$

Such that $A_1^* = A_i - \gamma I, i = 1,2,3$

Where;

$$A_1^* = \begin{bmatrix} (1-\gamma) & \lambda_2 & \lambda_2 & \lambda_2 \; \dots \; \lambda_2 \\ \lambda_2 & (3\lambda_4 - \gamma) & \lambda_4 & \lambda_4 \; \dots \; \lambda_4 \\ \lambda_2 & \lambda_4 & (3\lambda_4 - \gamma) & \dots \; \lambda_4 \\ & & . & \\ & . & & . \\ . & & & . \\ \lambda_2 \; \lambda_4 & \lambda_4 & \lambda_4 & \dots \; (3\lambda_4 - \gamma) \end{bmatrix}_{[k+1]\times[k+1]}, \qquad (3.25)$$

$$A_2^* = \begin{bmatrix} (\lambda_2 - \gamma) \; 0 \; 0 \dots 0 \\ 0 \; (\lambda_2 - \gamma) \; 0 \dots 0 \\ . \\ . \\ . \\ 0 \; 0 \; 0 \dots (\lambda_2 - \gamma) \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]}, \qquad (3.26)$$

and

$$A_3^* = \begin{bmatrix} (\lambda_4 - \gamma) \; 0 \; 0 \dots 0 \\ 0 \; (\lambda_4 - \gamma) \; 0 \dots 0 \\ . \\ . \\ . \\ 0 \; 0 \; 0 \dots (\lambda_4 - \gamma) \end{bmatrix}_{\left[\frac{k}{2}\right]\times\left[\frac{k}{2}\right]}. \qquad (3.27)$$

3.4.3 The Determinant Criterion, D-Criterion

The D-Criterion is defined as;

$$\emptyset_0(M) = (\det M)^{\frac{1}{s}}, \qquad (3.28)$$

Such that

$$s = \begin{bmatrix} k+2 \\ 2 \end{bmatrix} \text{ and M is the moment matrix.} \qquad (3.29)$$

Where s is the number of coefficients of the model and k is the number of factors. The D-Criterion maximizes the determinant of the moment matrix. The determinant of the moment matrix obtained from (3.16) is given by;

$$|M| = \{|A_i|; i = 1,2,3\}^{\frac{1}{s}} . \tag{3.30}$$

Where,

$$|A_1| = [3\lambda_4 - \lambda_2^2 + (\mathrm{k}-1)(\lambda_4 - \lambda_2^2)](2\lambda_4)^{(k-1)}$$

$$|A_2| = \lambda_2^{[k]} \quad \text{and} \quad |A_3| = \lambda_4^{[k]} \tag{3.31}$$

Therefore, substituting the variables in (3.31) to (3.30) gives D- Criterion as;

$$\emptyset_0(\mathrm{M}) = \{[3\lambda_4 - \lambda_2^2 + (\mathrm{k}-1)(\lambda_4 - \lambda_2^2)](2\lambda_4)^{(k-1)}[\lambda_2^{[k]}][\lambda_4^{[k]}]\}^{\frac{1}{s}}. \tag{3.32}$$

3.4.4 Average Variance Criterion, A-Criterion.

The average variance is defined as:

$$\emptyset_{-1}(M) = (\frac{1}{s}\ \mathrm{trace} M^{-1})^{-1} \tag{3.33}$$

A- Criterion minimizes the sum or average of the variances of the parameter estimates. From (3.20), the trace of the sub-matrices of M^{-1} was given by;

$$tr(M^{-1}) = tr(A_1^{-1}) + tr(A_2^{-1}) + tr(A_3^{-1}) \tag{3.34}$$

Where;

$$tr(A_1^{-1}) = \frac{1}{[2[(\mathrm{k}+2)\lambda_4^2 - k\lambda_2^2\lambda_4]}[2(\mathrm{k}+2)\lambda_4^2 + \mathrm{k}[(\mathrm{k}+1)\lambda_4 - (\mathrm{k}-1)\lambda_2^2]] , \tag{3.35}$$

$$tr(A_2^{-1}) = \frac{k}{\lambda_2} \quad \text{and} \quad tr(A_3^{-1}) = \frac{k}{\lambda_4}. \tag{3.36}$$

The equations ;(3.35) and (3.36) were used in (3.34) to obtain;

$$\emptyset_{-1}(M) = \left[\frac{1}{s}\{\frac{1}{2[(k+2)\lambda_4^2 - k\lambda_2^2\lambda_4]}[2(k+2)\lambda_4^2 + k[(k+1)\lambda_4 - (k-1)\lambda_2^2]] + \left[\frac{k}{\lambda_2}\right] + \left[\frac{k}{\lambda_4}\right]\}\right]^{-1}. \tag{3.37}$$

3.4.5 The Eigen value Criterion

E- Criterion refers to the Smallest Eigen value criterion and is defined by

$$\emptyset_{-\infty}(M) = \gamma_{min}[(M)]. \tag{3.38}$$

E- Criterion reduces the variance of each individual parameter estimate. From (3.24), E- Criterion is given by;

$|M^{*}| = |A_1^{*}||A_2^{*}||A_3^{*}|$. (3.39)

Where;

$|A_1^{*}| = (1-\gamma)^2[(k+2)\lambda_4 - k\lambda_2^2 - \gamma](2\lambda_4 - \gamma)^{k-1}$,

$|A_2^{*}| = (\lambda_2 - \gamma)^k$ and $|A_3^{*}| = (\lambda_4 - \gamma)^k$. (3.40)

Substituting (3.40 to (3.39) and equating to zero gave a characteristic polynomial given as;

$$\emptyset_{-\infty}(M) = (1-\gamma)^2[(k+2)\lambda_4 - k\lambda_2^2 - \gamma](2\lambda_4 - \gamma)^{k-1}(\lambda_2 - \gamma)^k\,(\lambda_4 - \gamma)^k = 0.$$

(3.41)

Solving the characteristic polynomial and taking the smallest value of γ gave the E-Criterion.

3.4.6 The Trace Criterion, T-Criterion

The trace criterion is also known as the T- Criterion defined as;

$\emptyset_1(M) = \frac{1}{s}$ trace M (3.42)

T- Criterion is used for model discrimination and was obtained by adding the traces of the sub-matrices of the moment matrix given in (3.16), therefore,

$tr(M) = \frac{1}{s}[1 + tr(A_1) + tr(A_2) + tr(A_3)]$ (3.43)

Where;

$tr(A_1) = k\,(3\lambda_4)$,

$tr(A_2) = (\lambda_2)\,k$, (3.44)

and

$tr(A_3) = (\lambda_3)\,k$.

Substituting (3.44) to (3.43) gave;

$$\emptyset_1(M) = \frac{1}{s}[1 + \mathrm{k}(3\lambda_4) + (\lambda_2)\mathrm{k} + (\lambda_4)\mathrm{k}]. \quad (3.45)$$

3.5 Application of the Constructed Reduced Designs Points with Hypothetical Example.

Suppose an experimenter considers utilizing the existing second order rotatable design points given in (3.5) with twenty points to investigate the effects of three fertilizer ingredients on a new yield of hybrid maize` under field conditions. The fertilizers doses and the actual amount applied independently were Nitrogen (N), ranging from of 4.25mg/hole to 28.33 mg/hole; Phosphorus (P)ranging from 2.66mg/hole to 13.26mg/hole and Potassium (K) ranging from 2.78mg/hole to 18.99mg/hole. The response of interest was the average yield in mg per hole. The levels of Nitrogen, Phosphorus and Potassium were coded and the coded variables were defined as follows;

$$x_1 = \frac{(N)-16.29}{7.16},\ x_2 = \frac{(\mathrm{P})-7.96}{3.15},\ x_3 = \frac{(\mathrm{K})-10.89}{4.82}. \quad (3.46)$$

The values of 16.29, 7.96 and 10.89 are the center values of Nitrogen, Phosphorus and Potassium respectively. Suppose five levels of each of the variables were used in the design experiment. The coded and measured levels for the variables are listed in table 3.2 below.

Table 3.2: Coded values and corresponding actual values of yield of Hybrid maize

Fertilizer	**Levels**				
Coded values	**-1.682**	**-1.000**	**0.000**	**+1.000**	**+1.682**
N	4.25	9.13	16.29	23.45	28.33
P	2.66	4.81	7.96	11.11	13.26
K	2.78	6.07	10.89	15.71	18.99

The concept of generating experimental run by Parsad and Batra (2000) was used to generate experimental runs utilized for the analysis of response surface designs using Mintab version 17 in this study.

CHAPTER FOUR
RESULTS AND DISCUSSIONS

4.1 Introduction

This chapter presents the results on construction of optimal modified second order rotatable designs with reduced number of points in three, four and five dimension from existing second order rotatable designs and evaluation of alphabetic optimality criteria for the constructed designs.

4.2. Construction of SORD with reduced number of points from existing SORD in k-dimensions

This section presents results for reduced designs in three four and five dimensions.

4.2.1 Construction of SORD with reduced number of designs points in three dimensions

Consider the existing designs in (3.5) reduced to give new designs given in table 4.1 below together with their respective excess functions.

Table 4.1 Summary of reduced generated points sets in three dimensions

Reference.	J_{31}	J_{32}	J_{33}
Set composition of class	$\frac{1}{2}s(a,a,a)$ + s(c_1, 0,0) + $s(c_2, 0,0)$	$\frac{1}{2}$ s(a_1, a_1, a_1) + $\frac{1}{2}$ s(a_2, a_2, a_2) +s(c, 0,0)	$\frac{1}{2}$s(f,f,0) + s(c_1, 0,0) + s (c_2, 0,0)
Number of points	16	14	18
$\sum_{u=1}^{N} x_{iu}^2$	$4a^2$+$2c_1^2$+$2c_2^2$	$4a_1^2$+$4a_2^2$+$2c_1^2$	$4f^2$+$2c_1^2$+$2c_2^2$
$Ex(J_{ij}) = \sum_{u=1}^{N} x_{iu}^4$- $3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2$	$2c_1^4$+$2c_2^4$- $8a^4$	$2c^4$-$8a_1^4$- $8a_2^4$	$2c_1^4$+$2c_2^4$-$2f^4$

4.2.1.1 Sixteen points reduced SORD from twenty points SORD

From table 4.1, the set of points denoted by J_{31} forms a second order rotatable arrangement in three dimensions if the moment conditions given in (3.2) hold as follows;

(i) $\sum_{i=1}^{16} x_{iu}^2 = 4a^2$+ $2c_1^2$ +$2c_2^2$ = $16\lambda_2$,

(ii) $\sum_{i=1}^{16} x_{iu}^4 = 4a^4 + 2c_1^4 + 2c_2^4 = 48\lambda_4$, (4.1)

(iii) $\sum_{u=1}^{16} x_{iu}^2 x_{ju}^2 = 4a^4 = 16\lambda_4$.

From (3.3) and table 4.1 the excess function for J_{31} denoted by *Ex* [J_{31}] was given by;

$Ex\left\{\frac{1}{2}s\ (a, a, a) + s\ (c_1, 0, 0) + s\ (c_2, 0, 0)\right\} = 0$

Therefore,

$\sum_{i=1}^{16} x_{iu}^4 - 3\sum_{u=1}^{16} x_{iu}^2 x_{ju}^2 = c_1^4 + c_2^4 - 4a^4 = 0$ (4.2)

Letting $c_1^2 = xa^2$ and $c_2^2 = ya^2$ in (4.2) yielded; (4.3)

$x^2 + y^2 - 4 = 0$

$\Rightarrow y = \sqrt{4 - x^2}$,

$-2 \leq x \leq 2$

Let $x = 1$ then $y = \sqrt{3}$

The values of x and y are chosen such that they are real, positive and exist within the design existence interval. Substituting the values of y and x to (4.3) gave;

$c_1 = a$ and $c_2 = 1.316074a$ if $a = 1$ (4.4)

Therefore,

$J_{31} = \frac{1}{2}s(1,1,1) + s(1,0,0) + s(1.316074,0,0)$.

The points set J_{31} forms a second order rotatable arrangement in three dimensions. Substituting the variables in (4.4) to the conditions for rotatability in (4.1) yielded;

$\lambda_2 = 0.5915063a^2$ and $\lambda_4 = 0.25a^4$. (4.5)

Substituting λ_2 and λ_4 in (4.5) to (3.4) gave as follows;

$\frac{\lambda_4}{\lambda_2^2} = 0.714531 > \frac{k}{k+2} = 0.6$. (4.6)

This satisfies the non-singularity condition and therefore J_{31} formed a second order rotatable design in three factors.

4.2.1.2 Fourteen points reduced SORD from twenty two points SORD

From table 4.1 the set of points J_{32} forms a second order rotatable arrangement in three dimensions if the moment conditions hold as given in (3.2) as follows;

(i) $\sum_{u=1}^{14} x_{iu}^2 = 4a_1^2 + 4a_2^2 + 2c^2 = 14\lambda_2$,

(ii) $\sum_{u=1}^{14} x_{iu}^4 = 4a_1^4 + 4a_2^4 + 2c^4 = 42\lambda_4$, (4.7)

(iii) $\sum_{u=1}^{14} x_{iu}^2\, x_{ju}^2 = 4a_1^4 + 4a_2^4 = 14\lambda_4$.

From table (4.1), the excess function for J_{32} is given by;

$$\sum_{i=1}^{14} x_{iu}^4 - 3\sum_{u=1}^{14} x_{iu}^2\, x_{ju}^2 = c^4 - 4a_1^4 - 4a_2^4 = 0 \quad (4.8)$$

Let $a_1^2 = xc^2$ and $a_2^2 = yc^2$ in (4.8) (4.9)

Substituting the variables in (4.9) to equation (4.8) gave;

$$1 - 4x^2 - 4y^2 = 0$$

$$\Rightarrow y = \sqrt{\frac{1}{4} - x^2},$$

Where $-0.5 \leq x \leq 0.5$

Le $x = 0.4$ then $y = 0.3$.

The value of x is chosen from the first quadrate of the ellipse such that it's real, positive and exist within the design existence interval. Substituting the values of $y = 0.4$ and $x = 0.3$ to (4.9) gave;

$$a_1 = 0.632456c \text{ and } a_2 = 0.547723c, \text{ where } c = 1 \quad (4.10)$$

The Points set J_{32} forms a second order rotatable arrangement for the values given in (4.10) in three dimensions.

The points set J_{32} forms a rotatable design if the conditions given in (4.7) are satisfied. Thus, substituting the variables in (4.10) to (4.7) gave;

$\lambda_2 = 0.342857c^2$ and $\lambda_4 = 0.071429c^4$. (4.11)

Therefore, from (3.4);

$\frac{\lambda_4}{\lambda_2^2} = 0.607643 > \frac{k}{k+2} = 0.6$ (4.12)

From (4.12), the non-singularity condition was satisfied hence J_{32} formed a second order rotatable design in three dimensions.

4.2.1.3 Eighteen points reduced SORD from twenty four points SORD

From table 4.1 above, the set of points J_{33} forms a second order rotatable arrangement in three dimensions if the moment conditions hold as given in (3.2) as follows;

(i) $\sum_{u1}^{18} x_{iu}^2 = 4f^2+2c_1^2+2c_2^2= 18\lambda_2,$

(ii) $\sum_{u=1}^{18} x_{iu}^2 = 4f^4+2c_1^4+2c_2^4=54\lambda_4,$ (4.13)

(iii) $\sum_{u=1}^{18} x_{iu}^2 x_{ju}^2 =2f^4= 18\lambda_4.$

From table 4.1 the excess function for J_{33} denoted by;

$Ex\left\{\frac{1}{2}\text{s}\,(f, f, 0) + \frac{1}{2}\text{s}\,(c_1, 0,0) + \text{s}\,(c_2, 0\,,0)\right\} = 0$

Such that;

$\sum_{i=1}^{18} x_{iu}^4 - 3\sum_{u=1}^{18} x_{iu}^2\, x_{ju}^2 = c_1^4+c_2^4-f^4= 0$ (4.14)

Let $c_1^2 = xf^2$ and $c_2^2 = yf^2$ in (4.14) so that to give; (4.15)

$x^2 + y^2 - 1 = 0$

$\Rightarrow$ y $=\sqrt{1 - x^2}$,

Where,

$-1 \leq x \leq 1$,

Let $x = 0.5$ then $y = 0.866025$.

The value of x is chosen such that it lies within the design existence interval.

Substituting the values of x and $y = 0.866025$ to (4.15) gave;

$$c_1 = 0.25f \text{ and } c_2 = 0.930604f \quad \text{where}, = 1 . \tag{4.16}$$

The Points set J_{33} forms a second order rotatable arrangement for the constant values given in (4.13) in three dimensions. The points set J_{33} forms a rotatable design if the conditions given in (4.13) are satisfied. Thus, substituting the variables in (4.16) to (4.13) gave;

$$\lambda_2 = 0.325392f^2 \text{ and } \lambda_4 = 0.111111c^4 \tag{4.17}$$

Substituting (4.17) in (3.4) gave;

$$\frac{\lambda_4}{\lambda_2^2} = 1 > \frac{k}{k+2} = 0.6. \tag{4.18}$$

From (4.18), the non-singularity condition was satisfied therefore J_{33} formed a second order rotatable design in three dimensions.

4.2.2 Construction SORD with reduced number of points in four dimensions

Suppose an experimenter wanted to perform an experiment by starting with three factors. If after performing the experiment in three factors he/she feels that a fourth factor was needed, then a design in four dimensions becomes of great interest. For example, suppose the experimenter wanted to estimate the maximum yield of a crop to various fertilizer doses of potassium, calcium and sodium. After soil investigation, the experimenter discovers deficiency of phosphorus mineral element in the soil. This therefore necessitates the experimenter to append a fourth factor to the soil which is phosphorus fertilizer.

Consider the existing design in (3.6). The generated points set for the reduced designs given in table 4.2 below together with their designs points.

Table 4.2 Summary of reduced generated points set in four dimensions

Reference.	J_{41}	J_{42}
Set composition of class	$\frac{1}{2}s(a,a,a,a)$ + $s(c_1,0,0,0)$ + $s(c_2,0,0,0)$	$\frac{1}{2}s(a_1,a_1,a_1,a_1)$ $+\frac{1}{2}s(a_2,a_2,a_2,a_2)$ $+s(c,0,0,0)$
Number of points	24	24
$\sum_{u=1}^{N} x_{iu}^2$	$8a^2$+$2c_1^2$+$2c_2^2$	$8a_1^2$+$8a_2^2$+$2c^2$
$Ex(J_{ij}) = \sum_{u=1}^{N} x_{iu}^4$- $3\sum_{u=1}^{N} x_{iu}^2\, x_{ju}^2$	$2c_1^4$+$2c_2^4$- $16a^4$	$2c^4$-$16a_1^4$- $16a_2^4$

4.2.2.1 Twenty four points reduced SORD from thirty two points SORD

The set of points J_{41} from table 4.2 form a second order rotatable arrangement in four dimensions if the moment conditions in (3.2) holds given as follows;

(i) $\sum_{u=1}^{24} x_{iu}^2$ =$8a^2$+ $2c_1^2$ +$2c_2^2$ = 24 λ_2,

(ii) $\sum_{u=1}^{24} x_{iu}^4$= $8a^4$+ $2c_1^4$ +$2c_2^4$ =72 λ_4, (4.19)

(iii) $\sum_{u=1}^{24} x_{iu}^2\, x_{ju}^2$= $8a^4$ = 24 λ_4.

From table 4.2 the excess for J_{41} is given by;

c_1^4 +c_2^4 – $8a^4$ = 0 (4.20)

Letting c_1^2 = xa^2 and c_2^2 = ya^2 in (4.20) yielded (4.21)

x^2 + y^2 - 8= 0

Therefore,

y =$\sqrt{8-x^2}$,

Where $-2.8 \leq x \leq 2.8$

The values of x and y are chosen such that they are real, positive and exist within the design existence interval.

Let $= 2$, $y = 2$.

Substituting the values of y and x to (4.21) gave;

$c_1 = 1.414213a$ and $c_2 = 1.414213a$,where a =1. (4.22)

Substituting (4.22) in (4.19), yielded;

λ_2= 0.666666a^2 and λ_4= 0.333333a^4 (4.23)

Substituting λ_i $(i = 2{,}4)$ in (4.23) to (3.4) gave;

$\frac{\lambda_4}{\lambda_2^2} = 0.75 > \frac{k}{k+2}$ =0.66666667. (4.24)

Thus, from (4.24) the non-singularity condition was satisfied therefore J_{41} formed a second order rotatable design in four dimensions.

4.2.2.2 Twenty four points reduced SORD from forty points SORD

The set of points J_{42} from table 4.2 above forms a second order rotatable arrangement in four dimensions if the moment conditions in (3.2) are satisfied given as follows;

(i) $\sum_{u=1}^{24} x_{iu}^2 = 8a_1^2 + 8a_2^2 + 2c^2 = 24\lambda_2$,

(ii) $\sum_{u=1}^{24} x_{iu}^4 = 8a_1^4 + 8a_2^4 + 2c^4 = 72\lambda_4$, (4.25)

(iii) $\sum_{u=1}^{24} x_{iu}^2\, x_{ju}^2 = 8a_1^4 + 8a_2^4 \quad = 24\lambda_4$.

From table 4.2, the excess function for J_{42} is given by;

$c^4 - 8a_1^4 - 8a_2^4 = 0$ (4.26)

Letting $a_1^2 = xc^2$ and $a_2^2 = yc^2$ in (4.26) yielded; (4.27)

$1 - 8x^2 - 8y^2 = 0$

Implying that;

$y = \sqrt{0.125 - x^2}$

Where, $-0.35355 < x < 0.35355$

Let $x = 0.3$ then $y = 0.187083$. (4.28)

Substituting (4.28) to (4.27) gave;

$a_1 = 0.547722c$ and $a_2 = 0.432509c$, where $c = 1$ (4.29)

The Points set J_{42} forms a second order rotatable arrangement for the constant values given in (4.29) in four dimensions.

Substituting (4.29) to (4.25) yielded;

$\lambda_2 = 0.245694c^2$ and $\lambda_4 = 0.041667c^4$ (4.30)

Substituting λ_2 and given λ_4 in (4.30) to (3.4) gave;

$\frac{\lambda_4}{\lambda_2^2} = 0.690245 > \frac{k}{k+2} = 0.666667.$ (4.31)

From (4.31), the non-singularity condition was satisfied thus J_{42} formed a second order rotatable design in four factors.

4.2.3 Construction of reduced SORD in five dimensions

Suppose again the experimenter realizes that after years of cultivation of the crop the soil has been ruined and a fifth mineral element is deficiency in the soil most probably Nitrogen. This necessitates the experimenter to append a fifth factor into the soil which compels an experimenter to consider a design in five dimensions.

Consider the existing design in (3.7) with generated points set for the reduced designs given in table 4.3 below together with their designs points.

Table 4.3 Summary of reduced generated set of points in five dimensions

Reference.	J_{51}	J_{52}
Set composition of class	$\frac{1}{2}s(a,a,a,a.a)$ + $s(c_1,0,0,0,0)$ + $s(c_2,0,0,0,0)$	$\frac{1}{2}s(a_1,a_1,a_1,a_1,a_1)$ + $\frac{1}{2}s(a_2,a_2,a_2,a_2,a_2)$ + $s(c_1,0,0,0,0)$
Number of points	36	42
$\sum_{u=1}^{N} x_{iu}^2$	$16a^2$+$2c_1^2$+$2c_2^2$	$16a_1^2$+$16a_2^2$+$2c_1^2$
$Ex(J_{ij}) = \sum_{u=1}^{N} x_{iu}^4$- $\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2$	$2c_1^4$+$2c_2^4$- $32a^4$	$2c^4$-$32a_1^4$- $32a_2^4$

4.2.3.1 Thirty six points reduced SORD from fifty two points SORD

The set of thirty six points denoted by J_{51} from table 4.3 forms a second order rotatable arrangement in five dimensions if the moment conditions given in (3.2) holds as follows;

(i) $16a^2$+ $2c_1^2$ +$2c_2^2$ = 36 λ_2,

(ii) $16a^4$+ $2c_1^4$ +$2c_2^4$ =108 λ_4, (4.32)

(iii) $16a^4$ = 36 λ_4.

For all i$\neq j = 1,2,3$ and all other sums and products and powers up to and including order four are zero.

From table 4.3, the excess of J_{51} is given by;

c_1^4+c_2^4–$16a^4$=0 (4.33)

Letting c_1^2= xa^2 and c_2^2 = ya^2 in (4.33) yielded; (4.34)

x^2 + y^2 - 16= 0

$\Rightarrow$ y =$\sqrt{16 - x^2}$

Where $-4 < x < 4$

The value of x is chosen such that it lies within the design existence interval.

Let $= 3, y = 2.64575$. (4.35)

Substituting the values of x and y to (4.34) gave;

$c_1 = 1.41421a$ and $c_2 = 1.62658a$, where $a = 1$ (4.36)

The Points set J_{51} forms a second order rotatable arrangement for the constant values given in (4.36) in five dimensions.

Substituting (4.33) in (4.34) gave;

λ_2=0.702541a^2 and λ_4= 0.444444a^4 (4.37)

Again, substituting $\lambda_2$and λ_4 given in (4.37) to (3.4) gave;

$\frac{\lambda_4}{\lambda_2^2}$ = 0.900479 > $\frac{k}{k+2}$ =0.714285. (4.38)

The non-singularity condition was satisfied therefore J_{51} formed a second order rotatable design in five dimensions.

4.2.3.2 Forty two points reduced SORD from seventy four points SORD

The moment conditions given in (3.2) were used to test rotatability for the design points J_{52} given in table 4.3 as follows;

(i) $16a_1^2$+ $16a_2^2$ +2c^4= $42\lambda_2$.

(ii) $16a_1^4$+ $16a_2^4$ +2c^4=$126\lambda_4$, (4.39)

(iii)16c^4+ $16a_2^4$ = $42\lambda_4$.

For all i≠ $j = 1$,2,3 and all other sums and products and powers up to and including order four are zero.

From table 4.3 the excess of J_{52} was given by;

c^4-16a_1^4-16a_2^4=0 (4.40)

Letting a_1^2=xc^2 and a_2^2=yc^2 in (4.40) gave; (4.41)

$1 - 16x^2 - 16y^2 = 0$

$\Rightarrow y = \sqrt{0.0625 - x^2}$

Where $-0.25 < x < 0.25$.

Let $x = 0.2$ then $y = 0.15$ (4.42)

The value of x was chosen such that it's real, positive and exist within the design existence interval. Substituting the values of x and y in (4.42) to (4.41) gave;

$a_1 = 0.447213c$ and $a_2 = 0.387298c$, where $c = 1$. (4.43)

The Points set J_{52} forms a second order rotatable arrangement for the constant values given in (4.43) in five dimensions.

Substituting (4.43) in (4.39) gave;

λ_2= 0.180952c^2 and λ_4= 0.023809c^4 (4.44)

Substituting $\lambda_2$and λ_4 given in (4.44) to (3.4) gave;

$\frac{\lambda_4}{\lambda_2^2}$ = 0.727134 > $\frac{k}{k+2}$ = 0.71428571 (4.45)

Thus, from (4.45) the non-singularity condition was satisfied therefore J_{52} formed a second order rotatable design in five dimensions.

4.3 Construction of modified second order slope rotatable central composite designs from existing modified slope central composite rotatable designs

Here results on construction of reduced modified SOSCCRD for 4 and 5 factors from existing modified SOSRD designs constructed by Vicorbabu (2005) were presented.

4.3.1 Construction of thirty two points second order SRCCD from sixty four points in four dimensions

Consider the existing design given in (3.8) (i).Using the rotatable conditions given by Victorbabu (2005) in (3.9) given as;

i. $\sum_{u=1}^{N} x_{iu}^2 = 8 + 2n_a a^2 = N\lambda_2$,

ii. $\sum_{u=1}^{N} x_{iu}^4 = 8 + 2n_a a^4 = cN\lambda_4$,where c and N are given in (3.9) ;

iii. $\sum_{u=1}^{N} x_{iu}^2 x_{iu}^2 = 8 = N\lambda_4$.

Applying the condition for rotatability to (4.46), i.e., $\sum_{u=1}^{N} x_{iu}^4 = 5\sum_{u=1}^{N} x_{iu}^2 x_{iu}^2$, resulted to;

$8 + 2n_a a^4 = 40$

$n_a a^4 = 16$, let $n_a = 1 \Rightarrow a^2 = 4$ (4.47)

The number of replications denoted by n_a was chosen such that 'a^2' is an integer.

Using the modified condition (3.11) and substituting the values of n_a and a^2 obtained in (4.47) to the relation in (i) and (iii) of (4.46) gave;

$\lambda_2^2 = \left(\frac{16}{N}\right)^2$ and $\lambda_4 = \frac{8}{N}$, (4.48)

Therefore, using (4.48) in (3.11) resulted to,

N = 32. (4.49)

Next, using (4.49) in (4.48), the following were obtained as;

$\lambda_2 = 0.5$ and $\lambda_4 = 0.25$ (4.50)

The values of v and c given in (3.8)i and (3.9)ii respectively together with the values in (4.50) satisfied (3.10). Hence, the 32 points design satisfied the rotatablity conditions.

Further, the variances and covariances of the parameter estimates were obtained by using (4.49) and (4.50) together with v= 4 and c=5 in(3.11) resulted in;

$V(\hat{b}_0)$ = 0.0625

$V(\hat{b}_i)$ = 0.0625

$(\hat{b}_{ij})$ = 0.125 (4.51)

$V(\hat{b}_{ii})$ = 0.03125

$Cov(\hat{b}_0, \hat{b}_{ii})$ = - 0.015625

$Cov(\hat{b}_{ii}, \hat{b}_{jj})$ = 0 and other covariances are zero.

Again using (4.49) and (4.50) in (3.13) gave the variance of the estimated response given as;

$V\left(\frac{\delta\hat{Y}}{\delta x_i}\right) = 0.0625 + 0.125d^2$,$(d^2 = \sum_{i=1}^{v} x_{iu}^2)$. (4.52)

Where d^2 is the function of the distance of points from the design center.

4.3.2 Construction of thirty two points reduced SRCCD from sixty four points in five dimensions

Consider the existing design given in (3.8) (ii).Using the rotatable conditions given by Victorbabu (2005) in (3.9) given as;

i. $\sum_{u=1}^{N} x_{iu}^2 = 8 + 2n_a a^2 = N\lambda_2$,

ii. $\sum_{u=1}^{N} x_{iu}^4 = 8 + 2n_a a^4 = cN\lambda_4$, (4.53)

iii. $\sum_{u=1}^{N} x_{iu}^2 x_{iu}^2 = 8 = N\lambda_4$.

Applying the condition for rotatability to (4.56) i.e. $\sum_{u=1}^{N} x_{iu}^4 = 5\sum_{u=1}^{N} x_{iu}^2 x_{iu}^2$, gave;

$n_a a^4 = 16$

Let $n_a = 1$, where $n_a$is chosen such that 'a^2' is an integer. (4.54)

$\Rightarrow a^2 = 4$

Using (3.11) and (4.54) in (4.53) part (i) and (iii) resulted to;

$\lambda_2^2 = \left(\frac{16}{N}\right)^2$ and $\lambda_4 = \frac{8}{N}$ (4.55)

Therefore; using (4.55) in (3.11) resulted in,

N = 32. (4.56)

Next, using (4.56) in (4.55), the following were obtained as;

λ_2 = 0.5 and λ_4 = 0.25 (4.57)

The values of v and c given in $(3.8)ii$ and $(3.9)ii$ respectively together with the values in (4.56) satisfied (3.10). Hence, the 32 points design satisfied the rotatablity conditions

Using (4.56) and (4.57) together with v=5 and c=5 in (3.12) resulted in;

$V(\hat{b}_0)$ = 0.0703

$V(\hat{b}_i)$ = 0.0625

$(\hat{b}_{ij})$ = 0.125

$V(\hat{b}_{ii})$ = 0.03125 (4.58)

$Cov(\hat{b}_0, \hat{b}_{ii})$ = = - 0.015625

$Cov(\hat{b}_{ii}, \hat{b}_{jj})$ = 0

Again Using (4.56) and (4.57) in (3.13) gave the estimated variance of the response as;

$V\left(\frac{\delta \hat{Y}}{\delta x_i}\right)$ = 0.0625+0.125d^2, $(d^2 = \sum_{i=1}^{v} x_{iu}^2)$ (4.59)

From the above two examples of construction of reduced modified second order slope rotatable designs using central composite designs, further construction of reduced modified slope rotatable designs from the existing designs presented in table 3.1 for 4≤v≤10 were done to give new reduced designs given in table 4.4 below.

Table 4.4 A list of reduced modified SOSRCCD for 4≤ v ≤10

No. of Factors (v)	t(v)-1	n_a'	$a^{2'}$	n_0'	N	N' (reduced)	$V\left(\frac{\delta\hat{Y}}{\delta x_i}\right)\delta^{-2'}$	% reduction
4	3	1	4	16	64	32	$0.0625+0.125d^2$	50
5	3	1	4	14	64	32	$0.0625+0.125d^2$	50
6	4	2	4	24	72	64	$0.0313+0.0625d^2$	11
7	5	4	4	40	144	128	$0.0156+0.0313d^2$	11
8	5	4	4	32	144	128	$0.0156+0.0313d^2$	11
9	6	2	8	44	200	144	$0.0139+0.0278d^2$	28
10	6	2	8	40	200	144	$0.0139+0.0278d^2$	28

From the existing designs of both second order rotatable designs (SORD) and modified second order rotatable designs (MSORD) utilized in this study, assuming the cost of experimentation at any point was constant, the percentage reduction of the existing design points considered ranged from 11% to 50% .This could potentially reduce the experimentation cost equivalent to the proportion of the reduced number of design points.

4.4 Evaluation of optimality criteria for the reduced designs

The results for optimality criteria for the constructed reduced designs in three, four and five dimensions are discussed.

4.4.1 Evaluation of optimality criteria for the reduced designs in three dimensions

Optimality criteria for reduced design denoted by J_{31}, J_{32} and J_{33} with 16, 14 and 18 points respectively were evaluated.

Consider the 16 points denoted by J_{31}. For $k = 3, s = 10$ and $\lambda_i (i = 2,4)$ obtained from (4.5), where k, s and λ_i represents the number of factors, number of the coefficients of the model and the moments respectively. The alphabetic optimality criteria for J_{31} are considered independently.

4.4.1.1 The Determinant Criterion

The D- Criterion is obtained by evaluating the s^{th} root of the determinant of the moment matrix. Substituting for k and λ_i to the formula given in (3.31) yielded the D-optimal value as;

$\emptyset_0(M)$= 0.417748.

The D- optimal obtained maximizes the determinant of the moment matrix.

4.4.1.2 The Average Variance Criterion

The A- Optimality was obtained by determining the trace of the inverse of the moment matrix. Substituting for k, s and λ_i to the Average Variance Criterion formula given in (3.37), yielded the A- optimal value as;

$\emptyset_{-1}(M)$= 0.194949.

The A- optimal obtained minimizes the sum or average of the variances of the parameter estimates.

4.4.1.3 The Eigen value Criterion

The Eigen value Optimality Criterion was determined by taking the smallest Eigen value of the moment matrix by substituting k and λ_i to the E- Criterion formula given in (3.41) and evaluating the characteristic polynomial yielded the smallest Eigenvalue as;

$\emptyset_{-\infty}(M)$ = 0.200159.

The E- optimal seeks to maximize the minimum Eigenvalue of the moment matrix.

4.4.1.4 The Trace Criterion

The trace of the moment matrix was obtained by substituting k, s and λ_i to the T-Criterion formula given in (3.45) to yield;

$\emptyset_1(M)$= 0.577469.

The T- Optimal obtained maximizes the trace of the moment matrix.

For the other remaining two design points in three dimensions (14 and 18) with their respective $\lambda_i (i = 2,4)$ obtained from (4.11) and (4.17), their optimality criteria were obtained using the same approach. Table 4.5 below presents summary of the evaluated optimality criteria for the three designs point in three dimensions.

Table 4.5 Summary of optimality criteria in three dimensions

Design	**Dimension**	**Points**	**D**	**A**	**E**	**T**
1	3	16	0.36367	0.194949	0.200159	0.577469
2	3	14	0.129695	0.005069	**0.004492**	0.288572
3	3	18	0.236837	0.023801	0.111111	0.330951

From table 4.5, taking the least value of the optimality criteria among the three constructed reduced designs considered gave an optimal design. Therefore, J_{32} with 14 points was E- optimal design with an optimal eigenvalue of 0.004492 when compared with other design points. The individual particular optimality criteria exhibit different properties. For example, in circumstances where the experimenter is only interested in a D- optimal design, then from table 4.5 among the three designs the design with 14 points with an optimal value of 0.129695 would be of great interest. Also with regard to A-, E- and T- optimality criteria, the design with 14 points with respective optimal values of 0.005069, 0.004492 and 0.28857 respectively was optimal when selected independently across the three designs considered.

Using the same approach used for evaluating optimality criteria in three dimension table 4.6 and 4.7 presents results for optimality criteria in four and five dimensions given below.

4.4.2 Evaluation of optimality criteria for the reduced designs in four dimensions

Results for optimality criteria for SORD denoted by J_{41} and J_{42} each with 24 points and a Modified SRCCD with 32 points respectively were presented and discussed in table 4.6 below.

Table 4.6 Summary of optimality criteria in four dimensions

Design	Dimension	Points	D	A	E	T
4	4	24	0.558527	0.537633	0.222223	0.599999
5	4	24	0.13051	**0.001437**	0.008539	0.176629
6	4	32	0.477422	0.280374	0.25000	0.466667

From table 4.6 above, the determination of an optimal design was done by taking a design with least optimality criterion among the three designs considered. Therefore, J_{42} with 24 points was A- optimal with a trace of 0.001437. In situations where the experimenter wishes to choose a design from each particular optimality criterion independently since each optimality criterion has a specific goal that it achieves then a design with 24 points with optimal values of 0.13051, 0.001437**,** 0.008539 and 0.176629 was D-, A-, E- and T- optimal design when selected independently.

4.4.3 Evaluation of optimality criteria for the reduced designs in five dimensions

Results on evaluation optimality criteria for reduced design denoted by J_{51} and J_{52} each with 36 and 42 and modified SRCCD with 32 points respectively were presented in table 4.7 below.

Table 4.7 Summary of optimality criteria in five dimensions

Design	Dimension	Points	D	A	E	T
7	5	36	0.725727	1.583654	0.444444	0.638171
8	5	42	0.115975	**0.000357**	0.002945	0.113378
9	5	32	0.516779	0.353933	0.25000	0.404762

From table 4.7 in five dimensions, J_{52} was A- optimal with 42 points having a least optimal value of 0.000357 when compared with other designs. The D, A, E and T with optimal values of 0.115975, 0.000357**,** 0.002945 and 0.113378 was optimal with 42 points when chosen independently.

From the above evaluated alphabetic optimality criteria, it was noted that each of the optimality criteria evaluated exhibits unique properties and has an experimental goal associated with it that achieves a specific property for the final fitted model. For instance according to Atkinson *et al.,* (2007), the D- Optimum designs minimize the content of the ellipsoidal confidence region for the parameters of the linear model. Eigen-values minimize the generalized variance of the parameter estimates. A-optimality minimizes the sum or average of the variance of parameter estimates.

4.5 Application of the Constructed Reduced Designs Points with Hypothetical Example.

From table 4.1, utilizing the reduced design points denoted by J_{31} and using (3.46) and table 3.2 gave 16 experimental runs. Table 4.8 below gives the design settings of x_1, x_2 and x_3 of the transformed coded variables and the natural variables or actual variables of N, P and K and the yield with 16 experimental runs.

Table 4.8: Hybrid Maize experimental Data

Run	Coded variables x_1	x_2	x_3	Natural variables N	P	K	Yield
1	1	1	1	23.45	11.11	15.71	49.63
2	-1	1	1	9.13	11.11	15.71	52.87
3	1	-1	-1	23.45	4.81	6.07	37.96
4	-1	-1	-1	9.13	4.81	6.07	50.76
5	1.682	0	0	28.33	7.96	10.89	35.41
6	-1.682	0	0	4.26	7.96	10.89	35.41
7	0	1.682	0	16.29	13.26	10.89	49.77
8	0	-1.682	0	16.29	2.66	10.89	54.36
9	0	0	1.682	16.29	7.96	18.99	46.93
10	0	0	-1.682	16.29	7.96	2.78	35.91
11	1.682	0	0	28.33	7.96	10.89	35.41
12	-1.682	0	0	4.26	7.96	10.89	35.41
13	0	1.682	0	16.29	13.26	10.89	49.77
14	0	-1.682	0	16.29	2.66	10.89	54.36
15	0	0	1.682	16.29	7.96	18.99	46.93
16	0	0	-1.682	16.29	7.96	2.78	35.91

Let the letters N, P and K represent the different types of fertilizers used in this experiment. The data in table 4.7 was used in the analysis of a response surface design using Minitab version 17 and the outputs of the results were presented in tables; 4.9, 4.10 and 4.11 below.

4.5.1 Model fit for Hybrid maize

Table 4.9 below gives the coefficients, standard errors, *t*-values and *p*-values of the Hybrid Maize model with natural values.

Table 4.9: Model for Hybrid Maize

Term	Coeff.	SE Coeff.	T-Value	P-Value
Constant	19560	4984	3.92	0.006
N	-963	247	-3.90	0.006
P	-2252	574	-3.93	0.006
K	-1285	328	-3.91	0.006
N^2	29.53	7.57	3.90	0.006
P^2	152.8	39.0	3.92	0.006
K^2	65.2	16.7	3.91	0.006

NP	0.1060	0.0342	3.10	0.017
PK	-16.85	4.39	-3.83	0.006

From table 4.9, at a level of 5% significance, the *p*-values reveals that the main effects (N, P and K), the pure quadratic (N^2, P^2 and K^2) and the interaction effects (NP and PK) were all significant. This implies that the independent variables contribute significantly to the yield of hybrid maize. The constant term coefficient reveals that even if all the independent variables were set to zero, the experimenter would still have a yield of 19560.The interaction effect of NK was not estimated because of lack of central points in the experimental runs. The fitted model therefore with significant factors is given as;

$$\hat{y} = 19560 - 963N - 2252P - 1285K + 29.53N^2 + 152.8P^2 + 65.2K^2 + 0.1060NP - 16.85PK$$

4.5.2 The Analysis of Variance for Hybrid Maize

Table 4.11 below gives the output of analysis of variance.

Table 4.10: Analysis of Variance for Hybrid Maize

Source	Df	Adj. SS	Adj. MSS	F Value	P Value
Model	8	899.832	112.479	47.16	0.000
Linear	3	663.861	221.287	92.79	0.000
Pure Quadratic	3	605.323	201.774	84.60	0.000
Two way interactions	2	57.911	28.956	12.14	0.005
Error	7	16.694	2.385		
Total	15	916.526			

From table 4.10, the F-statistic value was found to be 47.16 with (*p*=0.000). These small *p*-values indicate that the model was reliable and adequately represents the yield of Hybrid Maize. The small *p*-values for linear, pure quadratic and two way interactions also show that they have a significant effect on the response.

Table 4.12: Model summary for the yield of Hybrid Maize

Model Summary		
S	R.sq	R.sq (adj)
1.54432	98.18%	96.10%

From table 4.12, the adjusted R^2 indicate that 96.10% of the variation in the response was explained by the model. This shows that the second order model adequately represent the yield of hybrid maize with a reliability of 96.10% and would provide useful information about hybrid maize yield.

CHAPTER FIVE
CONCLUSIONS AND RECOMMENDATIONS

5.1 Introduction
This section gives conclusions and recommendations derived from the study findings.

5.2 Conclusions
The existing second order rotatable designs points considered in this study were all reducible and rotatable in three, four and five dimensions.

The constructions of modified slope rotatable designs with reduced design points from existing modified designs were also all rotatable in four and five dimensions.

The percentage reduction of these existing design points considered ranged from 11% to 50% .This potentially minimizes the experimentation cost equivalent to the proportion of the reduced number of design points.

From the evaluation of optimality criteria in three dimensions, J_{32} with fourteen points was E- optimal when compared to other design points. In four dimensions, J_{42} with twenty four points was A- and in five dimensions J_{52} was A- optimal with forty two points.

5.3 Recommendations

For cost effectiveness in designing of experiments for production processes in agricultural and industrial processes, the study recommends practical application of optimal modified second order rotatable designs constructed.

It would be also important if combination of the optimality criteria is done to obtain compound optimality criteria for optimal modified designs for researchers who are interested in more than one optimal measure in a design to give balance when any two or more alphabetic optimality criteria are combined.

The study recommends further construction of optimal modified rotatable designs in higher orders if the second order designs for some experiments are established to be inadequate.

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