Universal Optimality Designs

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Abstract

In the design of experiments for estimating statistical models, optimal designs allow parameters to be estimated without bias and with minimum variance. A non-optimal design on the other hand requires a greater number of experimental runs to estimate the parameters with the same precision as an optimal design. Thus in practical terms, optimal experiments can reduce the costs of experimentation. We construct an optimal design d^* among a family of designs, $d \in \Omega_{t,b,k}$, with b=5 blocks of size k=5 and with t=5 treatments. We demonstrate that such a design is optimal under all the optimality criteria considered by Kiefer (1975). It is thus universally optimal.

Introduction

Many of the current statistical approaches to designed experiments originate from the work of R. A. Fisher in the early part of the 20th century. Fisher demonstrated how taking the time to seriously consider the design and execution of an experiment before trying it helped avoid frequently encountered problems in analysis. Key concepts in creating a designed experiment include blocking, randomization and replication. In the design of experiments, optimal designs are a class of experimental designs that are optimal with respect to some statistical criterion. In the design of experiments for estimating statistical models, optimal designs allow parameters to be estimated without bias and with minimum variance. A non-optimal design on the other hand requires a greater number of experimental runs to estimate the parameters with the same precision as an optimal design. Thus in practical terms, optimal experiments can reduce the costs of experimentation.

Since the optimality criterion of most optimal designs is based on some function of the information matrix, the 'optimality' of a given design is modeldependent. While an optimal design is best for that model, its performance may deteriorate on other models. On other models, an optimal design can be either better or worse than a non-optimal design. Therefore, it is important to benchmark the performance of designs under alternative models.

The choice of an appropriate optimality criterion requires some thought, and it is useful to benchmark the performance of designs with respect to several optimality criteria. Cornell writes that since the [traditional optimality] criteria are variance-minimizing criteria, a design that is optimal for a given model using one of the criteria is usually near-optimal for the same model with respect to the other criteria.

Indeed, there are several classes of designs for which all the traditional optimality-criteria agree, according to the theory of "universal optimality" of kiefer (1975). The experience of practitioners like Cornell and the "universal optimality" theory of Kiefer suggest that robustness with respect to changes in the optimality-criterion is much greater than is robustness with respect to changes in the model.

The Model

The concept of experimental designs may be applied to field trials and of major concern are parameter estimates of yield response models. A commonly estimated additive model is one that assumes the yield is explained by treatment, block effects and some error term. Optimal designs to estimate treatment and block effects can then be developed. Other than these three factors, plot yields can also be affected by interference effects. Interference arise when response to treatments in plots are influenced by treatment applied in neighboring plots (Oliver and Kempton, 1996). We assume an experiment with b blocks of size k and t treatments. A design for such an experiment is a mapping $d: (1,2,\dots,b) \times (1,2,\dots,k) \rightarrow (t_1, t_2, \dots, t_t)$ that assigns treatment $d(i,j)$ to plot (i,j) of the field. The set of all possible designs d for such an experiment is denoted $\Omega_{t,b,k}$. We consider an interference model with different left and right neighbor effects, as in Kurnet and Martin (2000). We model the observed response y_{ij} at plot (i,j) as;

$$y_{ij} = \mu + \beta_i + \tau_{d(i,j)} + \lambda_{d(i,j-1)} + \rho_{d(i,j+1)} + \varrho_{ij}$$

Where μ denotes istheeffectgeneral β_i ofthei-thblock, mean, $\tau_{d(i,j)}$, the direct effect of the treatment applied to plot (i,j), $\lambda_{d(i,j-1)}$ and $\rho_{d(i,j+1)}$ are the left and right neighbor effects, respectively and ϱ_{ij} is the random error. We postulate that there are no guard plots so that $\lambda_{d(i,0)} = \rho_{d(i,k+1)} = 0$ for all $i =$ 1,2 , … b.

Classical optimality criteria

A-optimality

TheA-optimality (*"average" or trace*) is one criterion which seeks to minimize the trace of the inverse of the information matrix. This criterion results in minimizing the average variance of the estimates of the regression coefficients. If the coefficient matrix is partitioned into n columns, $L = (r_1, r_2, r_3, \dots, r_n)$, then the inverse $\frac{1}{\emptyset_{-1}}$ can

$$\frac{1}{\emptyset_{-1}(A_L(C))} = \frac{1}{n} tr A_L(C)^{-1} = \frac{1}{n} tr\, L' C^{-} L = \frac{1}{n} \sum_{j \le n} r_j' C^{-} r_j$$

This is the average of the standardized variances of the optimal estimators for the scalar parameter systems $r_1'\theta, r_2'\theta, \dots, r_n'\theta$ formed from the columns of L. We can pass back and forth between the information point of view and the dispersion point of view. Maximizing the average–variance criterion among information matrices is the same as minimizing the average of the variances given above, Pukelsheim, F. (2006).

C-optimality

The C-optimality is the other criterion which minimizes the variance of a best linear unbiased estimator of a predetermined linear combination of model parameters.

D-optimality

The determinant criterion $\emptyset_0(C)$ differs from the determinant (detC) by taking the s-th root whence both functions induce the same pre-ordering among information matrices. From a practical point of view one may therefore dispense with the s-th root and consider the determinant directly. However, the determinant is positively homogeneous of degree s, rather than 1. For comparing different criteria, and for applying the theory of information functions, the version $\emptyset_0(C) = det(C)^{1/s}$ is appropriate. Maximizing the determinants of information matrices is the same as minimizing the determinants of dispersion matrices, because of the formula $(\det C)^{-1} = det(C^{-1})$. Thus the D-optimality criterion seeks to minimize $|(X'X)^{-1}|$, or equivalently maximize the determinant of the information matrix X'X of the design. This criterion results in maximizing the differential shannon information content of the parameter estimates, Pukelsheim, F. (2006).

E-optimality

The E-optimality (*eigen-value*) is a criterion which maximizes the minimum eigenvalue of the information matrix. It is the same as minimizing the largest eigen-value of the dispersion matrix,

$$\frac{1}{\emptyset_{-\infty}(L_k(C))} = \lambda_{max}(L_k(C)^{-1}) = \max_{z \in \mathbb{R}^s : \|z\| = 1} z'K'A^{-}Kz$$

Minimizing this expression guards against the worst possible variance among all one-dimensional subsystem $z'K'\theta$, with a vector z of norm 1. In terms of variance, it is a minimax approach, in terms of information a maximin approach. The E-optimality criterion need not be differentiable at every point. Such E-optimal designs can be computed using methods of convex minimization that use sub-gradients rather than gradients at points of non-differentiability, Pukelsheim, F. (2006).

T-optimality

The other extreme member of the $\emptyset_p$ family is the trace criterion $\emptyset_1$. By itself the trace criterion is rather meaningless. A criterion ought to be concave so that information cannot be increased by interpolation. The trace criterion is linear, and this is so weak that interpolation becomes legitimate. Yet trace optimality has its place in the theory, mostly accompanied by further conditions that prevent it from going astray. An example is the Kiefer optimality of Balanced Incomplete Block Designs. In general the T-optimality criterion maximizes the trace of the information matrix.

Other optimality-criteria are concerned with the variance of predictions:

G-optimality

The G-optimality criterion seeks to minimize the maximum entry in the diagonal of the hat matrix $X(X'X)^{-1}X'$. This has the effect of minimizing the maximum variance of the predicted values.

I-optimality

The I-optimality (*integrated*) is a second criterion on prediction variance, which seeks to minimize the average prediction variance over the design space.

V-optimality

The V-optimality (*variance*) is a criterion, which seeks to minimize the average prediction variance over a set of m specific points.

Treatment sequences for a universally optimal design

Denote the set of all sequences of treatments applied to k plots in a block
by

$$\Psi(k,t) = \{s = (t_1, t_2, \ldots, t_k) : t_j \in [1,2,\ldots,t] \forall j = 1,2,\ldots,k\}$$

. We refer to two treatment sequences s and $\bar{s}$

as equivalent if one sequence can be transformed into the other by relabeling. Thus a set, $\Psi(k,t),$ of all treatment sequences, can be divided into Ϗclasses of equivalent sequences. The partial design matrices T_s, L_s and R_s of all sequences from one equivalence class are equal up to the permutations of the columns. For a sequence $\{s = (t_1, t_2, \ldots, t_k) \in \Psi(k,t)\}$ the symmetric compliment $s' = \{t'_1, t'_2, \ldots, t'_k \in \Psi(k,t)\}$ is defined by $t'_j = t_{k-j+1} \quad 1 \le j \le k$. The pair (s, s') is a symmetric pair. If $s = s'$, then s is symmetric. The symmetric complements of all sequences from one equivalence class U all lie in the same equivalence class U'. If U=U' then class U is symmetric.

In the case of our consideration that $t \ge k \ge 5$, the least number of possible treatment sequences per block is $t^k = 625,$ i.e when $t = k = 5.$ We consider the following sequence classes $U_1 = s_1 = (t_1, t_2, \ldots t_k)$, $U_2 = s_2 = (t_1, t_1, t_2, \ldots, t_{k-1})$, $U_3 = s_3 = (t_1, t_2, \ldots, t_{k-1}, t_{k-1})$ and $U_4 = s_4 = (t_1, t_1, t_2, \ldots, t_{k-3}, t_{k-2}, t_{k-2})$. We notice that U_3 is the symmetric complement of U_2. For the case that $t = k = 5$, the sequence classes of interest will be $U_1 = s_1 = (t_1, t_2, t_3, t_4, t_5)$, $U_2 = s_2 = (t_1, t_1, t_2, t_3, t_4)$, $U_3 = s_3 = (t_1, t_2, t_3, t_4, t_4)$ and $U_4 = s_4 = (t_1, t_1, t_2, t_3, t_3)$. The treatment effects, left effects and right effects incidence matrices for all designs falling in equivalence class U_1 are given as;

$$T_{s_1}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\qquad L_{s_1}=VT_{s_1}=\begin{bmatrix}0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$$

$$R_{s_1}=V'T_{s_1}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$$

where $V=\begin{bmatrix}0&0&0&0&0\\1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\end{bmatrix}$

For every class $U\in\Psi(k,t)$ the $c_{ij}(U)'s$ are given as follows;

$$c_{11}(U)=tr(T_s'B_kT_s) \qquad (4.1.1)$$

$$c_{12}(U)=tr(T_s'B_kL_s) \qquad (4.1.2)$$

$$c_{13}(U)=tr(T_s'B_kR_s) \qquad (4.1.3)$$

$$c_{22}(U)=tr(B_tL_s'B_kL_sB_t) \qquad (4.1.4)$$

$$c_{23}(U)=tr(B_tL_s'B_kR_sB_t) \qquad (4.1.5)$$

$$c_{33}(U) = tr(B_t R_s' B_k R_s B_t) \qquad (4.1.6)$$

where $B_k = I_k - \frac{1}{k} 1_k 1_k'$. The tables below shows the four equivalence classes of interest with $t = k = 5$, $t = k = 6$, and $t = k = 7$,with their corresponding $c_{ij}'s$.

Table 4.1.2: The classes U_i of sequences and corresponding $c_{ij}s$ for the case $t = k = 5.$

U	Equivalence Sequence	$c_{11}(U)$	$c_{12}(U)$	$c_{13}(U)$	$c_{22}(U)$	$c_{23}(U)$	$c_{33}(U)$
1	$(t_1, t_2, t_3, t_4, t_5)$	4	-0.8	-0.8	76	-14	76
2	$(t_1, t_1, t_2, t_3, t_4)$	0.6	-1.2	4	-11	9	0
3	$(t_1, t_2, t_3, t_4, t_4)$	3.6	-1	0.8	50	0	64
4	$(t_1, t_1, t_2, t_3, t_3)$	0.2	-1.4	3.6	12	3	-2

In the general case, corresponding with the four equivalence classes of interest, we obtain the treatment effects incidence matrices alongside the left and right effect matrices as follows;

Case I: A Sequence of treatments belonging to an equivalence class $U_1 = s_1 = (t_1, t_2, \dots t_k)$,

$$T_{s_1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad L_{s_1} = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$R_{s_1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

Case II: A sequence of treatments belonging to an $U_2 = s_2 = (t_1, t_1, t_2, \dots, t_{k-1})$

$$T_{s_2} = \begin{bmatrix} 1 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad L_{s_2} = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 0 & 0 \end{bmatrix},$$

$$R_{s_2} = \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 \\ & & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix}$$

Case III: A sequence of treatments belonging to an equivalence class $U_3 = s_3 = (t_1, t_2, \dots, t_{k-1}, t_{k-1})$

$$T_{s_3} = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \qquad L_{s_3} = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix},$$

$$R_{s_3} = \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \end{bmatrix}$$

Case IV: A sequence of treatments belonging to an equivalence class

$$U_4 = s_4 = (t_1, t_1, t_2, \dots, t_{k-3}, t_{k-2}, t_{k-2})$$

$$T_{\square_4} = \begin{bmatrix} 1 & 0 & 0 & & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & & 1 & 0 & 0 \end{bmatrix},$$

$$L_{s_4} = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & & 0 & 1 & 0 & 0 \end{bmatrix} \quad R_{s_4} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & & \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & 0 \end{bmatrix}$$

Universally Optimal Designs for models of blocks of size five with independently identically distributed error terms

From 3.8.2 we note that a design d is universally Optimal in approximate design theory provided that such a design satisfies the linear equations 3.8.2.1 – 3.8.2.4. For the case of $t \geq k \geq 5$ we obtain the following linear equations;

$$\sum_{s_i \in U} \pi_{s_i} \left[T'_{s_i} C^{\sim} T_{s_i} + x_1 T'_{s_i} C^{\sim} R_{s_i} + x_1 T'_{s_i} C^{\sim} L_{s_i} \right] = \frac{b}{t(t-1)} (tI_t - J_t)$$

$$\sum_{s_i \in U} \pi_{s_i} [R'_{s_i} C^{\sim} T_{s_i} + L'_{s_i} C^{\sim} T_{s_i} + x_1 R'_{s_i} C^{\sim} R_{s_i} + x_1 L'_{s_i} C^{\sim} R_{s_i} + x_1 R'_{s_i} C^{\sim} L_{s_i} + x_1 L'_{s_i} C^{\sim} L_{s_i}] = 0$$

$$C^{\sim} \left[\sum_{s_i \in U} \pi_{s_i} \left(T_{s_i} + x_1 R_{s_i} + x_1 L_{s_i} \right) \right] = 0$$

$$\pi_{s_i} = 0 \qquad\qquad if \quad s_i \notin U$$

Considering Models where error terms are identically and independently distributed within and among blocks then $C^{\sim} = I_{bk}$. The condition on the linear equations reduces to;

$$\sum_{s_i \in U} \pi_{s_i} \left[T'_{s_i} T_{\square_i} + x_1 T'_{s_i} R_{s_i} + x_1 T'_{s_i} L_{s_i} \right] = \frac{b}{t(t-1)} (tI_t - J_t)$$

$$\sum_{s_i \in U} \pi_{s_i} [R'_{s_i} T_{s_i} + L'_{s_i} T_{s_i} + x_1 R'_{s_i} R_{s_i} + x_1 L'_{s_i} R_{s_i} + x_1 R'_{s_i} L_{s_i} + x_1 L'_{s_i} L_{s_i}] = 0$$

$$\left[\sum_{s_i \in U} \pi_{s_i} \left(T_{s_i} + x_1 R_{s_i} + x_1 L_{s_i} \right) \right] = 0$$

$$\pi_{s_i} = 0 \qquad\qquad if \quad s_i \notin U$$

Considering treatment sequences $s'_i s$ which belong to universal classes $U_i, \ 1 \leq i \leq 4,$ then the equations above yield the following results;

$$\pi_1\begin{bmatrix}1 & 0 & 0 & & 0\\ 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & & 0\\ & \vdots & & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix} \qquad \pi_2\begin{bmatrix}1 & 0 & 0 & & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & & 0 & 0\\ & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 1\\ 0 & 0 & 0 & & 1 & 1\end{bmatrix}$$

$$\pi_3\begin{bmatrix}1 & 0 & 0 & & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & & 0 & 0\\ & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & & 1 & 0\end{bmatrix} + \pi_4\begin{bmatrix}2 & 0 & 0 & & 0 & 0\\ 0 & 2 & 0 & \cdots & 0 & 0\\ 0 & 0 & 2 & & 0 & 0\\ & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & & 0 & 0\end{bmatrix} +$$

$$\pi_1 x_1\begin{bmatrix}0 & 1 & 0 & 0 & & 0 & 0\\ 1 & 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & 1 & & 0 & 0\\ & \vdots & & & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & 0 & & 1 & 0\end{bmatrix} + \pi_2 x_1\begin{bmatrix}1 & 0 & 0 & 0 & & 0 & 0\\ 1 & 1 & 0 & 0 & \cdots & 0 & 0\\ 0 & 2 & 0 & 0 & & 0 & 0\\ & \vdots & & & \ddots & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 0 & 0 & & 0 & 0\end{bmatrix} +$$

$$\pi_3 x_1\begin{bmatrix}1 & 0 & 0 & & 0 & 0\\ 1 & 0 & 0 & \cdots & 1 & 0\\ 0 & 0 & 1 & & 1 & 0\\ & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 2 & 0\\ 0 & 0 & 0 & & 1 & 0\end{bmatrix} \qquad \pi_4 x_1\begin{bmatrix}2 & 1 & 0 & & 0 & 0\\ 1 & 1 & 1 & \cdots & 0 & 0\\ 0 & 1 & 0 & & 0 & 0\\ & \vdots & & \ddots & & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & & 1 & 0\end{bmatrix}$$

$$\frac{b}{t(t-1)}\begin{bmatrix}(t-1) & -1 & -1 & & -1 & -1\\ -1 & (t-1) & -1 & \cdots & -1 & -1\\ -1 & -1 & (t-1) & & -1 & -1\\ & \vdots & & \ddots & & \vdots\\ -1 & -1 & -1 & \cdots & (t-1) & -1\\ -1 & -1 & -1 & & -1 & (t-1)\end{bmatrix}$$

$$\pi_1\begin{bmatrix}0&1&0&&0&0\\1&0&1&\cdots&0&0\\0&1&0&&0&0\\&\vdots&&\ddots&\vdots&\\0&0&0&\cdots&0&1\\0&0&0&&0&0\end{bmatrix}\qquad \pi_2\begin{bmatrix}2&0&&0&0&0\\1&1&\cdots&0&0&0\\0&1&&0&0&0\\\vdots&&\ddots&&\vdots&\\0&0&&1&0&0\\0&0&\cdots&1&0&0\\0&0&&0&0&0\end{bmatrix}$$

$$\pi_3\begin{bmatrix}0&1&0&&0&0&0\\1&0&1&\cdots&0&0&0\\0&1&0&&0&0&0\\&\vdots&&\ddots&&\vdots&\\0&0&0&\cdots&1&2&0\\0&0&0&&0&0&0\end{bmatrix}+\pi_4\begin{bmatrix}2&1&0&&0&0&0\\1&1&0&\cdots&0&0&0\\0&1&1&&0&0&0\\&\vdots&&\ddots&&\vdots&\\0&0&0&\cdots&1&0&0\\0&0&0&&1&0&0\end{bmatrix}+$$

$$\pi_1x_1\begin{bmatrix}1&0&1&0&&0&1&0\\0&1&0&1&\cdots&0&0&0\\0&1&1&0&&0&0&0\\&\vdots&&&\ddots&&\vdots&\\0&0&0&0&\cdots&0&1&1\\0&0&0&0&&0&0&1\end{bmatrix}+\pi_2x_1\begin{bmatrix}2&0&1&0&&0&0&0\\1&2&0&1&\cdots&0&0&0\\0&1&2&0&&0&0&0\\&\vdots&&&\ddots&&\vdots&\\0&0&0&0&\cdots&2&0&1\\0&0&0&0&&0&1&2\end{bmatrix}+$$

$$\pi_3x_1\begin{bmatrix}1&0&1&0&&0&0&0\\0&2&1&0&\cdots&0&0&0\\0&0&2&1&&0&0&0\\&\vdots&&&\ddots&&\vdots&\\0&0&0&0&&2&1&0\\0&0&0&0&\cdots&0&1&0\\0&0&0&0&&0&0&0\end{bmatrix}+\pi_4x_1\begin{bmatrix}2&1&0&&0&0&0\\1&2&0&\cdots&0&0&0\\0&1&2&&0&0&0\\&\vdots&&\ddots&&\vdots&\\0&0&0&\cdots&2&1&0\\0&0&0&&0&0&0\end{bmatrix}=$$

$$\begin{bmatrix}0&0&\cdots&0&0\\0&0&&0&0\\\vdots&&\ddots&&\vdots\\0&0&\cdots&0&0\\0&0&&0&0\end{bmatrix}\qquad 4.3.10$$

$$\pi_1\begin{bmatrix}1&0&0&\cdots&0&0\\0&1&0&\cdots&0&0\\0&0&1&&0&0\\&\vdots&&\ddots&&\vdots\\0&0&0&\cdots&0&1\end{bmatrix}+\pi_2\begin{bmatrix}1&0&0&&0&0\\1&0&0&\cdots&0&0\\0&1&0&&0&0\\0&0&1&&0&0\\&\vdots&&\ddots&&\vdots\\0&0&0&\cdots&1&0\end{bmatrix}+\pi_3\begin{bmatrix}1&0&0&&0&0&0\\0&1&0&\cdots&0&0&0\\0&0&1&&0&0&0\\&\vdots&&\ddots&&\vdots&\\0&0&0&&1&0&0\\0&0&0&\cdots&0&1&0\\0&0&0&&0&0&0\end{bmatrix}+$$

$$\pi_4\begin{bmatrix}1&0&0&&0&0&0\\1&0&0&\cdots&0&0&0\\0&1&0&&0&0&0\\&\vdots&&\ddots&&\vdots&\\0&0&0&&1&0&0\\0&0&0&\cdots&1&0&0\\0&0&0&&0&0&0\end{bmatrix}\qquad \pi_1x_1\begin{bmatrix}0&1&0&0&&0&0\\1&0&1&0&\cdots&0&0\\0&1&0&1&&0&0\\&\vdots&&&\ddots&\vdots&\\0&0&0&0&\cdots&0&1\\0&0&0&0&&1&0\end{bmatrix}$$

$$\pi_2 x_1 \begin{bmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \qquad \pi_3 x_1 \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & & 1 & 0 \end{bmatrix}$$

$$\pi_4 x_1 \begin{bmatrix} 1 & 0 & 0 & 0 & & & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & & & 0 & 0 & 0 \\ & \vdots & & & \ddots & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & & 0 & 0 \end{bmatrix}$$

4.3.11

On evaluating these equations, we obtain some of the solutions for the proportion of treatment sequences as;

$$\pi_1 = \frac{10 - 6k + t - 6kt + k^2 t}{2kt - 4tx_1 - 3t}$$

$$\pi_2 = \frac{8kt - 4tx_1 - 4t + 6k - k^2 t - 10}{4kt - 8tx_1 - 6t}$$

$$\pi_3 = \frac{8kt - 4tx_1 - 4t + 6k - k^2 t - 10}{4kt - 8tx_1 - 6t}$$

$$\pi_4 = 0$$

Other solutions are;

$$\pi_1 = \frac{t(k-2) - (5 - 3k - t - 3kt + k^2 t)x_1}{2x_1(5 - 3k - t - 3kt + k^2 t) - t}$$

$$\pi_2 = 0$$

$\pi_3 = 0$

$$\pi_4 = \frac{(5 - 3k - t - 3kt + k^2t)x_1 + t - tk}{2x_1(5 - 3k - t - 3kt + k^2t)}$$

Thus a design composed of treatments that belong to the equivalence U_1, U_2, U_3 and U_4 in proportions of π_1, π_2, π_3 and π_4 respectively and such that $C_{d\mu v}, 1 \leq \mu, v \leq 2$ are reasonably symmetric, is universally optimal.

Construction of some optimal designs

From the general results obtained we construct some universally optimal designs for the case of t=5, k=5 and b=5. Using results in 4.3 and 4.2.16, we obtain the treatment proportions as follows;

$\pi_1 = 0.6, \quad \pi_2 = 0, \quad \pi_3 = 0, \quad \pi_4 = 0.4$. Thus we take any three treatment sequences belonging to equivalence class U_1 and two treatment sequences belonging to equivalence class U_4. The following are the possible competing designs;

$$d_1 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_1 & t_2 & t_3 & t_3 \end{bmatrix}, \quad d_2 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \end{bmatrix},$$

$$d_3 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \end{bmatrix}, \quad d_4 = \begin{bmatrix} t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \end{bmatrix},$$

$$d_5 = \begin{bmatrix} t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & \square_3 & t_4 & t_5 \end{bmatrix}, \quad d_6 = \begin{bmatrix} t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{bmatrix},$$

$$d_7 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{bmatrix}, \quad d_8 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & \square_5 \end{bmatrix},$$

$$d_9 = \begin{bmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{bmatrix}, \quad d_{10} = \begin{bmatrix} t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_1 & t_2 & t_3 & t_3 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 \end{bmatrix}$$

Results

The construction of universally optimal designs in this paper involves the use of different treatment sequences that must necessarily belong to equivalence classes (S_1, S_2, S_3, S_4) assigned to the b blocks of the experimental layout. The allocation of these treatments according to proportions $\pi_i, \; i = 1,2,3,4.$ In the case that t=k=b=5, three treatments sequences belonging to equivalence class S_1, and two belonging to equivalence class S_4 were used resulting in ten competing designs. The information matrices of all the competing designs are symmetric. Designs d_1, d_2, d_3, d_4 and d_9 have unique information matrices whereas d_5 and d_6 have similar information matrices. Similarly d_7, d_8 and d_{10} have same information matrix. In terms of $\phi -$optimality, a universally optimal design will attain a trace bound of 18.1. Both designs d_1, d_2, d_3, d_4 have a trace of 16.96, with an efficiency of 0.937. Designs d_5 and d_6 have trace of 17.28 with an efficiency of 0.955 while designs d_7, d_8 and d_{10} have a trace of 17.8 with an efficiency of 0.955 as well. Though the off diagonal elements of design d_9 are different from those of d_7, d_8, d_{10}, presenting the information matrix of d_9 to be different, the trace in still 17.28 with an efficiency of 0.955. In the summative, designs $d_5, d_6, d_7, d_8, d_9, d_{10}$ are universally optimal though they do not attain the trace bound$a^*_{5,5,5,} = 18.1$. They are superior to designsd_1, d_2, d_3, d_4 .

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