NUMERICAL RANGES AND SPETRA OF INNER PRODUCT TYPE INTEGRAL TRANSFOMERS ON HILBERT SPACES

BY

PRISCAH MORAA OHURU

A Thesis Submitted in Fulfilment of the Requirement for the Award of the Degree of Doctor of Philosophy in Pure Mathematics, School of Science and Aerospace Studies, Department of Mathematics, Physics and Computing

MOI UNIVERSITY

Declaration

This thesis is my own work and has not been presented for a degree award in any other institution.

PRISCAH MORAA OHURU

PHD/PM/5547/21

Signature .... Date 27/11/2025.................................

This thesis has been submitted for examination with our approval as the university supervisors.

1. Dr. David Ambogo

Department of Pure and Applied Mathematics

Maseno University, Ke

Signature .... Date 26/11/2025...................................

2. Prof. Dr. Fredrick Nyamwala

Department of Mathematics, Physics and Computing

Moi University, Eldoret, Kenya

Signature .... Date 26/11/2025...................................

Acknowledgments

First, I thank God for His marvelous grace that has brought me this far. I am grateful for the good health, strength, and sound mind that have enabled me to complete this study successfully. I extend my heartfelt appreciation to my supervisor, Dr. David Ambogo, due to his dedication to this work, his availability for consultations, his motivation, and his timely guidance, all of which played a crucial role in developing this thesis. I also appreciate Prof. Dr. Nyamwala for his understanding, encouragement, insightful comments, and unwavering support, which enabled me to complete my study. To my family, especially my husband, Obed, thank you for always being there for me, encouraging me to do my best, and uplifting me whenever I felt like giving up. I am also deeply grateful to my friends and colleagues, especially my Dean, Dr. Obogi, for the day-to-day reminders that kept me focused and determined to complete this work. Finally, I extend my sincere appreciation to DAAD for awarding me a scholarship, which provided me with the opportunity to complete my PhD without financial constraints. May God bless you.

Dedication

To my husband Obed Monari and my children Kenan and Yuviah.

Abstract

The theory of linear operators in Hilbert spaces has seen substantial progress, particularly for integral operators of inner product-type, which are known to satisfy fundamental inequalities like Cauchy-Schwarz and Landau within unitarily invariant norm ideals. While these inequalities provide explicit formulas for operator norms, they do not inherently characterize or exclude the study of numerical range and spectrum - rather, they offer complementary information that bounds these spectral properties without fully describing them. This reveals a significant gap: explicit norm formulas exist but leave the numerical range, spectrum, and their relationship uncharacterized. This research addresses these gaps by systematically investigating the numerical range and spectrum of inner product-type integral operators. Using operator-theoretic techniques including rank-one operators and tensor products, we construct the algebraic numerical range. Through spectral analysis employing abelian properties, commutators, and von Neumann algebra methods, we classify the spectrum into point, continuous, and residual components. Convexity arguments via the Toeplitz-Hausdorff theorem and norm inequalities establish bounds for the numerical range and ensure spectral inclusion. Our results demonstrate that the numerical range maintains essential classical properties, with the spectrum contained within its closure. The applications are substantial in quantum mechanics, providing a rigorous framework for characterizing quantum states and observables, with direct implications for evaluating system stability and dynamical behavior. We recommend future work develop more refined techniques - specifically, sharper spectral decomposition and numerical range characterization - to enable precise quantum dynamical predictions and extend these structures to open quantum systems.

Contents

Index of Notations

Chapter 1

INTRODUCTION

The analysis of integral transformers holds a significant importance across multiple areas of Mathematics, especially in fields such as functional analysis, operator theory, and Mathematical Physics. Among these, the inner product type integral transformer is a significant tool, providing valuable insights into Hilbert and other Banach spaces. This chapter lays the groundwork for understanding the construction and applications of such transforms, offering essential background information to support the objectives of this study.

1.1 Mathematical Background

The significance of Hilbert spaces is deeply rooted in the field of operator theory. The core aim of this field is to categorize and analyze all linear operators within Hilbert and other Banach spaces. In Algebra, the classification dilemma is effectively tackled using general situations and issues related to a bounded linear operator's minimal and characteristic polynomials, eigenvalues and eigenspaces, and culminating in a normal canonical form of any linear operator (Kowalski, 2009). Significant progress has been made in the fields of linear operator theory within Hilbert spaces and the study of Mathematical inequalities, more generally, issues related to an operator that is bounded and linear can be resolved through the reduction method, utilizing the polar decomposition theorem for operators that are positive on Hilbert spaces and the theory of the spectrum (Sunder, 2000).
The development of functional calculus for normal operators has led to significant progress

in applications across quantum mechanics, signal processing, and numerical analysis (Colbrook, 2019). The spectral theorem, which characterizes normal operators via a spectral measure, remains a fundamental result for understanding Hilbert spaces. Advancements in the study of compact operators, which play a crucial role in functional analysis, have yielded new insights into singular value decomposition and its applications in data science and machine learning. These developments have been instrumental in addressing ill-posed problems in inverse theory, particularly in medical imaging and geophysics (Pedersen, 2020).

Another key area of development in Hilbert space theory is the significance of unbounded operators, particularly in quantum mechanics and differential operators. The theory of self-adjoint extensions of symmetric operators has seen renewed interest, with applications in quantum field theory and partial differential equations (Reed & Simon, 2021). Moreover, studies have explored the interplay between Hilbert space operators and optimization problems, leading to new techniques in convex analysis and optimization theory. Additionally, there has been growing research on the interplay between operator theory and machine learning, particularly in the study of kernel methods and reproducing kernel Hilbert spaces (RKHS). These spaces provide a powerful framework for analyzing high-dimensional data and designing efficient learning algorithms (Scholkopf & Smola, 2023). The application of RKHS in signal processing and control theory has also expanded, offering new tools for stability analysis and system identification. Furthermore, in Banach spaces, advancements in studying non-commutative operator theory have contributed to the generalization of classical results from Hilbert spaces. The study of operator algebras, particularly C*-algebras and Von Neumann algebras, continues to provide deep insights into both pure and applied Mathematics (Takesaki, 2019).

The spectral theorem, which establishes the circumstances under which an operator can be broken down into more basic components using its spectrum, is one of the foundational elements of operator theory. According to this theorem, compact self-adjoint operators can be represented in terms of their eigenvalues and matching eigenvectors, creating an orthonormal basis for the Hilbert space. This result is crucial in quantum mechanics, functional analysis, and Mathematical Physics. Additionally, the study of unbounded operators, which frequently arise in differential equations and quantum mechanics, has led to significant developments in understanding self-adjoint, normal, and unitary operators. The concepts of essential spectrum and continuous spectrum have further enhanced the comprehension of how these operators behave under perturbations. In recent advancements, the interplay between Hilbert space operators and Mathematical inequalities has become a focal point of research. Key results in majorization theory, trace inequalities, and operator convexity have provided deeper insights into norm inequalities and their applications in quantum information theory and probability. Furthermore, the study of compact operators in Hilbert spaces, particularly those of finite rank, has facilitated progress in Fredholm theory. The notion of index theory, introduced by Fredholm as illustraed in (Atiyah & Singer, 1969), has been extended to address more complex scenarios, including applications in partial differential equations and K-theory.

The foundation of spectral theory of linear operators comes from the theory of matrices and equations of the integral type. Spectral theory of compact operators in Hilbert spaces has been described as richer and closely related to integration theory and measure theory. In Hilbert spaces, significant work has been done concerning the spectral theory of operators. Toeplitz (Toeplitz, 1918) introduced spectral theory for self-adjoint bounded operators, and the theory has been extended to self-adjoint unbounded operators where the unbounded self-adjoint operator is rewritten in terms of bounded unitary operators using the Cayley transform (Weckman, 2020). The study of spectral theory has laid a foundation for operator analysis and exploring the geometric properties of operators. In their book, (Vladimir & Birkhauser, 2003) attempted to organize the recent progress in spectral theory, particularly the results concerning various types of spectra in a unified, axiomatic manner within a Banach algebra setting.

Spectral theory plays a fundamental role in functional analysis and mathematical physics, especially in quantum mechanics, where operators represent physical observables. The spectrum of an operator generalizes the concept of eigenvalues of a matrix, encompassing not only discrete spectra but also continuous and residual spectra. Further advancements in spectral theory include Gelfand's work on spectral decomposition in Banach algebras and the spectral theorem for normal operators in Hilbert spaces (Gelfand & Neumark, 1943), which provides a representation of such operators in terms of projection-valued measures. This result is essential in quantum mechanics, where self-adjoint operators correspond to measurable quantities.

Another significant development is the study of compact operators, which resemble finite-dimensional operators in many respects. The spectral properties of compact operators in Hilbert spaces lead to results such as the Hilbert-Schmidt theorem and the Fredholm alternative (Reed & Simon, 1980), which provide a classification of spectra in terms of eigenvalues with finite multiplicities. The development of spectral theory has also influenced the study of differential operators and integral equations, where spectral methods provide tools for solving boundary value problems. This connection has applications in Mathematical Physics, signal processing, and numerical analysis. Recent research has focused on spectral properties of non-self-adjoint operators, perturbation theory, and spectral asymptotics (see Locker (2000), Makin (2024) and references therein), which have implications in stability analysis and dynamical systems. The interplay between spectral theory and other Mathematical disciplines continues to be an active area of study, leading to deeper insights into the structure of operators in various functional spaces.

Linear operator theory in Hilbert spaces has grown exponentially in the last decade. Operators are classified into differential operators and integral type operators on functional spaces. In (Kreyszig , 1978), an integral operator $\pi : \mathfrak{B}[0,1] \to \mathfrak{B}[0,1]$ is defined as $\zeta = \pi\eta$, where $\zeta(s) = \int_0^1 \mu(s,\tau)\eta(\tau)d\tau$, where μ is the *kernel* of L and on the closed square $\mathcal{S} = \mathcal{Y} \times \mathcal{Y}$, it is assumed to be continuous in the $s\tau$-plane, and $\mathcal{Y} = [0,1]$. This operator is linear and bounded. Furthermore, the spectral properties of such operators are of great interest. If π is a compact integral operator, then its spectrum consists of a sequence of eigenvalues that accumulate only at zero. This follows from the compactness theorem, which states that compact operators on infinite-dimensional Banach spaces have at most countably many nonzero eigenvalues (Amonyela, Nyongesa , & Wanambisi, 2022).

An important result in the spectral theory of compact integral operators is the Hilbert-Schmidt theory. If μ satisfies the Hilbert-Schmidt condition: $\int_0^1 \int_0^1 |\mu(s,\tau)|^2 ds d\tau < \infty$, then π is a Hilbert-Schmidt operator, which ensures that its singular values (the square roots of the eigenvalues of $\pi^*\pi$) decay to zero (Balazs, 2008).

Integration is a fundamental tool in Mathematical analysis and plays a crucial role in defining the inner product of continuous functions. According to Satya (2017), let

$\mathcal{F} = U[x, y]$ be the vector space of all continuous functions $\mathfrak{s} : [\zeta, \eta] \longrightarrow \mathbb{R}$. For $y, z, C[\zeta, \eta]$, the inner product is given as: $\langle y, z \rangle = \int_\zeta^\eta y(x)z(x)dx$.
This definition satisfies the properties of an inner product space. Similarly, double integration of continuous functions can also be defined, and the inner product can be written in terms of the norm. For this study, we were interested in integral-type operators, particularly inner product-type integral transformers. Integral transforms are mathematical operators that map a function into another function, typically involving an inner product integral to simplify complex problems or offering insights into underlying structures of operators. One type of integral transform, the inner product-type integral transform, has gained attention due to its potential applications in various fields, including physics, engineering, and signal processing, making it a useful tool in functional analysis, differential equations, and applied mathematics. The following are examples of the integral transforms.

1.1.1 Examples of Integral Transforms

Example 1.1 (The Fourier Transform (Kellaway, 1969))**.** The Fourier transform is among the most commonly used integral transforms in mathematics and engineering. It provides a powerful method for analyzing functions by transforming them from the time domain into the frequency domain. Consider a periodic function $g(t)$ with period 2π. Its Fourier series representation can be expressed as:

$$y_0 + \sum_{\zeta=1}^{\infty} \left[s_\zeta \cos(\zeta t) + r_\zeta \sin(\zeta t) \right],$$

where the coefficients are determined by the following integrals:

$$\begin{aligned} y_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t)\, dt, \\ s_\zeta &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos(\zeta t)\, dt, \\ r_\zeta &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin(\zeta t)\, dt. \end{aligned}$$

The Fourier transform has applications in physics, image processing, and signal processing. Notable properties include linearity, homogeneity, additivity, and invertibility.

Example 1.2 (The Laplace Transform (Kellaway, 1969))**.** The Laplace transform, introduced by Pierre-Simon Laplace in the 18th century, is a powerful mathematical tool extensively applied in engineering and systems analysis for solving differential equations. The transform is defined as:

$$L(k) = A\{b(s)\} = \int_0^\infty b(s)e^{-ks}ds,$$

where $k = \rho + i\omega$ is complex, with ρ and ω being real numbers. The Laplace transform simplifies operations like differentiation and integration. Its properties include linearity, time shifting, and convolution.

Example 1.3 (The Mellin Transform (Kellaway, 1969))**.** The Mellin transform generalizes the Fourier and the Laplace transforms and is primarily used in asymptotic analysis and scaling problems. It is given by:

$$M\{b(s)\}(a) = \int_0^\infty b(s)s^{a-1}ds,$$

where s is a real and positive variable, and a is complex. The Mellin transform is linear, satisfies the convolution theorem, and is invertible. It has applications in the study of the zeta function, number theory, aerodynamics, and fluid mechanics.

Example 1.4 (The Fredholm Integral Operators (Kress, 2014))**.** Fredholm integral operators involve integral equations with fixed limits of integration. They are categorized into:

1. **The Fredholm Integral Equation of the First Kind:**

$$g(k) = \int_m^n K(k,l)h(l)dl,$$

where $K(k,l)$ is the kernel function.

2. **Fredholm Integral Equation of the Second Kind:**

$$g(k) = s(k) + \zeta \int_m^n K(k,l)h(l)dl.$$

Fredholm integral equations are fundamental in functional analysis, leading to results such as the Fredholm alternative, which provides conditions for the existence of solutions.

Example 1.5 (Inner Product Type Integral Transform)**.** Integral transforms of the inner product type are closely related to inner product spaces. In an inner product space, the inner product is a fundamental operation that combines two vectors or functions to yield a scalar. For two given functions $m(x)$ and $n(x)$, an integral transform representing an inner product can be expressed as:

$$\mathcal{T}\{m,n\} = \int_a^b \int_a^b Y(x,y)\, m(x)\, n(y)\, dx\, dy,$$

where $Y(x,y)$ is a suitable kernel function defining the transform. These transforms have applications in quantum mechanics, signal processing, and operator theory.

More details on the inner product type integral transformers are discussed in the subsequent setions.

Recent advancements in integral transforms have significantly expanded their theoretical foundations and practical applications across various scientific and engineering disciplines. These developments include the introduction of new transforms, deeper analysis of existing ones, and innovative applications in solving complex problems. Abdallah et al. (2024) introduced the Abdallah Group Transform, a novel integral transform designed to address differential, partial, and integral equations. This transform simplifies complex mathematical models and has been recognized as a valuable tool in applied mathematics and engineering. Similarly, Zhao & Lee (2024) proposed a New Integral Transform that has proven effective in solving initial value problems, particularly ordinary differential equations, offering alternative methods for tackling engineering and physics challenges.

Further advancements include the introduction of generalized integral transforms, which broaden the applicability of existing methods. Ahmed & Patel (2022) formulated a new generalized integral transform that encompasses many known transforms as special cases. Their study established fundamental properties such as existence theorems, scaling properties, and inversion formulas, demonstrating the transform's effectiveness in solving initial boundary value problems. Additionally, Jackson & Wang (2024) presented new identities for classical transforms, including the Laplace, Glasser, and Widder potential transforms. Their research has facilitated complex integral evaluations and provided novel representations for special functions like the error function and the Riemann zeta function.

The emerging applications of integral transforms continue to grow, particularly in fractional calculus and operational calculus, where they play a crucial role in handling irregular operations and extending distribution theory. These contributions are essential for modeling phenomena with memory effects or hereditary properties, as highlighted by Martinez et al. (2023). Furthermore, integral transforms have found increasing use in signal processing and quantum mechanics, where they assist in signal analysis and reconstruction. In quantum mechanics, their role has been explored in operator theory and spectral analysis, providing valuable tools for understanding the behavior of quantum systems (Nelson & Roberts, 2024). These advancements underscore the dynamic and evolving nature of integral transform theory, highlighting its expanding influence and applicability in addressing contemporary challenges across various scientific and engineering fields.

Another significant transformation that has been studied is the Aluthge transform, initially introduced and analyzed by Aluthge (1990). Since then, several researchers have explored various properties of this transformation, making it an essential tool in operator theory. Jung, Ko & Pearcy (2005) examined the convergence properties of the Aluthge iteration and its implications for normal operators, further refining the understanding of this transformation. Later, Okubo & Ando (2015) explored applications of the Aluthge transform in perturbation theory, particularly in the spectral analysis of bounded linear operators. The Banach algebra setting of these transformations has been extensively analyzed in works such as Vahid (2012), where the Gelfand transform was shown to be a key tool in spectral theory. In particular, Rickart (1960) studied the Gelfand representation in the study of commutative Banach algebras, while a research by Arveson (1998) linked the Gelfand transformation to noncommutative harmonic analysis and C*-algebra representations.

The concept of the inner product-type integral transformation was initially explored by Jocic (Jocic, 2005). Its formulation is derived from the Gel'fand integral, also known as the weak*-integral, which applies to operator-valued functions.

To start with, consider a separable complex Hilbert space K and let $\mathcal{L}(K)$ denote the space of all bounded linear operators on K. Suppose (Ψ, Σ, γ) is a measure space, where Ψ is a compact set, Σ is the σ- algebra, γ is a measure and $P : \Psi \to \mathcal{L}(K)$ is an operator-valued function defined by $s \mapsto P_s = P(s)$, where $s \in \Psi$. This function is said to be *weakly*-measurable* (or *weakly*-integrable*) if, for every pair of vectors $\mathfrak{q}, \mathfrak{r} \in K$, the scalar function $s \mapsto \langle P_s \mathfrak{q}, \mathfrak{r} \rangle$ is measurable (or integrable, respectively). A key step in proving the non-trivial aspect of this result involves the closed graph theorem, which guarantees that the operator-valued integral linked to a sesquilinear form remains bounded. If the function $s \mapsto \langle P_s \mathfrak{q}, \mathfrak{q} \rangle$ is integrable for all $\mathfrak{q} \in K$, then the mapping $\mathfrak{q} \mapsto \int_\Psi \langle P_s \mathfrak{q}, \mathfrak{q} \rangle d\gamma(s)$ defines a quadratic form of a unique bounded operator

$$\int_\Psi P_s d\gamma(s) \tag{1.1.1}$$

satisfying

$$\left\langle \int_\Psi P_s d\gamma(s) \mathfrak{q}, \mathfrak{r} \right\rangle = \int_\Psi \langle P_s \mathfrak{q}, \mathfrak{r} \rangle \, d\gamma(s), \quad \forall \mathfrak{q}, \mathfrak{r} \in K.$$

Furthermore, the trace property holds:

$$\operatorname{tr}\left\{ \int_\Psi P_s d\gamma(s) G \right\} = \int_\Psi \operatorname{tr}\{P_s G\} d\gamma(s), \quad \forall G \in L(K),$$

where G is a rank-one operator of the form $G = \eta^* \otimes \eta$ with $\eta \in \mathcal{D}_P$, and $\mathcal{D}_P$ is the underlying domain given by

$$\mathcal{D}_P = \left\{ \mathfrak{q} \in K : \int_\Psi \|P_s \mathfrak{q}\|^2 d\gamma(s) < \infty \text{ and } (P\mathfrak{q})(s) = P_s \mathfrak{q} \right\}.$$

This formulation provides a rigorous framework for studying operator-valued integral transformations and their applications in functional analysis and operator theory.

The bounded operator described by Equation 1.1.1, is known as the *Gelfand integral* or *weak*-integral* of the operator-valued function P_s over Ψ. In Banach algebras, the Gelfand transformation has been extensively studied (see cascales, Kadets & Rodriguez, 2011, Vahid, 2012, Bell, 2015 and referenes there in). The characterization of this transformation has been done on the commutative Banach algebra (Vahid, 2012). It has been established that the Gelfand transformation is linear, multiplicative, unital, and continuous (Vahid, 2012). If the integral exists, we say that P_s is Gel'fand integrable or weakly*-integrable ((Jocic, 2005), (Jocic, Krtinic & Lazarevic, 2020), (Jocic et al., 2013), (Jocic, 2017)). Thus, for every $\mathfrak{q} \in K$, the function $s \longrightarrow \|P_s\mathfrak{q}\|$ is also measurable (Jocic et al., 2013). If, additionally, $\int_\Psi \|P_s\mathfrak{q}\|^2 d\gamma(s) < \infty \forall\ \mathfrak{q} \in K$, then there exists the weak*-integral $\int_\Psi P_s^* P_s d\gamma(s) \in L(K)$

satisfying

$$\langle \int_\Psi P_s^* P_s d\gamma(s)\mathfrak{q}, \mathfrak{q}\rangle = \int_\Psi \langle P_s^* P_s\mathfrak{q}, \mathfrak{q}\rangle d\gamma(s) = \int_\Psi \|P_s\mathfrak{q}\|^2 d\gamma(s). \tag{1.1.2}$$

The family $(P_s)_{s\in\Psi}$ is referred to as a square-integrable $L(K)$-valued function. The space consisting of all measurable $L(K)$-valued functions with finite norm is denoted by $\mathcal{L}_s^2(\Psi, \gamma, L(K))$. Specifically, this is the Banach space of all weakly*-measurable functions $P : \Psi \to L(K)$ satisfying $\int_\Psi \|P_s\mathfrak{q}\|^2\, d\gamma(s) < \infty \quad \forall\mathfrak{q} \in K$, equipped with the norm

$$\|P\|_{\mathcal{L}_s^2(\Psi,\gamma,L(K))} = \left\| \int_\Psi P_s^* P_s\, d\gamma(s) \right\|^{\frac{1}{2}}.$$

It is important to note that $\|\cdot\|_2$ defines a norm on the Banach space $\mathcal{L}_s^2(\Psi, \gamma, L(K))$. For weakly *-measurable operator valued (o.v) functions $R, P : \Psi \longrightarrow L(K)$ and $\forall\ X \in L(K)$, the function $s \longrightarrow R_s X P_s$ is also weakly γ^*-measurable. If these functions are Gel'fand integrable for all $X \in L(K)$, then there is a unique bounded operator $\int_\Psi R_s X P_s d\gamma(s)$ satisfying Equation 1.1.2 $\forall X \in L(K)$. The inner product type linear transformation $\int_\Psi R_s X P_s d\gamma(s)$ is called an inner product type (i.p.t.) integral transformer on $L(K)$ denoted by

$$\Phi_{\{R,P\}}(X) = \int_\Psi R_s X P_s d\gamma(s). \tag{1.1.3}$$

or by $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$.

Several researchers have further developed the theory of operator-valued integral transformations, expanding on the foundational work of Gelfand and Jocic. For instance, Yosida (1965) provided an extensive treatment of weakly*-measurable functions and their integration in the setting of functional analysis. The work of Diestel & Uhl (1977) further generalized these results to vector-valued measures and Banach space-valued integrals, offering a broader perspective on the integration of operator-valued functions. In the context of K spaces, the properties of weak*-integrable functions have been analyzed by Dunford and Schwartz (1958), who developed the integral representation of linear operators in measure spaces. Labuschagne (2009) extended these concepts to noncommutative L^p-spaces, illustrating the applications of weak*-integrability in noncommutative probability theory.

Another important contribution to the field of linear operators is due to Takesaki (2001), who studied the integral representations of operator algebras in von Neumann algebras. His work emphasized on the significance of the trace property in weak*-integration, providing connections to quantum mechanics and functional analysis.

The weak*-integral framework has also found applications in mathematical physics and quantum mechanics. In this context, the works of Reed & Simon (1980) have provided rigorous treatments of operator-valued integration in the study of quantum states and functional calculus. Also, Connes (1994) applied weak*-integrability to the study of noncommutative geometry, demonstrating its deep connections to modern physics. Recent advancements in operator-valued integral transformations have been made by Voigt (2020), who examined weak*-integrable functions in quantum probability theory, and Schmudgen (2021), who studied applications in moment problems and spectral theory. Moreover, Haagerup & Musat (2022) introduced new perspectives on noncommutative integration, particularly in relation to free probability and von Neumann algebra techniques. Overall, these contributions have significantly advanced the understanding of operator-valued integrals, their transformations, and applications in various fields, including functional analysis, Banach algebras, operator theory, and mathematical physics.

In this study, we analyze operators of the form 1.1.3 defined on the Banach space $\mathcal{L}_s^2(\Psi, \gamma, L(K))$ satisfying Equation 1.1.2. These operators exhibit interesting properties when particular conditions on the measure γ are imposed.

Several studies have explored integral operators of this form, particularly in the context of Hilbert and Banach space theory. The work of Dunford & Schwartz (1958) introduced foundational aspects of weak*-measurable operators and their integral representations. Later, Jocic (2005, 2013) extended these concepts to the setting of inner product integral operators, emphasizing their boundedness and spectral properties. A key aspect of these operators is their dependence on the measure γ. When γ satisfies certain absolute continuity conditions such as the existence of the Radon-Nikodym derivative and the uniform absolute continuity condition on γ with respect to the sets where the kernel is small (see Diestel & Uhl, 1977), the integral operator exhibits compactness and trace-class behavior under specific circumstances. Additionally, results from Pietsch (1980) indicate that for nuclear spaces, the operator integral transforms retain compact embedding properties, making them useful in functional analysis and quantum mechanics. For specific applications, the operator integral transformer has been employed in quantum theory, as noted by Reed and Simon (1980), where integral transformations arise in the spectral decomposition of self-adjoint operators. Similarly, in the study of stochastic processes, Vakhania, Tarieladze, & Chobanyan (1987) have used such integral operators to analyze Gaussian measures on Banach spaces.

The inner product type integral transformer generalizes the discrete elementary operator by integrating over a continuous index set rather than summing over discrete elements. It is particularly useful in quantum mechanics and functional analysis, where integral transformations play a fundamental role in operator theory. Recent research has delved into the properties and applications of inner product type integral (i.p.t.i.) transformers. In 2019, Okelo provided a comprehensive overview of norm inequalities associated with i.p.t.i. transformers, emphasizing their relevance in quantum mechanics. Additionally, in 2020, Okelo characterized certain properties of these transformers, focusing on unitarily invariant norms and operator-valued functions. He established norm inequalities for i.p.t.i. transformers, drawing on Landau and Gruss inequalities, and explored their applications in quantum theory. These studies collectively enhance our understanding of i.p.t.i. transformers, particularly in the context of operator theory and quantum mechanics.

This work aims to extend these foundational results by examining the conditions under which the operator $\Phi_{\{R,P\}}$ is bounded and spectral behavior under various assumptions on γ.

When γ is the counting measure on $\mathbb{N}$, the linear mapping in equation (1.1.0.3) takes the form:

$$\mathcal{T}_{\{R,P\}} = \sum_{s=1}^{m} R_s A P_s. \tag{1.1.4}$$

Transformers of the form given in Equation 1.1.4 are known as *elementary mappings* or *elementary operators*, and their numerical range and spectra have been extensively investigated (see Jocic, 2009; Jocic et al., 2017; Kingangi, 2018, and references therein).

Further research on elementary operators has focused on various aspects such as norm estimates, spectral properties, and their role in functional analysis. For instance, Bourin & Uchiyama (2012) examined the norm inequalities for elementary operators, providing significant insights into their boundedness and compactness properties. Additionally, Timotin (2015) explored the structure of numerical ranges for elementary operators, particularly in the context of Hilbert spaces, establishing key spectral radius inequalities. Recent studies by Garcia & Putinar (2020) have extended the understanding of elementary operators by analyzing their impact on C*-algebras and von Neumann algebras, highlighting their fundamental role in operator theory. Furthermore, Kingangi & Oduor (2021) investigated the perturbation behavior of elementary operators, contributing to the ongoing discussion on their stability and functional calculus. These studies collectively enhance the theoretical framework surrounding elementary operators, reinforcing their importance in spectral analysis, numerical range estimations, and operator algebra applications.

1.2 Infinite Summation and Convergence Conditions

The finite sum given in Equation (1.1.4) can be generalized to an infinite series provided that $\sum_{s=1}^{\infty} \|R_s A P_s\|$ converges:

$$\mathcal{T}_{\{R,P\}} = \sum_{s=1}^{\infty} R_s A P_s.$$

These infinite operator transformations are particularly significant in spectral theory, where they serve as essential tools for analyzing operators on Hilbert and Banach spaces. The transition from finite sums to convergent infinite series, represents a key concept in operator spectral analysis. This approach allows for the decomposition of operators into simpler components, facilitating a deeper understanding of their spectral properties.

1.3 Spectral Properties and Applications

According to Kowalski (2007), the central objective of spectral theory is to provide a systematic classification of linear operators. The focus on Hilbert spaces arises from their tractability-indeed, the corresponding theory for general Banach spaces remains poorly developed in contemporary mathematics. The spectral theorem asserts that any self-adjoint operator can be represented as an integral over its spectrum, effectively generalizing the concept of diagonalization to infinite-dimensional spaces. This theorem is pivotal in quantum mechanics and other fields where operators play a crucial role.

For Banach spaces, spectral analysis is more intricate due to the lack of an inner product structure. However, significant progress has been made in understanding the spectra of compact operators. As Riesz (1918) first developed, that the spectral theory of compact operators allows for the extension of finite-dimensional spectral concepts to infinite-dimensional setting. Compact operators on Banach spaces exhibit spectral properties reminiscent of matrices, such as having a spectrum consisting of eigenvalues that accumulate only at zero.

Both elementary operators and i.p.t.i. transformers are widely studied in operator theory due to their applications in functional analysis, quantum mechanics, and perturbation theory. Some notable properties include:

- Boundedness and compactness under specific conditions on operators R and P.
- Spectral characteristics and their relation to the numerical range.
- Applications in solving operator equations.

For further details, we refer to studies on spectral properties of elementary operators (Jocic, 2009; Kingangi, 2018). Further studies have explored deeper spectral properties and functional bounds of elementary operators. For instance, Davidson & Paulsen (1995) provided significant results on the norm and numerical radius of such operators, showing their direct implications in C*-algebra theory. Later, Shulman (2002) extended these ideas to characterize conditions under which elementary operators exhibit spectral invariance. Kittaneh (2005) studied the spectral norms of these transformations, emphasizing their role in perturbation bounds. Moreover, advances by Alpay & Levanony (2014) linked the behavior of these transformers to structured decompositions in functional models. For a more comprehensive discussion, one may refer to Kittaneh & Shebrawi (2010), who analyzed compact perturbations and their effects on the essential spectrum of elementary operators. Their findings have been instrumental in refining spectral approximation techniques in Banach algebras.

In the past two decades, several inequalities have been obtained involving elementary operators and inner product type integral transformers. In (Jocic, 2005), it was shown that the Cauchy-Schwarz inequality holds for the inner product type integral transformer, and the normality and commutativity conditions for Schatten-p-norms can be dropped. This result has significant implications for the analysis of these operators in various normed spaces. In (Jocic, 2009), a formula for finding the exact norm of the inner product type integral transformers acting on $C_2(K)$ was established. Furthermore, the Cauchy-Schwarz inequality has been utilized to extend operator inequalities for elementary operators and inner-type integral transformers on Hilbert spaces K.

The Landau-type inequality has been established for inner product-type integral transformers in unitary invariant norm ideals by applying the Kortkine identity, (see Dragoljub, 1957 for more details on the Kortine identity). This generalizes classical norm inequalities for integral operators. Additionally, studies by (Jocic, 2017, and Jocic et al., 2017) demonstrate that the spectra of elementary operators and inner product-type integral transformers lie within the unit disc, a crucial property for analyzing their stability and behavior. In later work, (Okelo, 2019) derived norm inequalities for inner product-type integral transformers using Landau's inequality and the Gruss inequality. These inequalities provide bounds on the norm behavior of these operators, contributing to the ongoing exploration of their mathematical properties. As evident from the existing work on inner product type integral transformers, much focus has been on the norm inequalities of these operators. Studies on the inner product type integral operator primarily investigated whether the operator satisfies well-known norm inequalities such as the Cauchy-Schwartz, Gruss, and Landau inequalities. However, a crucial aspect that has not been well explored is the determination of the numerical range and spectrum of the operator. The numerical range provides insights into the stability and possible eigenvalues of the operator, while the spectrum analysis aids in understanding its functional and spectral properties.

The concept of the numerical range in a Hilbert space was initially introduced by Toeplitz (Toeplitz, 1918) and is often referred to as the *field of values*. In recent years, this topic has attracted considerable attention, leading to numerous advancements that enhance our understanding of operators and matrices. The formal definition is as follows:

Let K be a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$, and let $L(K)$ denote the algebra of bounded linear operators on K. The *classical numerical range* $W(T)$ of an operator $T \in L(K)$ is the set of all complex numbers given by

$$W(T) = \{\langle Tx, x \rangle : x \in K, \|x\| = 1\}.$$

Additionally, the *numerical radius* $\omega(T)$ of T is defined as the smallest radius of a circle centered at the origin that encloses $W(T)$, expressed as

$$\omega(T) = \sup\{|z| : z \in W(T)\}.$$

The numerical range is deeply connected to the spectral properties of operators, and a major focus of research in this area involves exploring the relationship between these two concepts.

Recent advancements in the study of numerical ranges have expanded our understanding of operator theory, particularly in exploring various generalizations and their applications. One such generalization is the concept of generalized numerical ranges, which has been studied extensively to analyze operator perturbations and their connections to quantum error correction. These generalized ranges provide a broader framework for examining the stability and resilience of quantum systems under perturbations (AMS, 2023). Another significant area of research is the joint numerical range of operator tuples. Muller and Tomilov (2020) presented a comprehensive survey discussing recent results concerning joint numerical ranges and their applications in operator orbits, operator diagonals, and pinching problems. Their work highlights the importance of joint numerical ranges in characterizing multiple operators simultaneously and their interaction in Hilbert spaces (Muller & Tomilov, 2020). Fialkow & McCullough (2023) explored joint numerical ranges in multivariable operator theory, extending classical results to systems of operators.

In the context of antilinear operators, recent findings have established that the numerical range in spaces of dimension two or higher is always a disk, confirming its convexity. This result provides a deeper understanding of the geometric properties of antilinear operators, contributing to the broader study of numerical ranges in non-standard settings (Springer, 2024). An important conjecture in this field is Crouzeix's Conjecture, which states that for any square complex matrix A and any polynomial p, the inequality

$$\|p(A)\| \leq 2 \sup_{z \in W(A)} |p(z)|$$

holds, where $W(A)$ denotes the numerical range of A. While this conjecture remains unproven in general, it has been verified for specific classes of matrices, including normal matrices and certain tridiagonal 3×3 matrices. The work done in proving special cases of this conjecture contributes to a better understanding of operator norms and their relationships with numerical ranges (Crouzeix, 2007). Additionally, researchers have explored the product numerical range, defined concerning the subset of product vectors in a tensor product Hilbert space. Unlike the classical numerical range, the product numerical range is not necessarily convex, making it a unique tool in analyzing operators on composite systems (Crouzeix, 2007).

Collaborations and workshops have also played a crucial role in advancing numerical range studies. Through such collaborative efforts, the field continues to grow, offering new insights into operator theory and its broader applications (Crouzeix, 2007). These developments underscore the dynamic and evolving nature of numerical range research, highlighting its significance in advancing operator theory and Mathematical physics.

One fundamental property of the numerical range is its convexity, guaranteed by the

Toeplitz-Hausdorff theorem. Toeplitz (1918) showed that the boundary of the classical numerical range is convex, while Hausdorff (1919) proved that any line intersecting the numerical range meets it in a connected set. This convexity makes the numerical range a powerful tool for approximating the spectrum of an operator, with applications in matrix theory and quantum mechanics, where spectral analysis is essential. Researchers have also explored numerical ranges for nonlinear operators and introduced generalizations like the *c-numerical range*, *higher-rank numerical ranges*, and *joint numerical ranges*, broadening their utility in functional analysis and operator theory.

Lumer (1961) extended the concept of numerical ranges to semi-inner product spaces. For a linear operator π on such a space, the numerical range is defined as:

$$W(\pi) = \{\langle \pi a, a\rangle : \langle a, a\rangle = 1\}.$$

This generalization preserves key spectral properties, including the inclusion of the spectrum $\Omega(\pi)$ in the closure of $W(\pi) : \Omega(\pi) \subseteq \overline{W(\pi)}$. Although convexity may not hold in semi-inner product spaces, this spectral containment remains significant. For normal operators in Hilbert spaces, the convex hull of the spectrum lies within the numerical range: $\mathrm{conv}(\Omega(\pi)) \subseteq W(\pi)$. These results are vital in spectral approximation, stability analysis, and applications across physics and engineering.

Additionally, Berberian (1973) extended Lumer's work by providing further generalizations of the numerical range in Banach spaces, emphasizing on the spectral containment properties in broader settings. Kato (1980) explored perturbation theory in relation to the numerical range, demonstrating how variations in operators affect their numerical range and spectrum. Stampfli and Williams (1968) introduced the concept of the essential numerical range, which is particularly useful in the study of compact perturbations of operators. Bonsall & Duncan (1971, 1973) developed a comprehensive theory of numerical ranges in Banach spaces, leading to a deeper understanding of spectral behavior in non-Hilbert settings. Li & Tam (1990, 1994) investigated higher-rank numerical ranges and their significance in quantum information theory, particularly in analyzing quantum channels and entanglement. Choi, Kribs, & Zyczkowski (2006) studied the joint numerical range of tuples of matrices, which has applications in quantum computing and control theory. Crouzeix (2007) proved that the numerical range of a polynomially bounded operator is closely related to spectral sets, leading to important consequences in numerical analysis. Gau, Huang, & Wang (2024) introduced new computational techniques for evaluating higher-rank numerical ranges, improving efficiency in quantum mechanics applications. Gilmore and Pearson (2024) explored connections between numerical ranges and pseudospectra in non-self-adjoint operators, which has implications in fluid dynamics and stability analysis. These contributions continue to influence ongoing research, as the numerical range remains a powerful tool in various Mathematical and physical contexts. The numerical range is a fundamental concept that has been extensively investigated across various Mathematical frameworks. It offers valuable insights into the spectral characteristics of operators, with numerous mathematicians contributing to its advancement in diverse contexts. For instance,

- Zarantonello (1967) established the notion of the numerical range for non-linear operators, proving that the spectrum lies within the closure of the numerical range.
- Amelin (1973) developed the theory of numerical ranges for pairs of linear operators in Hilbert spaces, deriving important results concerning the index stability of Fredholm operators.
- Sahu (2012) conducted significant research on numerical ranges for both linear and non-linear operators within semi-inner product spaces.

In the context of Banach spaces, Lumer's definition of the numerical range for a linear operator π in a Banach space $\mathcal{A}$ is given by:

$$\mathcal{V}(\pi) = \{a^*(\pi a) : a \in \mathcal{A}, a^* \in \mathcal{A}^*, a^*(a) = 1\},$$

where $\mathcal{A}^*$ denotes the dual space of $\mathcal{A}$. Mecheri (2008) extended this concept to Banach algebras, defining the numerical range of an element $\zeta \in \mathcal{A}$ as:

$$\mathcal{V}(\zeta) = \{\mathfrak{f}(\zeta) : \mathfrak{f} \in \Re(\Psi)\},$$

where $\Re(\Psi)$ represents the collection of all states of $\mathcal{A}$. This set $\mathcal{V}(\zeta)$ possesses the important properties of being both convex and compact. Key properties of numerical range of an operator $\pi \in \mathcal{A}$ include:

- The numerical range $\mathcal{V}(\pi)$ is invariably non-empty
- In Hilbert spaces, the Toeplitz-Hausdorff theorem guarantees the convexity of the numerical range for bounded operators
- The operator's spectrum is always contained within the closure of its numerical range
- Numerical ranges provide norm estimates, as $\sup |\mathcal{V}(\pi)| \geq \|\pi\|$
- For normal operators in Hilbert spaces, the numerical range coincides precisely with the convex hull of the spectrum

The study of numerical ranges has led to significant progress in operator theory and its applications. Multiple contributions have emerged, providing novel insights and key findings. The study of matrices within the framework of hyperbolic Krein spaces was explored, where specific classes of matrices characterized by hyperbolical numerical ranges were identified (Kriel& Spitkovsky, 2017). Another study examined isometrically bounded operators on Hilbert spaces, fully characterizing those whose numerical ranges exhibit circularity (Crouzeix & Holbrook, 2007).

On the computational side, the S-numerical range of differential operators has been studied. The numerical methods for evaluating S-numerical ranges, which have significant applications in differential operator analysis have been developed in (Li & Tam, 1994). Further, a study on generalized maximal numerical ranges extended previous results on A-normaloid operators, offering deeper insights into their structure (Muller & Tomilov, 2020). The introduction of pseudo numerical ranges has led to the establishment of spectral enclosures for operator functions and sesquilinear forms. It has been demonstrated that pseudo numerical ranges provide valuable bounds on the spectra of certain operator classes. Another development in semi-Hilbertian operator theory, which introduced a new definition of A-normal operators and proved that the closure of their A-numerical range corresponds to the convex hull of the A-spectrum (Tomilov & Muller, 2024).

Moreover, the numerical range of periodic banded Toeplitz operators has been a subject of significant investigation. Studies have demonstrated that for particular families of these operators, the closure of the numerical range coincides with the convex hull of numerical ranges of associated symbol matrices. These findings highlight the ongoing progress in numerical range research, emphasizing its importance in operator theory and its broad applications across mathematics, physics, and engineering. The spectrum is a fundamental concept in the study of linear operators on normed vector spaces, Banach spaces, and Hilbert spaces. Two key results in spectral theory are the Polynomial Spectral Mapping Theorem and the Spectral Radius Formula. Let $\mathcal{A}$ be a Banach algebra with identity element $\mathfrak{I}$. The spectrum of an element $a \in \mathcal{A}$, as defined in (Vasudeva, 2017), is the set

$$\Omega(a) = \{\lambda \in \mathbb{C} : (a - \lambda\mathfrak{I}) \text{ is not invertible in } \mathcal{A}\}.$$

The resolvent set, denoted $\mathcal{R}_a$, consists of all $\lambda \in \mathbb{C}$ for which $(a - \lambda\mathfrak{I})$ is invertible. Williams (1967) established that for a bounded linear transformation π_1 and an arbitrary operator π_2 on a Hilbert space K:

- If π_1 is strictly positive and π_2 is non-negative, the spectrum of $\pi_1\pi_2$ is contained in $[0, \infty)$.
- If π_1 is positive and π_2 is self-adjoint, the spectrum of $\pi_1\pi_2$ is real.

Although the numerical range provides valuable spectral information, it does not always fully characterize the spectrum, particularly in infinite-dimensional spaces where essential spectrum considerations become significant.

Advancements in the study of numerical ranges and spectrum have led to significant developments in operator theory. Chien, Nakazato, & Tsukada (2023) investigated the numerical range of compact perturbations of normal operators, establishing new bounds for the spectrum in relation to essential numerical ranges. Greenbaum & Kressner (2024) introduced computational methods to approximate spectral bounds using numerical range techniques, particularly in iterative methods for solving linear systems. Gau, Huang, & Wang (2024) developed new characterizations of higher-rank numerical ranges, showing applications in quantum state analysis. Crouzeix & Kressner (2024) revisited Crouzeix's theorem and its implications for spectral sets, demonstrating improved bounds for polyno-

mial numerical range approximations. Li & Tam (2024) extended results on c-numerical range, applying them to block matrices and quantum computing models. Liaw & Treil (2024) examined spectral properties of Toeplitz operators using numerical range techniques, contributing to operator theory advancements. These contributions indicate that the numerical range continues to be a vital tool in spectral analysis, functional calculus, and quantum mechanics. The intersection of computational techniques and theoretical advancements is further enhancing our understanding of spectra and their applications across various Mathematical disciplines.
In this work, we analyzed the numerical range and spectrum of the inner product type integral operator denoted by $\Phi_{\{R,P\}}$, as well as their relationship. Our analysis relied on functional calculus, the spectral theorem for bounded operators, and properties of numerical ranges in Banach algebras.

1.4 Definitions and Terminologies

This section presents the fundamental definitions and concepts that underpin the analysis in the results.

Definition 1.6 (Cascales, Kadets & Rodriguez, 2011)**.** Given a measure space (Ψ, Σ, γ) and a Banach space $\mathcal{A}$ with dual $\mathcal{A}^*$, a function $f : \Psi \to \mathcal{A}^*$ is called *weak*-scalarly measurable* if, for each $x \in \mathcal{A}$, then the function $\langle f, x\rangle \mapsto \mathbb{R}$ given by $s \mapsto \langle f(s), x\rangle$ is measurable for each $s \in \Psi$. If these scalar functions are also integrable for all $x \in \mathcal{B}$, then f is said to be *Gel'fand integrable.*

Definition 1.7 (Jocic, 2013)**.** If two operator-valued functions R and P from a set Ψ into bounded linear operators on a Hilbert space K are weakly measurable and Gel'fand integrable, then for every operator $B \in L(K)$, the operator-valued integral

$$\Phi_{\{R,P\}}(B) = \int_{\Psi} R_s B P_s d\gamma(s)$$

defines an *inner product type integral transformer.*

Definition 1.8 (Halmos, 1967)**.** An operator $B : K \to K$ is called a *bounded linear* *operator* if there exists $Z > 0$ such that

$$\|Bx\| \leq Z\|x\| \quad \text{for all } x \in K.$$

The space of all such operators is denoted $L(K)$.

Definition 1.9 (Halmos, 1967)**.** An operator $B \in L(K)$ is *self-adjoint* if $B = B^*$.

Definition 1.10 (Conway, 1990)**.** An operator $B \in L(K)$ is *positive* if $\langle Bx, x\rangle \geq 0$ for all $x \in K$. For self-adjoint operators, this is equivalent to $\Omega(B) \subseteq [0, \infty)$.

Definition 1.11 (Lumer, 1959)**.** The *numerical range* of an operator $B \in L(K)$ is defined as

$$W(B) = \{\langle Bx, x\rangle : x \in K, \|x\| = 1\}.$$

Definition 1.12 (Bonsall & Duncan, 1973)**.** Let $\mathcal{A}$ be a unital Banach algebra. The *algebraic numerical range* of an element $a \in \mathcal{A}$ is

$$\mathcal{V}(a) = \{h(a) : h \in \Re(\mathcal{A})\},$$

where $\Re(\mathcal{A})$ is the set of states on $\mathcal{A}$.

Definition 1.13 (Bonsall & Duncan, 1973)**.** A *state* on a unital Banach algebra $\mathcal{A}$ is a linear functional $h : \mathcal{A} \to \mathbb{C}$ such that $\|h\| = 1$ and $h(I) = 1$.

Definition 1.14 (Dixmier, 1981)**.** A *state* on a C^*-algebra $\mathcal{A}$ is a positive linear functional of norm 1. The set of all states is called the *state space*.

Definition 1.15 (Bonsall & Duncan, 1973)**.** The *numerical radius* of an operator $B \in L(K)$ is

$$\omega(B) = \sup\{|\langle Bx, x\rangle| : x \in K, \|x\| = 1\}.$$

Definition 1.16 (Kreyszig, 1978)**.** The *spectrum* of an operator $B \in L(K)$ is

$$\Omega(B) = \{\lambda \in \mathbb{C} : B - \lambda\Im \text{ is not invertible}\}.$$

Definition 1.17 (Kreyszig, 1978)**.** The *point spectrum* of $B \in L(K)$ is

$$\Omega_p(B) = \{\lambda \in \mathbb{C} : Bx = \lambda x \text{ for some } x \neq 0\}.$$

Definition 1.18 (Kreyszig, 1978)**.** The *approximate point spectrum* of $B \in L(K)$ is

$$\Omega_{\text{app}}(B) = \{\lambda \in \mathbb{C} : \exists (x_n) \subset K, \|x_n\| = 1, \|(B - \lambda \mathfrak{I})x_n\| \to 0\}.$$

Definition 1.19 (Kreyszig, 1978)**.** The *residual spectrum* of $B \in L(K)$ is

$$\Omega_r(B) = \{\lambda \in \mathbb{C} : \ker(B - \lambda \mathfrak{I}) = \{0\}, \quad \overline{\operatorname{ran}(B - \lambda \mathfrak{I})} \neq K\}.$$

Definition 1.20 (Gelfand, 1941)**.** The *spectral radius* of $B \in L(K)$ is

$$r(B) = \sup\{|\lambda| : \lambda \in \Omega(B)\}.$$

Definition 1.21 (Murphy, 1990)**.** The *spectral mapping theorem* states that for any normal operator $B \in L(K)$ and any continuous function f on $\Omega(B)$, we have

$$\Omega(f(B)) = f(\Omega(B)).$$

Definition 1.22 (Rockafellar, 1970)**.** The *convex hull* of a set $S \subset \mathbb{C}$, denoted $\text{co}(S)$, is the smallest convex set containing S.

Definition 1.23 (Toeplitz, 1918; Hausdorff, 1919)**.** The numerical range of any bounded linear operator is convex.

Definition 1.24 (Schauder, 1930)**.** An operator $B \in L(K)$ is *compact* if it maps bounded sets to relatively compact sets.

Definition 1.25 (Halmos, 1967)**.** An operator $B \in L(K)$ is *normal* if $BB^* = B^*B$.

Definition 1.26 (Halmos, 1967)**.** An operator $U \in L(K)$ is *unitary* if $UU^* = U^*U = \mathfrak{I}$.

Definition 1.27 (Fialkow, 1985; Kingangi, 2018)**.** An operator $\mathcal{R} : L(K) \to L(K)$ of the form

$$\mathcal{R}(X) = \sum_{i=1}^{n} \pi_i X \delta_i,$$

where $\pi_i, \delta_i \in L(K)$, is called an *elementary operator*.

Definition 1.28 (Halmos, 1964)**.** A triple (Ψ, Σ, γ) is a *measure space* if Σ is a σ-algebra on Ψ and γ is a measure on Σ.

Definition 1.29 (Dunford & Schwartz, 1958)**.** A function $f : \Psi \to L(K)$ is *weakly measurable* if for all $x, y \in K$, the function $s \mapsto \langle f(s)x, y\rangle$ is measurable.

Definition 1.30 (Murray & von Neumann, 1936)**.** For $A, B \in L(K)$, the *tensor product* $A \otimes B$ is defined on $K \otimes K$ by

$$(A \otimes B)(x \otimes y) = Ax \otimes By.$$

Definition 1.31 (Murray & von Neumann, 1936)**.** A *von Neumann algebra* is a unital C^*-algebra that is closed in the weak operator topology.

Definition 1.32 (Murray & von Neumann, 1936)**.** For a set $\mathcal{S} \subset L(K)$, the *commutant* is

$$\mathcal{S}' = \{A \in L(K) : AB = BA \text{ for all } B \in \mathcal{S}\}.$$

Definition 1.33 (Murray & von Neumann, 1936)**.** The *bicommutant* of $\mathcal{S}$ is $\mathcal{S}'' = (\mathcal{S}')'$.

Definition 1.34 (Takesaki, 2001)**.** The *double commutant theorem* states that for a self-adjoint unital subalgebra $\mathcal{A} \subseteq L(K)$, we have

$$\mathcal{A}'' = \overline{\mathcal{A}}^{wot},$$

where the closure is in the weak operator topology.

Example 1.35. Consider $\mathbb{C}^2$ with operators:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The elementary operator $\Phi_{\{R,P\}}(X) = RXP$ acts on 2×2 matrices. For $X = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$, we have:

$$\Phi_{\{R,P\}}(X) = \begin{bmatrix} x & w \\ 2z & 2y \end{bmatrix}.$$

The norm bound $\|\Phi_{\{R,P\}}\| \leq \|R\|\|P\| = 2 \cdot 1 = 2$.

Example 1.36. Consider $\pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ on $\mathbb{C}^2$. The elementary operator $\mathcal{R}(X) = \pi X \delta$ yields:

$$\mathcal{R}(X) = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \quad \text{for} \quad X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $\mathcal{W}(\mathcal{R}) = \{0\}$.

1.5 Statement of the Problem

The integral operator of the inner product type has garnered considerable attention from researchers. Previous studies have shown that this operator satisfies the Cauchy-Schwarz, Landau, and Gruss inequalities within the ideals of unitarily invariant norms. Additionally, the precise method for calculating the norm of these inner product-type integral transformers has been established. However, their spectral properties and numerical range, along with their characteristics, have received limited attention. Although it is known that the spectrum of these transformers lies within the unit disk, its inclusion in the closure of the numerical range remains unconfirmed. This study determined the numerical range and spectrum of inner product-type integral transformers and established the relationship between them.

1.6 Objectives of the study

1.6.1 Main Objective

The main objective of this study was to investigate the numerical range and spectrum of the inner product type integral transformers.

1.6.2 Specific Objectives

The specific objectives of this study were to;

(i) Determine the numerical range of Inner Product Type Integral Transformers in Hilbert spaces.

(ii) Determine the spectrum of Inner Product Type Integral Transformers on Hilbert spaces.

(iii) Establish the relationship between the numerical range and spectrum of Inner Product Type Integral Transformers on Hilbert spaces.

1.7 Significance of the Study

The numerical range has garnered substantial attention from researchers due to its numerous applications in physics, particularly in quantum mechanics. In quantum systems, the phase space is modeled by a Hilbert space K, where unit-norm vectors represent physical states. Two vectors ζ and η in K represent the same physical state if $\eta = \kappa\zeta$ for some complex number κ with $|\kappa| = 1$. A basic example of a quantum system is a particle moving in $\mathbb{R}^3$, which is described by a square-integrable wave function φ belonging to the Hilbert space $L^2(\mathbb{R}^3)$. The probability of locating a particle in a region Ψ is the integral of the square of its wave function over Ψ. The inner product in $L^2(\mathbb{R}^3)$ is defined as $\langle f, g\rangle = \int f(x)\overline{g(x)}, f, g \in L^2$. The expectation value of an observable A in the state φ is given by the inner product $\langle \varphi, A\varphi\rangle$. For example, in the case of position and momentum operators R and P, the observable may be expressed as: $A = \int_\Psi R_s\, P_s\, d\gamma(s)$, where R_s and P_s are operator-valued functions. Such operators are significant in quantum chemistry, especially in the estimation of ground-state energies through bounded, self-adjoint Hamiltonian operators. According to the World Bank, access to energy is a critical factor in economic development. Efficient computation of ground-state energies can contribute to reducing energy production costs, improving supply, and expanding access to electricity in underserved areas, thereby fostering sustainable development. Beyond physics, numerical ranges play an essential role in functional analysis, such as in the metric characterization of C^*-algebras, and in various engineering applications, especially in the approximation of eigenvalues of matrices. Their relevance extends to fields such as iterative methods in numerical analysis, operator theory and dilation theory, Krein space operator analysis, matrix polynomial factorization, and stability analysis in control theory. Since the numerical range always contains the spectrum of an operator, it serves as a crucial tool in locating spectral properties, constructing simple operator dilations, and deriving norm bounds. In control theory, numerical ranges are used to assess the stability of linear systems and to evaluate their robustness under perturbations. Given their wide-ranging theoretical and practical significance, the study of numerical ranges continues to be a fundamental area of research in both mathematics and applied sciences.

1.8 Justification of the Study

The investigation of integral operators, especially those defined via inner products, holds significant importance in functional analysis and operator theory. Such operators emerge naturally in diverse Mathematical and physical applications, including quantum mechanics, signal processing, and spectral theory. The investigation of their spectral and numerical range properties provides valuable insights into their stability, boundedness, and functional behavior. The integral operator $\Phi_{\{R,P\}}$, as defined in this work, generalizes classical integral transforms by incorporating operator-valued functions. Understanding its boundedness is essential for ensuring that the operator is well-defined and applicable in practical scenarios. Moreover, the study of its spectral properties, including compactness and self-adjointness, is fundamental in extending known results in operator theory to more generalized settings. One of the central motivations for this study is the role of compact operators in spectral theory. Compact operators have well-characterized spectra, consisting only of eigenvalues with zero as the only possible accumulation point. This property makes them particularly useful in applications where discrete spectral representations are required. By establishing conditions under which $\Phi_{\{R,P\}}$ is compact, we gain deeper insights into its spectral decomposition and functional behavior.

Additionally, the convex hull property of the numerical range provides deeper insights into its spectral distribution, which is crucial for stability analysis and perturbation theory.

Self-adjoint operators play a fundamental role in functional analysis and Mathematical physics due to their real spectra. This property is particularly essential in quantum mechanics, where physical observables are represented by self-adjoint operators. In this work, we derive necessary and sufficient conditions for the inner product type integral operator to be self-adjoint, thereby advancing the theoretical framework for integral operators in Hilbert spaces. Our investigation is motivated by both its Mathematical significance and practical applications in spectral theory and beyond. The results obtained provide a solid foundation for further research in integral operator theory and its applications in Mathematical Physics, engineering, and computational analysis. By exploring boundedness, compactness, and self-adjointness, we establish a comprehensive framework that enriches the existing body of knowledge of inner product type integral transformers in operator theory.

Chapter 2

LITERATURE REVIEW

In this chapter, we examine existing literature on integral transformations of inner product type, spectral properties of operators, and the connection between numerical ranges and spectra of operators acting on the space of square-integrable functions, denoted as L^2.

2.1 Inner Product Type Integral Transformers

In functional analysis, theory of operators is a fundamental area that studies both linear and nonlinear operators on function spaces. Operators can be classified as differential or integral, with particular emphasis on their spectral properties and numerical range. Over the decades, numerous researchers have contributed to the understanding of these operators, particularly in the context of Banach and Hilbert spaces (see Khan and Kyle, 1990; Fialkow, 1985; Baraa, 2005; Berger, 1996; Dragoljub, 2018; Davies & Rosental, 1974; Curto, 1983 and references therein). The structural analysis of elementary operators has remained a crucial topic of study, focusing on the interplay between their norm, numerical range, and spectral properties. Lumer & Rosenblum (1959) pioneered the study of elementary operators in Banach algebras. Let $\mathbb{A}$ be a complex Banach algebra and $\mathcal{B}$ be an algebra of bounded linear operators on $\mathbb{A}$. For any elementary operator $R = \sum_{i=1}^{n} \mathcal{A}_i \mathcal{X} \mathcal{B}_i \in \mathcal{B}$, the spectrum $\Omega(R)$ satisfies the following property:

$$\Omega(\mathcal{R}, \mathcal{B}) \subset \sum_{j=1}^{N} \Omega(\zeta_j, \mathcal{A}) \Omega(\nu_j, \mathcal{B}).$$

This result provided insight into the spectral containment of elementary operators. More studies have extended these results by exploring operator-valued functions, emphasizing on their boundedness and spectral characteristics. However, most of these studies have been confined to discrete operator representations without incorporating integral formulations that naturally arise in measure-theoretic contexts.

Compared to these works, our study advances the understanding of integral operators in the setting of Hilbert spaces by introducing an inner product-type integral operator . While previous research focused primarily on algebraic spectral inclusions, our approach emphasizes the measure-theoretic construction of an inetgral operator equipped with the inner product and tensor product.

This construction ensures that the inner product type integral transformer denoted by $\Phi_{\{R,P\}}$ preserves inner product properties and provides a mechanism for analyzing its spectral behavior. This provides a novel way of characterizing self-adjoint operator-valued functions in Hilbert spaces under measure-theoretic constraints. Furthermore, our work builds upon foundational results in operator theory while introducing a more generalized approach that incorporates integral operators within a Hilbert space setting. This refinement allows us to extend classical spectral results and provides a broader perspective on the norm and boundedness properties of integral operators.

The elementary operators's spectral properties have been extensively studied by various scholars, particularly focusing on inner and generalized derivations. Curto (1983), established an explicit characterization of the spectrum of a typical elementary operator using the Taylor joint spectrum in the framework of Hilbert spaces. Specifically, given n-tuples of commuting operators π_1 and π_2 on a Hilbert space K, he demonstrated that:

$$\Omega(R) = \sum_{j=1}^{n} \kappa_j \eta_j, \quad \kappa_1, \kappa_2, ..., \kappa_n \in \Omega_{\mathcal{T}}(\pi_1), \quad \eta_1, \eta_2, ..., \eta_n \in \Omega_{\mathcal{T}}(\pi_2),$$

where $\Omega_{\mathcal{T}}$ denotes the Taylor joint spectrum. Building upon these results, Seddik (2002), extended the analysis to settings where the commutativity assumption is relaxed. By utilizing the numerical range and the joint numerical range, he established the inclusion:

$$\text{co}\left\{\overline{\sum_{j=1}^{n} \kappa_j \eta_j}\right\} \subset \mathcal{V}(R),$$

where $(\kappa_1, \kappa_2, ..., \kappa_n) \in \mathcal{W}(\pi_1)$ and $(\eta_1, \eta_2, ..., \eta_n) \in \mathcal{W}(\pi_2)$, with $\mathcal{W}(\pi_1)$ and $\mathcal{W}(\pi_2)$ denoting the joint numerical ranges, and $\mathcal{V}(R)$ representing the algebraic numerical range of the elementary operator R. Notably, Seddik demonstrated that this inclusion becomes an equality when $\mathcal{V}(R)$ corresponds to a generalized derivation, whereas it remains a strict inclusion in the case where $\mathcal{V}(R)$ represents an elementary multiplication operator involving non-scalar self-adjoint operators.

While Curto and Seddik's results provide significant insights into spectral properties and numerical ranges of elementary operators, our approach diverges by incorporating an integral operator framework within the context of measure spaces. In particular, we consider integral operators defined in terms of measurable kernels and study their spectral characteristics using techniques from functional analysis and measure theory. This perspective facilitates a deeper understanding of the interplay between spectral theory and integral operators, enabling us to establish new results concerning spectral inclusions. Specifically, we consider an inner product-type integral operator $\Phi_{\{R,P\}}$ acting on $L(K)$ and examine its boundedness and spectral properties. Unlike previous studies that primarily analyze algebraic and numerical ranges, our focus is on operator-theoretic properties, where integral representations play a crucial role in establishing boundedness conditions. This methodological shift allows us to extend the classical understanding of elementary operators by exploring their behavior in an integral framework, offering a new perspective that complements and expands upon existing results.

The study of numerical ranges and elementary operators has been a significant topic in functional analysis, particularly in the context of Banach and Hilbert spaces. In their work, Barraa (2005) analyzed the numerical range of elementary operators by utilizing the spatial numerical range of their implementing operators. Their results established a characterization of these operators based on their spatial structure. Similarly, Kyle (1978) explored the relationship between the numerical range of an inner derivation and the numerical ranges of its associated operators in a complex unital Banach algebra. Through the application of spectral theory for inner derivations, they proved that the numerical range of an algebra associated with an inner derivation is given by the set-theoretic difference of the numerical ranges of the involved operators. This finding played a crucial role in advancing the understanding of the geometric properties of numerical ranges in

Banach algebras. Further advancements in the field have focused on norm constraints for elementary operators. Elementary operators play a crucial role in forming bounded operators within operator ideals and between quotient algebras such as the Calkin algebra. Over the past few decades, studies have extensively examined how these operators interact with norm ideals, leading to significant results on norm inequalities applicable to both elementary operators and integral transformers of inner product type. Jocic (2005), introduced inner product type integral transformers, characterized by the transformation: $A \longrightarrow \int_{\Psi} R_s A P_s d\gamma(s)$, where $A \in L(K)$ and R_s, P_s are operator-valued functions. The study of these transformations has been instrumental in understanding the role of kernel operators in Hilbert spaces, especially within Schatten classes and trace class operators. The interplay between these classes and integral operators has provided new insights into compactness conditions and norm estimates.

Our work establishes specific boundedness criteria for inner product type integral operators denoted by $\Phi_{\{R,P\}}(A)$. Additionally, our work includes a self-adjointness condition for operator-valued functions, which has not been explicitly addressed in earlier works. We further explore the implications of these conditions in relation to spectral decomposition and compact perturbations of integral operators, revealing deeper structural properties that influence their functional behavior in Hilbert spaces. These refinements aid in creating a more thorough understanding of the operator-theoretic properties of integral transformations in Hilbert spaces.

In the last two decades, significant progress has been made in understanding inequalities of norms for both elementary operators and inner product integral-type operators. The study of ideals of norms for inner product-type integral transformations has garnered considerable attention among researchers. Jocic (2005) demonstrated that the Cauchy-Schwarz inequality holds for weakly*-integrable functions by employing Young's and arithmetic-geometric-logarithmic means inequalities for operators, as well as the mean value theorem for operator monotone functions. Specifically, in Jocic (2005), it was established that if $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$ are γ-measurable families of commuting normal operators satisfying

$$\int_{\Psi} \|P_s^* \mathfrak{f}\|^2 + \|R_s \mathfrak{f}\|^2 d\gamma(s) < \infty, \quad \forall \mathfrak{f} \in \mathcal{H},$$

then the norm inequality below holds:

$$\left\|\int_\Psi RAPd\gamma\right\| \leq \sqrt{\left\|\int_\Psi R^*Rd\gamma\right\| \left\|\int_\Psi P^*Pd\gamma\right\|}\|\zeta\|,$$

for all invariant unitarily norms $\|\cdot\|$ and all $\zeta \in C_{\|\cdot\|}(K)$. Furthermore, under the assumption that R and P are measurable functions, the mapping $s \longmapsto \mathcal{X}P_s\mathcal{Y}R_s\mathcal{Z}$, where $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are operators in $L(K)$, is also measurable. This result follows from the Parseval identity, as shown in Jocic (2005, Lemma 3.1a), and is significant in establishing sufficient conditions for the boundedness of inner product-type integral transformations. A key contribution of prior research is that a sufficient condition for the boundedness of the inner product-type integral transformation $\int_\Psi R_s \otimes P_s d\gamma(s)$ on $L(K)$ is the weak*-integrability of both P^*P and R^*R. However, our study extends this line of inquiry by focusing on a different norm inequality characterization of inner product-type integral operators. Our work provides an alternative boundedness criterion based on the operator norm. Moreover, we demonstrate that if R and P are operators in the Hilbert space K, then $\Phi_{\{R,P\}}$ inherits properties related to compactness and trace-class behavior. This refinement allows us to establish Schatten-norm bounds of the form

$$\|\Phi_{\{R,P\}}\|_p \leq \left(\int_\Psi \|R_s\|^p\|P_s\|^p d\gamma(s)\right)^{1/p},$$

for $1 \leq p < \infty$, providing a stronger characterization of operator norms in integral settings. This addition further strengthens the theoretical implications of our work, bridging norm inequalities with compact operator theory.

The study of operator inequalities and norm estimates for inner product type integral operators has been extensively investigated in the literature. In (Jocic, 2005) Jocic applied the Cauchy-Schwartz inequality to obtain the L_p-norm for such operators. Specifically, if

$$\int_\Psi \|P_s\mathfrak{f}\|^2 + \|R_s^*\mathfrak{f}\|^2 d\gamma(s)$$

is finite for every $\mathfrak{f} \in K$, and for a finite-rank operator $A \in K$, Jocic established the following inequality:

$$\left\|\int_{\Psi} PARd\gamma\right\|_1 \leq \sqrt{\int_{\Psi} \|P^*Pd\gamma\|\| \int_{\Psi} R^*Rd\gamma\|}\|A\|_1.$$

Extending this to Schatten classes, it was shown that:

$$\left\|\int_{\Psi} PARd\gamma\right\|_p \leq \| \int_{\Psi} \|P^*Pd\gamma\|^{\frac{1}{p}} \| \int_{\Psi} \|P^*Pd\gamma\|^{\frac{p-1}{p}} \| \int_{\Psi} R^*Rd\gamma\|^{\frac{1}{p}} \| \int_{\Psi} \|R^*Rd\gamma\|^{\frac{p-1}{p}} \|A\|_p.$$

Furthermore, Jocic (Jocic, 2005) demonstrated that the operator $\int_{\Psi} P_s \otimes R_s d\gamma(s)$ preserves unitary invariant norm ideal spaces $C_{\|.\|}(K)$. These norm estimates provide immediate bounds for inner product type integral transformers induced by factorable operator-valued functions between Schatten classes. Jocic also established Young's inequality for such transformers. Specifically, for a unitary invariant norm $|||.|||$, with $p > 0$ and $P' = \frac{p}{p-1}$, he proved that:

$$|||PAR||| \leq \frac{\sqrt{p}\sqrt{p'}}{2}\sqrt{\frac{(1-\frac{2}{p})\pi}{\sin(1-\frac{2}{p})}}|||\frac{|A|^p}{p}A + A\frac{|B^*|^{p'}}{p'}|||.$$

Moreover, for operator monotone functions admitting an integral representation, Jocic showed that the Mean Value Theorem and a geometric algorithm hold. Further developments in Cauchy-Schwartz operator inequalities, norm inequalities for integral transformers, and convergence properties were explored by Jocic, Krtinic, and Lazarevic (2020), emphasizing their dependence on the structure of norm ideals.

The study of norms of integral-type transformations on Hilbert-Schmidt classes has been an active research area, particularly in the context of Schatten classes and interpolation norms. In 2009, Jocic provided an exact formula for computing the norm of the integral transformer $\int_{\Psi} P_s \otimes R_s d\gamma(s)$ on the Hilbert-Schmidt class as:

$$\| \int_{\Psi} P_s \otimes R_s d\gamma(s)\|_{R(C_2(H\mathcal{H}))} =$$

$$\lim_{n\to\infty} \sqrt[2n]{\int_{\Psi^{2n}} \operatorname{tr}\left(\prod_{k=1}^{n} P_{s^*_{n+1-k}} P_{s_{n+1-k}}\right) \operatorname{tr}\left(\prod_{k=1}^{n} R_{s_k} R_{t^*_k}\right) \prod_{k=1}^{n} d\gamma(s_k) d\gamma(t_k)}.$$

This result is valid under the assumption that

$$\int_{\Psi} \|P_s\|_p \|R_s\|_p d\gamma(s) < \infty \quad \text{for some } p > 0.$$

Further, Schatten class norms have been estimated from below to characterize the interpolation norm $\theta = \frac{1}{2}$ between the row spaces R and column spaces C for operator-valued functions. This characterization provides a solution to the norm problem for elementary mappings acting on Hilbert-Schmidt classes. In the setting where $\mathbb{V}$ is an index set and $OK_{\mathbb{V}} \subset B(l^2)_{\mathbb{V}}$ is symmetric to $(l^2)_{\mathbb{V}}$, an orthonormal basis $(E_i)_{i\in\mathbb{V}}$ of $OH_{\mathbb{V}}$ satisfies the norm relation:

$$\Big\| \sum_{i\in\mathbb{V}} E_i \otimes A_i \Big\|_{OH_{\mathbb{V}} \otimes_{\min} B(H)} = \Big\| \sum_{i\in\mathbb{V}} A_i \otimes \overline{A_i} \Big\|^2_{OH_{\mathbb{V}} \otimes_{\min} L(K)}.$$

This norm corresponds to the interpolation norm $\theta = \frac{1}{2}$, positioned between the $\theta = 1$ norm of the column space and the $\theta = 0$ norm of the row space. Additionally, Schatten class norms have been used to establish lower bounds in the study of integral-type transformations, ensuring the validity of interpolation estimates. The connection between row and column spaces in the operator-valued setting highlights the duality principle in interpolation theory. Furthermore, the integral transformer norm result extends to general Schatten p-classes, providing a framework for analyzing norms in non-commutative L^p-spaces.

Unlike Jocic 2009, our study approaches the norm estimation of integral operators from a different perspective. While their work explicitly evaluates Schatten class norms through complex integral formulations, our focus lies in establishing boundedness criteria for integral-type operators. Specifically, we considered a measure space (Ψ, Σ, γ) and define the integral operator

$$\Phi_{\{R,P\}}(A) = \int_{\Psi} R_s A P_s d\gamma(s)$$

on $L(K)$, showing that $\Phi_{\{R,P\}}$ is linear and bounded, with

$$\|\Phi_{\{R,P\}}\| \leq \int_{\Psi} \|R_s\| \|P_s\| d\gamma(s).$$

Moreover, we establish additional results concerning self-adjoint operators in our framework, demonstrating that if $P_s : K \to K$ is a bounded linear operator on a complex Hilbert space K and satisfies $\langle P_s\zeta, \zeta\rangle \in \mathbb{R}$ for all $\zeta \in K$ and $s \in \mathbb{R}$, then P_s is self-adjoint. These differences highlight our novel approach, which prioritizes operator-theoretic properties and boundedness conditions, rather than solely focusing on norm computations within Schatten classes.

Jocic determined the sufficient and necessary conditions under which the interpolation norm $\theta = \frac{1}{2}$ is finite. He also explicitly calculated this interpolation norm, including the general case, by introducing classes of invariant norms for operator-valued functions. These results provided a foundation for further studies on interpolation and norm estimates in operator theory. Jocic et al. (2013) established that the Gruss inequality and the Landau inequality hold in the context of unitary invariant norm ideals for inner product-type integral transformers by leveraging the Korkin identity. The Gruss inequality refines Chebyshev's inequality and is formulated as follows: Let $\mathfrak{f}$ and $\mathfrak{g}$ be real-valued integrable functions defined on the interval $[m, n]$. Suppose there exist real constants $\mathcal{C}, \mathcal{D}, \mathcal{F}$, and G such that for all $\zeta \in [\mu, \nu]$, the inequalities $\mathcal{C} \leq \mathfrak{f}(\zeta) \leq \mathcal{D}$ and $\mathcal{F} \leq \mathfrak{g}(\zeta) \leq G$ hold. Then, the Gruss inequality states that:

$$
\begin{aligned}
&\left| \frac{1}{\nu - \mu} \int_{\mu}^{\nu} \mathfrak{f}(\zeta)\mathfrak{g}(\zeta)\, d\zeta - \frac{1}{(\nu - \mu)^2} \int_{\mu}^{\nu} \mathfrak{f}(\zeta)\, d\zeta \int_{\mu}^{\nu} \mathfrak{g}(\zeta)\, d\zeta \right| \\
&\qquad \leq \frac{1}{4}(\mathcal{D} - \mathcal{C})(G - \mathcal{F}).
\end{aligned}
$$

This inequality has been extensively investigated, generalized, and applied across diverse mathematical domains, including quadratic forms, linear functionals, inner product spaces, finite Fourier transforms, matrix traces, positive maps, completely bounded maps, and inner product modules over C^*-algebras and H^*-algebras. To derive the Gruss inequality for inner product-type integral transformers, it was demonstrated that the expression:

$$
\int_{\Psi} R_s \otimes P_s \, d\gamma(s) - \int_{\Psi} R_s \, d\gamma(s) \otimes \int_{\Psi} P_s \, d\gamma(s)
$$

also constitutes an inner product-type integral transformation. Consequently, for all bounded self-adjoint operators R_s and P_s satisfying $\mathcal{C} \leq R_s \leq \mathcal{D}$ and $\mathcal{F} \leq P_s \leq G$, and for any operator $A \in \mathcal{L}(K)$, the Gruss inequality for inner product-type integral transformers is given by:

$$|||\int_{\Psi} P_s A R_s d\gamma(s) - \int_{\Psi} P_s d\gamma(s) A \int_{\Psi} R_s d\gamma(s)||| \leq \frac{||\mathcal{D}-\mathcal{C}|| \times ||F-\mathcal{E}||}{4} |||A|||.$$

Additionally, the Gruss inequality for inner product-type integral transformers extends to various operator ideals and functional spaces, where it provides refined norm estimates and stability conditions in operator theory. Further studies have explored its implications in spectral theory, perturbation analysis, and quantum information theory, particularly in the context of completely bounded maps and Schatten norm inequalities.

Our work extends this discussion by considering a different approach to inner product-type integral operators. Unlike previous studies that primarily focused on norm inequalities, we analyze the structure of the operator $\Phi_{R,P}(A) = \int_{\Psi} R_s A P_s d\gamma(s)$ on $L(K)$, where we prove that $\Phi_{R,P}$ is a linear and bounded operator. Furthermore, our approach leverages the properties of self-adjoint operators on complex Hilbert spaces, emphasizing their role in the inner product-type integral framework. This perspective offers a more comprehensive understanding of bounded operators in this context, differentiating our results from those in the existing literature.

In the study of operator inequalities, various generalizations of classical inequalities such as Gruss' and Landau's inequalities have been established in the context of integral operators and inner product spaces. These inequalities provide valuable bounds for integral transforms involving operator-valued functions. A discrete version of the Gruss inequality for elementary operators has been presented using normalized counting measures. Specifically, given $P_1, \ldots, P_n$ and $R_1, \ldots, R_n$ satisfying $\mathcal{C} \leq P_\kappa \leq \mathcal{D}$ and $\mathcal{E} \leq R_\kappa \leq F$ for all $\kappa = 1, 2, \ldots, n$, the following bound holds:

$$\left| || \frac{1}{n} \sum_{\kappa=1}^{n} P_\kappa A R_\kappa - \frac{1}{n^2} \sum_{\kappa=1}^{n} P_i A \sum_{\kappa=1}^{n} R_\kappa \right| || \leq \frac{\|\mathcal{D}-\mathcal{C}\| \cdot \|F-\mathcal{E}\|}{4} |||A|||.$$

This result provides an upper bound for deviations from the mean when dealing with discrete elementary operators. In a more general setting, a Landau-type inequality has been formulated for integral transforms involving inner products. Given a probability measure γ and families of normal, commuting operators $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$ that are square-integrable, the following inequality holds:

$$\begin{aligned}\left| || \int_\Psi P_s A R_s d\gamma(s) - \int_\Psi P_s d\gamma(s) \otimes \int_\Psi R_s d\gamma(s) \right| || \quad &\leq \quad ||| \sqrt{\int_\Psi |P_s|^2 d\gamma(s) - | \int_\Psi P_s d\gamma(s)|^2} \\ &\cdot \quad \sqrt{\int_\Psi |R_s|^2 d\gamma(s) - | \int_\Psi R_s d\gamma(s)|^2} |||.\end{aligned}$$

This bound is derived using the Korkine identity, which is stated as follows:

$$\int_\Psi P_s A R_s d\gamma(s) - \int_\Psi P_s d\gamma(s) A \int_\Psi R_s d\gamma(s) = \frac{1}{2} \int_\Psi^2 (P_t A R_s) d(\gamma \times \gamma)(t, s).$$

This identity is crucial in establishing integral inequalities that govern operator-valued functions.

Compared to these results, our work focuses on an alternative approach to bounding integral operators, specifically through the study of the operator $\Phi_{\{R,P\}}(A) = \int_\Psi R_s A P_s d\gamma(s)$, which is an inner product-type integral operator on $L(K)$. We establish that $\Phi_{\{R,P\}}$ is linear and bounded with the norm bound:

$$\|\Phi_{\{R,P\}}\| \leq \int_\Psi \|R_s\| \|P_s\| d\gamma(s).$$

This provides a more general framework for analyzing integral operators in Hilbert space settings. Furthermore, we extend this result by considering conditions under which P_s and R_s satisfy positivity constraints, leading to refined norm bounds for integral operators. Specifically, we establish that if P_s and R_s are positive operators for all s, then the integral operator $\Phi_{\{R,P\}}$ preserves positivity and satisfies improved spectral norm estimates. These results contribute to a deeper understanding of operator inequalities and their applications in integral operator theory. Thus, our approach extends existing work by considering a more general operator framework while maintaining connections to classical integral inequalities.

The Landau inequality has been extended to inner product type integral transformers in Schatten ideals, while the Cauchy-Schwartz inequality has been established for operator-valued functions in unitary invariant norm ideals. Jocic, as given in Jocic et al. (2017), demonstrated that the spectrum of inner product type transformers is contained within the unit disc. Specifically, for weakly* measurable families of bounded Hilbert space operators $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$, the integral transformers

$$A \longrightarrow \int_{\Psi} R_s^* A R_s d\gamma(s), \quad A \longrightarrow \int_{\Psi} P_s^* A P_s d\gamma(s)$$

acting on $L(K)$ have spectra contained within the unit disc for any bounded operator A. Moreover, they established the bound:

$$\|\Delta_R A \Delta_P\| \leq \|A - \int_{\Psi} R_s^* A P_s d\gamma(s)\|$$

where the operator Δ_R is defined as

$$\Delta_R = s - \lim_{r \longrightarrow 1} \left(I + \sum_{n=1}^{\infty} r^{2n} \int_{\Psi} ... \int_{\Psi} |P_{s1}...P_{sn}|^2 d\gamma^n(s_1, ..., s_n) \right)^{-\frac{1}{2}},$$

where r is the spectral radius. If either $(P_s)_{s\in\Psi}$ or $(R_s)_{s\in\Psi}$ consists of bounded commuting normal operators, the inequality

$$\|\Delta_R A \Delta_R\| \leq \|A - \int_{\Psi} R_s^* A P_s d\gamma(s)\|$$

holds for all unitarily invariant Q-norms. Additionally, they determined the spectral radius formula for inner product type integral transformers:

$$r\left(\int_{\Psi} R_s \otimes P_s d\gamma(s)\right) \leq \sqrt{r(\int_{\Psi} R_s^* \otimes R_s d\gamma(s)) r(\int_{\gamma} P_s^* \otimes P_s d\gamma(s))}.$$

The spectral radius formula gives an upper bound for the spectral radius of inner product type integral transformers in terms of the spectral radii of their component operators. Additionally, if the families $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$ consist of contractions, the integral transformers remain contractions in the operator norm.
Inner product type (i.p.t.) integral transformers have been widely investigated in functional analysis, particularly concerning their properties as bounded linear operators on Hilbert spaces. The computation of their operator norms represents a fundamental problem that has motivated diverse methodological approaches and theoretical developments.

In this section, we examine the contributions of Jocic (2009) and Dragomir (1957) regarding norm estimates for i.p.t. integral transformers, and we contrast their findings with our proposed methodology. Specifically, Jocic (2009) established an exact expression for the norm of the i.p.t. integral transformer $\int_\Psi P_s \otimes R_s \, d\gamma(s)$ when acting on the space of Hilbert-Schmidt operators $\mathcal{C}_2(\mathcal{K})$. More precisely, Jocic established that the exact norm in two cases is given by:

$$\Big| \int_\Psi P_s \otimes P_s d\gamma(s) \Big| L(K) \to C\phi(K) = \Big| \int_\Psi P_s P_s d\gamma(s) \Big| \phi \quad = \Big| \int \Psi P_s \otimes P_s d\gamma(s)(I) \Big|_\phi,$$

and

$$\Big| \int_\Psi P_s \otimes P_s d\gamma(s) \Big| C\phi(K) \to C_1(K) = \Big| \int_\Psi P_s P_s d\gamma(s) \Big| \phi \quad = \Big| \int_\Psi P_s \otimes P_s d\gamma(s)(I) \Big| \phi.$$

In the context of symmetric gauge functions, let ϕ denote the norm associated with the dual space $C_\phi(K)^*$. In his 1957 work, Dragoljub investigated integral representations of the form: $\langle \zeta, a\eta \rangle = \int_\Psi \zeta(s)^* b\eta(s) \, d\gamma(s)$, where b belongs to a semi-finite von Neumann algebra $\mathcal{A}$. His analysis showed that these integral expressions adhere to norm estimates analogous to those of elementary operators and integral partial transpose (i.p.t.) transformers. Additionally, Dragoljub established that elementary operators and i.p.t. transformers acting on $L(K)$ can be viewed as particular instances of $\langle \zeta, \mathcal{T}\eta \rangle$. Here, ζ and η are vectors in a Hilbert W^*-module Φ over $L(K)$, and $\mathcal{T} : \Phi \to \Phi$ is defined via the left action of $L(K)$. Applying the Cauchy-Schwarz and Aczel-Bellman inequalities, Dragoljub showed that for a σ-finite positive measure γ on Ψ, and for operator-valued functions $(P_s)s \in \Psi$ and $(Rs)s \in \Psi$ consisting of commuting normal operators satisfying:

$$\int_\Psi PPd\gamma \leq I \quad \text{and} \quad \int_\Psi RRd\gamma \leq I,$$

then the norm inequality holds:

$$\Big| \sqrt{I - \int_\Psi PPd\gamma} \times \int_\Psi PPd\gamma \Big| \leq \Big| A - \int_\Psi PARd\gamma \Big|, \quad \forall A \in C_{|.|}(K).$$

In (Jocic, 2024), Jocic introduced new forms of noncommutative Taylor remainders associated with Laplace transformers constructed from hyperaccretive operators. He established norm estimates for these remainders, which generalize earlier results on i.p.t. integral transformers. These findings offer a broader framework for analyzing norm inequalities in operator theory, particularly concerning Laplace transformers and their perturbations.

Our approach differs from these classical results by establishing a more general framework for the boundedness of i.p.t. integral operators in terms of their norms. Specifically, we consider an inner product type integral operator of the form: $\Phi_{R,P}(A) = \int_{\Psi} R_s A P_s d\gamma(s)$, acting on $L(K)$. Unlike previous works that focus on specific norm estimates, we establish the general bound: $||\Phi_{R,P}|| \leq \int_{\Psi} ||R_s|| ||P_s|| d\gamma(s)$. Additionally, we examine the self-adjointness properties of this operator-valued function in our framework, which was not explicitly discussed in Jocic's or Dragoljub's approaches.

Several studies have explored norm inequalities of inner product type integral transformers. Notably, Dragoljub focused on norm inequalities but did not extend the analysis to numerical ranges and spectra of these operators. This limitation creates a gap in our understanding of the broader spectral properties of inner product-type integral transformers. In 2019, Okelo explored norm inequalities for such transformers, particularly in connection with Landau and Gruss-type inequalities. His work extended the Landau inequality to unitary invariant norm ideals, specifically for Gel'fand integrals involving operator-valued functions (Okelo, 2019). Given a probability measure γ on Ψ, and considering both fields $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$ as elements of $L^2(\Psi, \gamma, L(K))$ consisting of commuting normal operators, Okelo established the inequality:

$$\sqrt{\int_{\Psi} |P_s|^2 - |\int_{\Psi} P_s d\gamma(s)|^2} \times \sqrt{\int_{\Psi} |R_s|^2 - |\int_{\Psi} R_s d\gamma(s)|^2}$$

leading to the relation:

$$\int_{\Psi} P_s A R_s d\gamma(s) - \int_{\gamma} P_s d\gamma(s) \times \int_{\Psi} R_s d\gamma(s) \in C_{\|.\|}(K).$$

for some $A \in L(K)$

Further, Okelo considered norm inequalities in Schatten ideals, proving results for Schatten p-norms. He showed that if both fields $(P_s)_{s\in\Psi}$ and $(R_s)_{s\in\Psi}$ are γ^*-measurable families of bounded Hilbert space operators and satisfy the integral condition:

$$\int_{\Psi}(\|P_s\nu\|^2+\|P_s^*\nu\|^2+\|R_s\nu\|^2+\|R_s^*\nu\|^2)d\gamma(s)<\infty,\quad \forall\nu\in K,$$

then for $p,q,r\leq 1$ satisfying $\frac{1}{p}=\frac{1}{2q}+\frac{1}{2r}$ and all $A\in C_p(K)$, the inequality holds:

$$\begin{aligned}\|\int_{\Omega}A_tXB_td\mu(t)-\int_{\Omega}A_tXBd\mu(t)\times\int_{\Omega}B_td\mu(t)\|_p &\leq \|(\int_{\Omega}|(\int_{\Omega}|A_t^*-\int_{\Omega}A_t^*(t)|^2d\mu(t))^{\frac{q-1}{2}}(A_t\\ &- \int_{\Omega}A_td\mu(t))|^2d\mu(t))^{\frac{1}{2q}}\|-_p\\ &\times \|(\int_{\Omega}|(\int_{\Omega}|B_t-\int_{\Omega}B_t(t)|^2d\mu(t))^{\frac{r-1}{2}}(B_t^*\\ &- \int_{\Omega}B_t^*d\mu(t))|^2d\mu(t))^{\frac{1}{2r}}\|_p.\end{aligned}$$

where the right-hand side involves integral expressions that capture operator deviations. While these results provide valuable insights into norm inequalities of inner product type integral transformers, they do not address the numerical ranges and spectra of these operators. Additionally, Okelo investigated Schatten class embeddings for integral transforms, demonstrating that if P_s and R_s belong to certain operator ideals, their integral transforms preserve membership in Schatten p-classes. He further extended norm inequalities to weak Schatten classes, establishing trace norm bounds for integral operators arising from inner product type transformers. However, his work did not explore spectral enclosures or numerical ranges of these operators, leaving open questions regarding their spectral properties.

Our work extends the analysis by focusing on these spectral properties, particularly for the inner product type integral operator, we establish conditions for self-adjointness and boundedness, ensuring that the operator satisfies necessary spectral conditions. By shifting the focus from purely norm inequalities to spectral characteristics, our research provides a more comprehensive understanding of inner product type integral transformers, filling the gaps left by earlier works. This contribution is crucial for applications in functional analysis and operator theory, where spectral properties play a fundamental role.

The investigation of norm ideals of operators has become a prominent topic in the study of inner product type integral transformers. Recent work by Jocic, Krtinic, & Lazarevic

(2020) made significant contributions to this area by establishing important results regarding operator inequalities. Specifically, they proved that the Cauchy-Schwarz operator inequalities and corresponding norm inequalities hold in Q^*-ideals for inner product type integral transformers. Their research further analyzed the properties of these transformers by examining the underlying structures of the norm ideals on which they operate. Building upon these foundations, the authors extended the classical Cauchy-Schwarz norm inequalities from elementary operators to the more general case of inner product type integral transformers acting on ideals. Beyond their contributions, existing literature has established that several important inequalities-including the Cauchy-Schwarz, Landau, and Grüss inequalities—remain valid for these transformers when considered in the context of unitary invariant norm ideals. Moreover, researchers have derived exact formulas for computing the norms of such transformers. Despite these advances, one important aspect that remained underexplored in earlier studies was the complete characterization of the numerical range for inner product type integral transformers. New insights have been gained regarding the geometric properties of these transformers. Moreover, extensions of classical inequalities in norm ideals have been explored in relation to numerical range estimates. These developments provide a more comprehensive explanation of the interplay between norm structures and spectral properties of integral transformers.

In this work, we advance the current understanding of integral operators by analyzing the numerical range and spectral characteristics of inner product-type integral transformers. Our main results precisely characterize the connection between the operator spectrum and its numerical range. The presented methodology offers new insights into how these essential operator-theoretic properties interact in the context of integral transformations defined through inner products.

2.2 The Numerical Range of an Operator in Hilbert Spaces

In the study of Hilbert spaces, the concept of the *numerical range* (also referred to as the *field of values*) serves as a powerful tool for understanding the behavior and characteristics of linear operators. It provides key information about an operator's nature, including its boundedness, positivity, and spectral properties. Notably, the numerical range is always a convex subset of the complex plane and contains the entire spectrum of the operator, making it instrumental in various analytical investigations. This notion finds extensive application in several branches of mathematics and physics, including functional analysis, operator theory, quantum mechanics, and the analysis of differential and integral equations. The idea of the numerical range was initially formulated by Toeplitz in 1918. Let π be a bounded linear operator on a Hilbert space K, that is, $\pi \in L(K)$. The numerical range of π is defined by: $W(\pi) = \{\langle \pi x, x\rangle \in \mathbb{C} : x \in K, \|x\| = 1\}$. This set $W(\pi)$ captures important operator characteristics and plays a foundational role in many theoretical developments related to linear operators.

This definition establishes a fundamental tool for studying operator properties, particularly in the spectral analysis of normal and self-adjoint operators. The numerical range has been extensively studied due to its deep connections with matrices, C^*-algebras, dilation theory, iteration methods, and Krein space operators (Mecheri, 2008). The numerical range can be interpreted using the quadratic form associated with the sesquilinear form φ_π defined as: $\varphi_\pi(\zeta, \eta) = \langle \pi\zeta, \eta\rangle, \quad \forall \zeta, \eta \in K$, and the corresponding quadratic form is given by $\widehat{\varphi_\pi}(\zeta) = \varphi_\pi(\zeta, \zeta) = \langle \pi\zeta, \zeta\rangle, \quad \zeta \in K$.

It is a well-established result, known as the Toeplitz-Hausdorff Theorem, that the numerical range of a bounded linear operator on a Hilbert space is always a convex set. This property has significant implications in operator theory. In particular, if π is a bounded linear operator on a Hilbert space, then its numerical range not only exhibits convexity but also includes the spectrum of π. Moreover, the numerical range serves as a useful tool for estimating both the norm of the operator and bounds related to its spectral properties. Additionally, for compact and normal operators, the numerical range provides critical information about eigenvalues and spectral distributions. The study of numerical ranges has been extended to tensor product operators, integral operators, and other functional transformations, providing a broad framework for applications in functional and harmonic analysis.

The numerical range (also referred to as the field of values) of an operator in a Hilbert space remains a central topic in functional analysis and operator theory, owing to its rich structure and significant theoretical implications. In recent years, researchers have made notable progress by extending classical notions and exploring broader frameworks. In (Feki et al., 2024) , the concept of the joint q-numerical range for tuples of bounded linear operators on Hilbert spaces was studied. This generalization of the traditional numerical range accommodates multiple operators simultaneously, offering fresh insights into their combined behavior. The authors also developed several inequalities associated with the q-numerical radius, and they extended their results to semi-Hilbert spaces. An important outcome of their work is the proof of convexity of the q-numerical range in these generalized settings. Wojcik and Puchala (2024) explored numerical ranges for antilinear operators in both Hilbert and Banach spaces, discovering that these ranges can lack convexity and exhibit behaviors distinct from those of linear operators. Parallel to this, Carvalho, Diogo, and Mendes (2024) investigated the numerical ranges of antilinear operators-operators that reverse the scalar multiplication through complex conjugation. These operators appear naturally in various physical and mathematical contexts, particularly in quantum mechanics. Their analysis, conducted in both Hilbert and Banach space settings, sheds light on the spectral characteristics of antilinear operators and their potential applications. Kolaczek& Muller (2024) investigated antilinear operators, showing that in Hilbert spaces of dimension greater than two, the numerical range is a convex disc. They provided the definition of antilinear operators and their numerical ranges, proving that the numerical radius satisfies the inequality $0 \leq \omega(\pi) \leq \|\pi\|$. Additionally, they explored the numerical ranges for antilinear operators in Banach spaces, defining the numerical range using specific elements from the dual space. Barasa, Chikamai, & Aywa (2024) conducted a study focusing on the geometric structure of numerical ranges for isometrically bounded operators. They identified specific conditions under which the numerical range forms a circular disc centered at the origin, thereby offering a complete characterization of such operators. This work contributes significantly to understanding the geometry of operator action through its numerical range. One of the outstanding open problems in matrix analysis related to numerical ranges is known as *Crouzeix's conjecture.* This conjecture states that for any complex square matrix π and any polynomial p, the inequality $\|p(\pi)\| \leq 2 \sup_{z \in \mathcal{W}(\pi)} |p(z)|$ holds, where $\mathcal{W}(\pi)$ denotes the numerical range of π. The conjecture remains a pivotal challenge, motivating much of the ongoing research in this domain. While the conjecture remains unproven in general, progress has been made in special cases, and it continues to stimulate extensive research in the field. These developments underscore the dynamic and evolving nature of research on numerical ranges in Hilbert spaces, highlighting their central role in advancing operator theory and related mathematical disciplines.

While previous studies have focused on the classical numerical range of bounded and linear self-adjoint operators, our work extends these ideas to the setting of integral transforms and inner product-type operators. Specifically, we analyzed the algebraic numerical range of operators of the form $\phi_{\{R,P\}} = \int_{\Psi} R_s \otimes P_s d\gamma(s)$, and established new convexity properties, algebraic structures, and numerical radius estimates within this framework. Unlike traditional approaches, our results incorporate integral transforms and tensor products, leading to novel insights into the numerical range properties of these operators. These developments have implications in spectral theory, functional analysis, and operator approximation methods. Further details of our results are provided in Chapter 4, where we rigorously develop the algebraic numerical range properties of integral-type operators and compare them to classical numerical range concepts.

The study of the numerical range of matrices dates back to the early 20th century and has attracted significant attention over the decades. A major breakthrough was made by Toeplitz in 1918, who showed that the numerical range of a matrix not only includes all its eigenvalues but also possesses a convex boundary. This foundational work was further extended in 1919 through the celebrated Toeplitz-Hausdorff theorem, which establishes that the numerical range $\mathcal{W}(\pi)$ of any bounded linear operator π on a Hilbert space K is a convex subset of the complex plane (see also Shapiro, 2003; Skoufranis, 2014). Hausdorff also contributed a notable result by proving that the intersection of a line with the numerical range is always connected or non-empty. These classical results paved the way for a deeper understanding of the structure and behavior of the numerical range. Various researchers, including Igunza (2005) and Sadia (2005), have explored and documented the fundamental properties of $\mathcal{W}(\pi)$, which include the following:

(i) $\mathcal{W}(\pi)$ is guaranteed to be non-empty.

(ii) Unitary invariance: For any unitary operator U on K, $\mathcal{W}(U^*\pi U) = \mathcal{W}(\pi)$.

(iii) $\mathcal{W}(\pi)$ is contained within a closed disc centered at the origin with radius $\|\pi\|$.

(iv) All eigenvalues of π are elements of $\mathcal{W}(\pi)$.

(v) The numerical range of the adjoint satisfies: $\mathcal{W}(\pi^*) = \{\overline{\zeta} : \zeta \in \mathcal{W}(\pi)\}$.

(vi) For the identity operator $I \in L(K)$, we have $\mathcal{W}(I) = \{1\}$.

(vii) Linearity: For scalars $\zeta, \eta \in \mathbb{K}$, $\mathcal{W}(\zeta\pi + \eta I) = \zeta\mathcal{W}(\pi) + \eta$.

(viii) Subadditivity: $\mathcal{W}(\pi_1 + \pi_2) \subseteq \mathcal{W}(\pi_1) + \mathcal{W}(\pi_2)$.

(ix) Convexity: $\mathcal{W}(\pi)$ is convex, as established by the Toeplitz-Hausdorff theorem.

(x) When K is finite-dimensional, $\mathcal{W}(\pi)$ is compact.

Remark 2.1. In finite-dimensional Hilbert spaces, the compactness of $\mathcal{W}(\pi)$ results from the compactness of the unit sphere in K and the continuity of the quadratic form associated with π.

Recent studies have expanded the classical numerical range to more generalized settings, including algebraic and integral operator frameworks. Unlike the classical numerical range, which primarily considers bounded operators on Hilbert spaces, our work explores numerical ranges in the context of integral transformations, particularly those involving tensor product operators and integral representations.

The numerical range is a fundamental concept in operator theory, particularly valued for its role in localizing the spectrum of operators. According to Tretter (2008), the convexity of the numerical range is a key feature that assists in confining the spectrum to a half-plane. Various studies have focused on the numerical range of distinct classes of operators. For example, Skoufranis (2014) analyzed the backward shift operator defined on a separable Hilbert space K with an orthonormal basis $\{\zeta_m\}_{m\geq 1}$. The operator π is defined via the action $\pi(\zeta_m) = \zeta_{m-1}$ for $m \geq 2$. It was shown that the numerical range $\mathcal{W}(\pi)$ in this case is the open unit disk centered at the origin. Interestingly, unlike many commonly encountered operators whose numerical ranges are closed, the numerical range of this backward shift operator is not closed. Skoufranis also considered diagonal operators on K, where an operator π is defined such that $\pi(\zeta_m) = b_m\zeta_m$ for a bounded sequence of scalars $\{b_m\}_{m\geq 1}$. In this context, the numerical range is described by the convex combination: $\mathcal{W}(\pi) = \left\{\sum_{m\geq 1} b_m c_m : c_m \geq 0, \sum_{m\geq 1} c_m = 1\right\}$. Additionally, it is known that if two operators π_1 and π_2 in K are approximately unitarily equivalent, then their numerical ranges have the same closure, that is, $\overline{\mathcal{W}(\pi_1)} = \overline{\mathcal{W}(\pi_2)}$. In finite-dimensional settings, Li (2005) formalized the numerical range for matrices. For a matrix $\pi_1 \in \mathbb{C}^{m\times m}$, the numerical range is defined by $\mathcal{W}(\pi_1) = \{\nu^*\pi_1\nu : \nu \in \mathbb{C}^m, \|\nu\| = 1\}$. As an example, consider the matrix

$$\delta = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},$$

whose numerical range is the closed unit disk $\mathcal{G} = \{\eta \in \mathbb{C} : |\eta| \leq 1\}$. More recent developments have broadened the investigation of numerical ranges. Muhammad, Azeez, & Taheri (2023) investigated S-numerical ranges for differential operators, developing numerical methods for approximating S-numerical ranges and enhancing spectral enclosures and stability analysis for differential equations. Additionally, further research confirms that for two approximately unitarily equivalent operators, their numerical ranges remain closely related, which has significant implications in perturbation theory and spectral stability. These developments highlight the evolving nature of numerical range research and its critical role in operator theory and applied mathematics.

Our work extends these classical results by investigating the algebraic numerical range properties of integral operators formed via tensor products. Specifically, while previous studies have primarily focused on individual operators or matrices, our approach involves integral transformations incorporating tensor products of operators. This provides a broader framework in which algebraic properties such as convexity, unitary invariance, and numerical radius estimates are established within an integral setting. Unlike Skoufranis (2014) and Li (2005), who mainly considered specific operators and finite-dimensional cases, our study generalizes the concept of the numerical range to a more abstract setting involving integral operators. The results we establish provide new insights into the algebraic numerical range, its convexity, and conditions under which it remains real or self-adjoint. These findings extend the theoretical understanding of numerical ranges beyond classical cases and offer potential applications in functional analysis and operator theory.

Various properties of the numerical range for matrices and operators have been extensively studied in the literature. In this section, we review relevant results from existing works, particularly those by Li (2005) and Gustafson and Rao (1997), and contrast them with our study. Li (2005), established fundamental properties of the numerical range for matrices as follows:

a. $\mathcal{W}(\eta\delta + \gamma I) = \mathcal{W}(\eta\delta) + \gamma, \eta, \gamma \in \mathbb{K}$.

b. $\mathcal{W}(S^*\delta S) = \mathcal{W}(\delta)$ for $\mathcal{S} \in \mathbb{C}^{m\times m}$, a unitary matrix.

c. If $\int \in \mathbb{C}^{m\times m}$ satisfies $\int^*\int$, then $\mathcal{W}(\int^*\delta\int) \subseteq \mathcal{W}(\delta)$.

d. The numerical range $\mathcal{W}(\delta)$ is compact.

e. $\mathcal{W}(\delta)$ is convex in $\mathbb{C}^{m\times m}$.

These properties provide a foundational understanding of the structure of the numerical range and its behavior under linear transformations. Gustafson and Rao (1997) presented illustrative examples concerning the numerical range:

Example 2.2. Consider the operator π in $\mathbb{C}^2$ given by

$$\pi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $\zeta = (\mu, \eta)$ with $\|\zeta\|^2 = |\mu|^2 + |\eta|^2 = 1$, it follows that $\mathcal{W}(\pi)$ is a full disk $\{\nu : |\nu| \leq \frac{1}{2}\}$.

Example 2.3. For a unilateral shift operator π in the l_2 space, it is shown that $\mathcal{W}(\pi)$ is contained in the unit disk $\{\zeta : |\zeta| < 1\}$.

Building upon the earlier contributions of Gustafson & Rao (1997), who characterized the numerical range of operators in two-dimensional settings as ellipses with their eigenvalues serving as foci, we delve further into the structural properties of such ranges. Their analysis revealed that for an extreme point $\zeta \in \mathcal{W}(\pi)$, the associated set $M_\zeta = \{\nu \in \mathcal{H} : \langle \pi\nu, \nu\rangle = \zeta\|\nu\|^2\}$ forms a linear subspace. In our study, we expand on these foundational insights by examining the algebraic numerical range characteristics specific to integral operators of the inner product type. Specifically, our work generalizes the classical numerical range by incorporating integral representations and tensor products of operators. Additionally, our results provide a broader framework for understanding the behavior of numerical ranges under integral transformations, which is not explicitly addressed in previous studies.

The numerical range of a matrix or linear operator plays a crucial role in understanding its structural and spectral characteristics. The relationship between the algebraic formulation of a matrix δ and the geometric nature of its numerical range $\mathcal{W}(\delta)$ has been extensively studied (see, for example, Li, 2005). Several fundamental properties of the numerical range are summarized below:

a. $\delta = \eta I$ if and only if $\mathcal{W}(\delta) = \{\eta\}$.

b. δ is self-adjoint, i.e., $\delta = \delta^*$, if and only if $\mathcal{W}(\delta) \subseteq \mathbb{R}$.

c. $\delta = \delta^*$ is positive definite if and only if $\mathcal{W}(\delta) \subseteq (0, \infty)$.

d. $\delta = \delta^*$ is positive semidefinite if and only if $\mathcal{W}(\delta) \subseteq [0, \infty)$.

The mapping $\mathcal{W} : \mathbb{C}^{m\times m} \to \mathcal{P}(\mathbb{C})$, which associates each matrix with its numerical range, satisfies the following essential properties:

1. For any matrix $\delta \in \mathbb{C}^{m\times m}$, the set $\mathcal{W}(\delta)$ is both compact and convex.
2. For any scalar $\zeta \in \mathbb{C}$ and any matrix $\mathcal{A}$, the numerical range satisfies the affine transformation property: $\mathcal{W}(\zeta\mathcal{A} + \nu I) = \zeta\mathcal{W}(\mathcal{A}) + \nu$.
3. The condition $\mathcal{W}(\delta) \subseteq \{\kappa \in \mathbb{C} : \kappa + \overline{\kappa} \geq 0\}$ holds if and only if δ is self-adjoint and positive semidefinite.

It is further known that $\mathcal{W}(\delta) = \mathcal{W}(\delta^*)$ for all $\delta \in \mathbb{C}^{m\times m}$. The function $\delta \mapsto \mathcal{W}(\delta)$ is continuous with respect to the matrix norm topology. Additionally, the numerical range always contains the spectrum of δ. When δ is a normal matrix, $\mathcal{W}(\delta)$ coincides with the convex hull of its eigenvalues. Interestingly, the numerical range of the product of two matrices does not, in general, adhere to a simple multiplicative relationship. Consider the following example: Given,

$$\delta_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then their product will be: $\delta_1\delta_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We observe that: $\mathcal{W}(\delta_1) = \mathcal{W}(\delta_2) = [-1, 1]$, but $\mathcal{W}(\delta_1\delta_2) = i[-1, 1]$.

This illustrates that $\mathcal{W}(\delta_1\delta_2)$ is not necessarily contained in the product set $\mathcal{W}(\delta_1)\mathcal{W}(\delta_2)$. Recent research has provided deeper insights into the properties and applications of numerical ranges in various contexts. The study by Li and Wu (2022), explores the behavior of numerical ranges under matrix multiplication. The authors examine conditions under which the numerical range of a product $\delta_1\delta_2$ relates to the numerical ranges of δ_1 and δ_2, highlighting cases where the expected inclusion fails. Meanwhile, Taki and Moktafi (2025) investigated the Maximal Numerical Range of Tensor Product of Two Operators, providing a detailed analysis of how tensor products influence numerical ranges, a topic highly relevant to quantum mechanics and operator theory. In another recent contribution by Muller and Tretter (2024), characterized conditions under which numerical ranges exhibit hyperbolical structures within Krein spaces, extending classical results in indefinite inner product spaces. Collectively, these studies expand the understanding of numerical ranges, reinforcing their significance in both mathematics and physics.

Compared to our approach, the existing literature primarily focuses on classical numerical ranges of operators, while our results extend these concepts to integral operators involving tensor products. Specifically, our results characterize the convex hull properties of integral transforms in a broader functional setting, as well as the numerical radius and adjoint properties of integral-type transformations. Our work generalizes existing results by incorporating operator-valued functions and measures, which provide a more extensive framework for numerical range analysis in infinite-dimensional spaces.

The concept of the numerical range for non-square matrices has been investigated by Aretaki and Maroulas in 2009. Their primary motivation was to explore how the numerical range for a rectangular matrix can be defined concerning the inner product and to

establish its fundamental properties. If $\Delta \in \mathbb{C}^{k\times m}$ is a complex matrix with $k \neq m$, define a sesquilinear form $h : \mathbb{C}^k \times \mathbb{C}^m \to \mathbb{C}$ by $h(a, b) = b^*\Delta a$, for vectors $a \in \mathbb{C}^k$ and $b \in \mathbb{C}^m$. The numerical range of the matrix Δ is the set $\mathcal{W}(\Delta) = \{b^*\Delta a : \|a\|_2 = \|b\|_2 = 1\}$.
The numerical range $\mathcal{W}(\Delta)$ of a non-square matrix Δ possesses the following properties:

(a) Scaling: For any scalar $\zeta \in \mathbb{C}$, $\mathcal{W}(\zeta\Delta) = \zeta\mathcal{W}(\Delta)$.

(b) Norm Bound: The numerical range is bounded in the complex plane by the spectral norm of Δ: $\mathcal{W}(\Delta) \subseteq \{z \in \mathbb{C} : |z| \leq \|\Delta\|_2\}$.

(c) Adjoint Symmetry:$\mathcal{W}(\Delta^*) = \mathcal{W}(\Delta)$.

(d) Submatrix Inclusion: If $\hat{\Delta}$ is any submatrix of Δ, then, $\mathcal{W}(\hat{\Delta}) \subseteq \mathcal{W}(\Delta)$.

(e) Block Diagonal Structure: For matrices $\Delta_1 \in \mathbb{C}^{m\times n}$ and $\Delta_2 \in \mathbb{C}^{n\times m}$, we have $\mathcal{W}\left(\begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}\right) = \max\{\mathcal{W}(\Delta_1), \mathcal{W}(\Delta_2)\}$.

(f) Unitary Invariance: If $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ are unitary matrices, then, $\mathcal{W}(A^*\Delta B) = \mathcal{W}(\Delta)$.

An extension of the numerical range, known as the *Q-numerical range*, incorporates a generalized inner product through a positive definite Hermitian matrix Q. It is defined as: $\mathcal{W}_Q(\Delta) = \big\{b^*\Delta a : a \in \mathbb{C}^k, b \in \mathbb{C}^m, a^*Qa = b^*Qb = 1\big\}$.

This generalization provides a structured framework for analyzing the numerical range under varying inner product conditions. In another significant development, Bebiano, Lemos, and Soares (2024) investigated matrices whose numerical ranges exhibit hyperbolical shapes in Krein spaces. They established necessary and sufficient conditions for low-dimensional tridiagonal matrices to possess this property, thereby enhancing the understanding of numerical ranges within indefinite inner product spaces. Additionally, Su (2023) extended the concept of stability from square to non-square matrices, focusing on their applications in distributed control systems. This research offers valuable insights into the stability analysis of non-square processes, which are essential in engineering and applied mathematics.
While Aretaki and Maroulas focused on the numerical range of non-square matrices in

the finite-dimensional setting, our work generalizes these ideas in the context of integral operators and tensor products of operators. Unlike Aretaki and Maroulas, we establish results for the algebraic numerical range involving integrals over function spaces rather than finite-dimensional matrices.

Shapiro (2003) examined the numerical range of operators that are unitarily invariant and established that this range coincides with the convex hull of their eigenvalues. Let π be a bounded linear operator on a Hilbert space K. Then there exists an orthonormal basis $\{\zeta_m\}_{m\geq 1}$ for K and a corresponding sequence of complex scalars $\{a_m\}$ such that $\pi\zeta_m = a_m\zeta_m$ for each m. The numerical range of π is defined as $\mathcal{W}(\pi) = \{\langle \pi b, b\rangle : b \in K, \|b\| = 1\}$.

With respect to the orthonormal basis $\{\zeta_m\}$, any unit vector $b \in K$ can be expressed as a linear combination $b = \sum_{m=1}^{\infty}\langle b, \zeta_m\rangle\zeta_m$, and hence the numerical range becomes

$$\mathcal{W}(\pi) = \left\{\sum_{m=1}^{\infty} a_m|\langle b, \zeta_m\rangle|^2 : \|b\| = 1\right\}.$$

Introducing $\zeta_m := |\langle b, \zeta_m\rangle|^2$, we note that $0 \leq \zeta_m \leq 1$ and $\sum_{m=1}^{\infty}\zeta_m = 1$, leading to the representation

$$\mathcal{W}(\pi) = \left\{\sum_{m=1}^{\infty} a_m\zeta_m : 0 \leq \zeta_m \leq 1, \sum_{m=1}^{\infty}\zeta_m = 1\right\}.$$

This shows that the numerical range $\mathcal{W}(\pi)$ is the infinite convex hull (denoted by conv_∞) of the eigenvalue set $\Lambda = \{a_m\}$: $\mathcal{W}(\pi) = \mathrm{conv}_\infty(\Lambda)$.

In the case of 2×2 complex matrices, the numerical range takes on a geometrically interpretable form, depending on the spectral and structural properties of the matrix:

1. If the matrix is a scalar multiple of the identity, then the numerical range is a single point (singleton).

2. If the matrix is normal with two distinct eigenvalues, the numerical range is the closed line segment joining the eigenvalues.

3. If the matrix is not normal and has two distinct eigenvalues, the numerical range is an elliptical disc whose foci are located at the eigenvalues.

This result is an extension of the classical Ellipse Theorem, which states that for a linear transformation π on $\mathbb{C}^2$, the numerical range $W(\pi)$ forms a (possibly degenerate) elliptical disc.

Gau and Wu (2006), investigated the number of line segments on the boundary of the numerical range of companion matrices. They established that for an $n \times n$ complex matrix, the number of line segments is at most n. Fiedler 1995, studied the geometric properties of the numerical range of matrices and derived the point equation of the associated algebraic curve. He demonstrated that, for any matrix π, the numerical range $\mathcal{W}(\pi)$ is the convex hull of the algebraic curvature $C(\pi)$, which is obtained through the dual line equation. Additionally, Fiedler obtained a formula for the curvature of $\mathcal{W}(\pi)$ at its boundary points. He also examined the relationship between the curvature of the boundary of $N(\pi)$ for a nonnegative matrix π and the Perron root of the symmetric part of π, as well as the curvature of the Levinger curve of π. Furthermore, he established that the numerical range $\mathcal{W}(\pi)$ possesses a corner point where the radius of curvature is zero and a flat point when π has multiple maximum eigenvalues.

Our work extends these classical results by considering the numerical range of integral operators . Our approach generalizes the numerical range to an algebraic setting, incorporating integral representations and tensor products, which were not addressed in Shapiro's framework. In particular, we establish results on the convexity, self-adjointness, and numerical radius properties of $\Phi_{\{R,P\}}$, highlighting distinctions from the finite-dimensional case analyzed by Shapiro. These findings contribute to the broader theory of numerical ranges in infinite-dimensional operator settings.

Shapiro also explored the quadratic form $\xi_\pi(a) = \langle \pi a, a\rangle$ for $a \in \mathbb{C}$, which maps the unit sphere in the Hilbert space $\mathcal{H}$ into the complex plane, thus generating the numerical range. He further established a connection between the numerical range and the operator norm, particularly for self-adjoint operators, by utilizing the polarization identity.

Remark 2.4. The foci of the numerical range $\mathcal{W}(\pi)$ of an operator $\pi \in L(K)$ coincide with the eigenvalues of π.

As demonstrated in Sadia (2005), if π is a 2×2 matrix with distinct eigenvalues a_1 and a_2, and corresponding unit eigenvectors g_1 and g_2, and if $\eta = |\langle g_1, g_2\rangle|$, then:

(a) The numerical range $\mathcal{W}(\pi)$ is generally an elliptical disk (which may be degenerate), with foci at a_1 and a_2.

(b) The eccentricity of $\mathcal{W}(\pi)$ is given by $\frac{1}{\eta}$.

(c) The major axis of $\mathcal{W}(\pi)$ has a length of $\frac{2|a_1-a_2|}{\sqrt{1-\eta^2}}$.

Remark 2.5. The size of the numerical range can be increased by applying similarity transformations to decrease the angle between the eigenvectors for given eigenvalues.

The classical results in the literature focus on the geometric properties of the numerical range of finite-dimensional operators, particularly 2×2 matrices. In contrast, our work extends these concepts to integral operator settings. Thus, while prior research has established foundational geometric and algebraic properties of the numerical range, our work provides novel insights by extending these ideas to integral operators, demonstrating structural properties that are not immediately evident in the finite-dimensional setting. In 1961, Lumer introduced the concept of the numerical range and extended it to semi-inner product spaces. According to (Lumer, 1961), suppose that$\mathcal{X}$ be a semi-inner product space, and let π be a linear operator on $\mathcal{X}$. The numerical range of π is defined by $\mathcal{W}(\pi) = \{\langle \pi\zeta, \zeta\rangle : \langle \zeta, \zeta\rangle = 1 \text{ for all } \zeta \in \mathcal{X}\}$.
Lumer also established some key properties of the numerical range in this context:

(a) The modulus of the numerical range satisfies $|\mathcal{W}(\pi)| \leq \|\pi\|$.

(b) The numerical range of a linear transformation π under a linear combination with a scalar is given by $\mathcal{W}(\kappa\pi + \eta I) = \kappa\mathcal{W}(\pi) + \eta$ for any $\kappa, \eta \in \mathbb{K}$.

(c) The numerical range of the sum of two operators is contained within the sum of their individual numerical ranges, i.e., $\mathcal{W}(\pi + \pi') \subset \mathcal{W}(\pi) + \mathcal{W}(\pi')$.

(d) The following properties hold for the numerical range: $|\mathcal{W}(\kappa\pi)| = |\kappa||\mathcal{W}(\pi)|$ and $|\mathcal{W}(\pi + \pi')| \leq \mathcal{W}(\pi) + \mathcal{W}(\pi')|$, meaning that the absolute value of the numerical range defines a semi-norm.

In the classical case, the Toeplitz-Hausdorff theorem establishes that the numerical range of an operator is convex. However, this convexity may not hold in semi-inner product spaces. The concept of the numerical range can also be extended to pairs of linear operators π_1 and π_2 acting between Hilbert spaces:

Definition 2.6. Let K_1 and K_2 be Hilbert spaces, and let $\pi_1, \pi_2 \in \mathcal{B}(K_1, K_2)$ be bounded linear operators. The numerical range of π_1 and π_2 is given by

$$\mathcal{W}(\pi_1, \pi_2) = \{\langle \pi_1\zeta, \pi_2\zeta\rangle : \|\pi_2\zeta\| = 1, \zeta \in \mathcal{D}(\pi_1) \cap \mathcal{D}(\pi_2)\},$$

where $\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$ represent the domains of the operators π_1 and π_2, respectively.

In the special case where $K_1 = K_2$ and π_2 is the identity operator, this definition reduces to the classical notion of the numerical range: $(\pi_1, I) = \{\langle \pi_1\zeta, \zeta\rangle : \|\zeta\| = 1\}$.
Furthermore, if $\pi_2^{-1} \in \mathcal{B}(K_1)$, then $\mathcal{W}(\pi_1, \pi_2^{-1}) = \mathcal{W}(\pi_1, \pi_2)$.
The concept of the numerical range has garnered significant attention in functional analysis, especially concerning operators on Hilbert spaces. In an effort to broaden the classical definition, numerous generalizations have been proposed to apply the concept to nonlinear mappings and integral operators. This review examines key contributions in this area and compares them with our findings. Amelin (1973) established that the numerical range $\mathcal{W}(\pi_1, \pi_2)$ of certain operators forms a convex set, as demonstrated in Lemma 1.2. Our research further expands these results by proving that the algebraic numerical range $\mathcal{V}(\Phi_{\{R,P\}})$ remains convex even when considering integral operators. This generalization applies to a broader class of operators, extending beyond the limitations of earlier studies. Zarantonello (1967) defined the numerical range of a nonlinear mapping π on a Hilbert space as: $\mathcal{W}(\pi) = \left\{ \frac{\langle \pi\zeta_1 - \pi\zeta_2, \zeta_1 - \zeta_2\rangle}{|\zeta_1 - \zeta_2|^2} : \zeta_1 \neq \zeta_2, \zeta_1, \zeta_2 \in \mathcal{D}(\pi) \right\}$.
Later, Nanda (2003) extended this definition to the case of two nonlinear operators in a semi-inner product space. Our work extends these ideas further by including integral operators and demonstrating that the algebraic numerical range retains its properties in this extended context. Skoufranis (2014) pointed out that the numerical range can reveal more about an operator than its spectrum. Specifically, for an operator $\pi \in L(K)$ and a scalar $\kappa \in \mathbb{C}$, the numerical range satisfies $\mathcal{W}(\pi) = \{\kappa\}$ if and only if $\pi = \kappa I_K$. Building on this, our findings establish new relationships between the algebraic numerical the spectrum of tensor product operators, offering fresh insights into spectral analysis. Bonsall (1973) provided an overview of several generalizations of the classical numerical

range. Our research adds to this body of work by examining the behavior of algebraic numerical ranges in a wider context, which includes integral operators and their application to spectral theory. Halmos (1964) defined the k-numerical range as

$$\mathcal{W}_k(\pi) = \left\{ \sum i = 1^k \langle \pi\zeta_i, \zeta_i \rangle : \zeta_1, ..., \zeta_k \text{ are orthonormal vectors in } K \right\},$$

and Westwick (1975) introduced the c-numerical range. While these generalizations apply to finite-dimensional cases, our study extends to integral operators, demonstrating that similar properties hold in infinite-dimensional spaces. Our research makes several distinct contributions:

1. We extend the numerical range to integral operators of the form $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$. Previous studies focused on elementary operators and bounded linear and nonlinear operators but did not address integral transformations.

2. We establish convexity results for the algebraic numerical range $\mathcal{V}(\Phi_{\{R,P\}})$, providing a new convexity theorem applicable to integral operators.

3. We prove that $\Phi_{\{R,P\}}$ is self-adjoint if and only if its numerical range is real, extending classical self-adjointness conditions to integral operator settings.

4. We derive properties of the numerical radius $\omega(\Phi_{\{R,P\}})$, including norm inequalities, demonstrating that these classical properties hold in our integral operator framework.

5. We confirm that unitary invariance holds for the algebraic numerical range, which was previously only established for finite-dimensional cases.

In the squel, we present a review of various numerical ranges studied in the literature, including the C-numerical range, q-numerical range, δ-numerical range, and maximal numerical range. These concepts have been investigated by several researchers, including Musundi (2008), Mecheri (2008), and Stampfli (1970). The discussions and results in these works provide a foundation for the study of the numerical range. Musundi (2008) introduced the algebraic C-numerical range, which is defined as

$$\mathcal{V}_C(S) = \overline{\{g(l) : g \in L(K)^*, g(I) = \|g\| = 1, g(S, S^*) = \sum_{i=1}^{r} C_i \|Sy_i\|^2\}}.$$

He established that the C-numerical range $\mathcal{W}_C(\mathcal{S})$ coincides with $\mathcal{V}_C(S)$. For $q \in \mathbb{C}$ with $|q| \leq 1$, Mecheri (2008) defined the q-numerical range as

$$\mathcal{W}_q(\pi) = \{\langle \pi\zeta, \eta \rangle : \zeta, \eta \in K, \langle \zeta, \zeta \rangle = \langle \eta, \eta \rangle = 1, \langle \zeta, \eta \rangle = q\}.$$

Musundi (2008) established several properties of $\mathcal{W}_q(K)$, including convexity, non-emptiness, unitary invariance, and its containment in a closed disc of radius $\|\pi\|$ centered at the origin. Additionally, it was shown that $\mathcal{W}_q(I) = \{q\}$ for the identity operator I, and that $\mathcal{W}_q$ contains all eigenvalues of π.

The concepts of the q-numerical range and rank-k numerical range have been explored by several researchers, including Chien et al. (2020), Tsing (1984), and Zahraei and Aghamollaei (2015). We highlight the distinctions between their results and our contributions, emphasizing the novelty of our approach. Chien et al. (2020) investigated the inverse of the q-numerical range for 2×2 matrices. The q-numerical range for an $n \times n$ matrix π is defined as follows:

Definition 2.7. Let π be an $n \times n$ matrix. For any complex value q with $|q| \leq 1$, the q-numerical range of π is given by $\mathcal{W}_q(\pi) = \{\mu^* \pi \zeta : \zeta, \mu \in \mathbb{C}^n, \zeta^* \zeta = \mu^* \mu = 1, \mu^* \zeta = q\}$.

Tsing (1984) established the convexity of the q-numerical range using concavity arguments. Chien et al. (2020) further explored the inverse q-numerical range for 2×2 matrices. Given a matrix π with eigenvalues λ and α such that $\lambda \geq \alpha$, they expressed π in terms of parameters $u = (\lambda + \alpha)/2$ and $v = (\lambda - \alpha)/2$ and derived the following characterization of the numerical range:

$$\mathcal{W}_q(\pi) = \left\{ (s, t) \in \mathbb{R}^2 : \frac{s^2}{(u + \sqrt{1 - q^2 v})^2} + \frac{t^2}{(v + \sqrt{1 - q^2 u})^2} \leq 1 \right\}.$$

They showed that any point in $\mathcal{W}_q(\pi)$ can be represented using unit vectors on a curve and that this quartic curve is contained in an elliptical disk equivalent to the classical numerical range of π. Additionally, they applied elimination methods to obtain necessary and sufficient conditions for the quartic curve to have a real singular point.

Stampfli (1970) introduced the maximal numerical range, defined as $\mathcal{W}_o(\pi) = \{\lambda : \langle \pi\zeta_n, \zeta_n\rangle \to \lambda, \|\zeta_n\| = 1, \|\pi\zeta_n\| \to \|\pi\|\}$.

He also proved the convexity of this set. Later, Kingangi (2018) applied Stampfli's maximal numerical range to determine the norm of an elementary operator of length two in a C^*-algebra $B(H)$. The δ-numerical range is defined as $\mathcal{W}_\delta(\pi) = C_o\{\langle \pi\zeta, \zeta\rangle : \pi \in K, \|\zeta\| = 1, \|\pi\zeta\| \geq \delta\}$.

In a notable contribution, Agure (1996) demonstrated the convexity of the δ-numerical range, thereby enriching the theoretical landscape of numerical range analysis. The concept of the numerical range in Banach spaces was formally introduced by Lumer in 1961. Let A be a Banach space and π a bounded linear operator acting on A. The numerical range of π, denoted by $\mathcal{V}(\pi)$, is defined as $\mathcal{V}(\pi) = \{\zeta^*(\pi\zeta) : \zeta \in S_A,\ \zeta^* \in S_{A^*},\ \zeta^*(\zeta) = 1\}$, where S_A and S_{A^*} denote the unit spheres in A and its dual space A^*, respectively. According to the duality representation theorem in Hilbert spaces, this definition is consistent with the classical numerical range used in Hilbert space operator theory. Building upon earlier work, Priscah, Musundi, and Ombaka (2018) explored further properties of the numerical range for bounded linear operators in Banach spaces. Their results provided a deeper understanding of the structure and behavior of such operators through the lens of the numerical range. These results provide foundational insights into the numerical range structure in Banach spaces. Our work extends these studies by considering integral operators of the form $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$ and analyzing their numerical range properties.

The concept of the numerical range of operators has attracted significant interest across various mathematical frameworks, including Banach spaces, Hilbert spaces, and semi-inner product spaces. We now provide a summary of notable advancements in this area and describe how our results build upon and extend the existing literature. In the setting of Banach spaces, Verma (1994) introduced a formulation of the numerical range for nonlinear operator π by utilizing a generalized Lipschitz norm. The numerical range, denoted by $\mathcal{V}_L(\pi)$, is defined as:

$$\mathcal{V}_L(\pi) = \left\{ \frac{\langle \pi\zeta, \zeta\rangle + \langle \pi\zeta - \pi\eta, \zeta - \eta\rangle}{\|\zeta\|^2 + \|\zeta - \eta\|^2} : \zeta, \eta \in \mathcal{D}(\pi), \zeta \neq \eta \right\}.$$

This construction led to several foundational properties, such as homogeneity, subadditivity, and invariance under certain translations and scalar shifts. Further developments were made by Sahu 2012, who examined the numerical ranges of pairs of linear operators within semi-inner product spaces, following the framework laid out by Amelin 1973. Their results established key characteristics including:

- Subadditivity: $\mathcal{W}(\pi_1 + \pi_2, \mathcal{A}) \subseteq \mathcal{W}(\pi_1, \mathcal{A}) + \mathcal{W}(\pi_2, \mathcal{A})$,
- Homogeneity: $\mathcal{W}(\kappa\pi, \mathcal{A}) = \kappa\mathcal{W}(\pi, \mathcal{A})$,
- Effect of scalar multiplication: $\mathcal{W}(\pi, \eta\mathcal{A}) = \overline{\eta}\mathcal{W}(\pi, \mathcal{A})$,
- Translation invariance: $\mathcal{W}(\pi - \zeta, \mathcal{A}) = \mathcal{W}(\pi, \mathcal{A}) - |\zeta|$.

Unlike Verma's and Sahu et al.'s studies, our work incorporates:

- Integration over a measurable set Ψ, leading to a broader class of numerical ranges.
- Tensor product structures, linking classical numerical ranges with integral operators.
- Properties specific to inner-product-type integral transforms, which do not appear in prior works.
- Self-adjointness conditions that characterize real numerical ranges in our setting.

These developments offer a fresh outlook on the numerical range by incorporating classical analytical concepts with operator-valued functions and integral transform techniques.
The notion of the coupled numerical range was introduced by Sahu, Nahak, and Nanda (2012), where they considered the numerical behavior of an operator π in relation to another operator $\mathcal{A}$. This was formally defined as

$$\mathcal{W}_{\mathcal{A}}(\pi) = \left\{ \frac{\langle \mathcal{A}\pi\zeta, \zeta \rangle}{\langle \mathcal{A}\zeta, \zeta \rangle} : \|\zeta\| = 1, \ \langle \mathcal{A}\zeta, \zeta \rangle \neq 0 \right\}.$$

Several foundational properties of this coupled numerical range were established, such as:

i. $\mathcal{W}_{\mathcal{A}}(\pi_1 + \pi_2) \subseteq \mathcal{W}_{\mathcal{A}}(\pi_1) + \mathcal{W}_{\mathcal{A}}(\pi_2)$,

ii. $\mathcal{W}_{\mathcal{A}}(\kappa\pi) = \kappa\mathcal{W}_{\mathcal{A}}(\pi)$ for any scalar $\kappa \in \mathbb{K}$,

iii. $\mathcal{W}_{\mu\mathcal{A}}(\pi) = \overline{\mu}\,\mathcal{W}_{\mathcal{A}}(\pi)$ for $\mu \in \mathbb{K}$,

iv. $\mathcal{W}_{\alpha\mathcal{A}}(\pi) = \mathcal{W}_{\mathcal{A}}(\pi)$ for non-zero real α.

Additionally, the same authors generalized the concept of the numerical range to a setting involving two nonlinear operators π_1 and π_2 acting on a Banach space. They proposed the following form:

$$\mathcal{V}_L(\pi_1, \pi_2) = \left\{ \frac{[\pi_1\zeta, \pi_2\zeta] + [\pi_1\zeta - \pi_1\eta, \pi_2\zeta - \pi_2\eta]}{\|\pi_2\zeta\|^2 + \|\pi_2\zeta - \pi_2\eta\|^2} : \zeta, \eta \in \mathcal{D}(\pi_1) \cup \mathcal{D}(\pi_2),\ \zeta \neq \eta \right\},$$

where $\mathcal{D}(\pi_1)$ and $\mathcal{D}(\pi_2)$ represent the domains of the respective operators. The structure of this set enjoys the following key properties:

i. $\mathcal{V}_L(\lambda\pi_1, \pi_2) = \lambda\mathcal{V}_L(\pi_1, \pi_2)$ for $\lambda \in \mathbb{K}$,

ii. $\mathcal{V}_L(\pi_1, \mu\pi_2) = \frac{1}{\mu}\mathcal{V}_L(\pi_1, \pi_2)$ for $\mu \in \mathbb{K} \setminus \{0\}$,

iii. $\mathcal{V}_L(\pi_1 + \pi_2, \mathcal{A}) \subseteq \mathcal{V}_L(\pi_1, \mathcal{A}) + \mathcal{V}_L(\pi_2, \mathcal{A})$,

iv. $\mathcal{V}_L(\pi_1 - \lambda\pi_2, \pi_2) = \mathcal{V}_L(\pi_1, \pi_2) - \lambda$.

Building on these foundational ideas, our current research examines the *integral numerical range* corresponding to integral operators constructed in the form

$$\Phi_{R,P} = \int_{\Psi} R_s \otimes P_s \, d\gamma(s),$$

where R_s and P_s are operator-valued functions defined on an index set Ψ, and γ is a suitable measure. The study explores newly identified convexity traits, algebraic properties, and conditions for self-adjointness within this framework. Furthermore, connections with traditional numerical ranges are analyzed to illustrate both generalization and refinement of classical results.

Extensions of the Numerical Range in Banach Spaces

The classical notion of the numerical range, typically associated with Hilbert spaces, has seen several meaningful extensions within the setting of Banach spaces. These generalized versions aim to explore and characterize the behavior of bounded linear operators from different viewpoints. Among these, the *intrinsic* and *spatial* numerical ranges introduced by Martin (2015) serve as central tools for operator analysis.

Intrinsic Numerical Range

Let A be a Banach space, with S_A denoting its unit sphere and A^* its dual space. For a bounded linear operator $\pi \in L(A)$, the intrinsic numerical range (also referred to as the algebraic numerical range) is defined by

$$\mathcal{V}(\pi) = \left\{ \Phi(\pi) : \Phi \in L(A)^*,\ \|\Phi\| = 1,\ \Phi(\mathrm{Id}) = 1 \right\},$$

where Id represents the identity operator on A.

Spatial Numerical Range

The spatial numerical range of an operator $\pi \in L(A)$ is given by

$$\mathcal{W}(\pi) = \{x^*(\pi x) : x \in S_A,\ x^* \in S_{A^*},\ x^*(x) = 1\}$$

.

Remark 2.8. When the underlying space A is a Hilbert space, the intrinsic and spatial numerical ranges coincide.

In the broader context of Banach spaces, however, the relationship between these two ranges changes. Specifically, the intrinsic numerical range coincides with the closed convex hull of the spatial numerical range: $\mathcal{V}(\pi) = \mathrm{co}\,\overline{\mathcal{W}(\pi)}$.

Function-Valued Extensions

The concept of numerical ranges has also been generalized to apply to Banach space-valued functions. Let Y be a Banach space, and let Γ be a non-empty index set. Suppose $g \in L_\infty(\Gamma, Y)$ is a function with $\|g\| = 1$. For a given $f \in L_\infty(\Gamma, Y)$, the intrinsic numerical range of f relative to g is defined as $V_g(f) = \{\Phi(f) : \Phi \in L_\infty(\Gamma, Y)^*,\ \|\Phi\| = 1,\ \Phi(g) = 1\}$.

Spatial Numerical Range of a Function

Similarly, the spatial numerical range of the function f relative to g is defined by $W_g(f) = \{y^*(f(t)) : y^* \in S_{Y^*},\ t \in \Gamma,\ y^*(g(t)) = 1\}$.

These function-based numerical ranges offer an enriched framework for analyzing vector-valued mappings and extend classical operator theory to a more flexible and abstract setting.

Approximated Spatial Numerical Range of a Function

Let Y be a Banach space and Γ a non-empty index set. Suppose $g \in L_\infty(\Gamma, Y)$ is fixed with $\|g\| = 1$. Then for each $f \in l_\infty(\Gamma, Y)$, we define the *approximated spatial numerical range* of f with respect to g as:

$$\widetilde{W_g(f)} = \bigcap_{\epsilon>0} \overline{\{y^*(f(t)) : y^* \in S_Y^*,\ t \in \Gamma,\ \operatorname{Re} y^*(g(t)) > 1 - \epsilon\}}.$$

Remark 2.9. The set $\widetilde{W_g(f)}$ is non-empty if and only if the image of g, that is $g(\Gamma)$, intersects the unit sphere S_Y of the Banach space Y.

Our work extends the theoretical framework initiated by Martin (2015) on numerical ranges in Banach spaces. While Martin primarily focused on pointwise numerical range constructions, our approach incorporates integral operator formulations and investigates their convexity and algebraic structures.

Essential Numerical Range

The concept of the essential numerical range has been investigated in various settings, including the work of Barraa (2014). Let $\mathcal{A}$ be an infinite-dimensional Banach space, and let $\pi \in L(\mathcal{A})$ be a bounded linear operator. The *essential numerical range* of π, denoted $\mathcal{V}_e(\pi)$, is defined by: $\mathcal{V}_e(\pi) = \bigcap_{k\in\mathcal{K}} \overline{\mathcal{V}(\pi + k)}$,

where $\mathcal{K}$ denotes the set of all compact operators on $\mathcal{A}$, and $\mathcal{V}(\cdot)$ represents the usual numerical range. This set has several important properties:

(i) $\mathcal{V}_e(\pi)$ is non-empty, compact, and convex.

(ii) The essential spectrum $\sigma_e(\pi)$ is always contained in $\mathcal{V}_e(\pi)$.

(iii) $\mathcal{V}_e(\pi) = \{0\}$ if and only if π is compact, i.e., $\pi \in \mathcal{K}(\mathcal{A})$.

(iv) $\mathcal{V}_e(\pi)$ can also be expressed as: $\mathcal{V}_e(\pi) = \bigcap_{K\in\mathcal{K}} (\mathcal{V}(\pi) + K)$, where K ranges over compact perturbations.

(v) An alternative representation is given by:

$$\mathcal{V}_e(\pi) = \left\{ f(\pi) : f \in L(\mathcal{A})^*,\ f(I) = 1,\ \|f\| = 1,\ \text{and } f|_{\mathcal{K}(\mathcal{A})} = 0 \right\}$$

.

(vi) The essential norm $\|\pi\|_e$ satisfies: $e^{-1}\|\pi\|_e \leq \sup\{|\lambda| : \lambda \in \mathcal{V}_e(\pi)\} \leq \|\pi\|_e$.

Unlike the standard essential numerical range, our results emphasize the role of integral structures and tensor products in defining and analyzing the numerical range. This provides a more generalized perspective, particularly in the study of integral transforms and their algebraic numerical properties. Additionally, we explore the conditions under which our integral operator remains self-adjoint and the implications of this property on the real-valued nature of its numerical range.

Muhati, Bonyo, and Agure (2017) extended these ideas by exploring the essential algebraic numerical range in Banach spaces. The essential algebraic numerical range $\mathcal{W}_e(\pi)$ of an operator $\pi \in L(\mathcal{A})$ is given by: $\mathcal{W}_e(\pi) = \{\mathfrak{f}(\pi) \mid \mathfrak{f} : L(\mathcal{A}) \to L(\mathcal{A})/R(\mathcal{A})\}$, where $R(\mathcal{A})$ denotes the set of compact operators. The essential norm $\|\pi\|_e$ is defined as:

$$\|\pi\|_e = \inf\{\|\pi + \kappa\| \mid \kappa \in R(\mathcal{A})\}.$$

The essential spatial numerical range of $\pi \in L(\mathcal{A})$ is the set of complex numbers ν for which there exist nets $(X_\mu) \subset \mathcal{A}$ and $(X_\mu^*) \subset \mathcal{A}^*$ satisfying $\|X_\mu\| = \|X_\mu^*\| = 1$, $\langle X_\mu, X_\mu^* \rangle = 1$, and X_μ tends to zero weakly while $\langle X_\mu, X_\mu^* \rangle$ tends to ν. Key properties of $\mathcal{W}_e(\pi)$ include:

- $\mathcal{W}_e(\pi)$ is nonempty and contains the essential spectrum of π.
- $\mathcal{W}_e(\pi) = \{0\}$ if and only if $\pi \in R(\mathcal{A})$.
- $\mathcal{W}_e(\pi_1 + \pi_2) \subset \mathcal{W}_e(\pi_1) + \mathcal{W}_e(\pi_2)$.

The essential algebraic numerical radius $Nr_e(\pi) = \sup\{|\nu| \mid \nu \in \mathcal{W}_e(\pi)\}$ satisfies properties such as:

- $Nr_e(\zeta\pi) = |\zeta|\, Nr_e(\pi)$.
- $Nr_e(\pi_1 + \pi_2) \le Nr_e(\pi_1) + Nr_e(\pi_2)$.

Additionally, the relationship between the essential spectrum and the essential algebraic numerical range was explored, yielding results such as:

- $Sp_e(\pi) = \mathcal{W}_e(\pi) = \{0\}$.
- $Sp_e(\pi_1 + \pi_2) = \mathcal{W}_e(\pi_1 + \pi_2) = \mathcal{W}_e(\pi_1) + \mathcal{W}_e(\pi_2)$.

These results characterize the essential spatial numerical range as a tool for studying operators in infinite-dimensional spaces.
The relationship between the essential numerical range and the classical numerical range has been the subject of significant study. Notably, work by John (1975) established that the essential numerical range $\mathcal{V}_e(\pi)$ is convex and satisfies the inclusion $\mathcal{V}_e(\pi) \subset \overline{\mathcal{W}(\pi)}$, where $\mathcal{W}(\pi)$ denotes the numerical range of the operator π, and the bar denotes the

closure in the complex plane. Earlier, in a 1970 study, Stampfli and Williams showed that the essential numerical range contains the essential spectrum of an operator and serves as the natural counterpart to the classical spectrum within the Calkin algebra, i.e., the quotient space $L(K)/\mathcal{K}$, where $L(K)$ is the algebra of bounded linear operators on a Hilbert space K and $\mathcal{K}$ denotes the ideal of compact operators. A fundamental characterization of points in the essential numerical range states that a complex number ζ lies in $\mathcal{V}_e(\pi)$ if and only if there exists a sequence of unit vectors $(a_m)_{m=1}^{\infty}$ in K, weakly converging to zero, such that $\langle \mathcal{T} a_m, a_m\rangle \to \zeta \quad \text{as } m \to \infty$.
Moreover, for any bounded operator π on K, it holds that $\operatorname{Ext}(\overline{\mathcal{W}(\pi)}) \subset \mathcal{V}_e(\pi) \cup \mathcal{W}(\pi)$, where $\operatorname{Ext}(\cdot)$ denotes the set of extreme points. Consequently, one has the relation

$$\overline{\mathcal{W}(\pi)} = \operatorname{conv}(\mathcal{V}_e(\pi) \cup \mathcal{W}(\pi)),$$

indicating that the closed numerical range is the convex hull of the union of the essential numerical range and the numerical range. In special cases, such as when $\mathcal{V}_e(\pi) \subset \mathcal{W}(\pi)$, it follows that the numerical range is already convex: $\mathcal{W}(\pi) = \operatorname{conv}(\mathcal{W}(\pi)) = \mathcal{W}(\pi)$.
It is also noteworthy that the essential numerical range of a compact operator reduces to the singleton $\{0\}$. Therefore, the numerical range of a compact operator is closed if and only if $0 \in \mathcal{W}(\pi)$. In cases where the numerical range lacks extreme points, the essential numerical range equals its closure: $\mathcal{V}_e(\pi) = \overline{\mathcal{W}(\pi)}$.

Joint Numerical Range

The concept of joint numerical range extends the notion of numerical range to tuples of operators. Let $\pi = (\pi_1, \ldots, \pi_m)$ be a tuple of bounded linear operators on a Hilbert space K. The joint numerical range of π is defined as follows:

Definition 2.10. Let K be a Hilbert space and $\pi = (\pi_1, \ldots, \pi_m)$ a collection of bounded operators on K. The joint numerical range $W(\pi)$ is the set of vectors $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{C}^m$ such that there exists a unit vector $a \in K$ ($\|a\| = 1$) with $\eta_i = \langle \pi_i a, a\rangle, \quad \text{for all } i = 1, \ldots, m$.

One of the key characteristics of the joint numerical range for a collection of commuting normal operators is its convexity. Given an M-tuple of commuting normal operators, the joint spectrum, denoted by $\Omega(\pi)$, is defined as the set of all tuples $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{C}^M$ such that each operator $(\pi_j - \eta_j I)$ is not invertible for some j. This concept is thoroughly discussed in the work by Muhammad and Saeed (2000). Their findings establish a fundamental relationship between the joint numerical range and the joint spectrum, expressed as: $\Omega(\pi_1, \ldots, \pi_m) \subseteq \overline{W(\pi_1, \ldots, \pi_m)}$, where $\overline{W(\pi_1, \ldots, \pi_m)}$ denotes the closure of the joint numerical range. For a general M-tuple of bounded linear operators $\pi = (\pi_1, \ldots, \pi_m)$ acting on a Hilbert space K, the joint spectral radius is defined by:$Nr(\pi) = \sup\{|\zeta| : \zeta \in \Omega(\pi)\}$, with the Euclidean norm on $\mathbb{C}^M$ given by $|\zeta| = \left(\sum_{j=1}^{m} |\zeta_j|^2\right)^{1/2}$. The joint normaloid of the M-tuple operator π is then defined as the norm of the tuple: $Nr(\pi) = \|\pi\|$, and this coincides with the joint spectraloid, denoted by: $\Omega(\pi) = Nr(\pi)$. Furthermore, Ando and Keeler (2020) presented a comprehensive survey on the joint numerical ranges, discussing recent developments and applications of joint numerical ranges in operator theory. Their work emphasized on the role of numerical ranges in analyzing operator orbits and diagonal entries, thereby enhancing the understanding of multi-operator systems and spectral characteristics.

Verma (1990) introduced the concept of the joint spatial numerical range for bounded operators on Banach spaces and showed that the joint spatial numerical range of the tensor product of such operators equals the Cartesian product of their individual spatial numerical ranges.

Joint Hermitian and Skew-Hermitian Operators

In the study of operator theory, it is often useful to decompose an operator into its Hermitian (self-adjoint) and skew-Hermitian components. For a tuple of operators, this concept extends naturally as follows:

Definition 2.11 (Joint Hermitian Components)**.** Let $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ denote a collection of bounded linear operators acting on a complex Hilbert space K. The Hermitian part of each operator π_j is defined by $\pi_{+_j} = \frac{1}{2}(\pi_j + \pi_j^*), \quad$ for $j = 1, 2, \ldots, m$, where π_j^* denotes the adjoint of π_j. The tuple $(\pi_{+_1}, \pi_{+_2}, \ldots, \pi_{+_m})$ is referred to as the joint

Hermitian component of π.

Definition 2.12 (Joint Skew-Hermitian Components)**.** Given the same tuple $\pi = (\pi_1, \pi_2, \ldots, \pi_m)$ of bounded linear operators on the Hilbert space K, the skew-Hermitian part of each operator π_j is given by $\pi_{-_j} = \frac{1}{2}(\pi_j - \pi_j^*)$, for $j = 1, 2, \ldots, m$. The collection $(\pi_{-_1}, \pi_{-_2}, \ldots, \pi_{-_m})$ is called the joint skew-Hermitian component of π.

The concept of the joint numerical range has been extensively studied in the literature, particularly in the context of Hermitian matrices. This section provides a summary of key results related to joint numerical ranges, focusing on the contributions of Szymanski, Weis, and Zyczkowski (2018), as well as the work of Calbeck (2008) on elliptic numerical ranges of companion matrices. Szymanski, Weis, and Zyczkowski (2018) define the joint numerical range of Hermitian matrices as follows:

Definition 2.13. (Szymanski, Weis, and Zyczkowski, 2018) Let π_n be a set of $n \times n$ matrices, and let $\pi_n^h = \{\eta \in \pi_n : \eta^* = \eta\}$ denote the real subspace of Hermitian matrices. For a tuple $\delta = (\delta_1, \delta_2, \ldots, \delta_m) \in (\pi_n^h)^m$, where $m \in \mathbb{N}$, the joint numerical range is defined as $\mathcal{W}(\delta) = \big\{(\langle \mu, \delta_1 \mu \rangle, \ldots, \langle \mu, \delta_m \mu \rangle) : \mu \in \mathbb{C}^d, \langle \mu, \mu \rangle = 1\big\} \subset \mathbb{R}^n$.

For $m = 1$ or 2, the joint numerical range reduces to the usual numerical range of matrices, a result that follows from the elliptical range theorem. When the dimension is two, they showed that the set $\mathcal{W}(\delta_1, \delta_2)$ can be a segment, a singleton, or a filled ellipse. Calbeck (2008) studied the elliptic numerical range of 3×3 companion matrices. A companion matrix is characterized by the roots of its characteristic polynomial, and its numerical range is determined by the location of these roots. Specifically, for a 3×3 matrix π with eigenvalues $\zeta_1, \zeta_2, \zeta_3$, they showed that the numerical range $\mathcal{W}(\pi)$ forms an ellipse with foci at ζ_1 and ζ_2 if the following conditions hold:

1. The eigenvalue ζ_3 lies within the ellipse defined by foci ζ_1 and ζ_2 and minor axis length $\sqrt{d}$.

2. The quantity d is given by $d = \mathrm{tr}(\pi\pi^*) - \sum_{r=1}^{3} |\zeta_r|^2 > 0$.

3. The third eigenvalue satisfies $\zeta_3 = \mathrm{tr}(\pi) + \frac{1}{d}\left(\sum_{r=1}^{3} |\zeta_r|^2 \zeta_r - \pi\pi^2\right)$.

The results in the aforementioned studies provide foundational insights into the structure of joint numerical ranges and elliptic numerical ranges of matrices. Our approach extends these concepts by considering the algebraic numerical range of integral transform operators and their convexity properties. In particular, our results investigate how the algebraic numerical range behaves under unitary transformations, addition, and scalar multiplication, as well as its relation to numerical radius and self-adjointness. Unlike previous works, which primarily focus on finite-dimensional matrices, our study explores integral operators and their associated numerical ranges in a functional analysis setting. This generalization provides new insights into operator theory and potential applications in functional analysis.

The literature primarily discusses the properties of the joint numerical range, joint spectrum, and related spectral quantities for commuting normal operators. The convexity property and the relationship between the joint spectrum and numerical range are well established. Additionally, the decomposition into Hermitian and skew-Hermitian parts provides a structured approach to analyzing operator tuples. However, our work extends these classical results by considering integral representations and inner product type integral transformers. Specifically, our focus on algebraic numerical ranges and their properties introduces novel insights. Unlike prior results that rely on operator decompositions and spectrum-based convexity, we establish convexity results via integral transformations, particularly in the context of $\Phi_{\{R,P\}}$ operators. Furthermore, we introduce results that demonstrate how algebraic numerical ranges interact with adjoints, unitary transformations, and linear combinations of operators, broadening the scope beyond standard joint numerical range discussions. These distinctions highlight the originality of our contributions and demonstrate how our findings extend and refine existing results in the literature.

The numerical range and numerical radius of operators have been extensively studied in various contexts, particularly in Hilbert and Banach spaces. Several key contributions have shaped the understanding of these concepts. Muhammed and Saeed in (Muhammad and Saeed, 2000) established that for a tuple of operators $\pi = (\pi_1, ..., \pi_m)$, if the numerical radius $Nr(\pi_1, ..., \pi_m)$ belongs to the numerical range $W(\pi_1, ..., \pi_m)$ and satisfies $Nr(\pi_j) = \|\pi_j\|$ for all $j = 1, ..., m$, then $Nr(\pi_1, ..., \pi_m)$ is a joint eigenvalue of

$\pi_1, ..., \pi_m$. Furthermore, if the joint spectral radius coincides with the joint eigenvalues, then $(\pi_1, ..., \pi_m)$ is a joint spectraloid. William in (Donoghue,1957) analyzed the geometric properties of numerical ranges in a two-dimensional Hilbert space. He introduced the angle θ between two subspaces spanned by vectors a and b in K, defined by

$$\cos\theta = \frac{|\langle a, b\rangle|}{\|a\|\|b\|},$$

and demonstrated that the closure of the numerical range $\mathcal{W}(\pi)$ is the smallest convex set containing the spectrum. His findings imply that $\mathcal{W}(\pi)$ is compact since $\langle \pi a, a\rangle$ is continuous on the compact unit sphere. Furthermore, he examined the numerical range structure when an operator has one or multiple eigenvalues. Specifically, he established that if π has a single eigenvalue ζ, then $\mathcal{W}(\pi)$ forms a circle centered at ζ with diameter $\|\pi - \zeta I\|$. When π has distinct eigenvalues ζ_1 and ζ_2, the numerical range takes the shape of an ellipse with eccentricity $\sin\theta$, where θ is the reduced angle between the corresponding eigenvectors. Bonsall (Bonsall, 1973), extended the concept of numerical ranges to Banach spaces by defining the lower numerical range of a bounded operator π in a Banach space A as $\mathcal{LW}(\pi) = \{(\pi\eta)(a) : (a, \eta) \in \prod(A), a, \eta \in A\}$, where $\prod(A) = A \times A'$ represents the product norm topology. While these studies have established fundamental results concerning the numerical range, numerical radius, and spectral properties of operators, our work extends these ideas to the algebraic numerical range of integral transformations of the form $\Phi_{\{R,P\}}$.

Bonsall, 1973 established the inclusion relations for the numerical range of an operator $\pi \in L(\mathcal{A}')$, showing that $\mathcal{LW}(\pi) \subset \mathcal{W}(\pi) \subset \overline{\mathcal{LW}(\pi)}$. Using the Bishop-Phelps-Bollobás theorem, he demonstrated that $\mathcal{W}(\pi) \subset \mathcal{W}(\pi^*) = \overline{\mathcal{W}(\pi)}$, leading to the result that $\mathcal{LW}(\pi^*) = \mathcal{W}(\pi)$. However, an example illustrated that $\mathcal{W}(\pi) \neq \mathcal{W}(\pi^*)$, distinguishing the numerical range from its closure. Joswig introduced the concept of the numerical range map, which applies to Hermitian operators uniquely determined in $A(\mathcal{L}^n)$. Given an operator $\pi = \pi_1 + i\pi_2$, the numerical range map is defined as:

$$\eta_\pi(a, b) = (\langle \pi_1(a + ib), a + ib\rangle, \langle \pi_2(a + ib), a + ib\rangle).$$

This formulation provides insight into the structure of numerical ranges in higher-dimensional settings.

While Bonsall and Joswig focused on traditional and mapped numerical ranges, our work extends these concepts to inner product-type integral transformers. In (Bouldin, 1970), the numerical range of the product of operators is determined in two principal ways. One common approach relies on results concerning the numerical radius, assuming commutativity of two operators, i.e., $\pi_1\pi_2 = \pi_2\pi_1$, or double commutativity, where $\pi_1\pi_2 = \pi_2\pi_1$ and $\pi_1\pi_2^* = \pi_2^*\pi_1$. However, (Bouldin, 1970) also introduces a second method that determines the numerical range of $\pi_1\pi_2$ in terms of the numerical range of π_1 and the numerical range of π_2, even in cases where $\pi_1 \neq \pi_2$. This approach is particularly useful because it simplifies the analysis; even if one of the operators is positive, broader results can be obtained with clear assumptions on the product $\pi_1\pi_2$. In (Chien, 1996), Chien investigated the numerical range of tridiagonal operators. An operator is said to be tridiagonal if there exists an orthonormal basis $(\nu_\kappa)_{\kappa\in\mathbb{N}}$ of K such that $\langle \mathcal{A}\nu_j, \nu_i\rangle = 0$ whenever $|i-j| > 1$. The diagonal entries $\langle \mathcal{A}\nu_i, \nu_i\rangle$, $i = 1, 2, ...$, constitute the main diagonal of $\mathcal{A}$. Marcus and Shure (1979) showed that the numerical range of $\mathcal{A}$ forms a circular disk with radius $\cos\left(\frac{\pi}{n+1}\right)$ centered at the origin. Additionally, it was established that the numerical range of $\mathcal{A}$ can be represented as an elliptical disk:

$$\{\zeta \in \mathbb{C}^n : \zeta = \cos\left(\frac{\pi}{n+1}\right)(\mathfrak{a}\nu^{i\theta} + \mathfrak{b}\nu^{i\theta}), 0 \leq \theta < 2\pi\}.$$

Chien further computed the numerical range of a tridiagonal operator $\mathcal{A}$ with a zero main diagonal and demonstrated that it is symmetric about the origin. Moreover, he showed that the numerical range of $\mathcal{A}$ is a closed elliptical disk centered at the origin.

Our study differs from the existing literature in several key ways. While previous works focus on specific operator classes such as tridiagonal matrices or impose commutativity conditions, our results extend the analysis to integral operators involving tensor products. Chien and Nakazato, (2001) generalized classical results on the numerical range to determine the c-numerical range of a tridiagonal matrix. They demonstrated that if δ_1 and δ_2 are tridiagonal matrices with entries $(\zeta_{i,j})$ and $(\eta_{i,j})$ respectively, then their c-numerical ranges are identical, i.e., $\mathcal{W}_c(\delta_1) = \mathcal{W}_c(\delta_2)$. Furthermore, they established that if δ_1 is a tridiagonal matrix with zero main diagonal, its c-numerical range forms a circular disk centered at the origin. They also provided conditions under which the c-numerical range of a tridiagonal matrix is an elliptic disc, thus generalizing classical numerical range results. Radl in (Radl, 2013) explored the numerical range of positive operators through the Perron-Frobenius theory. She showed that if K is a complex Hilbert space and $\pi \in L(K)$, then $|\langle \pi\zeta, \zeta\rangle| \leq Nr(\pi) \leq \langle \zeta, \zeta\rangle$,

where $Nr(\pi)$ denotes the numerical radius of π. She further established that if $\pi \in L(H)$ is normal or self-adjoint, then $\|\pi\| = r(\pi) = Nr(\pi)$, where $\|\pi\|$ is the norm, $r(\pi)$ is the spectral radius, and $Nr(\pi)$ is the numerical radius. Additionally, it was shown that the numerical range of π is symmetric with respect to the real axis. The properties of the numerical range of positive operators were also investigated, leading to the following results:

1. $Nr(\pi) = \sup\{\langle \pi\zeta, \zeta\rangle : \zeta \in K, \|\zeta\| = 1\}$.
2. $Nr(\pi) \in \overline{Nr(\pi)}$.
3. If $\kappa \in Nr(\pi)$ for some $\kappa \in \mathbb{K}$, then $Nr(\pi) \in \mathcal{W}(\pi)$.

The results from these studies provide a foundation for our investigation, which extends the numerical range analysis to inner product-type integral transformers. Unlike previous works focusing on tridiagonal matrices and positive operators, our study integrates integral transform techniques into the numerical range framework, offering a broader perspective on operator analysis.

Muhammad and Marletta (2013) examined the quadratic numerical range for Hain-Lust and Stoke type operators, showing it can have at most two components and is not necessarily convex. Ralph analyzed the numerical range of second-order elliptic operators with mixed boundary conditions in L^p spaces. Diogo (2015) explored operators whose numerical range closure contains zero, particularly for bounded linear operators on a Hilbert space K, and identified conditions for when $0 \in \overline{\mathcal{W}(\pi B)}$ holds for every $B \in \tau$. In the nonlinear case, Dorfner (1996) introduced the numerical range for smooth Banach spaces and used it to localize spectral sets for Lipschitz continuous transformers. Zarantonello's work in Hilbert spaces showed that if a number ζ is far from the numerical range, $(\zeta I - \pi)^{-1}$ is Lipschitz continuous. Dorfner demonstrated that for a Lipschitz continuous operator π in a smooth real Banach space, the closure of its numerical range includes its spectrum. In smooth complex Banach spaces, he showed that the spectrum of π lies within the smallest Jordan domain containing it. These studies provide significant insights into the properties of numerical ranges and their relation to operator spectra. Comparing with the previous studies that primarily investigated numerical ranges in linear settings or specific nonlinear cases, this study considered integral operators and their algebraic numerical range properties, including unitary invariance, spectral containment, and numerical radius estimates.

The numerical range of operators in Banach and Hilbert spaces has been widely studied, particularly with respect to its convexity. Mandal et al. (2020) demonstrated that while the numerical range in Banach spaces is generally not convex, certain types of operators, such as those of the form $\pi = \begin{pmatrix} 1 & w \\ 0 & -1 \end{pmatrix}$, exhibit convex numerical ranges. They extended this analysis to operators of the form $\pi = \begin{pmatrix} v & w \\ 0 & x \end{pmatrix}$, showing that these also have convex numerical ranges. However, they identified operators, such as $\pi = \begin{pmatrix} x & w \\ y & z \end{pmatrix}$, whose numerical range is not convex.

The study of numerical ranges and numerical radii of operators in Banach spaces has garnered attention, with the Bishop-Phelps-Bollobas theorem providing key perturbation results for elements in Banach spaces and their duals. For a Banach space $\mathcal{A}$ and $\epsilon > 0$, given (η, η^*) satisfying $\|\eta\| \leq 1$, $\|\eta^*\| = 1$, and $|\langle \eta, \eta^* \rangle - 1| < \frac{\epsilon^2}{4}$, there exists (ζ, ζ^*) such that $\|\zeta - \eta\| < \epsilon$ and $\|\zeta^* - \eta^*\| < \epsilon$. The numerical range $\mathcal{W}(\pi)$ of an antilinear operator π on a Banach space is convex, containing the origin, and for the space l_1, the numerical radius satisfies $Nr(\pi) = \|\pi\|$Hur and Lee 2021 introduced the numerical range of conjugations in Hilbert spaces. They defined a conjugation as an antilinear map $\mathfrak{D} : K \to K$ satisfying $\mathfrak{D}^2 = I$ and $\langle \mathfrak{D}\zeta, \mathfrak{D}\eta \rangle = \langle \eta, \zeta \rangle$ for all $\zeta, \eta \in K$. The numerical range of a conjugation $\mathfrak{D}$ is given by: $N(\mathfrak{D}) = \{\langle \mathfrak{D}\zeta, \mathfrak{D}\zeta \rangle : \zeta \in K, \|\zeta\| = 1\}$.

They established that $N(\mathfrak{D})$ is either:

1. $N(\mathfrak{D}) = \{\nu : |\nu| = 1\}$ when $K = \mathbb{C}$ (one-dimensional), or

2. $N(\mathfrak{D}) = \{\nu : |\nu| \leq 1\}$ when $\dim K \geq 2$.

This work highlighted the geometric structure of the numerical range of conjugations. Building on these ideas, Choa, Hurb, and Leec 2021 extended the numerical range of conjugations to Banach spaces. They defined a conjugation on a Banach space $\mathcal{A}$ as a map $\mathfrak{D} : \mathcal{A} \to \mathcal{A}$ satisfying:

$$\mathfrak{D}^2 = I, \quad \|\mathfrak{D}\| \leq 1, \quad \mathfrak{D}(\zeta + \eta) = \mathfrak{D}\zeta + \mathfrak{D}\eta, \quad \mathfrak{D}(\nu\zeta) = \overline{\nu}\mathfrak{D}\zeta,$$

for all $\zeta, \eta \in \mathcal{A}$ and $\nu \in \mathbb{C}$, ensuring that $\|\mathfrak{D}\zeta\| = \|\zeta\|$ for all $\zeta \in B$. The numerical range of $\mathfrak{D}$ in Banach spaces was defined as: $\mathcal{W}(\mathfrak{D}) = \{g(\mathfrak{D}\zeta) : (\zeta, g) \in \prod(\mathcal{A})\}$, where $\prod(\mathcal{A}) = \{(\zeta, g) \in \mathcal{A} \times B^* : \|g\| = g(\zeta) = \|\zeta\| = 1\}$. They proved that when $\dim B \geq 2$, $\mathcal{W}(\mathfrak{D})$ contains both 0 and 1 and remains connected in the complex plane.

In a related study, Harris (1971) investigated the numerical range of holomorphic functions in Banach spaces. A key result in his work was an inequality bounding each term in the Taylor series expansion of a holomorphic function in terms of its numerical radius.

Definition 2.14. (Harris, 1971) Let $\mathcal{A}$ be a Banach space and $\zeta \in \mathcal{A}$ with $\|\zeta\| = 1$. Define $S(\zeta)$ as a nonempty subset of the dual space B^* such that each $j \in S(\zeta)$ is a linear functional satisfying $\|j\| = j(\zeta) = 1$. If $h : \pi_1 \to \mathcal{A}$ is a holomorphic function with a continuous extension to $\mathrm{cl}(\pi_1)$, where π_1 is the open unit ball $\pi_1 = \{\zeta \in \mathcal{A} : \|\zeta\| < 1\}$, then the numerical range of h with respect to S is defined as:

$$\mathcal{W}(h) = \{j(h(\zeta)) \mid j \in S(\zeta), \|\zeta\| = 1\}.$$

The following fundamental properties of $\mathcal{W}(h)$ hold regardless of the choice of S:

1. $\mathcal{W}(\nu h) = \nu\mathcal{W}(h)$ for any scalar ν.
2. $\mathcal{W}(h_1 + h_2) \subseteq \mathcal{W}(h_1) + \mathcal{W}(h_2)$.
3. $\mathcal{W}(\nu I + h) = \nu + \mathcal{W}(h)$ for all $\nu \in \mathbb{K}$.

Harris also introduced the numerical radius of a holomorphic function h by defining

$h_t(\zeta) = h(t\zeta)$ for all $\zeta \in \mathrm{cl}(\pi_1)$ and $0 < t < 1$. The numerical radius is given by:

$$|Nr(h)| = \overline{\lim_{t\to 1^-}} \sup\{|\nu| : \nu \in \mathcal{W}(h_t)\}.$$

Harris' main result provided an upper bound for the nth term in the Taylor expansion of h at 0, denoted by Y_n: $|Y_n| \leq \kappa_n |\mathcal{W}(h)|$, where $\kappa_0 = 1, \kappa_1 = e$, and $\kappa_n = n^{n(n-1)}$ for $n \geq 2$.

Lastly, the connection between Birkhoff-James orthogonality and numerical range has been explored. Martin et al. (2023) provided characterizations of Birkhoff-James orthogonality in various Banach spaces through the lens of numerical ranges, offering valuable insights into operator geometry. These recent developments underscore the evolving nature of numerical range research, solidifying its importance in operator theory, spectral analysis, and functional analysis.

While previous studies, such as those by Harris (1971) who studied the numerical range for holomorphic functions and Hur and Lee (2021) who studied the numerical range of conjugations in Hilbert spaces, our research extends these concepts by considering integral transforms involving numerical range properties of operators.

Zahraei and Aghamollaei (2015) introduced the rank-k numerical range and the k-spectrum for rectangular complex matrices, investigating their algebraic and geometric properties. They defined the rank-k numerical range as follows:

Definition 2.15. Let $V_{n\times n}$ be the vector space of all $n \times n$ complex matrices, and let $\pi \in V_{n\times n}$. Then, the rank-k numerical range of π is defined by $\mathcal{W}_k(\pi) = \{\zeta \in \mathbb{C} : \tau\pi\tau = \zeta\tau\}$, where τ is a rank-k orthogonal projection on $\mathbb{C}^n$.

While Zahraei & Aghamollaei (2015) studied higher-rank numerical ranges, our work explores the algebraic numerical range properties for inner product-type integral transformers. Specifically, we establish convexity properties and characterize the self-adjoint nature of integral operators through their numerical range. Unlike previous studies that focused on explicit matrix forms, our work introduces integral representations, leveraging classical numerical range concepts and tensor products of inner product type integral transformers. This broader framework allows for deeper insights into numerical range structures in functional analysis.

The study of numerical ranges has evolved significantly, extending beyond classical numerical ranges to various operator classes. Several researchers have made contributions to the field, providing foundational results relevant to our study. Zahraei and Aghamollaei (see Theorem 2.6 in their work) demonstrated that under certain unitary matrices, the rank-k numerical range of rectangular matrices remains invariant, generalizing the known results for square matrices. This result laid the groundwork for further exploration into the numerical range of non-square matrices. The higher rank numerical range has been extended to Jordan-like matrices by Argerami & Mustafa (2021). Their work provides a framework for studying numerical ranges in a broader algebraic setting, demonstrating how these concepts extend beyond standard operator classes. Gunatillake, Jovovic, & Smith (2014) examined the numerical range of weighted composition operators on the Hardy-Hilbert space. They introduced the following definition:

Definition 2.16. (Gunatillake, Jovovic, & Smith, 2014) Let τ be a holomorphic self-map of an open disk δ, and let Δ be another holomorphic map of δ. If g is holomorphic on δ, then the operator mapping g to $\Delta \cdot g \circ \delta$ is called a weighted composition operator, denoted by $C_{\Delta,\delta}$.

In the special case where $\Delta = 1$, the operator simplifies to a composition operator. For specific choices of δ and bounded functions g, they derived explicit expressions for the numerical range $\mathcal{W}(C_{\Delta,\delta})$:

$$\mathcal{W}(C_{\Delta,\delta}) = \mathrm{Hull}(\delta(\tau)) \cup \nu\delta(\tau) \cup \cdots \cup \nu^{n-1}\delta(\tau),$$

where $\nu = \exp(2\pi i/n)$. Furthermore, they established spectral properties of weighted composition operators, showing that if $g \in K^{\infty}$ and $\delta(\zeta) = g(\zeta^n)$, then for $n > 1$, the spectrum satisfies:

$$\Omega(M_\delta | K_l) = \mathrm{Cl}(\delta(\tau)), \quad 0 \leq l < n.$$

Gunatillake, Jovovic, and Smith (2014) also related the numerical range of a shift operator to that of weighted composition operators. They introduced shift operators as follows:

Definition 2.17. (Gunatillake, Jovovic, and Smith, 2014) A bounded operator π on a Hilbert space K is called a shift operator if π is an isometry and $(\pi^*)^n$ converges to 0 strongly.

For a shift operator π, they established that $\mathcal{W}(\pi) = \tau$. Additionally, they demonstrated that if $C_{\Delta,\delta}$ is bounded on K^2, then 0 is contained in the closure of $\mathcal{W}(C_{\Delta,\delta})$.
In Hilbert spaces, Stampfli (1966) explored the normality of operators, establishing that an operator is normal if its spectrum lies on a curve in the complex plane and its numerical range satisfies certain conditions. Heydari (2012) extended these concepts to C^*-algebras by defining the C^*-numerical range, showing that if it forms a disk centered at the origin, a specific norm equality holds for all elements in the algebra. Paya-Albert (1982) investigated the relationship between the numerical range and projections in Banach spaces, characterizing the numerical range of operators in unital Banach algebras and demonstrating how projections affect the numerical range. These contributions collectively advanced the understanding of numerical ranges in various operator settings. Additionally, he showed that if $\mathcal{W}(\pi) = 0$, then π preserves the decomposition induced by ρ, meaning $\pi\rho = \rho\pi$.
The numerical range of uniformly continuous functions on the unit sphere of Banach spaces was studied by Palacios 2004. The concept of numerical range plays a crucial role in functional analysis and operator theory, providing insight into spectral properties and stability of operators. The concept of the numerical range in Banach spaces is introduced with respect to a unit sphere $S_{\mathcal{A}}$ and a dual space $\mathcal{A}^*$. For a point $u \in S_{\mathcal{A}}$, the set of states of $\mathcal{A}$ relative to u is defined as $D(\mathcal{A}, u) = \{\pi \in S_{\mathcal{A}^*} : \pi(u) = 1\}$, and for an element $\zeta \in \mathcal{A}$, the numerical range $\mathcal{W}(\mathcal{A}, u, \zeta)$ is the set $\{\pi(\zeta) : \pi \in D(\mathcal{A}, u)\}$. For a mapping $g : S_{\mathcal{A}} \to \mathcal{A}$, the spatial numerical range $\mathcal{W}(g)$ is defined as the union of numerical ranges over all points $\zeta \in S_{\mathcal{A}}$, specifically, $\mathcal{W}(g) = \bigcup_{\zeta \in S_{\mathcal{A}}} \mathcal{W}(\mathcal{A}, \zeta, g(\zeta))$. Alternatively, it can be written as $\mathcal{W}(g) = \{\pi(g(\zeta)) : (\zeta, \pi) \in \prod(\mathcal{A})\}$, where $\prod(\mathcal{A})$ denotes pairs $(\zeta, \pi) \in S_{\mathcal{A}} \times S_{\mathcal{A}^*}$ with $\pi(\zeta) = 1$. If the map g is bounded, its intrinsic numerical range is given by $\mathcal{W}(g) = \mathcal{W}(l_\infty(S_{\mathcal{A}}, \mathcal{A}), 1, g)$.
Our study extends these ideas by investigating the algebraic numerical range in the context of integral operators unlike Palacios in (Palacios, 2004) who focused on the spatial numerical range of mappings in Banach spaces .

The study of numerical ranges has been extensively explored in the literature for different classes of matrices and operators. Additionally, generalizations of numerical ranges in Banach spaces have been examined, yielding various properties applicable to different contexts. Despite these developments, there has been no discussion on the numerical range of inner product-type integral transformers in Hilbert spaces. This study aims to bridge this gap by establishing the numerical range of such integral transformers and examining their properties.
In the sequel, we review the related literature on the spectrum of various operators.

2.3 The Classical Spectral Theory

Spectral theory concerning linear operators on Hilbert spaces is foundational in various mathematical and physical advancements, particularly in quantum mechanics. The key elements of spectral theory include the spectrum of a linear operator, eigenvalues and eigenvectors, the spectral radius, and spectral integrals. These concepts have been extensively studied, leading to profound developments in functional analysis and operator theory. The Hilbert space laid the foundation for spectral theory by introducing fundamental concepts related to bounded self-adjoint operators (Yang, 1984). Neumann played a crucial role in the advancement of spectral theory for unbounded self-adjoint operators, driven by the requirements of quantum mechanics (Andrew and Green, 2002). He also contributed to the abstract theory of Hilbert space and linear operators, deepening the integration of spectral theory with measure and integration theory, as well as analytic functions (Tretter, 2008). Riesz expanded these results by generalizing spectral theory to include unbounded normal operators, thereby broadening its applicability (Tretter, 2008). Spectral theory of linear operators on Hilbert spaces is a cornerstone of functional analysis, with profound implications in quantum mechanics and other fields. Building upon the foundational work of Hilbert, Weyl, von Neumann, and Riesz, contemporary research continues to delve into various aspects of spectral theory, including rigged Hilbert spaces, operator classes, and spectral measures. The concept of rigged Hilbert spaces, or Gelfand triplets, has been instrumental in extending spectral theory to accommodate a broader class of operators, particularly unbounded ones. In recent studies, this framework has been applied to Schrodinger operators with exponentially decaying potentials. Notably, this approach provides a unified treatment of resonances (generalized eigenvalues) without resorting to spectral deformation techniques, thereby offering a more robust understanding of the spectral properties of such operators. Similarly, advancements in operator theory have focused on the spectral analysis of specific classes of operators. For instance, recent studies have examined bounded linear operators of the form $T = D + F$, where D is a diagonal operator and F is a finite-rank operator. This work utilizes the theory of Fredholm operators to derive significant results concerning the spectral properties of such operators, enhancing our understanding of their behavior in non-Archimedean Hilbert spaces.

Another key aspect of modern spectral theory research involves spectral measures and functional calculus. The spectral theorem, a pivotal result in spectral theory, asserts that self-adjoint operators can be represented via spectral measures, facilitating the definition of functions of operators through the Borel functional calculus. This theorem has been further explored to understand the decomposition of spectra and the role of projection-valued measures in the functional calculus, thereby deepening the integration of spectral theory with measure and integration theory. The field continues to evolve, with recent publications providing comprehensive treatments of spectral theory and operator analysis. For example, *Operator Theory by Example* by Garcia, Mashreghi, and Ross, (2023) offers an in-depth exploration of operator theory, including spectral aspects, through illustrative examples. Such works serve as valuable resources for both researchers and practitioners aiming to grasp the nuances of spectral theory in modern contexts. These developments underscore the dynamic nature of spectral theory research, highlighting its foundational importance and the continuous efforts to expand its applications across mathematics and physics.

While these classical results provide a robust theoretical framework, our study explores novel aspects of spectral theory by investigating the spectral properties of specific operator families in a Banach algebra setting. Unlike the classical results, which primarily focus on self-adjoint and normal operators, our approach considers commuting families of bounded self-adjoint operators in the Banach algebra $L(K)$. In particular, we analyze mappings within Banach algebras, the structural properties of operator-valued functions, and their spectral characteristics. This leads to new insights into the structure of operator spectra and their behavior under algebraic transformations.

Our work also diverges from traditional spectral theory by considering the mappings $^+$ and $^-$, demonstrating their bijectivity between Banach algebras $L(K)$ and $L(L(K))$. This result provides an extended algebraic framework for understanding spectral mappings, which is not explicitly covered in classical spectral theory. Additionally, by investigating the relationship between operator-valued functions P_s and their transformed versions P_s^+ and P_s^-, we established the novel spectral equivalences that offer deeper insight into operator spectra in Banach algebras. Overall, while our research builds upon the foundational results of Hilbert, Weyl, Neumann, and Riesz, it introduces new algebraic techniques and spectral characterizations that extend classical spectral theory into new domains. This contributes to the broader understanding of spectral properties in the context of Banach algebras and operator-valued function mappings.

Lankham (Lankham, 2007) explored the spectral theorem for normal linear maps, demonstrating that normal operators can be diagonalized with an orthonormal basis. Their work on unitary diagonalization led to a generalized singular-value decomposition, providing a framework for decomposing operators into simpler components. Tretter contributed to the characterization of the spectral radius of positive operators (Tretter, 2008). The origins of spectral theory for linear operators can be traced back to matrix theory and integral-type equations, which laid the groundwork for more advanced developments in Hilbert spaces. The spectral theory of compact operators in Hilbert spaces has been closely linked to Integration and Measure theories, fostering extensive research in the field.

While existing research has largely focused on the spectral properties of bounded and normal operators, our work extends these findings by considering the structure and spectral behavior of specific operator families in Banach algebras. In contrast to prior studies, our results explore the spectral characteristics of mappings within Banach algebras and introduce novel relationships between operator-valued functions and their spectral properties.

Spectral theory has its foundations in the works of Toeplitz (Toeplitz, 1918), who pioneered the study of spectral properties of bounded self-adjoint operators. Neumann later extended these results to unbounded self-adjoint operators (Tretter, 2008), laying the groundwork for spectral analysis in Hilbert spaces. The spectral theory for bounded linear operators in Hilbert spaces has been systematically developed, with fundamental contributions addressing spectral decomposition, the spectral radius, and the resolvent set (Davies, 2007).

Definition 2.18. (Davies, 2007) Let K be a complex Hilbert space with an inner product, and let π be a bounded linear operator on K. The spectrum of π is given by:

$$Sp(\pi) = \{\kappa \in \mathbb{C} : \pi - \kappa I \text{ is not invertible}\}.$$

The spectral radius of π is defined as: $r(\pi) = \sup\{|\kappa| : \kappa \in \Omega(\pi)\}$.
The resolvent set of an operator $\pi \in L(K)$ is the set of all complex numbers κ such that $\pi - \kappa I$ is bijective. It is denoted by: $\mathcal{R}(\pi) = \mathbb{C} - \Omega(\pi)$,

The spectrum possesses fundamental properties such as non-emptiness, boundedness, and compactness. When considering an $n \times n$ matrix δ with complex entries, the spectrum consists of its eigenvalues:

$$\delta\nu = \kappa\nu, \quad \nu \neq 0, \text{ where } \nu \text{ is the nonzero eigenvector and } \kappa \text{ is the eigenvalue.}$$

This leads to the classification of the spectrum as follows:

Definition 2.19. For a bounded linear operator δ on a Hilbert space K:

(a) The **point spectrum** consists of all $\kappa \in \Omega(\delta)$ for which $\delta - \kappa I$ is not injective (i.e., κ is an eigenvalue).

(b) The **continuous spectrum** consists of all $\kappa \in \Omega(\delta)$ where $\delta - \kappa I$ is injective but not surjective, with dense range.

(c) The **residual spectrum** consists of all $\kappa \in \Omega(\delta)$ where $\delta - \kappa I$ is injective but not surjective, and its range is not dense.

Remark 2.20. Since the spectrum $\Omega(\delta)$ is the complement of the resolvent set, which is open, it follows that:$\Omega(\delta) \subset \{\zeta \in \mathbb{C} : |\zeta| \leq \|\delta\|\}$.

Definition 2.21. The spectral radius of an operator δ is the radius of the smallest disk centered at zero containing $\Omega(\delta)$: $r(\delta) = \sup\{|\kappa| : \kappa \in \Omega(\delta)\}$.
If δ is a bounded operator, then:$r(\delta) = \lim_{m\to\infty} \|\delta^m\|^{\frac{1}{m}}$.
If δ is self-adjoint, then: $r(\delta) = \|\delta\|$.

While previous research, such as (Lankham, 2007) explored the spectral theorem for normal operators, proving unitary diagonalization and singular-value decomposition, our work takes a different direction. Instead of focusing on standard spectral decomposition, we investigate the structure of eigenspaces and spectral sets in the context of Banach algebras and operator-valued functions. In contrast to the classical spectral radius characterization by Tretter (Tretter, 2008), we establish novel results on the spectral properties of operator families in Banach algebras. Our key contributions include:

- A theorem characterizing the action of $\Phi_{\{R,P\}}$ on eigenspaces:

$$\Phi_{\{R,P\}}\zeta = \nu\zeta, \quad \text{where } \nu = \int_\Psi \kappa\xi_t d\gamma(s).$$

- A proof that $\Omega_r(\Phi_{\{R,P\}}) = \emptyset$, demonstrating spectral emptiness in specific cases.
- A mapping result showing that the operators $^+$ and $^-$ form bijections between Banach algebra $L(K)$ and $L(L(K))$.
- A spectral correspondence result:

$$\Omega(P_s, L(K)) = \Omega(P_s^+, L(L(K))) = \Omega(P_s^-, L(L(K))).$$

- A theorem establishing spectral integration over commuting families of bounded self-adjoint operators:

$$\Omega(\Phi_{\{R,P\}}) = \left\{\int_\Psi \kappa_s\nu_s d\gamma(s) : \kappa_s \in \Omega(P_s), \nu_s \in \Omega(R_s)\right\}.$$

These results extend classical spectral theory by introducing integral formulations and mappings in operator spaces. A crucial result in spectral theory is the spectral mapping theorem for polynomials, which states that for any polynomial $\mathfrak{p}$ with complex coefficients, $\Omega(\mathfrak{p}(\pi)) = \mathfrak{p}(\Omega(\pi))$.

This theorem has significant implications for spectral decomposition and operator analysis. Further developments in spectral theory include the study of $(\mathfrak{p}, k)$-quasiposinormal operators, particularly in relation to the Weyl spectrum, denoted $Sp_w(\pi)$, as investigated in (Senthilkumar et al., 2013). A bounded operator π is said to be $(\mathfrak{p}, k)$-quasiposinormal if:

$$\pi^{*k}(c^2(\pi^*\pi)^{\mathfrak{p}} - (\pi\pi^*)^{\mathfrak{p}})\pi^k \geq 0, \text{ for } 0 < \mathfrak{p} \leq 1,\ c > 0, \text{ and a positive integer } k.$$

The Weyl spectrum is defined as: $\Omega_w(\pi) = \{\kappa \in \mathbb{C} : \pi - \kappa \text{ is not Weyl}\}$.
Key results obtained in (Senthilkumar et al., 2013) include:

1. Weyl's theorem holds for $(\mathfrak{p}, k)$-quasiposinormal operators, i.e.,

$$\Omega(\pi) \setminus \Omega_w(\pi) = \pi_{00}(\pi), \text{ where } \pi_{00}(\pi) = \{\kappa \in Sp(\pi) : 0 < \zeta(\pi - \kappa) < \infty\}.$$

2. If π is $(\mathfrak{p}, k)$-quasiposinormal, then $\Omega(\pi) \setminus \{0\} = \Omega_e(\pi) \setminus \{0\}$, where $\Omega_e(\pi)$ is the compression spectrum.

3. Weyl's theorem holds for π if its adjoint π^* is $(\mathfrak{p}, k)$-quasiposinormal.

The study of spectral theory for bounded linear operators on complex unital Banach algebras reveals that the spectrum of an element π is the set of scalars $\kappa \in \mathbb{C}$ for which $\kappa I - \pi$ fails to be invertible. The spectrum is always non-empty and compact for nontrivial Banach spaces, while the resolvent set is non-empty and open. The spectral mapping theorem asserts that for any complex polynomial $\mathfrak{p}$, the spectrum of $\mathfrak{p}(\pi)$ equals the image of the spectrum of π under $\mathfrak{p}$. A special class of operators, called $(\mathfrak{p}, k)$-quasiposinormal operators, satisfies a specific operator inequality involving powers and adjoints. These operators have notable spectral properties, including conditions under which a point becomes a pole of the resolvent. It has also been shown that spectral continuity at the adjoint π^* implies spectral continuity at π. Furthermore, the spectrum of an element depends on the structure of the algebra and is guaranteed to be non-empty only when the scalar field is $\mathbb{C}$.

2.3.1 Numerical Radius

The numerical radius is an important concept in the study of bounded linear operators on Hilbert spaces. It provides an alternative way to measure the "size" of an operator and is closely related to the spectral radius and the operator norm. Let K be a nonzero complex Hilbert space and let $\pi \in L(K)$ be a bounded linear operator. The numerical radius of π is defined as $\omega(\pi) = \sup\{|\langle \pi a, a\rangle| : a \in K, \|a\| = 1\}$. This definition captures the maximum modulus of the inner product between πa and a over all unit vectors in K. Using the Cauchy-Schwartz inequality, it can be shown that $\omega(\pi) \leq \|\pi\|$.

The numerical radius defines a norm on $L(K)$, though it does not coincide with the operator norm in general. However, for normal operators, the numerical radius equals the operator norm. Moreover, if two operators are unitarily equivalent, their numerical radii are equal, reflecting the unitary invariance of this quantity. Our study extends the spectral analysis of operators in Banach spaces by investigating the spectral properties of the operator $\Phi_{\{R,P\}}$. Our key results include:

1. Characterization of eigenspaces: If G is an eigenspace with $\zeta \in G$, then $\Phi_{\{R,P\}}\zeta = \nu\zeta$, where:
$$\nu = \int_{\Psi} \kappa\xi_t d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}}).$$
2. The spectral set $\Omega_r(\Phi_{\{R,P\}})$ is empty.
3. Establishing one-to-one mappings from the Banach algebra $L(K)$ onto $L(L(K))$.
4. For an operator-valued function P_s on $L(K)$, the spectral sets satisfy:
$$\Omega(P_s, L(K)) = \Omega(P_s^+, L(L(K)))$$
and
$$\Omega(P_s, L(K)) = \Omega(P_s^-, L(L(K))).$$

Recent advancements in operator theory have led to significant improvements in the study of numerical radius inequalities. In a 2024 study, Altwaijry, Dragomir, and Feki presented novel upper and lower bounds for the Euclidean numerical radius associated with pairs of operators in Hilbert spaces. Their findings contribute to the refinement of existing bounds and offer enhanced estimates for both the numerical radius and the Davis-Wielandt radius. In related work, Audeh (2019) explored numerical radius inequalities within the context of operator matrices. His research provided more precise bounds concerning the sums and products of operators, thereby advancing the understanding of numerical range characteristics in operator theory. Spectral analysis in Banach spaces has also been an area of active research. Studies on compact linear operators have revealed that their spectral properties resemble those of finite-dimensional matrices, including the discreteness of the spectrum and the accumulation point at zero. Additionally, researchers have explored spectral sets of operator-valued functions, establishing relationships between the spectra of operators and their adjoints. These insights contribute to the broader understanding of spectral theory in functional analysis.

Another significant development concerns the product numerical range, which extends the traditional numerical range to operators on tensor product spaces. This concept is particularly relevant in quantum mechanics, where it helps describe operators acting on composite Hilbert spaces. By evaluating operators over product states, researchers have gained deeper insights into the spectral characteristics of multi-partite systems. These advancements highlight the ongoing exploration of numerical radius and spectral properties in operator theory. The refinement of inequalities, characterization of spectral sets, and extension of numerical range concepts continue to drive new discoveries in functional analysis and its applications in mathematical physics and quantum information theory.

Unlike previous works that focus on quasiposinormality and numerical radius properties, our study investigates the spectral structure and operator mappings within Banach algebras in a broader sense.

The study of operator norms, numerical radius, and spectral radius in Hilbert spaces has been extensively developed in previous research. In (Kubrusly, 2012), the relationship between these quantities is established through the following inequality:

$$0 \leq \Omega(\pi) \leq Nr(\pi) \leq \|\pi\| \leq 2Nr(\pi).$$

This result has led to various classifications of operators based on their numerical radius. An operator $\pi \in L(K)$ is termed normaloid if and only if $\omega(\pi) = \|\pi\|$, and it has been established that if there exists $\zeta \in \mathcal{W}(\pi)$ such that $|\zeta| = \|\pi\|$, then π is normaloid. Further, (Kubrusly, 2012) provides properties of the numerical radius, including subadditivity and homogeneity, as well as conditions under which an operator attains its numerical radius. Additionally, if π is a square matrix operator, it is shown that $\Omega(\pi) \subseteq \mathcal{W}(\pi) \subseteq \{\zeta \in \mathbb{C} : |\zeta| \leq \|\pi\|\}$. This leads to the introduction of radialoid and spectraloid operators (Li, 2005), where π is radialoid if $\Omega(\pi) = \|\pi\|$ and spectraloid if $\Omega(\pi) = \omega(pi)$.

The study of the numerical radius and its associated inequalities has been extensively explored in the literature (see Goldberg & Tadmor, 1982, Kittaneh, 2003, Moradi & Sababheh, 2020 and references therein). The numerical radius is known to satisfy the triangular inequality, a consequence of the sub-additivity of the norm and the axioms of the inner product in a Hilbert space. This property plays a crucial role in understanding the combined effect of two operators acting together. The numerical radius is also closely related to the Frobenius norm. The Frobenius norm, as introduced by Bhatia (Bhatia, 2013), is defined for a linear and bounded operator P on a Hilbert space K as follows:

Definition 2.22 (Frobenius Norm)**.** Let P be a linear and bounded operator on a Hilbert space K. The Frobenius norm, denoted by $\|P\|_F$, is given by

$$\|P\|_F = \left(\sum_{n,m} |b_{n,m}|^2 \right)^{\frac{1}{2}}.$$

Furthermore, the numerical radius satisfies the inequality: $Nr(P) \leq \|P\|_F$.

The existing literature has primarily focused on numerical radius inequalities and their applications in operator theory. Prior works have examined the interplay between numerical radius, norm inequalities, and various operator classes. However, our study extends these results by exploring the spectral properties and structure of operator families in Banach algebras. Specifically, our approach considers eigenspaces and the spectral mapping properties of operator-valued functions in the Banach algebra $L(K)$. In thus study, we analyze the structure of mappings in operator algebras and their implications for numerical range theory. Our results diverge from classical numerical radius studies by incorporating one-to-one and onto mappings in Banach algebras and examining their impact on operator families. Additionally, we establish novel conditions for the spectrum of operator-valued functions, particularly in the setting of commuting families of bounded self-adjoint operators. These distinctions mark a significant extension beyond traditional numerical radius inequalities, offering new insights into the interplay between operator spectra and Banach algebra structures as discussed in chapter 4.

2.3.2 The Relationship Between the Spectrum and the Numerical Range

This subsection explores existing research concerning the interplay between the numerical range and the spectrum of linear operators, particularly within Hilbert and Banach space settings. In the realm of functional analysis, significant attention is devoted to analyzing the spectral characteristics of operators, as these provide insights into their structure and behavior in infinite-dimensional contexts. Two critical constructs in this analysis are the spectrum and the numerical range. The spectrum of an operator encompasses essential information such as eigenvalues and other spectral components that define its action on the space. On the other hand, the numerical range serves as a valuable tool for estimating and localizing the spectrum, especially when dealing with complex or unbounded operators. Several studies have demonstrated that there exists a strong relationship between these two concepts. A prominent result in this regard asserts that the closure of the numerical range includes the spectrum of the operator. For a bounded linear operator $\pi \in L(K)$ on a Hilbert space K, it has been established that $\Omega(\pi) \subseteq \overline{\mathcal{W}(\pi)}$, where $\Omega(\pi)$ denotes the spectrum and $\mathcal{W}(\pi)$ the classical numerical range of π. Further generalizations of this relationship have been made in more abstract settings. For instance, in semi-inner product spaces, the notion of the numerical range has been adapted to suit the lack of a complete inner product structure. In this context, Lumer developed a modified definition of numerical range and showed that a similar inclusion result still holds, thereby extending the classical findings to a broader class of spaces. These foundational results provide a framework for deeper exploration of operator theory and help bridge various structural properties through the lens of the numerical range. Recent studies have investigated conditions under which the algebraic numerical range serves as a C-spectral set, refining the understanding of its spectral properties. Recent developments in spectral theory have revealed that the classical numerical range does not always guarantee a C-spectral set for certain operators, as demonstrated through various counterexamples. In response to these limitations, the concept of the essential numerical spectrum has been proposed to refine the classical numerical range. Notably, Altay et al. (2020) introduced this notion to offer a more accurate spectral enclosure, particularly in cases where the traditional numerical range fails to encapsulate the full spectrum of an operator in Banach spaces. In the realm of quaternionic Hilbert spaces, the spectral analysis of operators has taken a different turn due to the non-commutative nature of quaternions. Ghiloni et al. (2023) investigated the numerical range of bounded linear operators in this context and explored its relationship with the S-spectrum. Their findings confirmed that the closure of the numerical range encompasses the S-spectrum, thereby affirming the stability and generality of the numerical range concept within non-commutative settings. Moreover, the exploration of joint numerical ranges has significantly enriched the study of spectral properties in systems involving multiple operators. The work of Li and Tretter (2021) delved into both the geometric and algebraic aspects of joint numerical ranges, highlighting their usefulness in examining operator orbits and diagonal elements. Such investigations have deepened our comprehension of spectral behavior in multi-operator frameworks. These recent advancements highlight the ongoing development of research on numerical ranges and spectra, reinforcing their importance in operator theory. Continued exploration of these topics not only deepens theoretical understanding but also contributes to applications in mathematical physics and other fields that rely on spectral analysis of operators.

While these results provide fundamental insights into the relationship between numerical ranges and spectra, they do not directly address integral-type operators. Our approach extends these classical results by considering the spectrum and numerical range of inner product type integral transformers. Unlike previous work, our results apply to a broader class of integral operators by introducing a structured framework that incorporates inner product type integral transformers. This provides a more comprehensive understanding of the interplay between numerical ranges and spectra in this setting.

The interplay between spectral properties and numerical ranges of linear operators has been a central theme in functional analysis, offering profound insights into operator structure and behavior. A fundamental result in this direction establishes that for any bounded linear operator, its spectrum is always contained within the closure of its numerical range. This spectral inclusion principle, highlighted in the works of Kubrusly (2012) and later refined by Skoufranis (2014), demonstrates how the numerical range serves as a valuable geometric tool for approximating spectral characteristics. The relationship extends beyond standard operator theory, as shown by Lumer's investigation in semi-inner product spaces, where not only the spectrum but also the approximate point spectrum-including its boundary-lies within the numerical range closure. This reinforces the deep connection between spectral behavior and the geometric properties encapsulated by the numerical range. Further developments by Williams, (1967), revealed how spectral containment generalizes to products of operators, particularly when certain numerical range conditions are met, providing a framework for analyzing operator compositions. For the important class of normal operators, research by Meng, (1957) established a striking convexity property: the closed numerical range coincides exactly with the convex hull of the spectrum. This duality between spectral and numerical characteristics becomes even more pronounced when the numerical range is closed, as its extreme points correspond directly to eigenvalues. Complementing these results, Shapiro, (2003) derived spectral radius-based bounds, proving that operators themselves reside within their numerical range closures, thereby unifying spectral and operator norm perspectives. While these classical results primarily address individual operators, our work extends this theoretical foundation to integral transformers of inner product type. We demonstrate that the numerical range closure of such transformers is contained within an associated spectral set, and moreover, that their spectra inherit the classical inclusion property with respect to numerical ranges. This advancement bridges existing spectral theory with integral operator applications, enriching the understanding of spectral-numerical relationships in broader functional analytic contexts.

The study of spectral properties of linear operators is a central topic in functional analysis, with the spectrum and numerical range serving as fundamental tools for understanding operator behavior. The spectrum reveals essential features such as eigenvalues

and structural properties in infinite-dimensional spaces, while the numerical range provides a convex set that often contains the spectrum. A well-known result, established by Kubrusly (2012), asserts that for any bounded linear operator on a Hilbert space, the spectrum lies within the closure of its numerical range. Research on products of operators has also yielded significant insights. Williams, (1967) investigated spectral inclusion for operator products, demonstrating that under certain conditions, the spectrum of the product's inverse remains within a specific numerical range-derived set. Further contributions highlight important connections between numerical ranges, spectral radii, and operator norms. For instance, the numerical radius of an operator is always bounded by its norm, and the spectral radius remains invariant under similarity transformations. In the case of self-adjoint operators, the numerical range reduces to a real interval encompassing the spectrum, with extremal eigenvalues corresponding to the numerical range's bounds—a principle closely tied to the min-max theorem. These foundational results have been extensively discussed in works by Klaja (2014), Davies (2007) and others.
While existing literature primarily addresses standard operators in Hilbert and Banach spaces, this work extends these spectral-numerical range relationships to integral transformations involving commuting families of bounded self-adjoint operators. By analyzing inner product-type integral transformers, we provide a broader perspective on spectral behavior, bridging a gap in the current understanding of such operators. Our results generalize classical theorems, offering new insights into the structural connections between numerical ranges and spectra in this context.

2.4 Elementary Operators

The study of elementary operators, introduced by Lumer and Rosenblum (Lumer & Rosenblum, 1959), has been a central topic in functional analysis and operator theory. These operators, defined in the setting of Banach algebras, take the form: $X \longmapsto \sum_{i=1}^{n} \pi_i X \delta_i$, where $\pi_i, \delta_i \in \mathcal{A}$ are fixed elements of the Banach algebra $\mathcal{A}$, and X is an arbitrary element of $\mathcal{A}$. Various studies have explored the properties of these operators, including their spectra, numerical ranges, and norm estimates.
Several notable elementary operators have been investigated in the literature, see (Seddik,

2001):

- The *left multiplication operator* L_{S_1} defined by $L_{S_1}(X) = S_1X$ for all $X \in L(K)$.
- The *right multiplication operator* R_{S_2} defined by $R_{S_2}(X) = XS_2$ for all $X \in L(K)$.
- The *elementary multiplication operator* $M_{S_1,S_2}(X) = S_1XS_2$.
- The *inner derivation* $\Delta_{S_1}(X) = S_1X - XS_1$.
- The *generalized derivation* $\Delta_{S_1,S_2}(X) = S_1X - XS_2$.
- The *Jordan elementary operator* $U_{S_1,S_2}(X) = S_1XS_2 + S_2XS_1$.

Elementary operators have been extensively studied due to their connections with various areas of operator theory, including derivations, perturbation theory, and spectral theory. Early investigations into the numerical ranges of elementary operators primarily examined containment properties and convexity conditions. In particular, researchers established that for normal operators π_i and δ_i, the numerical range of an elementary operator typically lies within the convex hull of the numerical ranges of its components. Furthermore, spectral properties of these operators have been analyzed within the framework of Banach algebras and C^*-algebras, with notable attention given to estimating spectral radii. Subsequent work by Runji, Agure, and Nyamwala (2017) expanded upon these foundations by investigating the algebraic maximal numerical range of elementary operators. Their results demonstrated a connection between this range and the closed convex hull of the maximal numerical ranges of the implementing operators $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ defined on the algebra of bounded linear operators on a Hilbert space H. This contribution generalized earlier findings and provided deeper insights into the structure of numerical ranges for elementary operators.

The numerical range of elementary operators has attracted considerable attention in various mathematical settings. Seddik (2002) contributed significantly to this area by investigating the algebraic numerical range of elementary operators. He identified a relation involving the convex hull of bilinear combinations of vectors drawn from the joint spatial numerical ranges of the implementing operators. Although this convex set is contained in the algebraic numerical range of the operator, he observed that the inclusion can be strict, particularly when dealing with elementary multiplication operators induced by non-scalar joint operators. Shaw (1984) studied the generalized derivations restricted to invariant subspaces. He showed that the algebraic numerical range of such a derivation could be described as the difference between the algebraic numerical ranges of the associated implementing operators. This provided deeper insight into the structure of the numerical range in relation to the components defining the elementary operator. In a series of works (Seddik 2001, 2004), Seddik extended his analysis to include Banach spaces and specific operator classes such as the p-Schatten classes on Hilbert spaces, as well as norm ideals J in complex Banach algebras. He established that the algebraic numerical range of an elementary operator acting within these spaces contains the closed convex hull of bilinear forms involving elements from the joint algebraic numerical ranges of the implementing operators. Daniel, Musundi, & Kinyanjui (2003) examined elementary operators of length two and calculated their norms using Stampfli's approach, which leverages the maximal numerical range within tensor products. Their computations offered precise techniques for norm estimation in this context. Additional contributions have been made by various researchers including Mathieu (2001), Magjna (1987), and Barraa (2005, 2014), who have analyzed different aspects of the numerical range of elementary operators and enriched the theoretical framework surrounding these concepts. Their work has collectively advanced the understanding of operator theory in the context of algebraic and topological structures.

The spectrum of elementary operators has been widely investigated by various researchers (see Curto, 1983, Marvin, 1956 and references therein). For $\pi_1, \pi_2 \in L(K)$, the spectrum $\Omega(M)$ of the operator defined by $M\mathcal{X} = \pi_\infty \mathcal{X} \pi_\in$ was studied by Lumer and Rosenblum in (Lumer & Rosenblum, 1959). They proved that $\Omega(M) = \Omega(\pi_1) \cdot \Omega(\pi_2) = \{\kappa\eta : \kappa \in \Omega(\pi_1), \eta \in \Omega(\pi_2)\}$.

Davis and Rosenthal(Davis and Rosenthal, 1974) studied the spectrum $\Omega(\mathcal{R})$ of the elementary operator $\mathcal{R} : L(K) \to L(K)$ defined by $\mathcal{R}(\mathcal{X}) = \sum_{i=1}^{n} \pi_{1i}\mathcal{X}\pi_{2i}$, showing that for two commuting families $\{\pi_{1i}\}_{i=1}^{n}$ and $\{\pi_{2i}\}_{i=1}^{n}$,

$$\Omega_\delta(\mathcal{R}) \subset \{\textstyle\sum_{i=1}^{n} \kappa_i\eta_i : \kappa_i \in \Omega_\delta(\pi_{1i}), \eta_i \in \Omega_{app}(\pi_{2i})\},$$

where Sp_π and Sp_δ denote the approximate point spectrum and the defect spectrum, respectively. For two commuting families $\pi_1, \ldots, \pi_n$ and $\delta_1, \ldots, \delta_n$ of self-adjoint operators in $L(K)$, Khan and Kyle in (Khan and Kyle, 1990) showed that $\Omega(\mathcal{R}_{\pi,\delta}) = \{\kappa_i \mu_i : \kappa_i \in \Omega(\pi_i), \mu_i \in \Omega(\delta_i)\}$.

Recent advancements have provided deeper insights into the spectral properties of these operators. One important result concerns the scattered range problem for elementary operators on $B(H)$. A study by Chen, Li, & Wang (2024) demonstrated that for any elementary operator R on $B(H)$, the range of R cannot be scattered, meaning that its spectrum cannot be at most countable. This result provides structural limitations on the behavior of elementary operators (Chen 2024). Another recent study by Zhang & Xu (2024) focused on the norm properties of Jordan elementary operators, particularly in the tensor product of C^*-algebras. They established specific norm bounds, contributing to a broader understanding of operator behavior in complex algebraic structures (ZhangXu 2024).

Additionally, the conditions under which elementary operators are spectrally bounded have been explored by Alaminos, Bras, & Peral (2023). Spectral boundedness, which refers to the containment of an operator's spectrum within a bounded set, was analyzed in depth, with the authors providing necessary and sufficient conditions for an elementary operator on a Banach algebra to satisfy this property. Their findings offer a useful framework for studying operator spectra in different algebraic settings (Alaminos 2023). Furthermore, research by Davidson and Paulsen (1995) has investigated the structural properties of elementary operators, focusing on their spectral, algebraic, and metric characteristics. Their work delves into the interrelationships between these properties, enhancing our comprehension of elementary operators within different algebraic environments (Davidson, Paulsen 2023). These recent studies underscore the continued interest in understanding the spectral characteristics of elementary operators, building upon earlier foundational work by Lumer and Rosenblum (1959), Davis and Rosenthal (1974), and Khan and Kyle (1990). The contributions from contemporary researchers have further refined and expanded the field, providing new perspectives on the algebraic and spectral behavior of elementary operators.

While these studies have extensively investigated the spectral properties of elementary operators, the computation of the spectrum of inner product type integral transformers has not been explored. The norm inequalities of inner product type integral transformers have been studied extensively, particularly in relation to Landau's inequality, Cauchy-Schwarz inequality, and Gruss inequality. However, the numerical range and spectral properties of such integral transformers remain largely unexplored.

In this study, we focused on the numerical ranges and spectra of inner product type integral transformers in Hilbert spaces. Although significant work has been done on elementary operators, the numerical range and spectral properties of inner product-type integral transformers remain largely unexplored. Existing studies focus on norm inequalities, but a precise characterization of their numerical range, spectrum, and their relationship is lacking. This study fills this gap by investigating these properties using operator-theoretic and algebraic methods, providing new insights into the behavior of these operators in Hilbert spaces.

Chapter 3

RESEARCH METHODOLOGY

3.1 Introduction

This chapter gives the theoretical framework and analytical techniques employed to investigate the inner product-type integral operator $\Phi_{\{R,P\}}(A) = \int_\Psi R_s A P_s d\gamma(s)$. The methodology is structured around several core pillars: establishing foundational operator properties (linearity, boundedness, self-adjointness), constructing and analyzing its numerical range (both classical and algebraic), determining its spectral characteristics via two distinct methods, and exploring its behavior under specific conditions (compactness, normality). Each methodological component is directly linked to the proofs of the specific theorems and propositions presented in the results.

3.2 Foundational Operator Theory and Basic Inequalities

The initial analysis of $\Phi_{\{R,P\}}$ required establishing it as a bounded and linear operator. This foundational step was crucial for all subsequent spectral and numerical range analysis and relied on classical inequalities and definitions.

3.2.1 Linearity, Boundedness, and Norm Estimation

To prove that $\Phi_{\{R,P\}}$ is a bounded linear operator, the methodology involved a direct application of the definitions of linearity and operator norm. The linearity was verified by checking the property $\Phi_{\{R,P\}}(\zeta A+\eta B) = \zeta\Phi_{R,P}(A)+\eta\Phi_{\{R,P\}}(B)$ under the integral, leveraging the linearity of integration. Boundedness was established using the integral form of the triangle inequality and the Cauchy-Schwartz inequality for integrals: A sharper norm estimate was derived using the Cauchy-Schwartz inequality to the integral itself.

3.2.2 Self-Adjointness via Quadratic Forms

The proof that R_s and P_s are self-adjoint utilized the fundamental property that an operator is self-adjoint if and only if its quadratic form is real. This was shown using the conjugate symmetry of the inner product. This result was then directly applied to conclude that the inner product type integral transformer $\Phi_{\{R,P\}}$ is itself self-adjoint under these conditions.

3.3 Elementary Operators Framework

This framework provided intuition and results that were then generalized to the integral case $\Phi_{\{R,P\}}$ through limiting arguments and measure-theoretic considerations.
The analysis of the finite sum case $R(X) = \sum_{i=1}^{n} \pi_i X \delta_i$ provided a simplified model and concrete examples to illustrate the theory. The methodology here:

- Relied on the multiplicative property of spectra under the homomorphisms $^+$ and $^-$ in the abelian setting to establish spectral inclusion.
- Used the operator norm inequality to bound the numerical range.
- Applied Gelfand's formula of the spectral radius and the submultiplicativity of the operator norm to prove the spectral radius inequality.
- Deduced a numerical radius bound from the spectral radius inequality.

3.4 Algebraic Numerical Range Construction

A significant portion of the methodology focused on defining and analyzing the numerical range of $\Phi_{\{R,P\}}$. The algebraic numerical range $\mathcal{V}(\cdot)$ was the primary tool, constructed using states on the Banach algebra. The algebraic numerical range was defined using a state on the operator algebra. The key technique involved defining a specific state using rank-one operators and the trace functional (Fialkow, 1985): For any operator $A \in L(K)$ and for an element $a \in K$, the trace of A is obtained as: $g_{a\otimes a}(A) = \operatorname{tr}(A(a\otimes a)) = \langle Aa, a\rangle$. This technique was central to proving the initial inclusion $\overline{\operatorname{Co}\left\{\int_{\Psi} \kappa_s d\gamma(s)\right\}} \subset V(\Phi_0)$ for the simpler operator Φ_0 , which was then generalized to the operator $\Phi_{\{R,P\}}$.

3.4.1 Generalization to $\Phi_{\{R,P\}}$

The methodology was generalized to $\Phi_{\{R,P\}}$ by first defining its classical numerical range and its algebraic numerical range. The link between them was established using a state applied to the tensor product form $R_s \otimes P_s$. The convex hull inclusion, was then established by use of the compactness and convexity of the algebraic numerical range.

3.4.2 Properties of the Algebraic Numerical Range

The fundamental properties of the algebraic numerical range such as invariance under unitary transformations, subadditivity, among others were proven by systematically applying the properties of the state functional initially defined to the integral definition of $\Phi_{\{R,P\}}$.

3.4.3 Convexity via the Toeplitz-Hausdorff Theorem

The proof of convexity for $V(\Phi_{\{R,P\}})$ leveraged the classical Toeplitz-Hausdorff Theorem, which states that the classical numerical range of any bounded operator on a Hilbert space is convex. The methodology involved restricting $\Phi_{\{R,P\}}$ to a two-dimensional subspace and using the result that $W(P_S\Phi_{\{R,P\}}|_S) \subseteq W(\Phi_{\{R,P\}})$. Since the numerical range on a two-dimensional space is convex, the convex combination of any two points in $V(\Phi_{\{R,P\}})$ must also lie within it (Otae, 2017).

3.4.4 Numerical Radius Properties

The properties of the numerical radius were established by directly applying the classical definition of the numerical radius of operators in K , and using the properties of the state functional and the operator norm.

3.5 Spectral Theory Techniques

The methodology for determining the spectrum employed two distinct but complementary approaches.

Spectral Decomposition Method

The first approach involved a direct analysis using the standard partition of the spectrum into point spectrum , continuous spectrum, and residual spectrum .

Proving that the residual spectrum is empty by contradiction, relying on the fact that the algebraic numerical range of $\Phi_{\{R,P\}}$ is real. This led to the conclusion that the spectrum is equal to the point spectrum

Abelian Algebra and Commutants Method

The second, more powerful method for characterizing the entire spectrum of the inner product type integral transformer $\Phi_{\{R,P\}}$ utilized techniques from the theory of von Neumann algebras and commutants. The methodology was as follows:

Left/Right Multiplication Operators: Define the maps $^+$ and $^-$ from $\mathcal{L}(\mathcal{K})$ to $\mathcal{L}(\mathcal{L}(\mathcal{K}))$ by $R_s^+(X) = R_sX$ and $P_s^-(X) = XP_s$. These maps are isometric isomorphisms.

Commutativity: Note that P_s^+ and R_s^- commute: $P_s^+ R_s^-(X) = P_s^+(XR_s) = P_s X R_s = R_s^-(P_s X) = R_s^- P_s^+(X)$.

Abelian Algebra: The set $\xi = \{P_s^+, R_s^-\}$ generates an abelian von Neumann algebra ξ''

Spectral Mapping: The spectral identities $\Omega(P_s, \mathcal{L}(\mathcal{K})) = \Omega(P_s^+, \mathcal{L}(\mathcal{L}(\mathcal{K})))$.

Multiplicative Linear Functionals: Applying multiplicative linear functionals ϕ on this abelian algebra ξ'' to the operator $\Phi_{\{R,P\}} = \int_\Psi P_s^+ R_s^- d\gamma(s)$ yielded:

$$\phi(\Phi_{R,P}) = \int_\Psi \phi(P_s^+)\phi(R_s^-)d\gamma(s).$$

Spectral Identification: The values $\phi(P_s^+)$ and $\phi(R_s^-)$ are identified with points in the spectra $\Omega(P_s)$ and $\Omega(R_s)$, respectively,which helped to obtain the desired spectrum.

Spectral Radius

The spectral radius was defined using the classical definition of the spectral radius for operators in K.

3.6 Relationship and Inclusion Theorems

3.6.1 Numerical Range and Spectrum

The critical link between the numerical range and the spectrum was established using two key methodologies:

The methodology involved constructing a state h explicitly from a sequence of unit vectors proving that every point in the closure of the classical numerical range corresponds to a state, and hence belongs to the algebraic numerical range.

The relationship between the numerical of inner product type integral transformers and the spectrum relied on the fundamental Spectral Inclusion Theorem (Kubrusly, 2012), which holds for any bounded linear operator.

3.7 Analysis of Special Operator Classes

The methodology included analyzing $\Phi_{\{R,P\}}$ under specific constraints to derive stronger results.

3.7.1 Compact Operators

The results for compact operators are a direct application of the standard Riesz-Schauder Theory for compact operators. The methodology simply involved stating that if $\Phi_{\{R,P\}}$ is compact, then its spectrum is a countable set of eigenvalues with zero as the only possible accumulation point.

3.7.2 Normal and Self-Adjoint Operators

For normal operators, the methodology invoked the well-known result that the convex hull of the spectrum equals the closure of the numerical range.

Chapter 4

RESULTS AND DISCUSSION

4.1 Introduction

We present important findings on the inner product-type integral operator $\Phi_{\{R,P\}}$, focusing on its linearity, boundedness, and N_R properties. The elementary operators are analyzed, linking their numerical ranges and spectra to their component operators, providing a comprehensive framework for understanding integral operators in Hilbert and Banach spaces.We establishe conditions for self-adjointness, convexity of the algebraic N_R ,and the inclusion of the spectrum in its closure. Properties of the numerical radius are investigated, highlighting stability and spectral behavior. Additionally, compactness results indicate that the spectrum is composed entirely of eigenvalues, with zero being the only possible accumulation point.

4.2 Preliminary Results

We begin by establishing the fundamental definitions, notations, and concepts that are used throughout the analysis of the inner product type integral operator $\Phi_{\{R,P\}}$. These foundational concepts establish the essential mathematical framework required to analyze the properties of the N_R, spectral theory, and operator inequalities.
Let (Ψ, Σ, γ) represent a measure space, with Ψ denoting a set, Σ being a σ-algebra comprising of measurable subsets of Ψ, and γ defines a measure on Σ. Given a complex Hilbert space K, we consider operator-valued functions $R_s, P_s : K \to K$ that are linear

and bounded for each $s \in \Psi$. The inner product type integral operator is defined as

$$\Phi_{\{R,P\}}(A) = \int_{\Psi} R_s A P_s d\gamma(s),$$

for any $BLO\ \ A$ on K. The study of $\Phi_{\{R,P\}}$ involves analyzing its boundedness, numerical range, spectral properties, and convexity conditions.

The following theorem establishes the fundamental boundedness property of the inner product-type integral operator $\Phi_{\{R,P\}}$. It shows that under appropriate measurability conditions, this operator is bounded, with an explicit norm estimate that depends on the norms of the constituent operator-valued functions R_s and P_s.

Theorem 4.1. *Let (Ψ, Σ, γ) denote a measure space, and let $R, P \colon \Psi \to \mathcal{L}(K)$ be measurable operator-valued functions, where K is a Hilbert space. Define the integral operator $\Phi_{\{R,P\}} \colon \mathcal{L}(K) \to \mathcal{L}(K)$ by $\Phi_{\{R,P\}}(A) = \int_{\Psi} R_s A P_s \, d\gamma(s), s \in \Psi$. Then $\Phi_{\{R,P\}}$ is a linear and bounded operator with the norm*

$$\|\Phi_{\{R,P\}}\| \leq \sqrt{\int_{\Psi} \|R_s\|^2 d\gamma(s)} \sqrt{\int_{\Psi} \|P_s\|^2 d\gamma(s)}.$$

Proof. Define $\Phi_{\{R,P\}}(A) = \int_{\Psi} R_s A P_s d\gamma(s)$. To show that $\Phi_{\{R,P\}}$ is linear, let $A, B \in L(K)$ and $\zeta, \eta \in \mathbb{K}$, where $\mathbb{K}$ is the scalar field, then

$$\begin{aligned}
\Phi_{\{R,P\}}(\zeta A + \eta B) &= \int_{\Psi} R_s(\zeta A + \eta B) P_s d\gamma(s) \\
&= \int_{\Psi} (\zeta R_s A + \eta R_s B) P_s d\gamma(s) \\
&= \int_{\Psi} (\zeta R_s A P_s + \eta R_s B P_s) d\gamma(s) \\
&= \int_{\Psi} \zeta R_s A P_s d\gamma(s) + \int_{\Psi} \eta R_s B P_s d\gamma(s) \\
&= \zeta \int_{\Psi} R_s A P_s d\gamma(s) + \eta \int_{\Psi} R_s B P_s d\gamma(s) \\
&= \zeta \Phi_{\{R,P\}}(A) + \eta \Phi_{\{R,P\}}(B).
\end{aligned}$$

Hence, $\Phi_{\{R,P\}}$ is linear.

To demonstrate that $\Phi_{\{R,P\}}$ is bounded, it should be shown that $\exists Z > 0 : \|\Phi_{\{R,P\}}(A)\| \leq Z\|A\| \ \forall \ A \in K, Z \in \mathbb{R}$.

Now,

$$
\begin{aligned}
\|\Phi_{\{R,P\}}(A)\| &= \|\int_{\Psi} R_s A P_s d\gamma(s)\| \\
&\leq \int_{\Psi} \|R_s A P_s\| d\gamma(s) \\
&\leq \int_{\Psi} \|R_s\|\|A\|\|P_s\| d\gamma(s) \\
&= \|A\| \int_{\Psi} \|R_s\|\|P_s\| d\gamma(s) \\
&= Z\|A\|, where\ Z = \int_{\Psi} \|R_s\|\|P_s\| d\gamma(s)
\end{aligned}
$$

Thus, we have $\|\Phi_{\{R,P\}}(A)\| \leq Z\|A\|$ which implies that $\Phi_{\{R,P\}}$ is bounded.
The subsequent step involves calculating the norm of $\Phi_{\{R,P\}}$. The norm definition for linear and bounded operators is used to show this.

$$
\begin{aligned}
\|\Phi_{\{R,P\}}\|^2 &= \sup\left\{\frac{\|\Phi_{\{R,P\}}A\|^2}{\|A\|^2}, A \neq 0\right\} \\
&= \sup\left\{\frac{\| \int_{\Psi} R_s A P_s d\gamma(s)\|^2}{\|A\|^2}, A \neq 0\right\} \\
&\leq \sup\left\{\frac{\int_{\Psi} \|R_s\|^2 d\gamma(s) \int_{\Psi} \|P_s\|^2 d\gamma(s) \|A\|^2}{\|A\|^2}, A \neq 0\right\} \\
&= \sup\left\{\int_{\Psi} \|R_s\|^2 d\gamma(s) \int_{\Psi} \|P_s\|^2 d\gamma(s)\right\} \\
&= \int_{\Psi} \|R_s\|^2 d\gamma(s) \int_{\Psi} \|P_s\|^2 d\gamma(s).
\end{aligned}
$$

Hence,

$$
\|\Phi_{\{R,P\}}\| \leq \sqrt{\int_{\Psi} \|R_s\|^2 d\gamma(s)} \sqrt{\int_{\Psi} \|P_s\|^2 d\gamma(s)}. \tag{4.2.1}
$$

Since R_s and P_s are self-adjoint operators as we shall demostrate in Proposition 4.2, then from equation 4.2.1 it can be deduced that

$$
\|\Phi_{\{R,P\}}\| \leq \sqrt{\int_{\Psi} \|R_s^* R_s\| d\gamma(s)} \sqrt{\int_{\Psi} \|P_s^* P_s\| d\gamma(s)}. \tag{4.2.2}
$$

□

The following result establishes a fundamental characterization of self-adjoint operators in terms of their associated quadratic forms. It shows that an operator is self-adjoint if and only if its quadratic form yields real values for all vectors in the Hilbert space. This criterion is particularly useful in the analysis of integral transformers, as it allows us to deduce self-adjointness properties from the underlying operator-valued functions.

Proposition 4.2. *Let K be a complex Hilbert space, and let $P_s : K \to K$ be a bounded linear operator for each $s \in \Psi$. If the quadratic form $\langle P_s\nu, \nu\rangle$ is real for all $\nu \in K$ and all $s \in \Psi$, then P_s is self-adjoint.*

Proof. Assume $\langle P_s\nu, \nu\rangle \in \mathbb{R}$ for every $\nu \in K$ and $s \in \Psi$. Since the inner product of any vector with itself is real, we exploit the conjugate symmetry of the inner product to deduce: $\langle P_s\nu, \nu\rangle = \overline{\langle P_s\nu, \nu\rangle} = \langle \nu, P_s\nu\rangle$. By the definition of the adjoint operator P_s^*, we have $\langle P_s\nu, \nu\rangle = \langle \nu, P_s^*\nu\rangle$. Combining these two results yields: $\langle \nu, P_s^*\nu\rangle = \langle \nu, P_s\nu\rangle \quad \forall \nu \in K$. This implies that $\langle \nu, (P_s^* - P_s)\nu\rangle = 0$ for all $\nu \in K$. By the polarization identity or the fact that a sesquilinear form vanishing on the diagonal must vanish identically, we conclude that $P_s^* = P_s$. Hence, P_s is self-adjoint. □

Remark 4.3. Proposition 4.3 shows that the operator $\Phi_{\{R,P\}}$ is self-adjoint provided P_s and R_s are self-adjoint operators.

4.3 Elementary Operators

This section analyzes elementary operators, defined by $\mathcal{R}(X) = \sum_{i=1}^{n} \pi_i X \delta_i$ for $\pi_i, \delta_i \in \mathcal{L}(K)$, to establish foundational results that are directly instrumental in achieving this chapter's primary objectives regarding the integral transformer $\Phi_{R,P}$. As a discrete analogue of $\Phi_{R,P}$, the study of $\mathcal{R}$ provides the essential theoretical framework and proof techniques for the subsequent analysis. We will demonstrate that the numerical range of an elementary operator is contained within the convex hull of products of numerical range values from its constituent operators. Furthermore, we will establish that its spectrum is bounded by products of spectral values of the component operators. Under additional assumptions of self-adjointness and normality, explicit upper bounds for the numerical and spectral radii will be derived. These properties are crucial for quantifying the stability and spectral behavior of $\Phi_{R,P}$. Thus, the results herein are not presented in isolation but form the indispensable groundwork for the core analysis that follows.

Theorem 4.4. *Let $\pi_i, \delta_i \in L(K)$ be bounded linear operators on a Hilbert space K. For the elementary operator*

$$\mathcal{R}(X) = \sum_{i=1}^{n} \pi_i X \delta_i,$$

the numerical range of $\mathcal{R}$ satisfies the inclusion

$$\mathcal{W}(\mathcal{R}) \subseteq co\left\{ \sum_{i=1}^{n} \lambda_i \mu_i : \lambda_i \in \mathcal{W}(\pi_i), \mu_i \in \mathcal{W}(\delta_i) \right\},$$

where $co(S)$ denotes the convex hull of a set S.

Proof. For any $\mathcal{X} \in L(K)$ and any unit vector $v \in K$, we have

$$\langle \mathcal{R}(\mathcal{X})v, v\rangle = \sum_{i=1}^{n} \langle \pi_i X \delta_i v, v\rangle.$$

By the definition of numerical range, for each term, $\langle \pi_i X \delta_i v, v\rangle = \lambda_i \mu_i$, where $\lambda_i \in \mathcal{W}(\pi_i)$ and $\mu_i \in \mathcal{W}(\delta_i)$. Since the numerical range is convex and preserves convex combinations, the claim follows. $\square$

Example 4.5. Consider the operators:

$$\pi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which act on $\mathbb{C}^2$. Define the elementary operator R as $(\mathcal{R})(\mathcal{X}) = \pi \mathcal{X} \delta$. The numerical ranges of π and δ are given by: $\mathcal{W}(\pi) = [0, 1]$, $\mathcal{W}(\delta) = \{0\}$. The numerical range of $(\mathcal{R})$ satisfies:

$$\mathcal{W}((\mathcal{R})) \subseteq \text{co}\{\lambda\mu : \lambda \in \mathcal{W}(\pi), \mu \in \mathcal{W}(\delta)\} = \text{co}([0, 1] \cdot \{0\}) = \{0\}.$$

To verify this inclusion, let

$$\mathcal{X} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

Then, applying R gives:

$$(\mathcal{R})(\mathcal{X}) = \pi\mathcal{X}\delta = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}.$$

For any vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$, we compute:

$$\langle((\mathcal{R}))(X)v, v\rangle = ax\bar{y}.$$

If we choose X such that $a = 0$, then $\langle(\mathcal{R})(X)v, v\rangle = 0$, confirming that $\mathcal{W}((\mathcal{R})) = \{0\}$, which aligns with the theorem's prediction.

We now present a corresponding spectral inclusion property.

Proposition 4.6. *For any elementary operator $\mathcal{R} : L(K) \to L(K)$ of the form*

$$\mathcal{R}(\mathcal{X}) = \sum_{i=1}^{n} \pi_i \mathcal{X} \delta_i,$$

its spectrum satisfies the inclusion

$$\Omega(\mathcal{R}) \subseteq \left\{ \sum_{i=1}^{n} \lambda_i \mu_i : \lambda_i \in \Omega(\pi_i), \mu_i \in \Omega(\delta_i) \right\}.$$

Proof. Let $\lambda \in \Omega(\mathcal{R})$, so $\mathcal{R} - \lambda I$ is not invertible. Suppose $\mathcal{A}_i$ and $\mathcal{B}_i$ have spectral values λ_i and μ_i, respectively. Since spectrum is invariant under bounded linear transformations and multiplication preserves spectral properties, it follows that $\lambda = \sum_{i=1}^{n} \lambda_i \mu_i$. Thus, λ lies in the specified set. □

The next lemma establishes an upper bound on the N_R of elementary multiplication operators involving SA operators.

Lemma 4.7. *Let $\pi, \delta \in L(K)$ be self-adjoint operators with numerical ranges contained in an interval $[a, b]$. Then, for the elementary multiplication operator $M_{\pi,\delta}(\mathcal{X}) = \pi\mathcal{X}\delta$,* *we have* $\sup_{\|\mathcal{X}\|=1} |\langle M_{\pi,\delta}(\mathcal{X})v, v\rangle| \leq b^2$*.*

Proof. Since π and δ are SA, their N_R are real and bounded in $[a, b]$. For any unit vector $v \in K$, $|\langle \pi\mathcal{X}\delta v, v\rangle| \leq \|\pi\|\|X\|\|\delta\|$. Since $\|\pi\|, \|\delta\| \leq b$, it follows that $\sup_{\|\mathcal{X}\|=1} |\langle M_{\pi,\delta}(\mathcal{X})v, v\rangle| \leq b^2$.

□

We now present an important spectral radius inequality for elementary operators

Theorem 4.8. *Let* $\mathcal{R}(\mathcal{X}) = \sum_{i=1}^{n} \pi_i \mathcal{X}\delta_i$ *be an elementary operator on* $L(K)$*, where* π_i, δ_i *are normal operators. Then, the spectral radius of* $\mathcal{R}$ *satisfies the inequality*

$$r(\mathcal{R}) \leq \sum_{i=1}^{n} r(\pi_i) r(\delta_i),$$

where $r(\pi) = \sup\{|\lambda| : \lambda \in \Omega(\pi)\}$ *denotes the spectral radius.*

Proof. By Gelfand's formula for spectral radius,

$$r(\mathcal{R}) = \lim_{k\to\infty} \|\mathcal{R}^k\|^{1/k}.$$

Using sub-multiplicativity of the operator norm,

$$\|\mathcal{R}^k\| \leq \sum_{i=1}^{n} \|\mathcal{A}_i^k\|\|\mathcal{B}_i^k\|.$$

Taking the kth root and the limit, we obtain

$$r(\mathcal{R}) \leq \sum_{i=1}^{n} r(\pi_i) r(\delta_i).$$

□

Example 4.9. Consider the diagonal operators:

$$\pi = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \delta = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

which are normal matrices. Define the elementary operator R as:

$$(\mathcal{R})(\mathcal{X}) = \pi\mathcal{X}\delta$$

. The spectral radii of π and δ are:

$$r(\pi) = 1, \quad r(\delta) = 3.$$

By 4.8, we have the inequality:

$$r((\mathcal{R})) \leq r(\pi)r(\delta) = 3.$$

To verify this, let $\mathcal{X}$ be an arbitrary 2×2 matrix. Then, $(\mathcal{R})(\mathcal{X})$ remains a 2×2 matrix. The eigenvalues of $(\mathcal{R})(\mathcal{X})$, though potentially complex to compute in general, can be checked in this case. The largest absolute eigenvalue of $\mathcal{R}(\mathcal{X})$ is indeed at most 3, confirming the theorem's prediction. Furthermore, in this case, the bound is sharp, meaning equality holds, demonstrating the effectiveness of the inequality.

Remark 4.10. Examples 4.5 and 4.9 illustrate the key results of 4.7 and 4.8 and provide insight into the behavior of numerical ranges and spectral radii in elementary operator settings.

As a direct consequence of 4.8, we have;

Corollary 4.11. *If $\pi, \delta \in L(K)$ are such that $\mathcal{W}(\pi)$ and $\mathcal{W}(\delta)$ are subsets of the U , then the numerical radius of the elementary operator $\mathcal{R}(\mathcal{X}) = \pi\mathcal{X}\delta$ satisfies*

$$Nr(\mathcal{R}) \leq 1.$$

Proof. The numerical radius satisfies $Nr(\pi) \leq r(\pi)$. Given that

$$r(\pi) \leq 1 \quad \text{and} \quad r(\delta) \leq 1,$$

it follows from 4.8 that

$$r(\mathcal{R}) \leq 1.$$

Thus,

$$Nr(\mathcal{R}) \leq r(\mathcal{R}) \leq 1.$$

□

In the following section, the numerical range for inner product type integral transformers on K is determined, and the properties of the obtained numerical range are discussed. First, the numerical range for the transformer $\int_{\Psi} P_s d\gamma(s)$ is obtained, and the result is generalized to the transformer of the form $\Phi_{\{P,R\}} = \int_{\Psi} R_s A P_s d\gamma(s)$.

4.4 Numerical Range of the Inner Product Type Integral Transformers on K

The algebraic numerical range for the transformation $\int_{\Psi} P_s d\gamma(s)$ on Hilbert spaces is defined as follows:

Definition 4.12. Let K be a complex separable Hilbert space, and let $L(K)$ represent the space of all bounded linear operators. Consider a measure space (Ψ, Σ, γ), and let $P : \Psi \to L(K)$ be an operator-valued function mapping P to P_s. Define the operator $\Phi_0 = \int_{\Psi} P_s d\gamma(s)$, and let $\Re(\Psi)$ denote a collection of states on Ψ. The algebraic numerical range $\mathcal{V}$ of Φ_0 is given by:

$$\mathcal{V}(\Phi_0) = \{h(\Phi_0) : h \in \Re(\Psi)\}.$$

The following theorem establishes an important containment relationship for the integral transformer Φ_0. It shows that the convex hull of the closure of certain integral values derived from the numerical range of the operator-valued function P_s is contained within the algebraic numerical range of the transformer Φ_0. This result demonstrates how the geometric structure of the numerical range captures the integrated behavior of the operator family $\{P_s\}$.

Theorem 4.13. *Let* $\Phi_0 = \int_{\Psi} P_s d\gamma(s)$, *be an integral transformer, then*

$$Co\overline{\{\int_\Psi \kappa_s d\gamma(s)\}} \subset \mathcal{V}(\Phi_0),\ where\ \int_\Psi \kappa_s d\gamma(s) \in \mathcal{V}(\Phi_0).$$

Proof. First, a state is defined on $L(K)$ as follows. Let $\int_\Psi \kappa_s d\gamma(s) \in \mathcal{V}(\Phi_0)$. For any $a \in H$, with $\|a\| = 1$, define a linear form $g_{a\otimes a} \in L(K)$ by

$$g_{a\otimes a}(A) = tr(A_{a\otimes a}) = \langle Aa, a\rangle, \text{ where } A \text{ is the rank one transformation and } tr(.) \text{ is a linear structure trace .}$$

Then,

$$\begin{aligned}
\|g_{a\otimes a}(A)\| &\leq \|A_{a\otimes a}\| \\
&\leq \|A\|\|a\otimes a\| \\
&\leq \|A\|,
\end{aligned}$$
$$\|g_{a\otimes a}\| = 1.$$

Also, for any identity operator $\mathfrak{I}$, $g_{a\otimes a}(\mathfrak{I}_K) = tr(a\otimes a) = \langle a, a\rangle = \|a\|^2 = 1$, $g_{a\otimes a}$ becomes a state on $L(K)$, $\mathfrak{I}$ is the identity operator.
Next, using Definition 4.12 $g_{a\otimes a}(\Phi_0)$ is obtained as follows:

$$\begin{aligned}
g_{a\otimes a}(\Phi_0) &= g_{a\otimes a}\left(\int_\Psi P_s d\gamma(s)\right) \\
&= tr\left(\int_\Psi P_s d\gamma(s)(a\otimes a)\right) \\
&= \left\langle \int_\Psi P_s d\gamma(s)a, a\right\rangle \\
&= \int_\Psi \langle P_s a, a\rangle\, d\gamma(s) \\
&= \int_\Psi \kappa_s d\gamma(s), where\, \langle P_s a, a\rangle = \kappa_s.
\end{aligned}$$

Thus, $g_{a\otimes a}(\Phi_0) = \int_\Psi \kappa_s d\gamma(s) \subset \mathcal{V}(\Phi_0)$.
The algebraic numerical range is known to be compact and convex, therefore,

$$Co\overline{\{\int_\Psi \kappa_s d\gamma(s)\}} \subset \mathcal{V}(\Phi_0).$$

$\square$

The algebraic numerical range for the inner product type integral transformer $\Phi_{\{P,R\}}$ is defined using the classical definition of the algebraic numerical range for linear operators as follows;

Definition 4.14. For the operator $\Phi_{\{R,P\}}$, the algebraic numerical range of $\Phi_{\{R,P\}}$ is given as

$$\mathcal{V}(\Phi_{\{R,P\}}) = \left\{\eta(\Phi_{\{R,P\}}) : \eta \in \Re(\Psi)\right\}$$

where $\Re(\Psi)$ is the set of states in Ψ.

Using the classical definition of the usual numerical range, the following definition is obtained

Definition 4.15. Let $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$, then the usual numerical range $W(\Phi_{\{R,P\}})$ of the operator $\Phi_{\{R,P\}}$ is defined as

$$\begin{aligned}
W(\Phi_{\{R,P\}}) &= \left\{\langle(\int_\Psi R_s \otimes P_s d\gamma(s))\nu, \nu\rangle : \nu \in K, ||\nu|| = 1\right\} \\
&= \left\{\int_\Psi \langle(R_s \otimes P_s)\nu, \nu\rangle d\gamma(s)) : \nu \in K, ||\nu|| = 1\right\} \\
&= \left\{\int_\Psi \langle R_s\nu, \nu\rangle\langle P_s\nu, \nu\rangle d\gamma(s)) : \nu \in K, ||\nu|| = 1\right\} \\
&= \left\{\int_\Psi \kappa_s\eta_s d\gamma(s)), where\ \kappa_s = \langle R_s\nu, \nu\rangle,\ \ and\ \eta_s = \langle P_s\nu, \nu\rangle\right\}.
\end{aligned}$$

The following lemma establishes fundamental relationships between the numerical ranges of tensor product operators and their components. It shows that the product of numerical range values of the component operators belongs to the numerical range of their tensor product, and that the classical numerical range is contained within the algebraic numerical range for such operators..

Lemma 4.16. *Let $\kappa_s \in W(R_s)$ and $\zeta_s \in W(P_s)$, where $W(R_s)$ is the usual numerical range of R_s and $W(P_s)$ is the usual numerical range of P_s , then $\kappa_s\zeta_s \in W(R_s \otimes P_s)$ and $W(R_s \otimes P_s) \subset \mathcal{V}(R_s \otimes P_s)$*

Proof. Let ν, η be unit vectors such that $\kappa_s = \langle R_s\nu, \nu\rangle$ and $\zeta_s = \langle P_s\eta, \eta\rangle$. Then the tensor product $\nu \otimes \eta$ is a unit vector in the tensor product space, and:

$$\begin{aligned}
\kappa &= \langle (R_s \otimes P_s)(\nu \otimes \eta), \nu \otimes \eta\rangle, where \|\nu \otimes \eta\| = \|\nu\| = \|\eta\| = 1, with\, \eta, \nu \in K \\
&= tr\{\nu \otimes \eta(R_s \otimes P_s)\nu \otimes \eta\} \\
&= tr(R_s\nu \otimes P_s\eta)(\nu \otimes \eta) \\
&= tr(R_s\nu \otimes \nu)(P_s\eta \otimes \eta) \\
&= \langle R_s\nu, \nu\rangle\langle P_s\eta, \eta\rangle \\
&= \kappa_s\zeta_s
\end{aligned}$$

Thus $\kappa_s\zeta_s \in W(R_s \otimes P_s)$.

To establish the inclusion $W(R_s \otimes P_s) \subset \mathcal{V}(R_s \otimes P_s)$, we begin by defining a state function as follows:

$g_{\nu\otimes\eta}(I_K) = tr(\nu \otimes \eta)(\nu \otimes \eta) = tr(\nu \otimes \nu)(\eta \otimes \eta) = \langle \nu, \nu\rangle\langle \eta, \eta\rangle = \|\nu\|^2\|\eta\|^2 = 1$, then

$$\begin{aligned}
|g_{\zeta\otimes\eta}(A)| &\leq \|A_{(\zeta\otimes\eta)}(\zeta \otimes \eta)\| \\
&\leq \|\zeta \otimes \eta\| \times \|A(\zeta \otimes \eta)\| \\
&\leq \|A(\zeta \otimes \eta)\| \\
&\leq \|A\|
\end{aligned}$$

This implies that $\|g_{\zeta\otimes\eta}\| = 1$. Since $g_{\zeta\otimes\eta}(I_K) = tr(\zeta \otimes \eta)(\zeta \otimes \eta) = tr(\zeta \otimes \zeta)(\eta \otimes \eta) = \langle \zeta, \zeta\rangle\langle \eta, \eta\rangle = \|\zeta\|^2\|\eta\|^2 = 1$, then $\|g_{\zeta\otimes\eta}\| = g_{\zeta\otimes\eta}(I_K) = 1$; therefore $g_{\zeta\otimes\eta}$ is a state on the space of bounded linear operators on $L(K)$.

By Definition 4.12, replacing Φ_0 with $R_s \otimes P_s$, it deduces to

$$\begin{aligned}
g_{\nu\otimes\eta}(R_s \otimes P_s) &= tr\{\nu \otimes \eta(R_s \otimes P_s)\nu \otimes \eta\} \\
&= tr\{\langle R_s\nu, \nu\rangle(\eta \otimes P_s^*\eta)\} \\
&= \langle R_s\nu, \nu\rangle\langle P_s\eta, \eta\rangle \\
&= \kappa_s\zeta_s \\
&= \kappa, where\, \kappa = \kappa_s\zeta_s \\
&\Rightarrow \kappa \in \mathcal{V}(R_s \otimes P_s).
\end{aligned}$$

Therefore, $\{\kappa_s\zeta_s : \kappa_s \in W(R_s), \zeta_s \in W(P_s)\} \subset \mathcal{V}(R_s \otimes P_s)$.
Hence

$$W(R_s \otimes P_s) \subset \mathcal{V}(R_s \otimes P_s).$$

□

The following result establishes that the classical numerical range of the inner product type integral transformer $\Phi_{\{R,P\}}$ contains all integrals of the form $\int_\Psi \kappa_s\zeta_s d\gamma(s)$, where κ_s and ζ_s belong to the numerical ranges of R_s and P_s, respectively. This provides a concrete method for generating elements of $W(\Phi_{\{R,P\}})$ from the numerical ranges of the component operators.

Proposition 4.17. *Let $\kappa_s \in W(R_s)$ and $\zeta_s \in W(P_s)$, where $W(R_s)$ is the classical numerical range of R_s, then $\int_\Psi \kappa_s\zeta_s d\gamma(s) \in W(\Phi_{\{R,P\}})$.*

Proof. Using the state function as in 4.16, we have

$$\begin{aligned}
g_{\zeta\otimes\eta}(\int_\Psi R_s \otimes P_s d\gamma(s)) &= tr\{(\zeta \otimes \eta)(\int_\Psi R_s \otimes P_s d\gamma(s))(\zeta \otimes \eta)\} \\
&= \int_\Psi tr\{\langle R_s\zeta, \zeta\rangle(\eta \otimes P_s^*\eta)\}d\gamma(s) \\
&= \int_\Psi \langle R_s\zeta, \zeta\rangle\langle P_s\eta, \eta\rangle d\gamma(s) \\
&= \int_\Psi \kappa_s\nu_s d\gamma(s)
\end{aligned}$$

Since $\kappa_s \in W(R_s)$ and $\nu_s \in W(P_s)$, then $\int_\Psi \kappa_s\nu_s d\gamma(s) \in W(\Phi_{\{R,P\}})$. □

The following theorem establishes a fundamental convexity property of the algebraic numerical range for inner product type integral transformers. It demonstrates that under the assumption that certain integrated products of numerical range values lie within the algebraic numerical range, the closed convex hull of all such integrated products must also be contained within the algebraic numerical range. This result extends the classical Toeplitz-Hausdorff convexity theorem to the setting of integral transformers and provides a crucial tool for understanding the structural properties of these operators.

Theorem 4.18. *Let* $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$*, and suppose that*

$$\left\{ \int_\Psi \kappa_s \nu_s d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s) \right\} \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

Then, it follows that

$$\overline{Co\left(\left\{ \int_\Psi \kappa_s \nu_s d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s) \right\} \right)} \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

Proof. Let $\zeta, \eta \in K$ with $\|\zeta\| = \|\eta\| = 1$. Recall the definition of the state as given in 4.16, i.e define a linear functional g as

$$g_{\zeta\otimes\eta}(A) = tr\{(A_{(\zeta\otimes\eta)}(\zeta \otimes \eta)\}, A \in L(K)$$

whereby tr is the trace function, with $\|\zeta \otimes \eta\| = \|\zeta\| = \|\eta\| = 1$, then $g_{\zeta\otimes\eta}$ is a state on $L(L(K))$ as already been shown in the proof of Lemma 4.16.
For $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$, let $\mathcal{Q} = R_s \otimes P_s$, for the operator $\mathcal{Q}$, we have

$$\begin{aligned}
g_{\zeta\otimes\eta}(\mathcal{Q}) &= tr\{\mathcal{Q}_{\zeta\otimes\eta}(\zeta \otimes \eta)\} \\
&= tr\{(R_s \otimes P_s)_{\zeta\otimes\eta}(\zeta \otimes \eta)\} \\
&= tr\{(R_s\zeta \otimes P_s\eta)(\zeta \otimes \eta)\} \\
&= tr\{(R_s(\zeta \otimes \zeta))(P_s(\eta \otimes \eta))\} \\
&= \langle R_s\zeta, \zeta\rangle\langle P_s\eta, \eta\rangle
\end{aligned}$$

This implies that

$$\begin{aligned} g_{\zeta\otimes\eta}(\int_{\Psi} R_s \otimes P_s d\gamma(s)) &= \int_{\Psi} \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle d\gamma(s) \\ &= \int_{\Psi} \kappa_s \nu_s d\gamma(s). \end{aligned}$$

Thus,

$$\big\{\textstyle\int_{\Psi} \kappa_s\nu_s d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s)\big\} \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

The classic algebraic numerical range of a linear operator is convex and compact, therefore,

$$Co\overline{\big\{\textstyle\int_{\Psi} \kappa_s\nu_s d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s)\}\big\}} \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

□

In the next theorem, we show that the fundamental properties of the classical algebraic numerical range also hold for the operator $\Phi_{\{R,P\}} = \int_{\Psi} R_sAP_sd\gamma(s)$.

Theorem 4.19. *The algebraic numerical range of inner product type integral transformers satisfies the following properties:*

i. The algebraic N_R is closed under taking adjoints, i.e.,

$$\mathcal{V}(\Phi^*_{\{R,P\}}) = \overline{\mathcal{V}(\Phi_{\{R,P\}})},$$

where $\Phi^*_{\{R,P\}}$ denotes the adjoint of $\Phi_{\{R,P\}}$.

ii. The algebraic N_R remains invariant under unitary similarity transformations:

$$\mathcal{V}(Y^*\Phi_{\{R,P\}}Y) = \mathcal{V}(\Phi_{\{R,P\}}),$$

for any unitary operator Y.

iii. The algebraic N_R satisfies subadditivity:

$$\mathcal{V}(\Phi_{\{R,P\}} + N_{\{R,P\}}) \subseteq \mathcal{V}(\Phi_{\{R,P\}}) + \mathcal{V}(N_{\{R,P\}}).$$

iv. For scalars $\kappa, \nu \in \mathbb{K}$ and the identity operator I, the algebraic N_R satisfies the affine transformation property:

$$\mathcal{V}(\kappa\Phi_{\{R,P\}} + \nu I) = \kappa\mathcal{V}(\Phi_{\{R,P\}}) + \nu.$$

Proof. i. Let $\mathcal{Q} = R_s \otimes P_s$, then $\mathcal{Q}^* = P_s^* \otimes R_s^*$, and using the definition of a state as defined in Lemma 4.16, we have,

$$\begin{aligned}
g_{\zeta\otimes\eta}(\mathcal{Q}^*) &= tr\{\mathcal{Q}^*_{(\zeta\otimes\eta)}\zeta \otimes \eta)\} \\
&= tr\{(P_s^* \otimes R_s^*)_{(\zeta\otimes\eta)}(\zeta \otimes \eta)\} \\
&= tr\{(P_s^*\zeta \otimes R_s^*\eta)(\zeta \otimes \eta)\} \\
&= tr\{(P_s^*(\zeta \otimes \zeta))(R_s^*(\eta \otimes \eta))\} \\
&= \langle P_s^*\zeta, \zeta\rangle\langle R_s^*\eta, \eta\rangle \\
&= \langle R_s^*\eta, \eta\rangle\langle P_s^*\zeta, \zeta\rangle
\end{aligned}$$

This implies that

$$\begin{aligned}
g_{\zeta\otimes\eta}(\Phi^*_{\{R,P\}}) &= \int_\Psi \langle R_s^*\eta, \eta\rangle\langle P_s^*\zeta, \zeta\rangle d\gamma(s) \\
&= \int_\Psi \langle \eta, R_s\eta\rangle\langle \zeta, P_s\zeta\rangle d\gamma(s) \\
&= \int_\Psi \overline{\kappa_s\nu_s} d\gamma(s),
\end{aligned}$$

$\kappa_s = \langle \eta, R_s\eta\rangle$ and $\nu_s = \langle \zeta, P_s\zeta\rangle$.

Since $\left\{\int_\Psi \overline{\kappa_s\nu_s} d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s)\right\} = \overline{\mathcal{V}(\Phi_{\{R,P\}})}$, then $\mathcal{V}(\Phi^*_{\{R,P\}}) = \overline{\mathcal{V}(\Phi_{\{R,P\}})}$.

ii. Let $\mathcal{Q} = Y^*R_s \otimes P_sY$, then

$$\begin{aligned} g_{\zeta\otimes\eta}(\mathcal{Q}) &= tr\{\mathcal{Q}_{\zeta\otimes\eta}(\zeta\otimes\eta)\} \\ &= tr\{(Y^*R_s\otimes P_sY)_{\zeta\otimes\eta}(\zeta\otimes\eta)\} \\ &= tr\{(Y^*R_s\zeta\otimes P_sY\eta)(\zeta\otimes\eta)\} \\ &= tr\{(Y^*R_s(\zeta\otimes\zeta))(P_sY(\eta\otimes\eta))\} \\ &= \langle Y^*R_s\zeta,\zeta\rangle\langle YP_s\eta,\eta\rangle \end{aligned}$$

This implies that

$$\begin{aligned} g_{\zeta\otimes\eta}(\Phi_{\{R,P\}}) &= \int_\Psi \langle Y^*R_s\zeta,\zeta\rangle\langle YP_s\eta,\eta\rangle d\gamma(s) \\ &= \int_\Psi Y^*Y\langle R_s\zeta,\zeta\rangle\langle P_s\eta,\eta\rangle d\gamma(s) \\ &= \int_\Psi \langle R_s\zeta,\zeta\rangle\langle P_s\eta,\eta\rangle d\gamma(s) \quad (Since \;\; Y^*Y = YY^* = I) \\ &= \int_\Psi \kappa_s\nu_s d\gamma(s) \end{aligned}$$

It implies that $\left\{\int_\Psi \kappa_s\nu_s d\gamma(s) : \kappa_s \in W(R_s), \nu_s \in W(P_s)\right\} \subset \mathcal{V}(\Phi_{\{R,P\}})$.

iii. Let $\Phi_{\{R,P\}} = \int_\Psi R_s\otimes P_s d\gamma(s), S_{\{R,P\}} = \int_\Psi E_s\otimes F_s d\gamma(s)$, then

$$\begin{aligned} \Phi_{\{R,P\}} + S_{\{R,P\}} &= \int_\Psi R_s\otimes P_s d\gamma(s) + \int_\Psi E_s\otimes F_s d\gamma(s) \\ &= \int_\Psi R_s\otimes P_s + E_s\otimes F_s d\gamma(s) \end{aligned}$$

Let $\mathcal{G} = R_s \otimes P_s + E_s \otimes F_s$, then

$$\begin{aligned}
g_{\nu\otimes\eta}(\mathcal{G}) &= tr\{\mathcal{G}_{(\nu\otimes\eta}(\nu \otimes \eta)\} \\
&= tr\{(R_s \otimes P_s + E_s \otimes F_s)_{\nu\otimes\eta}(\nu \otimes \eta)\} \\
&= tr\{(R_s \otimes P_s(\nu \otimes \eta) + E_s \otimes F_s(\nu \otimes \eta))((\nu \otimes \eta)\} \\
&= tr\{((R_s\nu \otimes P_s\eta) + (E_s\nu \otimes F_s\eta))(\nu \otimes \eta)\} \\
&= tr\{(R_s\nu \otimes P_s\eta) + (\nu \otimes \eta)(E_s\nu \otimes F_s\eta)((\nu \otimes \eta)\} \\
&= tr\{(R_s\nu \otimes \nu)(P_s\eta \otimes \eta) + (E_s\nu \otimes \nu)(F_s\eta \otimes \eta)\} \\
&= \langle R_s\nu, \nu\rangle\langle P_s\eta, \eta\rangle + \langle E_s\nu, \nu\rangle\langle F_s\eta, \eta\rangle
\end{aligned}$$

This implies that

$$\begin{aligned}
g_{\nu\otimes\eta}(\Phi_{\{R,P\}} + S_{\{R,P\}}) &= \int_\Psi \langle R_s\nu, \eta\rangle\langle P_s\eta, \eta\rangle + \langle E_s\nu, \nu\rangle\langle F_s\eta, \eta\rangle d\gamma(s) \\
&= \int_\Psi \langle R_s\nu, \nu\rangle\langle P_s\eta, \eta\rangle d\gamma(s) + \int_\Psi \langle E_s\nu, \nu\rangle\langle F_s\eta, \eta\rangle d\gamma(s) \\
&= \int_\Psi \kappa_s\nu_s d\gamma(s) + \int_\Psi \mu_s\xi_s d\gamma(s)
\end{aligned}$$

where $\kappa_s = \langle R_s\nu, \nu\rangle, \nu_s = \langle P_s\eta, \eta\rangle, \mu_s = \langle E_s\nu, \nu\rangle$ and $\xi_s = \langle F_s\eta, \eta\rangle$

Which therefore means that

$$\begin{aligned}
&\left\{ \int_\Psi \kappa_s\nu_s \, d\gamma(s) \ : \ \kappa_s \in W(R_s), \quad \nu_s \in W(P_s) \right\} \\
+&\left\{ \int_\Psi \mu_s\xi_s \, d\gamma(s) \ : \ \mu_s \in W(E_s), \quad \xi_s \in W(F_s) \right\} \\
&\subset \mathcal{V}(\Phi_{\{R,P\}}) + \mathcal{V}(S_{\{R,P\}}).
\end{aligned}$$

iv. $\mathcal{V}(\kappa\Phi_{\{R,P\}} + \mu I) = \kappa\mathcal{V}(\Phi_{\{R,P\}}) + \mu$

Now, $\kappa\Phi_{\{R,P\}} + \mu I = \kappa \int_\Psi R_s \otimes P_s d\gamma(s) + \mu I$

Let $\mathcal{Q} = \kappa(R_s \otimes P_s) + \mu I$, then

$$
\begin{aligned}
g_{\nu\otimes\eta}(\mathcal{Q}) &= tr\{\mathcal{Q}_{(\nu\otimes\eta)}(\nu\otimes\eta)\} \\
&= tr\{(\kappa(R_s\otimes P_s)+\mu I)_{(\nu\otimes\eta)}(\nu\otimes\eta)\} \\
&= tr\{(\kappa(R_s\otimes P_s)_{\nu\otimes\eta}+\mu I_{\nu\otimes\eta})(\nu\otimes\eta)\} \\
&= tr\{(\kappa(P_s\nu\otimes P_s\eta)+\mu(\nu\otimes\eta))(\nu\otimes\eta)\} \\
&= tr\{\kappa(R_s\nu\otimes P_s\eta)(\eta\otimes\nu)+\mu(\nu\otimes\eta)(\eta\otimes\nu)\} \\
&= tr\{\kappa(R_s\nu\otimes\nu)(P_s\eta\otimes\eta)+\mu(\nu\otimes\nu)(\eta\otimes\eta))\} \\
&= \kappa\langle R_s\nu,\nu\rangle\langle P_s\eta,\eta\rangle+\mu\langle\nu,\nu\rangle\langle\eta,\eta\rangle \\
&= \kappa\langle R_s\nu,\nu\rangle\langle P_s\eta,\eta\rangle+\mu
\end{aligned}
$$

This implies that

$$
\begin{aligned}
g_{\nu\otimes\eta}(\kappa\Phi_{\{R,P\}}+\nu I) &= \int_{\Psi}\kappa\langle R_s\nu,\nu\rangle\langle P_s\eta,\eta\rangle d\gamma(s)+\mu \\
&= \int_{\Psi}\kappa\xi_s\zeta_s d\gamma(s)+\mu
\end{aligned}
$$

Hence

$$
\kappa\left\{\int_{\Psi}\xi_s\zeta_s d\gamma(s):\xi_s\in W(R_s),\zeta_s\in W(R_s)\right\}+\mu\subset\kappa\mathcal{V}(\Phi_{\{R,P\}})+\mu.
$$

□

The following theorem establishes an important geometric property of the algebraic numerical range of inner product type integral transformers. Specifically, it demonstrates that the set of all values obtained by applying states to the operator $\Phi_{\{R,P\}}$ forms a convex subset of the complex plane. This result extends the well-known Toeplitz-Hausdorff theorem for classical numerical ranges to the algebraic numerical range setting for integral operators of the form $\Phi_{\{R,P\}}(A)=\int_{\Psi}R_sAP_sd\gamma(s)$. The convexity property is fundamental for understanding the structural characteristics of these operators and their behavior under various functional-analytic operations.

Theorem 4.20. *The algebraic numerical range $\mathcal{V}(\Phi_{\{R,P\}})$ of $\Phi_{\{R,P\}}$ is convex.*

Proof. To establish that the algebraic numerical range $\mathcal{V}(\Phi_{\{R,P\}})$ of $\Phi_{\{R,P\}}$ is convex, we demonstrate that every line intersecting $\mathcal{V}(\Phi_{\{R,P\}})$ forms either a connected set or is empty. Since $\Phi_{\{R,P\}}$ is a linear and bounded operator on K (from Theorem 4.1), consider two distinct points in $W(\Phi_{\{R,P\}})$ given by

$$\int_\Psi \xi_s \nu_s d\gamma(s) = \langle \Phi_{\{R,P\}}\zeta, \zeta\rangle$$

and

$$\int_\Psi \eta_s \kappa_s d\gamma(s) = \langle \Phi_{\{R,P\}}\rho, \rho\rangle.$$

Here, ζ and ρ are distinct vectors in K. We claim that they are linearly independent. Suppose, for contradiction, that $\zeta = x\rho$ for some scalar $x \in \mathbb{C}$ with $|x| = 1$, where $\|\zeta\| = 1 = \|\rho\|$. Then,

$$\int_\Psi \xi_s \nu_s d\gamma(s) = \langle \Phi_{\{R,P\}}\zeta, \zeta\rangle = \langle \Phi_{\{R,P\}}x\rho, x\rho\rangle = \langle x\Phi_{\{R,P\}}\rho, x\rho\rangle = \int_\Psi \eta_s \kappa_s d\gamma(s).$$

This contradicts the assumption that these two points are distinct. Hence, ζ and ρ must be linearly independent. Define $S = \text{span}\{\zeta, \rho\}$, which forms a two-dimensional subspace of K. Let T_S be the projection onto S, with $T \in L(K)$. Since $\zeta, \rho \in S$, it follows that

$$\int_\Psi \xi_s \nu_s d\gamma(s), \int_\Psi \eta_s \kappa_s d\gamma(s) \in W(T_S M_{\{R,P\}}|_S) \subseteq W(\Phi_{\{R,P\}}) \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

Because S is a two-dimensional subspace of K, the set $W(T_S\Phi_{\{R,P\}}|_S)$ is convex. Consequently, for any $0 < \Upsilon < 1$,

$$\Upsilon \int_\Psi \xi_s \nu_s d\gamma(s) + (1-\Upsilon)\int_\Psi \eta_s \kappa_s d\gamma(s) \in W(T_S\Phi_{\{R,P\}}|_S) \subseteq W(\Phi_{\{R,P\}}) \subset \mathcal{V}(\Phi_{\{R,P\}}).$$

Since $\int_\Psi \xi_s \nu_s d\gamma(s)$ and $\int_\Psi \eta_s \kappa_s d\gamma(s)$ were arbitrary, it follows that $\mathcal{V}(\Phi_{\{R,P\}})$ is convex, as required. $\square$

To complete the discussion on the properties of the algebraic numerical range $\mathcal{V}(\Phi_{R,P})$ of the operator $M_{R,P}$, The following theorem establishes a fundamental characterization of self-adjointness for inner product type integral transformers in terms of their algebraic numerical ranges. Specifically, it shows that the operator $\Phi_{\{R,P\}}$ is self-adjoint if and only if its numerical range consists entirely of real numbers. This result connects the geometric property of the numerical range (being real) with the algebraic property of self-adjointness, providing a powerful criterion for identifying when these integral operators possess the symmetry properties characteristic of self-adjoint operators.

Theorem 4.21. *The operator* $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$ *is self-adjoint if and only if* $\mathcal{V}(\Phi_{\{R,P\}})$ *is real.*

Proof. First, we need to show that $\int_\Psi \overline{\kappa_s\xi_s} d\gamma(s) = \int_\Psi \kappa_s\xi_s d\gamma(s)$,

$$\begin{aligned}
\mathcal{V}(\Phi_{\{R,P\}}) &= \left\{ h\Big(\int_\Psi R_s \otimes P_s d\gamma(s)\Big) : h \in \Re(\Psi) \right\} \\
&= \left\{ \int_\Psi \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle d\gamma(s) \right\} \\
&= \left\{ \int_\Psi \kappa_s\xi_s d\gamma(s) : \kappa_s \in W(R_s), \xi_s \in W(P_s) \right\} \\
&= \int_\Psi \langle R_s^*\eta, \eta\rangle \langle P_s^*\zeta, \zeta\rangle d\gamma(s) \\
&= \int_\Psi \langle \eta, R_s\eta\rangle \langle \zeta, P_s\zeta\rangle d\gamma(s) \\
&= \left\{ \int_\Psi \overline{\kappa_s\xi_s} d\gamma(s) : \overline{\kappa_s} \in W(R_s^*), \overline{\xi_s} \in W(P_s^*) \right\}
\end{aligned}$$

Hence, we have that $\int_\Psi \overline{\kappa_s\xi_s} d\gamma(s) = \int_\Psi \kappa_s\xi_s d\gamma(s)$, which implies that $\mathcal{V}(\Phi_{\{R,P\}})$ is real. Conversely, from the definition of the algebraic numerical range, we have

$$h(\Phi_{\{R,P\}}) = \int_\Psi \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle d\gamma(s) = h(\Phi^*_{\{R,P\}}) = \int_\Psi \langle R_s^*\eta, \eta\rangle \langle P_s^*\zeta, \zeta\rangle d\gamma(s)$$

. This implies that, $h(\Phi_{\{R,P\}}) - h((\Phi_{\{R,P\}})^*) = \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle - \langle R_s^*\eta, \eta\rangle \langle P_s^*\zeta, \zeta\rangle$.

Since R_s and P_s are self-adjoint operators (from the proof of Proposition 4.2), we have

$$\begin{aligned}
\langle R_s\zeta, \zeta\rangle - \langle R_s^*\eta, \eta\rangle = 0, &\quad \text{for } \|\zeta\| = \|\eta\| = 1, \\
\langle P_s\zeta, \zeta\rangle - \langle P_s^*\eta, \eta\rangle = 0, &\quad \text{for } \|\zeta\| = \|\eta\| = 1.
\end{aligned}$$

Hence,

$$h(\Phi_{\{R,P\}}) - h(\Phi^*_{\{R,P\}}) = \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle - \langle R_s^*\eta, \eta\rangle \langle P_s^*\zeta, \zeta\rangle = 0.$$

This implies that $h(\Phi_{\{R,P\}}) - h(\Phi^*_{\{R,P\}}) = 0$. Thus, $h(\Phi_{\{R,P\}} - \Phi^*_{\{R,P\}}) = 0$. Since h is a linear functional on $L(K)$, it is not the zero function. Therefore, $\Phi_{\{R,P\}} - \Phi^*_{\{R,P\}} = 0$, which implies that $\Phi_{\{R,P\}} = \Phi^*_{\{R,P\}}$. Thus, we conclude that $\Phi_{\{R,P\}} = \int_\Psi R_s A P_s d\gamma$is self-adjoint. $\square$

Having determined the numerical range of the operator $\Phi_{\{R,P\}} = \int_\Psi R_s A P_s d\gamma(s)$, we can now obtain the numerical radius.

Definition 4.22. Let $\Phi_{\{R,P\}}$ be an operator. The numerical radius of $\Phi_{\{R,P\}}$, denoted by $\omega(\Phi_{\{R,P\}})$, is defined as

$$\omega(\Phi_{\{R,P\}}) = \sup \left\{ \left| \int_\Psi \kappa_s \xi_s \, d\gamma(s) \right| : \int_\Psi \kappa_s \xi_s \, d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}}) \right\},$$

where $\mathcal{V}(\Phi_{\{R,P\}})$ denotes the numerical range of the operator $\Phi_{\{R,P\}}$.

The numerical radius is a fundamental concept in operator theory that provides insight into the size and behavior of linear operators. For the inner product-type integral transformer $\Phi_{\{R,P\}}$, the numerical radius satisfies several essential properties that are analogous to those of classical numerical radii for bounded linear operators. The following theorem establishes these key properties, including positivity, homogeneity, the triangle inequality, and the relationship between the numerical radius and the operator norm. These results are crucial for understanding the quantitative aspects of $\Phi_{\{R,P\}}$ and will be used extensively in subsequent spectral and numerical range analyses.

Theorem 4.23. *The numerical radius $\omega(\Phi_{\{R,P\}})$ of $\Phi_{\{R,P\}}$ possesses the following properties:*

i. $\omega(\Phi_{\{R,P\}}) \geq 0$.

ii. $\omega(\Phi_{\{R,P\}}) = 0$ *if and only if* $\Phi_{\{R,P\}} = 0$.

iii. $\omega(\gamma\Phi_{\{R,P\}}) = |\gamma|\omega(\Phi_{\{R,P\}})$ *for any scalar* γ.

iv. $\omega(\Phi_{\{R,P\}} + S_{\{R,P\}}) \leq \omega(\Phi_{\{R,P\}}) + \omega(S_{\{R,P\}})$.

v. $\omega(\Phi_{\{R,P\}}) \leq \|\Phi_{\{R,P\}}\|$.

Proof.

i. This result directly follows from the definition of the numerical radius provided earlier 4.2.11.

$\omega(\Phi_{\{R,P\}}) = \sup\{\int_\Psi |\kappa_s\xi_s|d\gamma(s) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})\} \geq 0$ (from the property of absoluteness).

Hence, $\omega(\Phi_{\{R,P\}}) \geq 0$.

ii. Suppose $\omega(\Phi_{\{R,P\}}) = 0$

$$\begin{aligned}
&\Longrightarrow \quad \sup\{\int_\Psi |\kappa_s\xi_s|d\gamma(s) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(M_{\{R,P\}})\} = 0 \\
&\Longrightarrow \quad \int_\Psi |\kappa_s\xi_s|d\gamma(s) = 0 \\
&\Longrightarrow \quad \int_\Psi \kappa_s\xi_s d\gamma(s) = 0 \\
&\Longrightarrow \quad g_{\zeta\otimes\eta}(M_{\{R,P\}}) = \int_\Psi \langle R_s\zeta, \zeta\rangle\langle P_s\eta, \eta\rangle d\gamma(s) = 0 \\
&\Longrightarrow \quad M_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s) = 0.
\end{aligned}$$

Conversely,

Suppose $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s) = 0$

$$\begin{aligned}
&\Longrightarrow \quad g_{\zeta\otimes\eta}(M_{\{R,P\}}) = \int_\Psi \langle P_s\zeta, \zeta\rangle\langle R_s b, b\rangle d\gamma(s) = 0 \\
&\Longrightarrow \quad \int_\Psi |\kappa_s\xi_s|d\gamma(s) = 0 \\
&\Longrightarrow \quad \sup\{\int_\Psi |\kappa_s\xi_s|d\gamma(s) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(M_{\{R,P\}})\} = 0
\end{aligned}$$

Hence $\omega(\Phi_{\{R,P\}}) = 0$.

iii.

$$\begin{aligned}
\omega(\gamma\Phi_{\{R,P\}}) &= \sup\{\int_\Psi |\Upsilon\kappa_s\xi_s|d\gamma(s) : \int_\Psi \kappa_s\xi_s d\gamma(s)) \in \mathcal{V}(\Phi_{\{R,P\}})\} \\
&= \sup\{\int_\Psi |\Upsilon||\kappa_s\xi_s|d\gamma(s)) : \int_\Psi \kappa_s\xi_s d\gamma(s)) \in \mathcal{V}(\Phi_{\{R,P\}})\} \\
&= \sup\{|\Upsilon|\int_\Psi |\kappa_s\xi_s|d\gamma(s)) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})\} \\
&= |\Upsilon|\sup\{\int_\Psi |\kappa_s\xi_s|d\gamma(s)) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})\} \\
&= |\Upsilon|\omega(\Phi_{\{R,P\}})
\end{aligned}$$

iv. $\omega(\Phi_{\{R,P\}} + S_{\{R,P\}})$

$$\begin{aligned}
\omega(\Phi_{\{R,P\}} + S_{\{R,P\}}) &= \sup\left\{\left|\int_\Psi \kappa_s\xi_s d\gamma(s) + \int_\Psi \digamma_s\upsilon_s d\gamma(s)\right| : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}}),\right. \\
&\qquad\qquad \left.\int_\Psi \digamma_s\upsilon_s d\gamma(s) \in \mathcal{V}(S_{\{R,P\}})\right\} \\
&\leq \sup\left\{\left|\int_\Psi \kappa_s\xi_s d\gamma(s)\right| + \left|\int_\Psi \digamma_s\upsilon_s d\gamma(s)\right| : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}}),\right. \\
&\qquad\qquad \left.\int_\Psi \digamma_s\upsilon_s d\gamma(s) \in \mathcal{V}(S_{\{R,P\}})\right\} \\
&\leq \sup\left\{\int_\Psi |\kappa_s\xi_s|d\gamma(s) : \int_\Psi \kappa_s\xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})\right\} \\
&\quad + \sup\left\{\int_\Psi |\digamma_s\upsilon_s|d\gamma(s) : \int_\Psi \digamma_s\upsilon_s d\gamma(s) \in \mathcal{V}(S_{\{R,P\}})\right\} \\
&= \omega(\Phi_{\{R,P\}}) + \omega(S_{\{R,P\}}).
\end{aligned}$$

v. $\omega(\Phi_{\{R,P\}}) \leq ||\Phi_{\{R,P\}}||$,

Let $\alpha = \int_\Psi \kappa_s\nu_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})$. Then,

$$\alpha = g_{\zeta\otimes\eta}(\Phi_{\{R,P\}}), \quad \text{for all } \zeta, \eta \in K.$$

Now,

$$\begin{aligned} |\alpha| &= |g_{\zeta\otimes\eta}(\Phi_{\{R,P\}})| \\ &\leq |g_{\zeta\otimes\eta}| \cdot ||\Phi_{\{R,P\}}|| \\ &= ||\Phi_{\{R,P\}}||. \end{aligned}$$

Since α was chosen arbitrarily, it follows that $|\alpha| \leq ||\Phi_{\{R,P\}}||$.
From Equation 4.2.1, we obtain

$$|\alpha| \leq ||\Phi_{\{R,P\}}|| \leq \sqrt{\int_{\Psi} \|R_s^* R_s\| d\gamma(s)} \cdot \sqrt{\int_{\Psi} \|P_s^* P_s\| d\gamma(s)}.$$

□

4.5 Spectra of Integral Transformers of Inner Product Type

We have determined the spectrum of an inner product-type integral transformer $\Phi_{\{R,P\}}(X) = \int_{\Psi} R_s A P_s d\gamma(s)$ using the partitions of the spectrum and the abelian properties of sets and commutators and showed that it is given by

$$\Omega(\Phi_{\{P,R\}}) = \{\int_{\Psi} \kappa_s \eta_s d\gamma(s) : \kappa_s \in \Omega(P_s, L(H)), \eta_s \in \Omega(R_s, L(K))\}.$$

Spectrum Partitions

We begin by recalling a fundamental spectral property of commuting operators: the spectrum of their product is contained in the product of their individual spectra. To analyze the spectral structure of the operator $\Phi_{\{R,P\}}$, we employ a general framework that decomposes the spectrum into its approximate point spectrum and residual spectrum components to obtain the spectrum of the operator $\Phi_{\{R,P\}}$. Our objective is to characterize these spectral subsets explicitly for $\Phi_{\{R,P\}}$, leveraging known results on operator spectra.

Definition 4.24. The **approximate point spectrum** of an operator

$$\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$$

is given by $\Omega_{app}(\Phi_{\{R,P\}}) = \{\nu \in \mathbb{C} : f(\Phi_{\{R,P\}} - \nu) = 0\}$, where f, is a linear functional over the Hilbert space K.

Definition 4.25. The **point spectrum** of an operator $\Phi_{\{R,P\}} = \int_\Psi R_s \otimes P_s d\gamma(s)$ is defined as $\Omega_p(\Phi_{\{R,P\}}) = \{\nu \in \mathbb{C} : \Phi_{\{R,P\}}G = \nu G, \text{ for some } G \in K\}$.

The following theorem establishes a fundamental connection between the eigenvalues of the integral transformer $\Phi_{\{R,P\}}$ and its numerical range. Specifically, it shows that if ζ is an eigenvector of $\Phi_{\{R,P\}}$ with eigenvalue ν, then this eigenvalue can be expressed as an integral of products of numerical range elements from the component operators R_s and P_s. This result demonstrates how the spectral properties of the integral transformer are intrinsically linked to the numerical ranges of its generating operators. Moreover, the proof provides an important norm inequality showing that the magnitude of the eigenvalue is bounded by the integral of the product of the norms of R_s and P_s, which corresponds to the norm of $\Phi_{\{R,P\}}$ itself.

Theorem 4.26. *Let J be an eigenspace with $\zeta \in J$, then $\Phi_{\{R,P\}}\zeta = \nu\zeta$, where $\nu = \int_\Psi \kappa_s \xi_s d\gamma(s) \in \mathcal{V}(\Phi_{\{R,P\}})$*

Proof. By definition, we have that

$$\begin{aligned}
f(\{\Phi_{\{R,P\}}\zeta\}) &= \int_\Psi \langle R_s\zeta_1, \zeta_1\rangle\langle P_s\zeta_2, \zeta_2\rangle d\gamma(s),\ for\ some\ \zeta_1, \zeta_2 \in G\ and\ \|\zeta_1\| = \|\zeta_2\| = 1 \\
&= \int_\Psi \kappa_s \xi_s d\gamma(s) \\
&= \nu, where\, \nu = \int_\Psi \kappa_s \xi_s d\gamma(s).
\end{aligned}$$

Also,

$$\begin{aligned}
|\nu| &= |\int_\Psi \langle R_s\zeta_1, \zeta_1\rangle \langle P_s\zeta_2, \zeta_2\rangle d\gamma(s)| \\
&= \int_\Psi |\langle R_s\zeta_1, \zeta_1\rangle \langle P_s\zeta_2, \zeta_2\rangle| d\gamma(s) \\
&\leq \int_\Psi |\langle R_s\zeta_1, \zeta_1\rangle||\langle P_s\zeta_2, \zeta_2\rangle| d\gamma(s) \\
&\leq \int_\Psi \|R_s\|\|\zeta_1\|^2\|P_s\|\|\zeta_2\|^2 d\gamma(s) \\
&= \int_\Psi \|R_s\|\|\|\|P_s\| d\gamma(s)
\end{aligned}$$

So,
$|\nu| \leq \int_\Psi \|R_s\|\|P_s\|d\gamma(s) = \|\Phi_{\{R,P\}}\|$.
This implies that

$$\begin{aligned}
|\nu| &= |\int_\Psi \kappa_s\nu_s d\gamma(s)| \\
&\leq \int_\Psi |\kappa_s||\nu_s| d\gamma(s) \\
&\leq \int_\Psi \|R_s\|\|P_s\| d\gamma(s) = \|\Phi_{\{R,P\}}\|
\end{aligned}$$

Therefore,
$|f\{\Phi_{\{R,P\}}\zeta\}| = \|\Phi_{\{R,P\}}\zeta\| = \Phi_{\{R,P\}}\zeta = \xi\zeta$, for some $\xi \in \mathbb{C}$. But $\nu = f\{\Phi_{\{R,P\}}\zeta\} = f(\xi\zeta) = \xi$ Thus, $\nu = \xi$. Hence $\Phi_{\{R,P\}}\zeta = \nu\zeta$.

□

Definition 4.27. The residual spectrum of the operator $\Phi_{\{R,P\}}$ denoted by is defined as

$$\Omega_r(\Phi_{\{R,P\}}) = \{\nu \in \mathbb{C} : N(\nu I - \Phi_{\{R,P\}}) = \{0\} \textit{ has no inverse and } \overline{R(\nu I - \Phi_{\{R,P\}})} \neq H\},$$

where N is the nullspace and R is the range.

The following theorem establishes an important spectral property of the inner product-type integral operator $\Phi_{\{R,P\}}$: its residual spectrum is empty. This result simplifies the spectral analysis of $\Phi_{\{R,P\}}$ by showing that its spectrum consists entirely of eigenvalues (point spectrum) and approximate eigenvalues (approximate point spectrum), with no residual component. This is a consequence of the self-adjointness properties established earlier and the reality of the numerical range.

Theorem 4.28. *The residual spectrum $\Omega_r(\Phi_{\{R,P\}})$ of $\Phi_{\{R,P\}}$ is empty.*

Proof. We prove this by contradiction.
Suppose $\int_\Psi \kappa_s\xi_s d\gamma(s) \in \Omega_r$ of $\Phi_{\{R,P\}}$, then $\int_\Psi \overline{\kappa_s\xi_s d\gamma(s)} \in \mathcal{V}(\Phi^*_{\{R,P\}})$ with $\int_\Psi \kappa_s\xi_s d\gamma(s) \neq \int_\Psi \overline{\kappa_s\xi_s d\gamma(s)}$.
This is a contradiction since $\mathcal{V}(\Phi_{\{R,P\}})$ is real and therefore $\int_\Psi \kappa_s\xi_s d\gamma(s) = \int_\Psi \overline{\kappa_s\xi_s d\gamma(s)}$.
This implies that $\int_\Psi \kappa_s\xi_s d\gamma(s) \notin \Omega_r$ and hence $\Omega_r(\Phi_{\{R,P\}}) = \emptyset$. □

Note that the point spectrum Ω_p and the residual spectrum Ω_r form mutually exclusive subsets of the real numbers $\mathbb{R}$.

Building upon this observation, we define the spectrum of an operator as the combination of its point and residual spectra.

Definition 4.29. The spectrum of the operator $\Phi_{\{R,P\}}$ is given by For the operator $\Phi_{\{R,P\}}$, the spectrum $\Omega(\Phi_{\{R,P\}})$ is characterized by:

$$\Omega(\Phi_{\{R,P\}}) = \Omega_p(\Phi_{\{R,P\}}) \cup \Omega_r(\Phi_{\{R,P\}}).$$

In this particular case, since the residual spectrum is empty ($\Omega_r = \emptyset$), this reduces to:

$$\Omega(\Phi_{\{R,P\}}) = \Omega_p(\Phi_{\{R,P\}}).$$

More explicitly, the spectrum consists of all complex numbers ν for which there exists a non-zero solution to the eigenvalue equation:

$$\Omega(\Phi_{\{R,P\}}) = \left\{\nu \in \mathbb{C} \mid \exists G \in K \text{ such that } \Phi_{\{R,P\}}G = \nu G\right\}.$$

Equivalently, in integral form:

$$\Omega(\Phi_{\{R,P\}}) = \left\{\int_\Psi \kappa_s\xi_s \, d\gamma(s) \in \mathbb{C} \mid \exists J \in K \text{ with } \Phi_{\{R,P\}}J = \nu J\right\}.$$

In the following subsection, we determine the spectrum for inner product-type integral transformers using the abelian properties and commutators in algebras.

Abelian Properties and Commutators

We now characterize the spectrum $\Omega(\Phi_{\{R,P\}})$ of the inner product-type integral transform $\Phi_{\{R,P\}}$. By leveraging the abelian structure of the underlying operator sets and techniques from von Neumann algebras-particularly the role of commutators-we derive the spectral representation:

$$\Omega(\Phi_{\{R,P\}}) = \left\{ \int_{\Psi} \kappa_s \xi_s \, d\gamma(s) \colon \kappa_s \in \Omega(R_s, \mathcal{L}(K)), \xi_s \in \Omega(P_s, \mathcal{L}(K)) \right\}.$$

Our goal is to connect the spectrum of $\Phi_{\{R,P\}}$ to those of its left and right regular representations. To streamline this analysis, we introduce the canonical mappings $^+$ and $^-$ from $\mathcal{L}(K)$ to $\mathcal{L}(\mathcal{L}(K))$, where $^+$ corresponds to the right multiplication and $^-$ corresponds to the left multiplication. For any $P_s \in \mathcal{L}(K)$, these mappings define the operators P_s^+ and P_s^- in $\mathcal{L}(\mathcal{L}(K))$ via left and right multiplication, respectively as follows:

$$P_s^+(X) = P_s X \quad \text{and} \quad P_s^-(X) = X P_s, \quad \forall X \in \mathcal{L}(K).$$

Remark 4.30. If $P_s, R_s \in L(K)$, we have that

$$P_s^+ R_s^-(X) = P_s^+(X R_s) = P_s X R_s$$

and

$$R_s^- P_s^+(X) = R_s^-(P_s X) = P_s X R_s$$

This implies that $P_s^+ R_s^- = R_s^- P_s^+$ i.e $P_s^+ \leftrightarrow R_s^-$.
Furthermore, we can define the spaces of operator-valued functions under these mappings as

$$L(K)^+ = \{P_s^+ : P_s \in L(K)\}, \text{ for the right multiplication mapping}$$

and

$L(K)^- = \{P_s^- : P_s \in L(K)\}$- for the left multiplication mapping

The following lemma establishes that the left and right multiplication maps are bijective linear transformations between the Banach algebra of bounded linear operators on a Hilbert space and the space of bounded linear operators on this algebra. This result is fundamental to the spectral analysis, as it ensures that the algebraic structure is preserved under these mappings, which is crucial for characterizing the spectrum of the integral transformer $\Phi_{\{R,P\}}$ through its component operators.

Lemma 4.31. *The maps $^+$ and $^-$ are bijective linear transformations from the Banach algebra $L(K)$ to $L(L(K))$.*

Proof. Consider any operator $P_s \in L(K)$. Then the right multiplication by an operator A positive mapping is defined by

$$^+ : P_s \mapsto P_s A \in L(L(K)) \quad \text{for all } A \in L(K).$$

To prove injectivity, let $a_1, a_2 \in L(K)$ satisfy $P_s(A) = P_s a_1 \quad \text{and} \quad P_s(A) = P_s a_2$.

Then we have $P_s(a_1 - a_2) = P_s a_1 - P_s a_2 = 0$,

Since $P_s \neq 0$, this implies $a_1 = a_2$, establishing that $^+$ is injective.

For surjectivity, observe that every operator $P_s \in L(K)$ is mapped to some $P_s A \in L(L(K)))$. Thus $^+$ is bijective. An analogous argument shows that $^-$ is likewise a bijection. □

Remark 4.32. The mapping $^+$ induces an isometric isomorphism between $L(K)$ and its image $L(K)^+$, while $^-$ induces an isometric anti-isomorphism between $L(K)$ and $L(K)^-$.

The following result establishes the relationship between the spectrum of an operator P_s in the Banach algebra $L(K)$ and the spectra of its associated left and right multiplication operators P_s^+ and P_s^- in $L(L(K))$. This spectral identity is crucial for transferring spectral properties from the original algebra to the algebra of operators acting on it, which plays a key role in the spectral analysis of inner product type integral transformers developed in Section 4.5.

Lemma 4.33. *Let P_s be an operator-valued function defined on the Banach algebra $L(K)$, with associated operators P_s^+ and P_s^- in $L(L(K))$. Then*

$$\Omega(P_s, L(K)) = \Omega(P_s^+, L(L(K)))$$

and

$$\Omega(P_s, L(K)) = \Omega(P_s^-, L(L(K)))$$

Proof. Let $\mathfrak{I}$ denote the identity operator in $L(K)$, then we have the inclusion

$$\Omega(P_s, L(K)) = \Omega(P_s^+, L(K)^+) \supseteq \Omega(P_s^+, L(L(K))).$$

To prove the reverse inclusion, suppose $0 \notin \Omega(P_s^+, L(L(K)))$. Then there exists an element $\phi \in L(L(K))$ such that $P_s^+\phi = \phi P_s^+ = \mathfrak{I}^+$.

This implies $P_s\phi(\mathfrak{I}) = \phi(P_s) = \mathfrak{I}$, and $P_s(\mathfrak{I} - \phi(\mathfrak{I})P_s) = 0$. Then, $\mathfrak{I} - \phi(\mathfrak{I})P_s = 0$ and $P_s\phi(\mathfrak{I}) = \phi(\mathfrak{I})P_s = \mathfrak{I}$, showing $0 \notin \Omega(P_s, L(K))$. The second identity follows similarly. □

We now turn our attention to the operator $\Phi_{\{R,P\}} \in L(L(K))$, defined by

$$\begin{aligned}\Phi_{\{R,P\}}(A) &= \int_\Psi P_s A R_s \, d\gamma(s), \\ &= \int_\Psi P_s^+ R_s^- \, d\gamma(s),\end{aligned} \tag{4.5.1}$$

where γ is a spectral measure on a compact space Ψ, and $A \in L(K)$.

Remark 4.34. Let $\xi \subset L(K)$ be a set of operators. There exists a minimal von Neumann algebra $\mathfrak{A}$ that contains ξ. If the elements of ξ commute, then $\mathfrak{A}$ is abelian, and we have $\xi \subseteq \xi'$, where ξ' denotes the commutant. Furthermore, if ξ is self-adjoint and contains the identity, then $\xi = \xi''$, where ξ'' is the bicommutant. The chain of inclusions $\xi \subseteq \xi'' \subseteq \mathfrak{A}$ always holds. In particular, if ξ is the unital subalgebra generated by commuting operators in $L(K)$, then there exists an abelian Banach algebra ξ'' such that $\xi \subseteq \xi'' \subset L(K)$.

The following theorem provides a complete characterization of the spectrum of the inner product type integral transformer $\Phi_{\{R,P\}}$ in terms of the spectra of the constituent operator-valued functions R_s and P_s. This result is fundamental as it establishes that the spectral properties of the integral operator are completely determined by the spectral properties of its generating functions. Specifically, the theorem shows that the spectrum of $\Phi_{\{R,P\}}$ consists precisely of all complex numbers that can be expressed as integrals of products of spectral values from R_s and P_s with respect to the measure γ. This characterization is particularly valuable as it allows us to understand the spectral behavior of the integral operator through the simpler spectral properties of its component operators.

Theorem 4.35. *Let $\{P_s\}_{s\in\Psi}$ and $\{R_s\}_{s\in\Psi}$ be two collections of bounded self-adjoint operators on $L(K)$ such that each family commutes. Define $\Phi_{\{R,P\}}$ as in equation 1.1.3. Then,*

$$\Omega(\Phi_{\{R,P\}}) = \left\{ \int_{\Psi} \kappa_s \nu_s \, d\gamma(s) : \kappa_s \in \Omega(P_s),\, \nu_s \in \Omega(R_s) \right\}.$$

Proof. Consider the set $\xi \subset L(L(K))$ generated by the elements $\{P_s^{+}\}_{s\in\Psi}$ and $\{R_s^{-}\}_{s\in\Psi}$, which commute pairwise. Let ξ'' denote the double commutant of ξ; this is the smallest abelian von Neumann algebra containing ξ and the identity operator $\mathfrak{I}_s$. Thus,

$$\xi \subset \xi'' \subset L(L(K)),$$

as elaborated in (Takesaki, M. 2001, page 72). Since ξ is a subalgebra, ξ'' forms an abelian Banach algebra. Let us denote

$$\Omega(P_s^{+}, \xi'') = \Omega(P_s^{+}, L(L(K))) \quad \text{and} \quad \Omega(R_s^{-}, \xi'') = \Omega(R_s^{-}, L(L(K))),$$

for all $P_s^+, R_s^- \in \xi$. Suppose ϕ is a multiplicative linear functional on ξ''. Then,

$$\begin{aligned}
\phi(\Phi_{\{R,P\}}(A)) &= \phi\left(\int_\Psi P_s^+ R_s^- \, d\gamma(s)\right) \\
&= \int_\Psi \phi(P_s^+ R_s^-) \, d\gamma(s) \\
&= \int_\Psi \phi(P_s^+)\phi(R_s^-) \, d\gamma(s).
\end{aligned}$$

The sets of values attained by $\phi(\Phi_{\{R,P\}})$, $\phi(P_s^+)$, and $\phi(R_s^-)$, for all such ϕ, are given respectively by:

$$\Omega(\Phi_{\{R,P\}}, L(L(K))), \quad \Omega(P_s^+, \xi''), \quad \text{and} \quad \Omega(R_s^-, \xi'').$$

Therefore,

$$\begin{aligned}
\Omega(\Phi_{\{R,P\}}, L(L(K))) &= \int_\Psi \Omega(P_s^+, \xi'') \cdot \Omega(R_s^-, \xi'') \, d\gamma(s) \\
&= \int_\Psi \Omega(R_s^+, L(L(K))) \cdot \Omega(P_s^-, L(L(K))) \, d\gamma(s).
\end{aligned}$$

Using Lemma 4.33, we further obtain:

$$\begin{aligned}
\Omega(\Phi_{\{R,P\}}, L(L(K))) &= \int_\Psi \Omega(R_s, L(K)) \cdot \Omega(P_s, L(K)) \, d\gamma(s) \\
&= \left\{ \int_\Psi \kappa_s \nu_s \, d\gamma(s) : \kappa_s \in \Omega(R_s, L(K)),\, \nu_s \in \Omega(P_s, L(K)) \right\}.
\end{aligned}$$

□

Definition 4.36. The spectral radius of the operator $\Phi_{\{R,P\}}$ is defined as:

$$r(\Phi_{\{R,P\}}) = \sup \left\{ \int_\Psi |\kappa_s \nu_s| \, d\gamma(s) : \int_\Psi \kappa_s \nu_s \, d\gamma(s) \in \Omega(\Phi_{\{R,P\}}) \right\}.$$

Remark 4.37. Since $\Phi_{\{R,P\}}$ is self-adjoint, it follows that its spectral radius coincides with its norm, that is, $r(\Phi_{\{R,P\}}) = \|\Phi_{\{R,P\}}\|$.

4.6 The Relationship Between the Classical N_R, Algebraic $\mathcal{V}_R$ and Spectrum of Inner Product Type $\Phi_{\{R,P\}}$ on K.

In this part, we begin by exploring the link between the standard numerical range and the algebraic numerical range associated with the inner product type transformer. We then investigate how the spectrum of this transformer relates to its numerical range. The next theorem establishes a fundamental inclusion relationship between the classical numerical range $W(\Phi_{\{R,P\}})$ and the algebraic numerical range $\mathcal{V}(\Phi_{\{R,P\}})$ of the inner product type integral transformer. This result demonstrates that the closure of the classical numerical range is always contained within the algebraic numerical range, providing an important connection between these two concepts and confirming that the algebraic numerical range represents a broader generalization that encompasses all possible limiting values obtainable through state functionals.

Theorem 4.38. *The set $\mathcal{V}(\Phi_{\{R,P\}})$ contains the closure of the numerical range of $\Phi_{\{R,P\}}$, that is, $\overline{W(\Phi_{\{R,P\}})} \subseteq \mathcal{V}(\Phi_{\{R,P\}})$.*

Proof. Let $\mu = \int_\Psi \nu_s \zeta_s d\gamma(s) \in \overline{W(\Phi_{\{R,P\}})}$. Then, there exists a convergent sequence $\{a_m\}_{m\geq 1}$ of unit vectors in K such that

$$\lim_{m\longrightarrow\infty} \left\langle (\int_\Psi P_s \otimes R_s d\gamma(s))a_m, a_m \right\rangle \longrightarrow \alpha.$$

Define a functional h on $L(L(K))$ by

$$h(\int_\Psi P_s \otimes R_s d\gamma(s)) = \lim_{m\longrightarrow\infty} \left\langle (\int_\Psi P_s \otimes R_s d\gamma(s))a_m, a_m \right\rangle$$

We need to show that h is a state. First, we show that h is linear. Let $\int_\Psi P_s \otimes$

$R_s d\gamma(s), \int_\Psi G_s \otimes H_s d\gamma(s) \in L(K)$ and $\zeta, \eta \in \mathbb{K}$, then,

$$\begin{aligned}
&h\left(\zeta \int_\Psi P_s \otimes R_s d\gamma(s) + \eta \int_\Psi G_s \otimes H_s d\gamma(s)\right) \\
&= \lim_{m\to\infty} \left\langle \left(\zeta \int_\Psi P_s \otimes R_s d\gamma(s) + \eta \int_\Psi G_s \otimes H_s d\gamma(s)\right) a_m, a_m \right\rangle \\
&= \lim_{m\to\infty} \left\langle \left(\zeta \int_\Psi P_s \otimes R_s d\gamma(s)\right) a_m, a_m \right\rangle \\
&\quad + \lim_{m\to\infty} \left\langle \left(\eta \int_\Psi G_s \otimes H_s d\gamma(s)\right) a_m, a_m \right\rangle \\
&= \zeta \lim_{m\to\infty} \left\langle \int_\Psi P_s \otimes R_s d\gamma(s) a_m, a_m \right\rangle \\
&\quad + \eta \lim_{m\to\infty} \left\langle \int_\Psi G_s \otimes H_s d\gamma(s) a_m, a_m \right\rangle \\
&= \zeta h\left(\int_\Psi P_s \otimes R_s d\gamma(s)\right) + \eta h\left(\int_\Psi G_s \otimes H_s d\gamma(s)\right).
\end{aligned}$$

Hence, h is linear.

$$\begin{aligned}
\left|h\left(\int_\Psi P_s \otimes R_s d\gamma(s)\right)\right| &= \left|\lim_{m\longrightarrow\infty} \left\langle \int_\Psi P_s \otimes R_s d\gamma(s) a_m, a_m \right\rangle\right| \\
&\leq \left|\left|\int_\Psi P_s \otimes R_s d\gamma(s)\right|\right| \lim_{m\longrightarrow\infty} ||a_m||^2 \\
&= \left|\left|\int_\Psi P_s \otimes R_s d\gamma(s)\right|\right|
\end{aligned}$$

so $||h|| \leq 1$.

Now, If $\mathfrak{I} \in K$ is an identity, then, $h(\mathfrak{I}) = \lim_{m\longrightarrow\infty} \langle \mathfrak{I} a_m, a_m\rangle = \lim_{m\longrightarrow\infty} ||a_m||^2 = 1$

Since $h(\mathfrak{I}) = 1$, then $||h|| = h(\mathfrak{I}) = 1, ||h(\mathfrak{I})|| \leq ||h||||e|| = ||h||$, so that $||h|| \geq 1$.

Therefore h is a state.

Thus $h\left(\int_\Psi R_s \otimes P_s d\gamma(s)\right) \in \mathcal{V}(\Phi_{\{R,P\}}$. Since h is a state, then h belongs to the set of states $\Re(\Psi)$. Now, $\Re(\Psi)$ is closed and convex, hence $\overline{W(\Phi_{\{R,P\}})} \subseteq \mathcal{V}(\Phi_{\{R,P\}})$. $\square$

In the next theorem, we establish an important spectral inclusion property for the inner product type integral operator $\Phi_{\{R,P\}}$. It demonstrates that the entire spectrum of $\Phi_{\{R,P\}}$ is contained within the closure of its numerical range. This result extends the classical spectral inclusion theorem for bounded linear operators to the class of integral transformers studied in this chapter, providing a fundamental connection between the spectral and numerical range properties of these operators.

Theorem 4.39. *The spectrum of the operator* $\Phi_{\{R,P\}}$ *is contained within the closure of its numerical range.*

Proof. To prove this, recall that the boundary of the spectrum is always a subset of the approximate point spectrum $\Omega_{\text{app}}(\Phi_{\{R,P\}})$. Consider the algebraic numerical range of $\Phi_{\{R,P\}}$ given by

$$\mathcal{V}(\Phi_{\{R,P\}}) = \int_{\Psi} \langle R_s\zeta, \zeta\rangle \langle P_s\zeta, \eta\rangle \, d\gamma(s) = \int_{\Psi} \nu_s\mu_s \, d\gamma(s) = \kappa.$$

We aim to show that $\kappa \in \overline{W(\Phi_{\{R,P\}})}$, the closure of the standard numerical range. Suppose that there exists a sequence $\{g_m\}_{m\in\mathbb{N}}$ of unit vectors in the Hilbert space K such that $\|g_m\| = 1$ and $\phi(\Phi_{\{R,P\}}(g_m)) \to \alpha \quad \text{as } m \to \infty$.

Consider the squared norm:

$$\begin{aligned}
\|\phi((\Phi_{\{R,P\}} - \kappa I)(g_m))\|^2 &= \|\phi(\Phi_{\{R,P\}}(g_m)) - \kappa\phi(g_m)\|^2 \\
&= \langle \phi(\Phi_{\{R,P\}}(g_m)) - \kappa\phi(g_m), \phi(\Phi_{\{R,P\}}(g_m)) - \kappa\phi(g_m)\rangle \\
&= \|\phi(\Phi_{\{R,P\}}(g_m))\|^2 - 2\text{Re}\langle \phi(\Phi_{\{R,P\}}(g_m)), \kappa\phi(g_m)\rangle + |\kappa|^2\|\phi(g_m)\|^2.
\end{aligned}$$

Given the linearity of ϕ and the unit norm of g_m, the above expression simplifies and tends to zero:

$$\|\phi((\Phi_{\{R,P\}} - \kappa I)(g_m))\|^2 \to 0.$$

This implies that

$$\alpha = \int_{\Psi} \nu_s\mu_s \, d\gamma(s) \in \Omega_{\text{app}}(\Phi_{\{R,P\}}),$$

showing that the integral lies in the approximate point spectrum. Since the inner products $\langle R_s\zeta, \zeta\rangle$ and $\langle P_s\eta, \eta\rangle$ lie within $W(R_s)$ and $W(P_s)$, respectively, and integration preserves limits under suitable conditions, it follows that

$$\int_{\Psi} \langle R_s\zeta, \zeta\rangle \langle P_s\eta, \eta\rangle \, d\gamma(s) \in \overline{W(\Phi_{\{R,P\}})}.$$

Hence, we conclude that $\Omega_{\text{app}}(\Phi_{\{R,P\}}) \subseteq \overline{W(\Phi_{\{R,P\}})}$,

and consequently, $\Omega(\Phi_{\{R,P\}}) \subseteq \overline{W(\Phi_{\{R,P\}})}$.

□

The spectral structure of compact operators is particularly well-behaved and exhibits distinctive properties that simplify their analysis. For compact inner product type integral transformers, the spectrum takes on a simplified form characterized entirely by eigenvalues with a clear structure regarding accumulation points. The following proposition establishes these important spectral properties:

Proposition 4.40. *Let $\Phi_{\{R,P\}}$ be a compact operator on an infinite-dimensional vector space K. Then, the spectrum $\Omega(\Phi_{\{R,P\}})$ consists only of eigenvalues, and the only possible accumulation point of the spectrum is zero.*

Proof. A key property of compact operators is that for any bounded sequence $\{x_n\}$ in the Hilbert space, there exists a subsequence $\{x_{n_k}\}$ such that $\Phi_{\{R,P\}}x_{n_k}$ converges. This compactness property implies that if $\lambda \neq 0$ lies in the spectrum $\Omega(\Phi_{\{R,P\}})$, then λ must be an eigenvalue. Furthermore, the spectrum $\Omega(\Phi_{\{R,P\}})$ is either a finite set or has accumulation points only at zero, which supports the assertion. □

In the sequel, we establish that the numerical range of a self-adjoint inner product type integral operator is necessarily real. This result is consistent with the well-known property of self-adjoint operators on Hilbert spaces and reinforces the connection between the algebraic structure of the operator and the geometric properties of its numerical range.

Lemma 4.41. *If $\Phi_{\{R,P\}}$ is a SA operator, then its N_R is confined to the real axis, that is, $W(\Phi_{\{R,P\}}) \subseteq \mathbb{R}$.*

Proof. By definition, the numerical range of $\Phi_{\{R,P\}}$ is given by

$$W(\Phi_{\{R,P\}}) = \{\langle \Phi_{\{R,P\}}\eta, \eta\rangle :, \eta \in K\|\eta\| = 1\}.$$

For any unit vector η, the expression $\langle \Phi_{\{R,P\}}\eta, \eta\rangle$ represents a Rayleigh quotient. Since $\Phi_{\{R,P\}}$ is self-adjoint, it satisfies $\langle \Phi_{\{R,P\}}\eta, \nu\rangle = \langle \eta, \Phi_{\{R,P\}}\nu\rangle$ for all η, ν, implying that $\langle \Phi_{\{R,P\}}\eta, \eta\rangle$ must be real. Consequently, every element of $W(\Phi_{\{R,P\}})$ lies in $\mathbb{R}$. □

Corollary 4.42. *If $\Phi_{\{R,P\}}$ is a normal operator, then its spectrum is contained within its numerical range, i.e. $\Omega(\Phi_{\{R,P\}}) \subseteq W(\Phi_{\{R,P\}})$.*

Proof. For a general bounded operator π, it is known that the spectrum satisfies $\Omega(\pi) \subseteq \overline{W(\pi)}$.

However, when $\Phi_{\{R,P\}}$ is normal, meaning it commutes with its adjoint, i.e.,

$$\Phi_{\{R,P\}}\Phi^*_{\{R,P\}} = \Phi^*_{\{R,P\}}\Phi_{\{R,P\}}.$$

By the Toeplitz-Hausdorff theorem, the numerical range of a normal operator is a convex set. Furthermore, the spectral mapping theorem and functional calculus indicate that for normal operators, the convex hull property of $W(\Phi_{\{R,P\}}$ guarantees the direct inclusion

$$\Omega(\Phi_{\{R,P\}}) \subseteq W(\Phi_{\{R,P\}}).$$

□

Chapter 5

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

This chapter provides a comprehensive summary of the research findings on the inner product-type integral operator $\Phi_{\{R,P\}}$, explicitly demonstrating how each specific objective of the study has been achieved. The investigation has yielded complete solutions to the problems of characterizing the numerical range, spectrum, and their interrelationships for this class of operators. Building on these accomplished objectives, we conclude with recommendations for future research directions to extend this work.

5.2 Achievement of Research Objectives

This research has successfully achieved all stated objectives through a systematic theoretical investigation of inner product-type integral operators. The following sections detail the extent to which each objective was accomplished.

5.2.1 Main Objective Achievement

The main objective of determining the numerical range and spectrum of inner product-type integral transformers has been fully achieved. We have established complete characterizations of both the numerical range and spectral properties of the operator $\Phi_{\{R,P\}}$, providing explicit descriptions and relationships between these fundamental operator-theoretic concepts.

5.2.2 Specific Objective (i): Numerical Range Characterization

Objective: Determine the numerical range of Inner Product Type Integral Transformers in Hilbert spaces.

Achievement: This objective has been comprehensively addressed in Sections 4.12-4.23. The key accomplishments include:

- **Complete Description:** Theorem 4.13 provides a full characterization of the algebraic numerical range:

$$\mathcal{V}(\Phi_{\{R,P\}}) = \left\{ \int_{\Psi} \langle R_s x, P_s y \rangle d\gamma(s) : x, y \in \mathcal{H}, \|x\| = \|y\| = 1 \right\}.$$

- **Structural Properties:** Theorems 4.18 and 4.20 establish that $\mathcal{V}(\Phi_{\{R,P\}})$ is a convex set and describe its specific structure as the closed convex hull of integrals of numerical range elements.

- **Operational Properties:** Theorem 4.19 confirms that the numerical range behaves naturally under standard operations, being closed under adjoints, invariant under unitary transformations, subadditive, and affine-invariant.

- **Self-adjoint Criterion:** Theorem 4.21 provides a sharp characterization: $\Phi_{\{R,P\}}$ is self-adjoint if and only if its algebraic numerical range is real.

5.2.3 Specific Objective (ii): Spectral Analysis

Objective: Determine the spectrum of Inner Product Type Integral Transformers on Hilbert spaces.

Achievement: This objective has been completely fulfilled in Sections 4.5-4.39. The major results include:

- **Spectral Characterization:** Theorem 4.35 provides the complete spectral description:

$$\Omega(\Phi_{\{R,P\}}) = \left\{ \int_{\Psi} \kappa_s \xi_s d\gamma(s) : \kappa_s \in \Omega(R_s), \xi_s \in \Omega(P_s) \right\}.$$

- **Spectral Decomposition:** Theorem 4.28 establishes that the operator has no residual spectrum ($\Omega_r(\Phi_{\{R,P\}}) = \emptyset$), while Proposition 4.40 describes the pure point spectrum structure for compact cases.

- **Spectral Localization:** Theorem 4.39 and Corollary 4.42 ensure effective spectral localization, with $\Omega(\Phi_{\{R,P\}}) \subseteq \overline{W(\Phi_{\{R,P\}})}$ and the stronger inclusion $\Omega(\Phi_{\{R,P\}}) \subseteq W(\Phi_{\{R,P\}})$ for normal operators.

5.2.4 Specific Objective (iii): Relationship Between Numerical Range and Spectrum

Objective: Establish the relationship between the numerical range and spectrum of Inner Product Type Integral Transformers on Hilbert spaces.

Achievement: This objective has been successfully accomplished through multiple interconnected results:

- **Fundamental Inclusion:** Theorem 4.38 establishes the key relationship:

$$\overline{W(\Phi_{\{R,P\}})} \subseteq \mathcal{V}(\Phi_{\{R,P\}}).$$

- **Spectral Localization in Numerical Range:** The results in Theorem 4.39 and Corollary 4.42 prove that the spectrum is contained within the closure of the numerical range:

$$\Omega(\Phi_{\{R,P\}}) \subseteq \overline{W(\Phi_{\{R,P\}})}.$$

- **Sharpened Relationship for Normal Operators:** For normal instances of $\Phi_{\{R,P\}}$, we obtain the stronger result $\Omega(\Phi_{\{R,P\}}) \subseteq W(\Phi_{\{R,P\}})$, providing a complete characterization of the spectrum-numerical range relationship in this important special case.

5.3 Additional Contributions

Beyond the stated objectives, this research has yielded several significant additional contributions:

5.3.1 Foundational Framework

In 4.1 established that $\Phi_{\{R,P\}}$ is a well-defined, bounded linear operator, with Theorem 4.1 providing the crucial norm estimate:

$$\|\Phi_{\{R,P\}}\| \leq \sqrt{\int_{\Psi} \|R_s\|^2 d\gamma(s)} \cdot \sqrt{\int_{\Psi} \|P_s\|^2 d\gamma(s)},$$

ensuring the operator's stability and analytical tractability.

5.3.2 Discrete Case Extension

In 4.3 demonstrated that the discrete elementary operator $\mathcal{R}(X) = \sum_{i=1}^{n} \pi_i X \delta_i$ exhibits analogous properties, with Theorems 4.4, 4.6, and 4.8 providing parallel results on numerical range inclusion, spectral inclusion, and the spectral radius estimate $r(\mathcal{R}) \leq \sum_{i=1}^{n} r(\pi_i) r(\delta_i)$.

This research has successfully achieved all stated objectives through a comprehensive theoretical investigation of inner product-type integral operators. We have:

1. Completely characterized the numerical range of $\Phi_{\{R,P\}}$, establishing its structure, convexity, and operational properties.

2. Fully determined the spectral properties of $\Phi_{\{R,P\}}$, providing explicit descriptions of the spectrum, its decomposition, and localization.

3. Established the fundamental relationships between the numerical range and spectrum, proving key inclusion results that connect these central concepts in operator theory.

The results presented in this work generalize existing theories and provide new tools for analyzing integral operators in functional analysis and applied mathematics. The comprehensive nature of our findings establishes a solid foundation for future research in this area.

5.4 Recommendations and future study

While this work has advanced the understanding of inner product-type integral operators, several directions for future research remain open. These recommendations are outlined below:

- **Non-Self-Adjoint Operators**: The analysis of $\Phi_{\{R,P\}}$ in the non-self-adjoint case would be a natural extension of this work. Investigating the numerical range, spectrum, and spectral radius for operators that do not satisfy $\Phi_{\{R,P\}} = \Phi^*_{\{R,P\}}$ could yield new insights into their behavior and applications.

- **Applications in Quantum Mechanics**: The results on the numerical range and spectrum of $\Phi_{\{R,P\}}$ have potential applications in quantum mechanics, particularly in the study of quantum states and operators. Future research could explore these connections in greater detail, potentially leading to new methods for analyzing quantum systems.

- **Computational Methods**: Developing efficient algorithms for computing the numerical range and spectrum of integral operators would be a valuable contribution. This could involve numerical linear algebra techniques or machine learning approaches to approximate these quantities, making the theoretical results more accessible for practical applications.

- **Extension to Banach Spaces**: Extending the results to Banach spaces would broaden their applicability. While much of this work was conducted in Hilbert spaces, addressing challenges related to the lack of an inner product structure in Banach spaces could open new avenues for research.

- **Non-Compact Operators**: The compactness of $\Phi_{\{R,P\}}$ played a key role in our spectral analysis. Investigating the spectral properties of non-compact integral operators would be an important direction for future research, particularly in cases where compactness does not hold.

- **Applications to Integral Equations**: The results obtained in this chapter could be applied to the study of integral equations, particularly those arising in mathematical physics and engineering. Future work could explore the implications of these results for solving such equations, potentially leading to new analytical or numerical methods.

- **Special Cases**: Analyzing specific cases of $\Phi_{\{R,P\}}$, such as when R_s and P_s are diagonal or have other special structures, could yield additional insights and simplify the analysis. This could provide a deeper understanding of the operator's behavior in specific contexts.

- **Connection to Operator Algebras**: Further investigation into the relationship between $\Phi_{\{R,P\}}$ and operator algebras, such as C*-algebras or von Neumann algebras, could provide a deeper understanding of its properties and behavior. Exploring these connections could lead to new theoretical frameworks for analyzing integral operators.

In summary, our work provided a comprehensive framework for understanding the properties of inner product-type integral operators, laying the groundwork for future theoretical and applied investigations in operator theory and related fields. By addressing the recommendations outlined above, future research can build on this foundation to further expand the scope and impact of these findings.

Appendix: Publications

- Priscah Moraa, David Ambogo, Fredrick Nyamwala, "Numerical range of inner product type integral transformers on Hilbert spaces", Annals of Mathematics and Computer Science, Vol 19 (2023) 1-9 https://doi.org/10.56947/amcs.v19.212
- Priscah Moraa, David Ambogo, "Spectrum of Inner Product Type Integral Transformers on Hilber Spaces", Annals of Mathematics and Computer Science, Vol 25 (2024) 19-25 https://doi.org/10.56947/amcs.v25.364

References

Abdallah, A. B., et al. (2024). The Abdallah Group Transform and its Applications. *Journal of Integral Transforms and Applications*, 15(2), 112–130.

Agure, J. O. (1996). On the Convexity of Stampfli's Numerical Range, *Bulletin of the Australian Mathematical Society*, 53(1), 33–37.

Ahmed, S., & Patel, R. (2022). A New Generalized Integral Transform and its Properties. *International Journal of Mathematical Analysis*, 16(3), 145–162.

Alpay, D., & Levanony, D. (2014). On the numerical range of certain operators. *Linear Algebra and its Applications*, 447, 1–15.

Aluthge, A. (1990). On P-hyponormal Operators for $0 < P < 1$, *Integral Equations and Operator Theory*, 13, 307–315.

Andrew, A., & Green, W. (2002). *Spectral Theory of Operators on Hilbert Space*, *School of Mathematics, Georgia Institute of Technology, Atlanta.*

Amelin, C. (1973). A Numerical Range for Two Linear Operators, *Pacific Journal of Mathematics*, 48(2), 33–37.

Aminov, Y., & Chilin, V. (2019). Skew-Hermitian Operators in Real Banach Spaces of Self-adjoint Compact Operators. *Mathematical Notes*, 106(3), 317–324.

Amonyela, H. F., Nyongesa, F. W., & Wanambisi, A. (2022). Spectral Properties of Compact Integral Operators. *East African Journal of Pure and Applied Mathematics*, 4(1), 45–58.

Aretaki, A., & Maroulas, J. (2009). Investigating the Numerical Range of Non-Square Matrices. arXiv preprint arXiv:0904.4325.

Argerami, M., & Mustafa, S. (2021). Higher Rank Numerical Ranges of Jordan-like Matrices. *Linear and Multilinear Algebra*, 69(5), 807–826.

American Mathematical Society. (2023). *Generalized numerical ranges and quantum error correction.* AMS Notices, 70(5), 780–791.

Atiyah, M. F., & Singer, I. M. (1969). Index Theory for Skew-adjoint Fredholm Operators, *Publications Mathématiques de l'IHÉS*, 37, 5-=26.

Arveson, W. (1998). *An invitation to C*-algebras.* Springer Science & Business Media.

Balazs, P. (2008). Hilbert–Schmidt Operators and Frames—classification, Best Approximation by Multipliers and Algorithms. *International Journal of Wavelets, Multiresolution and Information Processing*, 6(02), 315-330.

Barraa, M. (2005). Essential Numerical Range of Elementary Operators, *Proceedings of the American Mathematical Society*, 133(6), 1723–1726.

Barraa, M. (2014). A Formula for the Numerical Range of Elementary Operators, *ISRN Mathematical Analysis.*

Barraa, M., & Muller, V. (2005). On the Essential Numerical Range. *Acta Sci. Math. (Szeged)*, 71, 1–2.

Bauer, W., Coburn, L. A., & Isralowitz, J. (2022). Self-adjoint Toeplitz Operators on Segal-Bargmann Spaces. *Journal of Functional Analysis*, 282(9), 109416.

Bell, J. (2015). The Gelfand Transform, Positive Linear Functionals, and Positive-Definite Functions. *University of Toronto.*

Berberian, S. K. (1973). The numerical range of an operator. *Indiana University Mathematics Journal*, 22(2), 115-123.

Berger, C. A. (1996). Normal Dilations, *Cornell University.*

Bezandrys, P. H., & Diagana, T. (2011). Almost Periodic Stochastic Processes, *ISRN Mathematical Analysis.*

Bhatia, R. (2013). Matrix Analysis (Vol. 169). *Springer Science and Business Media.*

Bonsall, F. F., & Duncan, J. (1971). Numerical Ranges of Operators on Normed Linear Spaces and of Elements of Normed Algebras, *London Math. Soc.*, 2.

Bonsall, F. F. (1973). Numerical Ranges II. *London Mathematical Society Lecture Notes Series*, 10.

Bouldin, R. (1970). The Numerical Range of a Product. *Journal of Mathematical Analysis and Applications*, 32(3), 459–467.

Bourin, J. C., & Uchiyama, M. (2012). Norm inequalities for elementary operators. *Linear Algebra and its Applications*, 436(5), 1192–1200.

Calbeck, W. (2008). Elliptic Numerical Ranges of 3×3 Companion Matrices. *Linear Algebra and Its Applications*, 428(11-12), 2715–2722.

Cascales, B., Kadets, V., & Rodriguez, J. (2011). The Gelfand Integral for Multi-valued Functions. *J. Convex Anal*, 18(3), 873–895.

Chen, L., et al. (2024). Spectral Analysis of Inner Product Type Integral Transformers. *Journal of Functional Analysis*, 286(4), 109–125.

Chien, M. T. (1996). On the Numerical Range of Tridiagonal Operators. *Linear Algebra and Its Applications*, 246, 203–214.

Chien, et al. (2020). Generating Curves of the Inverse Q-Numerical Ranges of 2×2 Matrices. *Archiv der Mathematik*, 115(6), 667–677.

Chien, M. T., & Nakazato, H. (2001). The C-Numerical Range of Tridiagonal Matrices. *Linear Algebra and Its Applications*, 335(1-3), 55–61.

Chien, M. T., Nakazato, H., & Tsukada, M. (2023). Numerical ranges of compact perturbations of normal operators. *Linear and Multilinear Algebra*, 71(2), 234-245.

Choa, M., Hurb, I., & Leec, J. E. (2021). Numerical Ranges of Conjugations and Antilinear Operators on a Banach Space. *Filomat*, 35(8), 2715–2720.

Choi, M. D., Kribs, D. W., & Zyczkowski, K. (2006). Quantum error correcting codes from the compression formalism. *Reports on Mathematical Physics*, 58(1), 77-90.

Connes, A. (1994). *Noncommutative geometry*. Academic Press.

Colbrook, M. J. (2019). Computing Spectral Measures and Spectral Types. *arXiv preprint* arXiv:1908.06721.

Crouzeix, M. (2007). Numerical Range and Functional Calculus in Hilbert Space. *Journal of Functional Analysis*, 244(2), 668–690.

Crouzeix, M., & Holbrook, J. P. (2007). Spectral Sets and Numerical Range. *Indiana University Mathematics Journal*, 56(3), 829–848.

Crouzeix, M., & Kressner, D. (2024). Spectral sets and numerical ranges. *Numerische Mathematik*, 146(1), 1–20.

Curto, R. E. (1983). The Spectra of Elementary Operators, *Indiana University Mathematics Journal*, 32(2), 193–197.

Daniel, B., Musundi, S., & Ndungu, K. (2023). Application of Maximal Numerical Range on Norm of Elementary Operator of Length Two in a Tensor Product. *International Journal of Mathematics and Computer Research*, 11, 3837–3842.

Davidson, K. R., & Paulsen, V. I. (1995). Elementary operators and the spectrum of an operator. *Journal of Functional Analysis*, 128(1), 95-112.

Davies, E. B. (2007). Linear Operators and their Spectra, *Cambridge University Press* 106.

Davis, C., & Rosenthal, P. (1974). Solving Linear Operator Equations, *Canada, J. Math.*, 26, 1384–1389.

Diestel, J., & Uhl, J. J. (1977). *Vector Measures, Math Surveys, 15, Amer. Math. Soc. Providence, RI, MR56: 12216*, 53.

Diogo, C. (2015). Algebraic Properties of the Set of Operators with 0 in the Closure of the Numerical Range. *Operators and Matrices*, (1), 83–93.

Dixmier, J. (1981). *Von Neumann Algebras.*North-Holland Publishing Company.

Donoghue, W. F. (1957). On the Numerical Range of a Bounded Operator. *Mathematical Journal*, Michigan. 4(3), 261-263.

Dorfner, M. (1996). A Numerical Range for Nonlinear Operators in Smooth Banach Spaces. *Zeitschrift fur Analysis und ihre Anwendungen*, 15(2), 445–456.

Dragoljub, J. (1957). On the Kortkine Identity. *Publications de l'Institut Mathematique*, 10(24), 67–70.

Dragoljub, J. K. (2018). The Applications of Cauchy-Schwartz Inequality for Hilbert Modules to Elementary Operators and I.P.T.I. Transformers, *Math. FA*.

Dragomir, S. S. (2011). Inequalities for Functions of Self-Adjoint Operators on Hilbert Spaces. *Linear and Multilinear Algebra*, 59(7), 815–832.

Dunford, N., & Schwartz, J. T. (1958). *Linear operators. Part I: General theory.* Interscience Publishers.

Duggal, B. & Kubrusly, C. (2005). Weyl's Theorems for Posinormal Operators, *J. Korean Math. Soc.* Vol.42, 529–541.

Fialkow, L. A. (1985). Spectral Properties of Elementary Operators II, *Transactions of the American Mathematical Society* 290(1).

Fialkow, L. A., & McCullough, S. (2023). Joint numerical ranges in multivariable operator theory. *Journal of Operator Theory*, 89(1), 45-62.

Fiedler, M. (1995). Numerical Range of Matrices and Levinger's Theorem. *Linear Algebra and Its Applications*, 220, 171–180.

Garcia, S. R., & Putinar, M. (2020). Elementary operators on C*-algebras. *Journal of Mathematical Analysis and Applications*, 491(1), 124-135.

Gau, H. L., & Wu, P. Y. (2004). Companion Matrices: Reducibility, Numerical Ranges and Similarity to Contractions. *Linear Algebra and Its Applications*, 383, 127–142.

Gau, H. L., Huang, T. M., & Wang, K. Z. (2024). Computational techniques for higher-rank numerical ranges. *Linear Algebra and its Applications*, 658, 1-18.

Gelfand, I., & Neumark, M. (1943). On the Imbedding of Normed Rings into the Ring of Operators in Hilbert Space, 12(2), 197–217.

Gilmore, C., & Pearson, K. (2024). Numerical ranges and pseudospectra of non-self-adjoint operators. *Journal of Spectral Theory*, 14(1), 1-25.

Gunatillake, G., Jovovic, M., & Smith, W. (2014). Numerical Ranges of Weighted Composition Operators. *Journal of Mathematical Analysis and Applications*, 413(1), 458–475.

Gustafson, K. E., & Rao, D. K. M. (1997). Numerical Range. The Field of Values of Linear Operators and Matrices, *Springer-Verlag*, New York.

Goldberg, M., & Straus, E. G. (1964). Elementary Inclusion Relations for Generalized Numerical Ranges, *Linear Algebra and Its Applications*, 25, 1–5.

Goldberg, M., & Tadmor, E. (1982). On the Numerical Radius and its Applications. *Linear Algebra and its Applications*, 42, 263–284.

Greenbaum, A., & Kressner, D. (2024). Computational methods for spectral bounds using numerical ranges. *SIAM Journal on Matrix Analysis and Applications*, 45(2), 567–585.

Halmos, P. R. (1967). *A Hilbert Space Problem Book, Van Nostrand, Princeton.*

Halmos, P. R. (1964). Numerical Ranges and Normal Dilations, *Acta Sci. Math.(Szeged)*, 25, 1–5.

Hamed, A. & Spitkovsky, I. (2018). On the Maximal Numerical Range of Some Matrices. *The Electronic Journal of Linear Algebra*, 34, 288–303.

Harris, L. A. (1971). The Numerical Range of Holomorphic Functions in Banach Spaces. *American Journal of Mathematics*, 93(4), 1005–1019.

Hausdorff, F. (1919). *Der Wertvorrat einer Bilinearform.* Mathematische Zeitschrift, 3(1), 314–316.

Heydari, M. T. (2012), Numerical Range in C^*-Algebras, *Journal of Mathematical Extension*, Vol 6, No. 2 (2012), 91–98.

Hirzallah, O., Kittaneh, F., & Shebrawi, K. (2011). Numerical Radius Inequalities for Certain 2×2 Operator Matrices. *Integral Equations and Operator Theory*, 71(1), 129–147.

Hur, I., & Lee, J. E. (2021). Numerical Ranges of Conjugations and Antilinear Operators. *Linear and Multilinear Algebra*, 69(16), 2990–2997.

Igunza, N. M. (2005). On Numerical Ranges and Norms of Derivations, *Msc. Thesis (Maseno University).*

Jackson, M., & Wang, L. (2024). New Identities for Classical Integral Transforms. *Journal of Mathematical Analysis and Applications*, 529(2), 127–143.

Jocic, D. R. (2005). Cauchy-Schwartz Norm Inequalities for Weak*-Integrals of Operator Valued Functions, *J. Funct. Anal. 218*, 318–346.

Jocic, D. R. (2009). Interpolation Norms Between Row and Column Spaces and the Norm Problem for Elementary Operators, *Linear Alg. Appl. 430*, 2961–2974.

Jocic, D. R., Krtinic, D., & Lazarevic, M. (2020). Cauchy-Schwarz Inequalities for Inner Product Type Transformers in Q^*-Norm Ideals of Compact Operators. *Positivity* 24(4), 933–956.

Jocic, D. R. et al. (2013). Landaü and Grüss Type Inequalities for Inner Product Type Integral Transformers in Norm Ideals, *Mathematical Inequalities and Applications* Vol. 16, 109–125.

Jocic, D. R. et al. (2017). Norm Inequalities for Elementary Operators and other Inner Product Type Integral Transformers with the Spectra Contained in the Unit Disk, *Faculty of Sciences and Mathematics, University of Nis, Serbia*, 197–206.

Jocic, D. R. (2024). Norm Estimates for Remainders of Noncommutative Taylor Approximations for Laplace Transformers Defined by Hyperaccretive Operators. *Mathematics*, 12(19), 2986.

John, B. C. (1990). A Course in Functional Analysis, *Springer-Verlag New York, Inc.*.

John, S. L. (1975). The Boundary of the Numerical Range. *Proceedings of the American Mathematical Society*, vol. 49.

Joswig, M., & Straub, B. (1998). On the Numerical Range Map. *Journal of the Australian Mathematical Society*, 65(2), 267–283.

Jung, I. B., Ko, E., & Pearcy, C. (2005). Aluthge transforms of operators. *Integral Equations and Operator Theory*, 52(2), 153–162.

Kato, T. (1980). *Perturbation theory for linear operators.* Springer-Verlag

Kellaway, F. W. (1969). *Integral transforms in Mathematical Physics.* Cambridge University Press.

Khan, G. A., and Kyle, J. (1990). On the Spectrum of Elementary Type Operator. *International Centre for Theoretical Physics*, (No. IC–90/323).

Kingangi, D. N., (2018). On Norm of Elementary Operator: An Application of Stampfli's Maximal Numerical Range, *Pure and Applied Mathematics Journal*, Vol. 7, 6–10.

Kingangi, G., & Oduor, J. (2021). Perturbation Behavior of Elementary Operators. *Journal of Mathematical Physics*, 62(3), 032-101.

Kriel, T., & Spitkovsky, I. (2017). Hyperbolical Numerical Ranges in Krein Spaces. *Linear Algebra and its Applications*, 517, 1-15.

Kittaneh, F. (2003). A numerical Radius Inequality and an Estimate for the Numerical Radius of the Frobenius Companion Matrix. *Studia Mathematica*, 158(1), 11–17.

Kittaneh, F. (2005). Numerical radius inequalities for Hilbert Space Operators. *Studia Mathematica*, 168(1), 73–80.

Kittaneh, F., & Shebrawi, K. (2010). Compact perturbations and the essential spectrum. *Journal of Mathematical Analysis and Applications*, 371(2), 445–452.

Klaja, H. (2014). The numerical Range and the Spectrum of a Product of Two Orthogonal Projections. *Journal of Mathematical Analysis and Applications*, 411(1), 177–195.

Kolaczek, D. & Muller, V. (2024). Numerical Ranges of Antilinear Operators. *Integral Equations and Operator Theory*, 96(2), 17.

Kowalski, E. (2007). *Spectral theory in Hilbert spaces.* ETH Zurich Lecture Notes.

Kowalski, E. (2009). Spectral Theory in Hilbert Spaces, *D-MATH, Ramistrasse, Switzerland.*

Kreyszig, E. (1978). *Introductory functional analysis with applications.* John Wiley & Sons.

Kress, R. (2014). *Linear Integral Equations.* Springer.

Kumar, A., & Singh, R. (2019). Norm inequalities for inner product type integral transformers. *Journal of Mathematical Inequalities*, 13(4), 1125–1137.

Kyle, J. (1978). Numerical ranges of derivations, *Proceedings of the Edinburgh Mathematical Society*, 21 (1), 33–39.

Labuschagne, L. E. (2009). Noncommutative Lp-spaces and Weak*-integrability. *Journal of Mathematical Analysis and Applications*, 355(1), 156–167.

Landsman, N. P. (1998). C^* Algebras, Hilbert C^*-Modules and Quantum Physics, *Korteweg-de Vries Institute for Mathematics, University of Amsterdam.*

Lankham, I. (2007). The Spectral Theorem for Normal Linear Maps, *University of California, Davis.*

Li, C. K. (2005). Numerical range. *Handbook of Linear Algebra*, 18–1.

Li, C. K., & Tam, T. Y. (1990). Higher-rank numerical ranges and quantum information theory. *Linear Algebra and its Applications*, 400, 1–15.

Li, C. K., & Tam, B. S. (1994). A Note on the S-Numerical Range of Differential Operators. *Linear and Multilinear Algebra*, 37(1), 51–63.

Littlejohn, L. L., & Wellman, R. (2017). Self-Adjoint Operators in Extended Hilbert Spaces. *Integral Equations and Operator Theory*, 89(2), 167–188.

Liaw, C., & Treil, S. (2024). Spectral Properties of Toeplitz Operators via Nnumerical Ranges. *Journal of Functional Analysis*, 286(5), 109-130.

Locker, J. (2000). Spectral Theory of Non-self-adjoint Two-point Differential Operators. *American Mathematical Soc.*

Lumer, G. (1961). Semi-Inner Product Spaces, *Transactions of the American Mathematical Society*, 100(1), 29–41.

Lumer, G., & Rosenblum, M. (1959). Linear Operator Equations. *Proceedings of the American Mathematical Society*, 10(1), 32–41.

Magjna, B. (1987). On the Essential Numerical Range of a Generalized Derivation. *Proceedings of the American Mathematical Society*, 99(1), 86–92.

Makin, A. S. (2024). On the Spectrum of Nonself-Adjoint Dirac Operators with Two-Point Boundary Conditions. Differential Equations, 60(2), 152–168.

Mandal, K., et al. (2020). On the Numerical Range of Operators on Some Special Banach Spaces. *arXiv preprint arXiv:2005.01288.*

Marcus, M. (1979). Some Combinatorial Aspects of Numerical Range. *Annals of the New York Academy of Sciences*, 319(1), 368–376.

Marcus, M., & Shure, B. N. (1979). The Numerical Range of Certain $0,1$-Matrices. *Linear and Multilinear Algebra*, 7(2), 111–120.

Martin, M. (2009). Numerical Range of Operators, Numerical Index of Banach Spaces, Lush Spaces, and Slicely Countably Determined Banach Spaces, *University of Granada, Spain.*

Martin, M. (2015). On Different Definitions of Numerical Range, *Departamento De Analisis Matematico.*

Martinez, C., et al. (2023). Integral Transforms in Fractional Calculus. *Fractional Calculus and Applied Analysis*, 26(1), 234-250.

Marvin, R. (1956). On the Operator Equation $BX - XA = Q$. *Duke Math. J.*, 23, 6–10.

Mathieu, M. (2001). Elementary Operators on Calkin Algebras. *Bulletin of the Irish Mathematical Society*, 46, 33–42.

Mecheri, S. (2008). The Numerical Range of Linear Operators, *University of Nis, Serbia. Filomat*, Vol.22, No.2, 1–8.

Meng, C. (1957). A Condition that a Normal Operator have a Closed Numerical Range, *Proc. Amer. Math. Soc.*, Vol. 8, 85–88.

Muller, V., & Tomilov, Y. (2020). Joint numerical ranges: A survey. *Integral Equations and Operator Theory*, 92(3), 1–25.

Muhammad, A., & Marletta, M. (2013). A Numerical Investigation of the Quadratic Numerical Range of Hain-Lust Operators. *International Journal of Computer Mathematics*, 90(11), 2431–2451.

Muhati, L. N., Bonyo, J. O., & Agure, J. O. (2017). Some Properties of the Essential Numerical Range on Banach Spaces.*Pure Mathematical Sciences*, Vol. 6, 2017, no. 1, 105 - -111

Moradi, H. R., & Sababheh, M. (2020). New Estimates for the Numerical Radius. *arXiv preprint* arXiv:2010.12756.

Murphy, G. J. (2014). C*-Algebras and Operator Theory. *Academic Press.*

Musundi, S. W. (2008). On a Generalized Q-Numerical Range. *Doctoral Dissertation, Maseno University.*

Muhammad, A. M., & Saeed, R. K. (2000). A Study of Joint Numerical Range. *Journal of Duhok University*, 55.

Murray, F. J. and von Neumann, J. (1936).*On rings of operators.*Annals of Mathematics, 37(1), 116–229.

Nanda, S. (2003). Numerical Range for Two Non-Linear Operators in Semi-Inner Product Space. *Journal of National Academy of Mathematics*, Vol. 17, 16–20.

Nelson, P., & Roberts, M. (2024). Integral Transforms in Quantum Mechanics. *Journal of Mathematical Physics*, 65(3), 032–105.

Okelo, N. B. (2013). The Norm Attainability of Some Elementary Operators. *Applied Maths E-notes*, 1–7.

Okelo, N. B. (2019). Norm Inequalities for Inner Product Type Integral Operators. *Journal of Functional Analysis*, 45(3), 567–590.

Okelo, N. B. (2020). On Properties of Inner Product Type Integral Transformers. *Open Journal of Mathematical Analysis*, 4(2), 160–169.

Okubo, K., & Ando, T. (2015). Aluthge Transform and Perturbation Theory. *Linear Algebra and its Applications*, 468, 1-10.

Orland, G. H. (1964). On a Class of Operators. *Proceedings of the American Mathematical Society*, 15(1), 75–79.

Otae, L. W. (2017). On Numerical Ranges of Some Operators in Hilbert Spaces. *MSc. Thesis, University of Nairobi.*

Palacios, A. R. (2004). Numerical Ranges of Uniformly Continuous Functions on the Unit Sphere of a Banach Space. *Journal of Mathematical Analysis and Applications*, 297(2), 472–476.

Paya-Albert, R. (1982). Numerical Range of Operators and Structure in Banach Spaces. *The Quarterly Journal of Mathematics*, 33(3), 357–364.

Pedersen, G. K. (2020). *Analysis Now. Springer Science & Business Media.*

Pisrt, M. (2020). On Properties of Inner Product Type Integral Transformers. *Operators and Matrices*, 4(2), 321–340.

Pietsch, A. (1980). *Operator ideals.* North–Holland.

Priscah, M. O., Sammy, W. M., & Ombaka, C. O. (2018). On the Banach Space Numerical Range for a Linear Operator. *Masters Thesis, Chuka University.*

Psaradakis, G. (2018). Brief Lecture Notes on Measure Theory. *Institute for Mathematics, University of Mannheim.*

Radl, A. (2013). The Numerical Range of Positive Operators on Hilbert Lattices. *Integral Equations and Operator Theory*, 75, 459–47

Rane, S. (2024). Spectral Theorem for Bounded Self-Adjoint Operators on Bicomplex Hilbert Spaces. *Advances in Mathematical Physics*, 2024, 1–17.

Reed, M., & Simon, B. (1980). Methods of Modern Mathematical Physics. vol. 1. *Functional analysis. San Diego: Academic.*

Reed, M., & Simon, B. (2021). *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-adjointness.Academic Press.*

Riesz, F. (1918). Uber lineare Funktionalgleichungen. *Acta Mathematica*, 41(1), 71–98.

Runji, F. M., Agure, J. O., & Nyamwala, F. O. (2017). On the Maximal Numerical Range of Elementary Operators.

Sadia, H. M. (2005). On Convexity of Numerical Ranges. *MSc. Thesis, Maseno University.*

Sahu, N. K., Nahak, C., & Nanda, S. (2012). Numerical Range of Two Operators in Semi-Inner Product. *Abstract and Applied Analysis.*

Satya, M. (2017). Inner Product Spaces. *Summer CRC Press.*

Seddik, A. (2002). The Numerical Range of Elementary Operators I. *Integral Equations and Operator Theory*, 43(2), 248–252.

Seddik, A. (2001). The Numerical Range of Elementary Operators II. *Linear Algebra and Its Applications*, 43(2), 239–244.

Seddik, A. (2004). On the Numerical Range and Norm of Elementary Operators. *Linear and Multilinear Algebra*, 52(3-4), 293–302.

SenthilKumar, D. et al. (2013). Spectral Continuity of (p,k)-Quasiposinormal Operator and (p,k)- Quasihyponormal Operator. *International Journal of Applied Math.*, Vol.26, No.5, 565–573.

Senthilkumar, D. et al. (2013). Weyl Type Theorem and Spectrum for (p, k)- Quasiposinormal Operators. *Banach J. Math. Anal.*, Vol.7, No.2, 30–41.

Schmudgen, K. (2021). *Unbounded operator algebras and representation theory.* Birkhauser.

Scholkopf, B., & Smola, A. J. (2023). *Learning with kernels: support vector machines, regularization, optimization, and beyond.* MIT Press.

Shaw, S. Y. (1984). On Numerical Ranges of Generalized Derivations and Related Properties. *Journal of the Australian Mathematical Society*, 36(1), 134–142.

Shapiro, J. (2003). Notes on the Numerical Range. *http://www.mth.msu.edu.*

Shulman, V. S. (2002). Spectral invariance of elementary operators. *Journal of Functional Analysis*, 186(2), 321–335.

Skoufranis, P. (2014). Numerical Ranges of Operators, *Lecture Notes. Department of Mathematics and Statistics, York University*, 34, 93–94.

Stampfli, J. G. (1970). The Norm Derivation, *Pacific Journal of Mathematics*, 33(3), 737–747.

Stampfli, J. G. & Williams, J. P. (1968). Growth Conditions and the Numerical Range in a Banach Algebra, *Tohoku, Math. J.*, Vol. 20, 417–424.

Stampfli, J. G. (1966). Normality and the Numerical Range of an Operator. *1021–1022.*

Sunder, V. S. (2000). Functional Analysis: Spectral Theory, *Institute of Mathematical Sciences.*

Szymanski, K., Weis, S., & Zyczkowski, K. (2018). Classification of Joint Numerical Ranges of Three Hermitian Matrices of Size Three. *Linear Algebra and Its Applications*, 545, 148–173.

Taki, Z., & Moktafi, H. (2025). Maximal numerical range of tensor product of two operators. Linear and Multilinear Algebra, 1-12.

Takesaki, M. (2001). *Theory of Operator Algebras I. Springer Berlin, Heidelberg.*

Takesaki, M. (2019). *Theory of operator algebras II. Springer Science & Business Media Berlin, Heidelberg.*

Taylor, A. (1958). Introduction to Functional Analysis. *Wiley, New York.*

Timotin, D. (2015). Numerical ranges and spectral radius inequalities. *Linear Algebra and its Applications*, 480, 1–10.

Toeplitz, O. (1918). Das algebraische Analogon zu einem Satze von Fejer. *Mathematische Zeitschrift*, 2(1-2), 187–197.

Tomilov, Y., & Muller, V. (2024). A-numerical ranges in semi-Hilbertian spaces. *Journal of Mathematical Analysis and Applications*, 529(1), 127–143.

Tretter, C. (2008). Spectral Theory of Block Operator Matrices and Applications, *Universitat Bern, Imperial College Press, London.*

Tsing, N. K. (1984). The Constrained Bilinear Form and the C-Numerical Range. *Linear Algebra and Its Applications*, 56, 195–206.

Vakhania, N. N., Tarieladze, V. I., & Chobanyan, S. A. (1987). *Probability distributions on Banach spaces.* D. Reidel Publishing Company.

Vahid, S. (Nov 2012). The Gelfand transform on commutative Banach algebras. *Journal of Algebra*, 361, 1–10.

Vasudeva, H. L. (2017). Elements of Hilbert Spaces and Operator Theory, *Springer Nature Singapore Pte Ltd.*

Verma, R. U. (1994). The Numerical Range of Nonlinear Banach Space Operators, *Acta Mathematica Hungarica*, 63(4), 305–312.

Verma, R. U. (1990). Joint Spatial Numerical Range of Tensor Products. *In Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, pp. 201–204. Royal Irish Academy.

Vladimir, M., & Birkhauser, B. (2003). *Spectral theory of operators in Banach spaces.* Birkhauser.

Voigt, J. (2020). Weak*-integrable functions in quantum probability. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 23(2), 1-15.

Walker, J. S. (2001). Fourier Series. *Encyclopedia of Physical Science and Technology*, 167–183.

Westwick, R. (1975). A Theorem on Numerical Range, *Linear and Multilinear Algebra*, 2(4), 311–315.

Weckman, T. (2020). The Cayley Transform and Unbounded Operators. *Journal of Operator Theory*, 83(1), 1–15.

Williams, J. P. (1967). Spectra of Products and Numerical Ranges, *Journal of Mathematical Analysis and Applications* Vol. 17, 214–220.

Yang, Y. (1984). A Note on the Numerical Range of an Operator, *Bull. Korean Math. Soc.*, 21, p. 33.

Yosida, K. (1965). *Functional analysis.* Springer-Verlag.

Yurii, A. (n.d.). *Spectral theory in Banach algebras.* Unpublished lecture notes.

Zahraei, M., & Aghamollaei, G. (2015). Higher Rank Numerical Ranges of Rectangular Matrices. *Ann. Funct. Anal.*, 6(2), 133–142.

Zarantonello, E. H. (1967). The Closure of the Numerical Range Contains the Spectrum, *Pacific Journal of Mathematics*, 22(3), 575–595.

Zhao, H., & Lee, J. (2024). A New Integral Transform for Initial Value Problems. *Journal of Computational and Applied Mathematics*, 418, 114-125.